MA 242 : Partial Differential Equations (August-December, 2018)

A. K. Nandakumaran, Department of Mathematics, IISc, Bangalore

Problem set 4

1. (a) Define the fundamental solution

$$\phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}}, & x \in \mathbb{R}^n, \ t > 0 \\ 0, & x \in \mathbb{R}^n, \ t = 0. \end{cases}$$

Show that ϕ satisfies $\phi_t - \Delta\phi = 0, \quad x \in \mathbb{R}^n, \ t > 0$ and

$$\lim_{(x,t)\to(x_0,0)} \phi(x,t) = 0 \ \text{ for } x_0 \neq 0.$$

(b) Show $\displaystyle\int_{\mathbb{R}^n} \phi(x,t)dx = 1 \ \forall t > 0.$

(c) For $\delta > 0$, show that

$$\lim_{t\to 0} \int_{|x-y|>\delta} \phi(x-y,t)dy = 0$$

2. Solve the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} & = 0, \quad x \in \mathbb{R}, \ t > 0 \\ u(x,0) & = f(x) \end{cases}$$

using Fourier transform (assume appropriate assumption on f).

3. Consider,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, \ t > 0$$

Find all the solution of the from $u(x,t) = \dfrac{1}{\sqrt{t}} v\left(\dfrac{x}{2\sqrt{t}}\right)$.

4. Discuss Irreversibility, Infinite speed of propagation and smoothing effects of the heat equation with examples. **(Write 2-pages independently and submit a typed report on or before 05, 2018).**

5. Let $E(x,t,r)$ be the heat ball and $E(1) = E(0,0,1)$. Show that

$$\iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4.$$

Use appropriate transformation to evaluate

$$\iint_{E(r)} \frac{|y|^2}{|s|^2} dyds,$$

where $E(r) = E(0,0,r)$.

6. Define

$$g(t) = \begin{cases} e^{-\frac{1}{t^\alpha}}, & t > 0 \\ 0, & t \leqslant 0, \end{cases}$$

with $\alpha > 1$ and $u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}}{(2k)!} x^{2k}$. Show that this gives infinitely many solutions to heat equation with zero boundary condition.

7. Find a sequence of solutions u_n of the 1-dimensional heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, 2\pi), t > 0 \\ &u(0,t) = u(2\pi, t) = 0, \end{aligned}$$

using variable separable form. Then formally construct a series of solution. Derive the condition so that the series solution also satisfies $u(x,0) = f(x)$.

8. Let u satisfies the heat equation $u_t - \Delta u = 0$ in $\Omega \times (0,T)$ and Γ_T is the parabolic boundary. Then following maximum principle holds,

$$\overline{\sup_{\Omega \times (0,T)}} u(x,t) = \sup_{\Gamma_T} u(x,t).$$

9. Let u satisfies

$$\begin{aligned} u_t - \Delta u &= u, \quad in \quad \Omega \times (0,T), \\ u(x,0) &= 0, \quad for \quad x \in \Omega, \\ u(x,t) &= 0 \quad on \quad \partial\Omega \times [0,T]. \end{aligned}$$

Then, show that $u(x,t) = 0$ in $\Omega \times (0,T)$.

10. Solve the following heat equation

$$\begin{aligned} u_t &= u_{xx}, \quad in \ x > 0, t > 0, \\ u(x,0) &= g(x), \quad for \ x > 0, \\ u(0,t) &= 0, \quad for \ t > 0. \end{aligned}$$

(Hint: use odd extension to write the equation in $\mathbb{R} \times (0, \infty)$).