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STEADY AND EVOLUTION STOKES EQUATIONS IN A POROUS MEDIA WITH NON-HOMOGENEOUS BOUNDARY DATA: A HOMOGENIZATION PROCESS

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Abstract. In this paper, we study the homogenization of the steady state and evolution Stokes equations with nonhomogeneous Dirichlet data on the boundary of the holes of a porous media Ω_ε, obtained from a domain Ω by removing a large number of holes of size ε ($\varepsilon > 0$, a small parameter), periodically distributed with period ε. In the homogenization process, we obtain a well defined system of equations involving both the 'slow' variable x and the 'fast' variable $y = \frac{x}{\varepsilon}$. We also derive the Darcy's law which contains an extra term and this additional term is the contribution due to the non-homogeneous data.

1. Introduction and the problem to be studied. We consider the steady state and evolution Stokes equation in a porous domain Ω_ε which is obtained from a domain Ω by removing a large number of holes of size ε (a small positive parameter) periodically distributed in the domain with period ε. We study the homogenization of the Stokes system with non-homogeneous Dirichlet condition on the boundary of the holes.

First we introduce the standard notations and then formulate the problems to be treated in this paper.

Notations. Let $Y = (0,1)^N$, $N \geq 2$, and T be an open set strictly contained in Y with smooth boundary S (the boundary S is a smooth manifold of dimension $N-1$) and $Y^* = Y \setminus \overline{T}$. Let $k \in \mathbf{Z}^N$, where $\mathbf{Z}$ is the set of all integers, and let

$$Y_k = Y + k, \quad T_k = T + k, \quad Y_k^* = Y^* + k, \quad S_k = S + k = \partial T_k.$$

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary Γ. Let $\varepsilon > 0$ be a small positive parameter. Consider the index sets

$$I_\varepsilon = \left\{k \in \mathbf{Z}^N : \varepsilon Y_k \subset \Omega\right\} \quad \text{and } J_\varepsilon = \left\{k \in \mathbf{Z}^N : \varepsilon Y_k \cap \Gamma \neq \emptyset\right\}.$$

Loosely speaking, $\{\varepsilon T_k, k \in I_\varepsilon\}$ are interior holes and $\{\varepsilon T_k : k \in J_\varepsilon\}$ are boundary holes and then define the perforations in Ω as follows:

$$T_\varepsilon = \bigcup_{k \in I_\varepsilon} \varepsilon T_k, \quad S_\varepsilon = \partial T_\varepsilon = \bigcup_{k \in I_\varepsilon} \partial\left(\varepsilon T_k\right).$$

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Now consider the perforated domain Ω_ε given by

$$\Omega_\varepsilon = \Omega \backslash \overline{T}_\varepsilon.$$

More precisely, Ω_ε is the set obtained from Ω by removing all holes of size ε from those cells εY_k completely contained in Ω. Note that we do not remove the holes intersecting the boundary Γ. Then

$$\overline{\Omega}_\varepsilon = \bigcup_{k\in I_\varepsilon} \varepsilon \overline{Y}_k^* \bigcup_{k\in J_\varepsilon} \left(\varepsilon \overline{Y}_k \cap \overline{\Omega}\right), \quad \partial\Omega_\varepsilon = \Gamma \cup S_\varepsilon.$$

The domain Ω_ε can be thought of as the part occupied by the fluid.

We also fix up notations for defining the evolution Stokes equations. Let $T > 0$ be any positive number. Let $\Omega_T = \Omega \times (0,T)$ and $\Omega_{\varepsilon T} = \Omega_\varepsilon \times (0,T)$, $\Gamma_T = \Gamma \times (0,T)$, $S_{\varepsilon T} = S_\varepsilon \times (0,T)$.

In addition to the standard Sobolev spaces L^2, H^1, H_o^1 etc., we also consider the following spaces: $H_p^1(Y)$ (resp. $C_p(Y)$) are $H^1(Y)$ (resp. $C(Y)$, the space of all continuous functions) functions which are Y-periodic and $L_p^2(Y)$ is the class of all functions in $L^2_{\text{loc}}(\mathbf{R}^N)$ which are Y-periodic. For any Banach space X and for any domain D, define the spaces $L^2(D,X)$, $L^\infty(D,X)$ and $C(\overline{D},X)$ as the set of all functions $f : D \to X$ which are square integrable, essentially bounded and continuous, respectively, and which are Banach spaces under the obvious norms. We denote by $\|\cdot\|_{\infty,2,\Omega}$ the norm given by

$$\|f\|^2_{\infty,2,\Omega} = \operatorname*{ess.\,sup}_{0\le t\le T} \int_\Omega |f(x,t)|^2\,dx.$$

Also, let $C_C(\overline{\Omega})$ be the set of all continuous functions with compact support in $\overline{\Omega}$ and $C_C(\overline{\Omega}, C_p(Y))$ be the functions $\psi : \overline{\Omega} \to C_p(Y)$ with continuous and compact support in $\overline{\Omega}$ and taking values in $C_p(Y)$.

