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AN INTRODUCTION AND APPLICATIONS TO HOMOGENIZATION

A. K. Nandakumaran
Department of Mathematics,
Indian Institute of Science,
Bangalore – 560 012, INDIA.
Email: nands@math.iisc.ernet.in

Abstract

Homogenization of partial differential equations is relatively a new area and has tremendous applications in various branches of engineering sciences like: material science, porous media, study of vibrations of thin structures, composite materials to name a few. Though the material scientists and others had reasonable idea about the homogenization process, it was lacking a good mathematical theory till early seventies. The first proper mathematical procedure was developed in the seventies and later in the last 30 years or so it has flourished in various ways both application wise and mathematically.

I plan to give an introductory presentation with the aim of catering to a wider audience. We go through few examples to understand homogenization procedure in a general perspective together with applications. We also present certain mathematical techniques available and if possible some details. *A possible definition of homogenization would be that it is a process of understanding a heterogeneous (in–homogeneous) media, where the heterogeneties are at the microscopic level, like in composite materials, by a homogeneous media. In other words, one would like to obtain a homogeneous description of a highly oscillating in–homogeneous media.* We also present other generalizations to non linear problems, porous media and so on. Finally, we will like to see a closely related issue of *optimal bounds* which itself is an independent area of research.

1 INTRODUCTION .

The mathematical theory of homogenization gained its significance in the seventies when rigorous mathematics to understand the procedure was introduced. The main motivation was from the study of composite materials, more generally any medium or domain which involves microstructures. Simply speaking, homogenization is a mathematical procedure to understand *heterogeneous materials (or media)* with *highly oscillating* heterogeneties (at the microscopic level) via a *homogeneous material.* Mathematically, it is a limiting analysis. The physical problems described on such materials leads to the study of mathematical equations like: differential or integral equations, optimization problems, spectral problems, and so on, will exhibit high oscillations in the coefficients present in the equation or in the domain. This high frequency , oscillations, in turn, will reflect in the solutions. Thus, even if the well posedness of the problems were guaranteed, a numerical computation (to predict the behaviour of such heterogeneous media) of such solutions will be highly non-trivial; in fact, it is almost impossible.

The homogenization deals with the study of asymptotic analysis of such solutions and obtain the equation satisfied by the limit. This limit equation will characterize the **bulk/ overall** behaviour of the material, which doesn't consists of microscopic heterogenities and can be solved or computed. This solved and computed solution will then, be a good approximation, in a suitable sense, to the original solution.

Composite Materials: These materials are obtained by **fine** mixing of two or more materials with different physical properties. The study of composite material is an important aspect in material science. The problems modeled on such materials lead to homogenization problems. Composite materials arise plenty in nature: e.g.; *wood, bone, lungs, soil, sand stone, granular media, any porous media* and so on. Moreover, the material scientists/engineers are engaged for many years in constructing composites (e.g. *concrete, reinforced concrete, plywood, steel* etc.) of desired properties. For example, one would like to construct good conducting materials in large quantities in an optimal way by mixing two or materials, where the availability of given good conducting material may be limited. Or one may prefer to develop elastic materials of a combination of contradictory properties, say *stiffness and softness* in different directions or high stiffness and low weight etc. The pore structure of the bonds, trunks of wood, leaves of trees provide examples when mixtures of stiff and soft tissues can be treated as a composite. The honey–comb structures are light and possess a high bending stiffness.

Porous Media: An example of another important microstructure is the porous media where the porosity is at a fine scale. Examples are fluid flow through porous media; like flow of ground water oil, flow of resin in mould industries. We will see the limiting equation depends heavily on the size of the porosity at the micro scale.

Layered materials: like plywood etc. This also can be viewed as a composite with oscillations only in one direction.

Micro-structure of phase transition: The crystal structure of materials changes at a critical temperature while cooling. This happens at the atomic level and the structure moves from *Austenite (high temperature)* state to *Martensite (low temperature)* state.

Analysis of vibrations of thin structures.

