MA 242 : Partial Differential Equations (August-December, 2018)

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Problem set 5

1. (a) Drive D'Alambert formula for the solution of

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{in } \mathbb{R}. \end{cases} \tag{1}$$

   Explain Domain of dependence, Range of influence, Domain of Determinacy.

   (b) Suppose $\text{supp}(f)$, $\text{supp}(g) \subset [a, b]$. Find the $\text{Supp}(u(\cdot, t_0))$ for any time t_0.

2. Derive the solution for the equation (1) in a semi-infinite string.

3. Apply Duhamel's principle to find the solution formula for

$$\begin{cases} u_{tt} - u_{xx} = h(x), \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{in } \mathbb{R}. \end{cases}$$

4. Solve the problem with two different characteristic speeds c_1 and c_2: that is

$$\begin{cases} \left(\frac{\partial}{\partial t} - c_1 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c_2 \frac{\partial}{\partial x}\right) u = 0, \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x), \; u_t(x,0) = g(x) \quad \text{in } \mathbb{R}. \end{cases}$$

   Study the formula for two different cases $c_1 \neq c_2$ and $c_1 = c_2$ and also derive D'Alambert formula as a special case. See, any of loss of regularity in one-dimensional case.

5. Integrate the wave equation $u_{tt} - u_{xx} = f(x,t)$ in the characteristic triangle $P(x,t)$, $Q(x - ct, 0)$, $R(x+ct, 0)$ to derive a formula for the solution (Hint: You may write $u_{tt} - u_{xx} = (u_t)_t - (u_x)_x$).

6. Solve the wave equation in the first quadrant with non-homogeneous Dirichlet boundary condition ; that is

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad \text{in } (0, \infty) \times (0, \infty) \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{in } \mathbb{R} \\ u(0,t) = h(t), \quad t > 0 \end{cases}$$

   using the general solution. Derive also the formula for $x < ct$ using the parallelogram identity.

7. Solve the above equation with the Neumann non-homogeneous boundary condition, that is $u(0,t) = h(t)$ is replaced by $u_x(0,t) = h(t)$.

8. Let $c_1, \cdots, c_k$ distinct positive real numbers. Show that the solution of the equation
$$(\partial_t^2 - c_1^2\partial_x^2)\cdots(\partial_t^2 - c_k^2\partial_x^2)u = 0,$$
can be written as
$$u(x,t) = \sum_{j=0}^{k} u_j(x,t)$$
where u_j satisfies $\partial_t^2 u_j - c_j^2\partial_x^2 u_j = 0$. The above is also true for n-dimensional case.

9. Let $n = 3$ and consider $(\partial_t^2 - c^2\partial_x^2)(\partial_t^2 - c^2\partial_x^2)u = 0$, $c > 0$. Taking smooth data for $\partial_t^j u(x,0) = f_j(x)$, $j = 0, 1, 2, 3$, write down the solution explicitly.

10. Consider the problem
$$\begin{cases} u_{tt} - \Delta u = 0, \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x) \ \text{ in } \ \mathbb{R}^n. \end{cases} \tag{2}$$
Let u^ϕ be the solution with initial data
$$u^\phi(x,0) = \phi(x), \quad u_t^\phi(x,0) = 0$$
and v^ψ be the solution with initial data
$$v^\psi(x,0) = 0, \quad v_t^\psi(x,0) = \psi(x).$$
Verify that the solution of (2) is given by
$$u(x,t) = u^\phi(x,t) + \int_0^t u^\psi(x,s)ds$$
and also
$$u(x,t) = v^\psi(x,t) + \frac{\partial}{\partial t}v^\phi(x,t).$$

11. Explain the domain of dependence, range of influence for dimension $n = 3$ and $n = 2$ by writing down the formula for the solution. Find $\text{supp}(u(\cdot, t_0))$ if $\text{supp}(f)$, $\text{supp}(g)$ $\subset B(x_0, \rho)$.

12. Derive the Poisson formula for $n = 2$ by the Hadamard's method of decent from the Kirchoff's formula for $n = 3$.

13. Explain Huygen's Principle for $n = 3$.

14. Find the solution
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \ \text{ in } \ (0,\infty)\times(0,\infty) \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad \text{in } \ \mathbb{R} \\ u_x(0,t) = 0, \quad \text{for } \ t > 0 \end{cases}$$
(Hint: Use even extension of g, h)

15. Let Ω is a bounded smooth open subset of $\mathbb{R}^n$. Consider the following wave equation
$$\begin{cases} u_{tt} - \Delta u = f \ \ in \ \ \Omega \times (0,T) \\ u = g \ \ on \ \ \Gamma_T \\ u = h \ \ on \ \ \Omega \times \{t = 0\}. \end{cases}$$
Define the "energy" as
$$e(t) = \int_\Omega u_t^2 + |\nabla u|^2 dx.$$
Using the energy, show that the above wave equation has at most one solution.