Problem formulation. First, we consider the steady Stokes equation. We look for the velocity $v_\varepsilon = (v_{\varepsilon 1}, \cdots v_{\varepsilon N}) \in H^1(\Omega_\varepsilon)^N$ and the pressure $p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbf{R}$ satisfying the equations

$$\begin{aligned} &\text{i)} \quad -\Delta v_\varepsilon + \nabla p_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ &\text{ii)} \quad \text{div } v_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ &\text{iii)} \quad v_\varepsilon = 0 \quad \text{on } \Gamma, \\ &\text{iv)} \quad v_\varepsilon = g^\epsilon \quad \text{on } S_\varepsilon. \end{aligned} \tag{1.1}$$

Here g^ϵ is given in the following way. Let $g \in H^1(Y)^N$ and g be Y-periodic and satisfy the compatibility condition

$$\int_S g\cdot\nu = 0, \tag{1.2a}$$

where ν is the outward unit normal to S. Then we extend g to all of $\mathbf{R}^N$ in a periodic manner and define $g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right)$. The sequence $\{f_\varepsilon\}$ is from $L^2(\Omega)$ such that

$$\varepsilon^2 f_\varepsilon \to f \quad \text{in } L^2(\Omega) \quad \text{strong.} \tag{1.2b}$$

For the variational formulation and the existence of solutions, one can refer to Temam [9]. We also introduce the evolution Stokes equation. We look for $v_\varepsilon(x,t) = (v_{\varepsilon_1}, \dots, v_{\varepsilon_N})$ and $p_\varepsilon(x,t)$ such that

$$\begin{aligned} &\text{i)} \quad \frac{\partial v_\varepsilon}{\partial t} - \Delta v_\varepsilon + \nabla p_\varepsilon = f_\varepsilon \quad \text{in } \Omega_{\varepsilon T}, \\ &\text{ii)} \quad \operatorname{div} v_\varepsilon = 0 \quad \text{in } \Omega_{\varepsilon T}, \\ &\text{iii)} \quad v_\varepsilon = 0 \quad \text{on } \Gamma_T, \\ &\text{iv)} \quad v_\varepsilon = g^\varepsilon \quad \text{on } S_{\varepsilon T}, \end{aligned} \tag{1.3}$$

and the initial condition

$$\text{v)} \quad v_\varepsilon(x,0) = v_o(x) \quad \text{in } \Omega_\varepsilon.$$

Here $g\colon Y \times (0,T) \to \mathbf{R}$ and is Y-periodic with the compatibility condition

$$\int_S g(\cdot,t)\nu \, ds = 0 \quad \text{for a.e. } t \in (0,T). \tag{1.4a}$$

We also assume g is smooth; i.e., g, $\frac{\partial g}{\partial t} \in C([0,T], L^2(Y))$ and define $g^\varepsilon(x,t) = g\left(\frac{x}{\varepsilon}, t\right)$. Also, $\{f_\varepsilon\}$ is a sequence from $L^2(\Omega_T)$ such that

$$\varepsilon^2 f_\varepsilon \to f \quad \text{in } L^2(\Omega_T) \quad \text{strong.} \tag{1.4b}$$

The initial data v_o satisfies

$$\operatorname{div} v_o = 0 \quad \text{in } \Omega. \tag{1.4c}$$

Our aim in this paper is to study the behaviour of the solutions of the problem (1.1) and (1.3) as $\varepsilon \to 0$. More precisely, we study the asymptotic behaviour of v_ε and p_ε as $\varepsilon \to 0$. We use the methods from the theory of homogenization for studying the problems (1.1) and (1.3). The homogenization process for the Stokes equation with zero Dirichlet condition on the boundary (i.e., the problem (1.1) with $g \equiv 0$) has been studied by Tartar [8] and the proof of convergence can be seen in [8] (see also [7]). The same problem with zero Dirichlet condition on the boundary of the holes, but with non-homogeneous data on the outerboundary Γ of Ω, has been studied by Mikelic [4], Mikelic-Aganovic [5] and the proof is essentially the same as in [8]. Later, we compare our results with the results from [8], [4] and [5]. See Remark 2.1 in §2.