The characteristic in all these structures is that locally inhomgeneous material behaves as a homogeneous medium when the size of the inclusions is much smaller than the size of the whole sample. In such a situation the properties of the composite can be described by the *effective modulli* by special kind of averaging of the properties of the components. We will see soon by a one dimensional example that the effective property of the medium is not the arithmetic average of the same properties of the components. *The branch of mathematics that study the effective behaviour of such phenomena is known as homogenization theory.*

Fore more details, we refer the readers to the literature: [4],[9],[20],[7],[11],[1],[14], [22].

2 EXAMPLES

Example 2.1 (One Dimensional case:) *Let $a^\epsilon \in L^\infty(c,d)$. Consider one-dimensional problem:*

$$\begin{cases} -\frac{d}{dx}(a^\epsilon \frac{du^\epsilon}{dx}) = f \quad in\ (c,d) \\ u_\epsilon(0) = u_\epsilon(1) = 0 \end{cases} \tag{2.1}$$

where $f \in L^2(C,d)$ is given.

Assume that

$$0 < \alpha \leq a^\epsilon(x) \leq \beta \quad a \cdot e \cdot x.$$

We set

$$\xi_\epsilon = a^\epsilon(x)\frac{du^\epsilon}{dx},$$

so that

$$-\frac{d\xi_\epsilon}{dx} = f. \tag{2.2}$$

Since a^ϵ is bounded in $L^\infty(c,d)$ and u_ϵ is bounded in $H_0^1(c,d)$, we have $\frac{du^\epsilon}{dx}$ is bounded in $L^2(c,d)$. Thus, ξ_ϵ is bounded in $L^2(c,d)$. We can derive upto a subsequence, $\frac{du^\epsilon}{dx} \rightharpoonup \frac{du}{dx}$ and $\xi_\epsilon \rightharpoonup \xi$ weakly in $L^2(c,d)$ and $a^\epsilon \rightharpoonup a^*$ in $L^\infty(c,d)$ weak*. Hence,we get the equation $-\frac{d\xi}{dx} = f$. It remains to find the relation between ξ and u to complete the analysis.

From the relation between ξ_ϵ, u_ϵ and a^ϵ we may tend to conclude that $\xi = a^*\frac{du_\epsilon}{dx}$. Unfortunately this is not true in general as the weak convergence does not preserve nonlinearities. This is the major issue not only in homogenization problems, but it is an issue in other non linear problems as well.

The present one dimensional situation can be handled with a simple trick. By (2.2), ξ_ϵ is bounded in $H^1(c,d)$. Thus, upto a subsequence, $\xi_\epsilon \rightharpoonup \xi$ weakly in $H^1(c,d)$ and so strongly in $L^2(c,d)$. We may write

$$\frac{du_\epsilon}{dx} = \frac{1}{a^\epsilon}\xi_\epsilon.$$

Note that $\{\frac{1}{a^\epsilon}\}$ is bounded in $L^\infty(c,d)$ and so converges upto a subsequence to some b in $L^\infty(c,d)$ weak*. Now, we can pass to the limit on the RHS as ξ_ϵ having strong convergence to get $\xi = \frac{1}{b}\frac{du_\epsilon}{dx}$. Finally we get the equation

$$\begin{cases} -\frac{d}{dx}(\frac{1}{b}\frac{du}{dx}) &= f \quad in\ (c,d) \\ u(0) = u(1) &= 0. \end{cases}$$

Remark 2.1 *Contrary to intuition, we got the limiting coefficient as $\frac{1}{b}$ and not the intuitive limit a^*. We remark that in general these two quantities are not equal.*

Periodic case: In this situation, one can explicitly compute the limiting coefficient. Periodic case is settled in general case as well, but the proof is different. Let $Y = [0,1]$ and let $a \in L^\infty(Y)$ be a Y-periodic function, i.e. $a(0) = a(1)$ and satisfies the ellipticity and boundedness. Using the scaling map $x \mapsto x/\epsilon$ from $[0,\epsilon] \to [0,1]$ we define $a_\epsilon(x) = a(x/\epsilon)$ on $[0,\epsilon]$ and then extend it periodically to all of $\mathbb{R}$. We continue to denote this function by $a(x/\epsilon)$. Then $a^\epsilon(x) = a(x/\epsilon)$ and $\frac{1}{a^\epsilon(x)} = \frac{1}{a}^\epsilon(x)$ are bounded in $L^\infty(c,d)$.

