MA 242 : Partial Differential Equations (August-December, 2018)

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Problem set 2 and QUIZ 2

Submit to me on or before September 17, 2018 by 1 p.m.

1. Consider the minimization problem from calculus of variation:

$$min \int_t^\tau \left(1 + \frac{1}{4}\dot{x}(s)^2\right) ds, \quad t \in [0,1],$$

over a suitable class of functions, say Lipschitz Functions. Here τ is the exit time of $(s, x(s))$ from the region $[0,1] \times [-1,1]$. Define $L(v) := L(t,x,v) = 1 + \frac{1}{4}v^2$, where v is any scalar. Let $x(.)$ be the trajectory with the initial value $x(t) = x$, then the *value function* is defined as

$$u(t,x) = \min\left\{\int_t^\tau L(t, x(s), \dot{x}(s))ds, x(.) \text{ Lipschitz}\right\}.$$

Show that u is given by

$$u(t,x) = \min_v\{(\tau - t)L(v)\},$$

where the minimization over reals. Further show that

$$v^* = \begin{cases} 2 & \text{if} \quad x \geq t \\ 0 & \text{if} \quad |x| < t \\ -2 & \text{if} \quad x \leq -t \end{cases},$$

is a minimizing solution and the corresponding value is given by

$$u(t,x) = \begin{cases} 1 - |x| & \text{if} \quad |x| \geq t \\ 1 - t & \text{if} \quad |x| \leq t \end{cases}$$

Find the differentiable region and show u satisfies the following Hamilton-Jacobi equation wherever it is differentiable:

$$-u_t(t,x) + (u_x(t,x))^2 - 1 = 0,$$

and satisfies conditions

$$u(t,1) = u(t,-1) = 0, t \in [0,1]; u(1,x) = 0, x \in [-1,1].$$

2. a) Consider a Lagrangian $L(q) = 1+\frac{1}{4}|q|^2, q \in \mathbb{R}^n$ and derive the Hamiltonian via the Legendre transformation.
   b) Now define the minimal value

$$u(x,t) = min \left[\int_0^t \left(1 + \frac{1}{4}\dot{w}(s)^2\right) ds + \frac{1}{2}w(0)^2 \right],$$

   where the minimum is taken over all smooth trajectories w satisfying $w(t) = x$. Using Hope-Lax formula find u explicitly, write down Hamilton Jacobi equation, show that u satisfies HJE and find the initial condition.

3. Consider the Burger's equation $u_t + uu_x = 0$ for $x \in \mathbb{R}$, $t > 0$ with the initial condition $u(x,0) = u_0(x)$, where is u_0 is given by

$$u_0(x) = \begin{cases} 1 & \text{if } \ x \le 0 \\ 1 - x & \text{if } \ 0 \le x \le 1 \\ 0 & \text{if } \ x \ge 1 \end{cases}$$

   a) Show that the characteristic curves do not meet till $t = 1$ by constructing it, draw the characteristics and solve the problem for $u(x,t)$ for all $x \in \mathbb{R}$ and $0 < t < 1$.
   b) Construct a curve of discontinuity $s(t)$ for $t \ge 1$ with $s(1) = 1$, define a discontinuous solution $u(x,t)$ for all $x < s(t)$, $x > s(t)$ and $1 \le t$ that satisfies Rankin-Hugnoit condition.