The first step in our problem is that we have to transform the problem (1.1) to another Stokes system in $(u_\varepsilon, p_\varepsilon)$ in which the solution u_ε will satisfy the homogeneous Dirichlet condition on the boundary of the holes. This can be achieved using a lemma (See Lemma 3.4) which we will be proving in §3, and this lemma is the main content of §3, in addition to a few preliminary lemmas. After this transformation, we study the behaviour of the solutions of the transformed problem as $\varepsilon \to 0$. Using a multi-scale expansion for u_ε involving both the slow variable x and the fast variable $y = \frac{x}{\varepsilon}$, one can obtain a well defined set of equations for the first term $u_o(x,y)$. This system admits a unique solution. The interesting aspect of our analysis is that we obtain the system satisfied by u_o not only via the formal asymptotic expansion, but also in a rigorous way (see Remark 2.2). We use certain convergence results given by Nguetseng [6] to achieve this. We also show that the weak limit $u(x)$ of $u_\varepsilon(x)$ is the averaged value of $u_o(x,y)$ with respect to y. So we can obtain the weak limit without the apriori knowledge of any test functions, which is the main ingredient in the energy method. Finally, we derive the Darcy's law using the test functions and the system satisfied by u_o. The Darcy's law contains an extra term which is the contribution due to the non-homogeneous data.

In §4, we transform the problem (1.1), and §5 is devoted to study the homogenization process. In the remaining Sections 6 and 7, we study the evolution case.

Before completing this introduction, we cite few references regarding the theory of homogenization. The general references are the books by Bensoussan-Lions-Papanicolaou [2], Attouch [1], Sanchez-Palencia [7], J.L. Lions [3] etc. For more references, one can refer to any one of these books.

2. Main results. In this section, we present the main results of this paper. First, we define the following problems.

Let e_k be the k^{th} unit vector in the canonical basis of $\mathbf{R}^N$ and v^k, q^k be the unique solution of the following problem:

$$\begin{aligned} &\text{i)} \quad -\Delta v^k + \nabla q^k = e_k \quad \text{in } Y^* \\ &\text{ii)} \quad \operatorname{div} v^k = 0 \quad \text{in } Y^*, \quad v^k = 0 \text{ on } S \\ &\text{iii)} \quad v^k, q^k \quad \text{are } Y\text{-periodic.} \end{aligned} \tag{2.1}$$

Now put

$$K_{ij} = \int_{Y^*} v_i^j(y)\,dy. \tag{2.2}$$

The problem (2.1) has a unique solution such that

$$\|v^k\|_{H^1(Y^*)} \le C \text{ and } \|q^k\|_{L^2(Y^*)} \le C. \tag{2.3}$$

The matrix $[K_{ij}]$ is symmetric and positive definite. For these results, the reader can refer to [7].

Now, we define a system in the domain $\Omega \times Y^*$. Let $v_o = v_o(x,y)$, $p = p(x)$, $p_1 = p_1(x,y)$ be the unique solution of the following system:

$$\begin{aligned} &\text{i)} \quad -\Delta_y v_o + \nabla_y p_1 + \nabla_x p = f \quad \text{in } \Omega \times Y^* \\ &\text{ii)} \quad v_o, p_1 \text{ are } Y\text{-periodic}, \; v_o = g \quad \text{on } S \\ &\text{iii)} \quad \operatorname{div}_y v_o = 0 \quad \text{in } \Omega \times Y^*, \quad \operatorname{div}_x \int_{Y^*} v_o(x,y) = 0 \quad \text{in } \Omega, \\ &\text{iv)} \quad \nu_x \cdot \int_{Y^*} v_o(x,y)\,dy = -\nu_x \cdot \int_S (g \cdot \nu_y) y\,ds \quad \text{on } \Gamma. \end{aligned} \tag{2.4}$$

Here ν_x, ν_y, are, respectively, the outward unit normal to Γ and S. The non-homogeneous terms f and g are given by (1.2).

The above problem (2.4) has a unique solution v_o, p_o, p_1 (p_o is unique up to an additive constant and p_1 is unique upto an additive function of x). The existence and uniqueness of the problem (2.4) with homogeneous boundary data, i.e., with $g = 0$, has been studied in Lions [3]. From this, one can immediately prove the uniqueness of our problem (2.4) because if v_o^1 and v_o^2 are two solutions of (2.4), then $v_o^1 - v_0^2$ is the unique solution of the problem (2.4) with $f = 0$, $g = 0$ and, hence, $v_o^1 - v_o^2 \equiv 0$. See the remark 5.1 for the existence result.

Now, we are in a position to state the main results.