We have the following general lemma and for a proof see [5].

Lemma 2.1 *Let $a^\epsilon(x) = a(\frac{x}{\epsilon})$ be as above, then $a^\epsilon(x) \rightharpoonup \int_0^1 a(y)dy$ in $L^\infty(c,d)$ weak*.*

From the above lemma for the periodic case, we get $b = \int_0^1 \frac{1}{a(y)}\, dy$. It is clear that in general $\int_0^1 \frac{1}{a(y)}\, dy \neq \frac{1}{\int_0^1 a(y)\, dy}$

Example 2.2 (General Periodic case): *The higher dimensional case is very delicate and as explained earlier, there is no explicit representation for the homogenized coefficients in the general case. However, when there is a periodic structure for the geometry, one can obtain the formula and prove it mathematically.*

Let $a^\epsilon_{i,j}(x) : \mathbb{R}^n \to \mathbb{R}^n$ be ϵ-periodic given by $a^\epsilon_{ij}(x) = a_{ij}(\frac{x}{\epsilon})$ where a_{ij} is defined on the unit cell $Y = [0,1]^n$ as a periodic function, that is it takes the same values on the opposite sides. Then extend a_{ij} periodically to all of $\mathbb{R}^n$ and we denote the extension by a_{ij} itself as it doesn't cause any confusion. Thus $a^\epsilon_{ij}(x)$ gives the material property tensor of the microstructure .

Remark 2.2 *The above Lemma 2.1 holds good in the higher dimensional case as well.*

In this example, as expected the limiting coefficients a^*_{ij} are constants indicating the homogeniety of the limit structure and is given by

$$a^*_{ij} = \frac{1}{|Y|}\int_Y \left[a_{ij}(y) - a_{ik}\frac{\partial \chi^j}{\partial y_k}(y)\right]. \tag{2.3}$$

The unknown functions χ^j for $1 \leq j \leq n$ are obtained by solving n elliptic problems in periodic cell Y as: for fixed k, define χ^k by

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial \chi^k}{\partial y_j}\right) = -\frac{\partial a_{ik}}{\partial y_i}, \quad \chi^k\ Y - \text{periodic}.$$

Equivalently

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial(\chi^k - y_k)}{\partial y_j}\right) = 0, \quad \chi^k\ Y - \text{periodic}.$$

We remark that the functions χ^k actually captures the oscillations in the solutions. This particular point may not be very transparent at this stage and requires further analysis. We do not go through the proof, however we do indicate some of the methods developed to study such problems.

Example 2.3 *(Layered Composite; Rank 1 materials): Indeed these composites can be treated as a special case, but it has its own practical significance. These are created by stacking together, say alternatively thin homogeneous materials of different physical properties. That is the material property changes rapidly only in one direction.*

By a suitable rigid motion, mathematically it amounts to the condition that the coefficients a_{ij} and a^{ϵ}_{ij} depends only on one variable; that is $a_{ij}(y) = a_{ij}(y_1)$. The homogenized coefficient has a very simple representation as follows:

$$\begin{aligned} \frac{1}{a^*_{11}} &= \mathcal{M}\left(\frac{1}{a_{11}}\right) \\ a^*_{1j} &= a^*_{11}\mathcal{M}\left(\frac{a_{1j}}{a_{11}}\right),\ a^*_{j1} = a^*_{11}\mathcal{M}\left(\frac{a_{j1}}{a_{11}}\right), \quad 2 \le j \le n \\ a^*_{1j} &= \frac{a^*_{1i}a^*_{j1}}{a^*_{11}} + \mathcal{M}\left(a_{ij} - \frac{a_{1i}a_{j1}}{a_{11}}\right), \quad 2 \le i,\ j \le n \end{aligned}$$

Here $\mathcal{M}(f)$ represents the average of the function f over the unit cell Y.

3 ELLIPTIC PROBLEMS

We now introduce homogenization problem related to a class of elliptic operators. We begin with the mathematical description and later we will see how such problems arise in literature. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Define, for $\alpha, \beta > 0$, the class of matrix functions:

$$E(\Omega) = E(\alpha,\beta,\Omega) = \{\, A = [a_{ij}(x)] : A \text{ is symmetric and satisfies (3.1)} \,\}.$$

We assume the matrix A satisfies

$$\alpha|\,\xi\,|^2 \le (A(x)\xi,\xi) = a_{ij}\xi_i\xi_j \le \beta|\,\xi\,|^2,\ \forall \xi \in \mathbb{R}^n. \tag{3.1}$$

The first inequality is nothing but the uniform ellipticity.

The aim is to introduce certain convergence in the above class relevant to the homogenization theory. This will become more clear as we go further. Given an element $A \in E(\Omega)$, introduce the elliptic boundary value problem

$$\begin{aligned} \mathcal{A}u &= f \ \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.2}$$

Here $\mathcal{A} = -\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_i})$ is the PDE operator associated to A.

Definition 3.1 *(G– convergence or H– convergence): We say a family$\{[a^\epsilon_{ij}]\}_{\epsilon>0}$, H–converges to $[a^*_{ij}]$ as $\epsilon \to 0$ if*

$$i)\ u_\epsilon \rightharpoonup u \text{ in } H^1_0(\Omega) \text{ weak},$$
$$ii)\ a^\epsilon_{ij}(x)\frac{\partial u_\epsilon}{\partial x_j} \rightharpoonup a^*_{ij}(x)\frac{\partial u}{\partial x_j} \text{ in } L^2(\Omega) \text{ weak}.$$

Here u^ϵ, u are, respectively, the solution of (3.2) corresponding to the operators $\mathcal{A}^\epsilon$, $\mathcal{A}^*$ and we write $[a^\epsilon_{ij}] \overset{H}{\rightharpoonup} [a^*_{ij}]$ or simply $\mathcal{A}^\epsilon \overset{H}{\rightharpoonup} \mathcal{A}^*$. ■

There is a very general compactness theorem (See [9], [11]) which is given below.

Theorem 3.1 *(Compactness Theorem) Let $[a^\epsilon_{ij}]_{\epsilon>0}, \epsilon \to 0$ be any family from $E(\Omega)$. Then there is a subsequence $\epsilon_n \to 0$ and a matrix $[a^*_{ij}] \in E(\Omega)$ such that*

$$[a^\epsilon_{ij}] \overset{H}{\rightharpoonup} [a^*_{ij}].$$

Remark 3.1 *If we do not include the symmetry assumption in the class $E(\Omega)$, one can still have a compactness theorem. In this case we can only conclude that $A^* \in E(\alpha, \frac{\beta^2}{\alpha}, \Omega)$. Observe that $\frac{\beta^2}{\alpha} \geq \alpha$.*

At first sight, from the above theorem, it may seem that the limiting analysis has been over. But, it is far from complete in the sense that, in general, we do not know the characterization of $[a^*_{ij}]$. The general theorem is not very useful as far as applications are concerned as the problems arising from physical and engineering models and needs to calculate the limiting coefficients which represents the physical modelling. However, there are some special cases, where a^*_{ij} can be explicitly characterized. For example, in the periodic case as well as in composites of thin sheets (layered materials). The following two questions are of utmost importance. *i) Either explicitly give methods to evaluate A^* or ii) Give appropriate lower and upper estimates (known as optimal bounds) on the limiting matrix.* Few examples where (i) can be answered will be given soon.

Composite of two materials; A mathematical modelling: Let two homogeneous materials with physical property (say conductivity) γ_1 and γ_2 occupies a fixed domain Ω which is a bounded set in $\mathbb{R}^n$. At a fine mixing of these two materials, it will occupy some parts of Ω which we denote by Ω^ϵ_1 and Ω^ϵ_2 , respectively. Here ϵ represents the micro geometry of the mixing which in general is unavailable or unknown, but fineness is indicated by the smallness of ϵ . The volume ratio of the materials fixed and constant, that is $\theta_1 = \frac{|\Omega^\epsilon_1|}{|\Omega|}$ and $\theta_2 = \frac{|\Omega^\epsilon_2|}{|\Omega|}$ are independent of ϵ . Of course one can think of more complicated problems where these quantities itself may change. The property of the medium at the micro level is then given by

$$\gamma^\epsilon(x) := \gamma_1\chi\Omega^\epsilon_1 + \gamma_2\chi\Omega^\epsilon_2.$$

Here χ_A is the characteristic function of the subset A of Ω. As is well known to the material scientists that for a fine mixing, that is when ϵ is small, the composite may behave like a homogeneous material with certain physical property (conductivity) γ . *The aim is to describe this γ as limit of γ^ϵ in some suitable sense which is precisely the homogenized limit.* In this case the elliptic equation, which is the modelling of various physical phenomena in such a composite, takes the form as in (3.2) with $A = A^\epsilon(x) = \gamma^\epsilon(x)I$.

Remark 3.2 *The two material described has an isotropic microstructure. But I would like to remind that the composite limit need not be isotropic in the sense that A^* need not be of the form γ I. This will be clear from the examples to be studied soon.*

More generally, an elliptic problem in a composite of two or more materials with isotropic or non isotropic can be written in the form as in (3.2), with appropriate boundary conditions.

We end this section with a notion of correctors.

Correctors: The compactness theorem tells us that the solution $u_\epsilon \rightharpoonup u$ in $H_0^1(\Omega)$ weak, which means that $u_\epsilon \rightharpoonup u$ strongly in $L^2(\Omega)$, but the ∇u_ϵ converges weakly in $L^2(\Omega)$. But the weak convergence is not good for numerical computations and the gradient is equally an important physical quantity. In general, the weak convergence cannot be improved and can be seen from the one dimensional example to be presented below. The aim of correctors is to add suitable lower order terms to obtain 'some sort of 'strong approximation to ∇u_ϵ which will be extremely useful in practical applications.

We now briefly describe various methods developed in the last 3 decades.

4 VARIOUS METHODS

4.1 Formal asymptotic expansion:

In any asymptotic problem, the first step is to look for a suitable asymptotic expansion and try to guess the correct limit from the formal analysis. The normal expansion like in any other asymptotic problems is as follows:

$$u_\epsilon(x) = u_0(x) + \epsilon u_1(x) + \dots .$$

Indeed this expansion leads to the anticipated, but wrong answer

$$\begin{cases} -\frac{d}{dx}(a^* \frac{du}{dx}) & = f \quad \text{in } (0,1) \\ u(0) = u(1) & = 0. \end{cases}$$

Keeping the particular problem in mind one looks for:

$$u_\epsilon(x) = u_0(x,y) + \epsilon u_1(x,y) + \dots .$$

where x is the **slow** variable and $y = \frac{x}{\epsilon}$ is the **fast** variable. Then, if possible, see that u_0 is independent of y and obtain the equation satisfied by u_0 ([4], [5]).

4.2 Energy method via test functions:

The idea is to construct suitable test functions having same oscillations as the solutions to control the trouble creating oscillating terms to pass to the limit. In the process, the energy of the original system converges to the energy of the homogenized system ([4], [5]). This was essentially carried out by J. L. Lions for the periodic case.

4.3 Compensated Compactness:

This method, actually, was introduced to pass to the limit in nonlinear problems under weak convergence. We have already remarked in general, we may not be able to conclude the convergence of $u_n v_n$ to uv from the weak convergence of u_n and v_n . This may be due to the oscillations in u_n and v_n and its interactions. But if u_n and v_n oscillate in transverse directions, then the nonlinear functional $u_n v_n$ behaves nicely. For example if u_n and v_n are functions on complementary variables i.e., $u_n = u_n(x')$ and $v_n = v_n(x'')$, where $x = (x', x'')$, then the convergence of $u_n v_n$ to uv can be concluded easily, i.e., one needs a sort of compensation to achieve the compactness. This is the basic motivation of compensated compactness, though the theory is much more involved (Ref: [6],[14],[23]). We state a fundamental lemma towards this direction. If we look at our problem, we see that our interest is in the convergence of $\sigma_i^\epsilon \frac{\partial u_\epsilon}{\partial x_j}$ and $\sigma_i^\epsilon = a_{ij}^\epsilon \frac{\partial u_\epsilon}{\partial x_j}$.

Lemma 4.1 *(Div–Curl Lemma): Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $L^2(\Omega)^N$ weakly. Further assume{ div u_n } and { curl v_n } remains in a compact subset of $H^{-1}(\Omega)$. Then*

$$u_n v_n \rightarrow uv \quad \text{in distribution.}$$

We remark that this can be used to prove H–compactness and the proof involves lot of specific details, in particular periodic case can be derived.

4.4 Gamma Convergence:

This is a variational convergence developed to study optimization problems. Gamma convergence is a very powerful notion introduced in the seventies and have applications in several problems including homogenization problems (Ref: [9]).

4.5 Two Scale (Multi–scale) Convergence:

This was specially introduced for studying homogenization problems. It makes the formal two scale asymptotic analysis mathematically rigorous. The two–scale limit captures the oscillations involved in a weakly convergent sequence. Here I would like to bring to the attention of the readers that rapid oscillations and concentrations are main cause which prevents the weakly convergent sequence to become strongly convergent. In this direction, we state the following theorem due to Nguetseng [18] (see also Allaire [1] and Nandakumaran [16]).

Lemma 4.2 *(Two-scale Convergence): Let $\{u_\epsilon\}$ be a uniformly bounded sequence in $L^2(\Omega)$. Then there is a subsequence of ϵ, denoted again by ϵ, and*

$$u_0 = u_0(x,y) \in L^2(\Omega, L^2_p(Y)).$$

such that

$$\int_\Omega u_\epsilon(x)\psi(x, \frac{x}{\epsilon} \to \int_{\Omega\times Y} u_0(x,y)\psi(x,y)\,dxdy \tag{4.1}$$

as $\epsilon \to 0$, for all $\psi \in C_c(\overline{\Omega}, C_p(Y))$. Moreover

$$\int_\Omega u_\epsilon(x)v(x)w(\frac{x}{\epsilon}) \to \int_{\Omega\times Y} u_0(x,y)v(x)w(y)\,dxdy \tag{4.2}$$

as $\epsilon \to 0, \forall v \in C_c(\overline{\Omega})$ and $\forall w \in L^2_p(Y)$. Further, if u is the L^2 weak limit of u_ϵ, then by taking $w \equiv 1$ in (4.2) we get

$$u(x) = \int_Y u_0(x,y)\,dy. \tag{4.3}$$

Here $L^2_p(Y)$ denotes the space of L^2- periodic functions and $C_p(Y)$ denotes the space of continuous periodic functions on Y.

4.6 Fourier (Bloch wave) method:

The latest addition is the Bloch wave method. Initially, problems from fluid- solid interaction were studied using bloch wave analysis (Ref: [8]). The basic idea is to work in phase space than in the physical space represented by x variable. Essentially one diagonalize the operator A^ϵ and transform the equations $A^\epsilon u^\epsilon = f$ into a sequence of scalar equations without the derivatives.The concept of Fourier decomposition when the medium is homogeneous is that the operator can be diagonalized in the basis of plane waves. In the current periodic situation, one requires Bloch waves.

5 A CAUCHY PROBLEM FOR A NONLINEAR PARABOLIC EQUATION

We now consider a specific initial-boundary value problem

$$\begin{aligned} \partial_t b(\frac{x}{\epsilon}, u_\epsilon) - \text{div}\, a(\frac{x}{\epsilon}, u_\epsilon, \nabla u_\epsilon) &= f(x,t) \quad \text{in } \Omega_T, \\ a(\frac{x}{\epsilon}, u_\epsilon, \nabla u_\epsilon).\nu &= 0 \quad \text{on } \Gamma_{2,T}, \\ u_\epsilon &= g \quad \text{on } \Gamma_{1,T}, \\ u_\epsilon(x,0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{5.1}$$

whose diffusion term is a monotone operator. Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary and let $T > 0$ be a constant. Let $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where it is assumed that Γ_1 has positive Hausdorff measure, $H^{n-1}(\Gamma_1)$. We will denote $\Omega \times [0,T]$ by Ω_T, and

$\Gamma_i \times [0,T]$ by $\Gamma_{i,T}, i = 1, 2$.

Physical Applications in:

- non steady filtrations
- heat propagation in composite materials at high temperature
- fast or slow diffusion through a non homogeneous medium etc.

Prototype of $b : b(s) = |s|^k \operatorname{sgn}(s),\ \ b(y,s) = c(y)|s|^k \operatorname{sgn}(s)$

For simplicity, we take the zero Dirichlet boundary condition.

Weak formulation gives the solution $u_\epsilon \in E := L^p(0,T;V)$, where $V = \{\, v \in W^{1,p}(\Omega) : v = 0 \ \text{ on } \Gamma_1 \,\}$ and let V^* be the dual of V. Moreover

$$b(\frac{x}{\epsilon}, u_\epsilon) \in L^\infty(0,T;L^1(\Omega)), \text{and}$$
$$\partial_t b(\frac{x}{\epsilon}, u_\epsilon) \in L^{p^*}(0,T;V^*).$$

We now state the homogenized equation and the main theorem whose details may be found in [17]

Homogenized Equation and the Main Theorem

Let u_ϵ be a family of solutions of (P_ϵ). Assume that there is a constant $C > 0$, such that

$$\sup_\epsilon \|u_\epsilon\|_{L_\infty(\Omega_T)} \le C. \tag{5.2}$$

Under the assumptions $(A1) - (A4)$, there exists a subsequence of ϵ , still denoted by ϵ , such that for all q with $0 < q < \infty$, we have,

$$\begin{aligned}
&u_\epsilon \to u \ \text{ strongly in } L^q(\Omega_T) \\
&\nabla u_\epsilon \rightharpoonup \nabla u \ \text{ weakly in } L^p(\Omega_T) \\
&b(\frac{x}{\epsilon}, u_\epsilon) - b(\frac{x}{\epsilon}, u) \to 0 \ \text{ strongly in } \ L^q(\Omega_T) \\
&b(\frac{x}{\epsilon}, u_\epsilon) - \bar{b}(u) \ \text{ weakly in } L^q(\Omega_T) \text{ for } q > 1,
\end{aligned}$$

and u solves,

$$\begin{aligned}
\partial_t \bar{b}(u) - \operatorname{div} A(u, \nabla u) = f \ \text{ in } \Omega_T, \\
A(u, \nabla u).\nu = 0 \ \text{ on } \Gamma_{2,T}, \\
u = g \text{ on } \Gamma_{1,T}, \\
u(x,0) = u_0 \ \text{ in } \Omega.
\end{aligned}$$

The assumption (5.2) is true in special cases and it is reasonable on physical grounds. The functions $\bar{b}$ and A are defined by .

$$\bar{b}(s) = \int_Y b(y,s)\, dy$$

and for $\mu \in \mathbf{R}, \lambda \in \mathbf{R}^n$,

$$A(\mu,\lambda) = \int_Y a(y,\mu,\lambda + \nabla\Phi_{\mu,\lambda}(y))\,dy,$$

where $\Phi_{\mu,\lambda} \in W^{1,p}_{per}(Y)$ solves the periodic boundary value problem

$$\int_Y a(y,\mu,\lambda + \nabla\Phi_{\mu,\lambda}(Y)).\nabla\phi(y)\,dy = 0$$

for all $\phi \in W^{1,p}_{per}(Y)$. Here $Y = (0,1)^n$.

Correctors: let $U_1 \in L^p(\Omega_T; W^{1,p}_{per}(Y))$ be the solution of the variational problem,

$$\int_{\Omega_T}\int_Y a(y,u,\nabla_x u + \nabla_y U_1(x,t,y)). \nabla_y\psi(x,t,y) = 0,$$

for all $\Psi \in L^p(\Omega_T; W^{1,p}_{per}(Y))$. If u, U_1 are sufficiently smooth, i.e. belong to $C^1(\Omega_T)$ and $C(\Omega_T; C^1_{per}(Y))$ then

$$u_\epsilon - u - \epsilon U_1(x,t,\frac{x}{\epsilon}) \to 0$$

and

$$\nabla u_\epsilon - \nabla u - \nabla_y U_1(x,t,\frac{x}{\epsilon}) \to 0,$$

strongly in $L^p(\Omega_T)$.

We have improved the convergence of ∇u_ϵ by adding the corrector term. If U_1 were to be differentiable in x, then $\nabla(u - \epsilon\nabla U_1(x,t,\frac{x}{\epsilon}))$ would approximate ∇u_ϵ.

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