Theorem 2.1. *Let v_ε, p_ε be the solution of the system* (1.1). *Then there exist extensions $\tilde{v}_\varepsilon$, $\tilde{p}_\varepsilon$ of v_ε, p_ε, respectively, such that*

$$\tilde{v}_\varepsilon \to v(x) \quad \text{in } L^2(\Omega) \quad \text{weak,} \tag{2.5}$$

$$\varepsilon^2 \tilde{p}_\varepsilon \to p(x) \quad \text{in } L^2(\Omega)/\mathbf{R} \quad \text{strong,} \tag{2.6}$$

$$v(x) = \int_{Y^*} v_o(x,y)\,dy + \int_S (g \cdot \nu_y)\, y\, ds. \tag{2.7}$$

Here v_o and p are given by the system (2.4)

Theorem 2.2. *Let v and p be as in Theorem* 2.1. *Then v and p are given by the unique solution of the following system:*

$$\begin{aligned} &\text{i)} \quad \operatorname{div} v = 0 \quad \text{in } \Omega \\ &\text{ii)} \quad v = K\,(f - \nabla p) + c \quad \text{in } \Omega \\ &\text{iii)} \quad v \cdot \nu_x = 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.8}$$

Here $K = [K_i j]$ is the matrix given by (2.2) *and c is the contant vector given by*

$$c_k = \int_S (g \cdot \frac{\partial v^k}{\partial \nu_y} - (g \cdot \nu_y)\, q^k) + \int_S (g \cdot \nu_y)\, y_k \tag{2.9}$$

System (2.8) has a unique solution since $[K_{ij}]$ is a symmetric positive definite matrix. The system (2.8) is referred to as Darcy's law. $\frac{\partial}{\partial \nu_y}$ is the normal derivative at S.

We also state the main theorem for the evolution Stokes equation.

Theorem 2.3. *Let v_ε and p_ε be the solution of the system* (1.3). *Then there exist extensions $\tilde{v}_\varepsilon$ and $\tilde{p}_\varepsilon$ of v_ε and p_ε, respectively, such that*

$$\begin{aligned} &\text{i)} \quad \tilde{v}_\varepsilon \to v \quad \text{in } L^2(\Omega_T) \quad \text{weak,} \\ &\text{ii)} \quad \varepsilon^2 \tilde{p}_\varepsilon \to p \quad \text{in } L^2\left(0,T,L^2(\Omega)\right) \quad \text{weak} \end{aligned} \tag{2.10}$$

and u, p is the unique solution of the following system:

$$\begin{aligned} &\text{i)} \quad \operatorname{div} v = 0 \quad \text{in } \Omega_T \\ &\text{ii)} \quad v = K\,(f - \nabla p) + \beta(t) \quad \text{in } \Omega_T \\ &\text{iii)} \quad v \cdot \nu_x = 0 \quad \text{on } \Gamma, \quad \text{a.e. } t \in [0,T] \end{aligned} \tag{2.11}$$

Here $\beta(t)$ is the vector given by

$$\beta_k(t) = \int_S (g \cdot \frac{\partial v^k}{\partial \nu_y} - g \cdot \nu_y q^k) + \int_S (g \cdot \nu_y)\, y_k \tag{2.12}$$

The proof of these results will be presented in §5. As a last part of this section, we make few remarks.

Remark 2.1. The problem (1.1) with homogeneous boundary data ($g \equiv 0$) has been studied by Tartar [8] and the weak limit is the solution of the system (2.8), but with $C = 0$. Similarly the problem (1.1) with non-homogeneous data on the outer boundary Γ (zero condition on the holes) has also been studied by Mikelic [4], Mikelic-Aganovic [5] and the resulting system is the same as above. In this case, boundary value of $u \cdot \nu$ on Γ is non-homogeneous, but without the extra term C in the equation (2.8, ii). In our case, the extra term C is the contribution due to the homogeneous boundary data.

Remark 2.2. Applying the two scale multi expansion for v_ε and comparing the terms, at least formally, it is possible to obtain the system (2.4). Also, Theorem 2.2 can be proved using the energy method with the help of the test functions v^k, q^k given by (2.1). But in this paper, we prove Theorem 2.1 and then derive Theorem 2.2 as a corollary of Theorem 2.1. As far as our problem is concerned, there is not much difference in either way of proving the result because it leads to the same results. The motivation behind doing this is that we can derive the system (2.4) (i.e., the system satisfied by the first term of the asymptotic expansion) without the apriori knowledge of the test functions and the weak limit v can be obtained as the average of the solution v_o with respect to y. Perhaps, it may be useful to study other problems when there is no apriori knowledge of test functions to be used.

A. Steady state Stokes equation.

3. Preliminary lemmas. In this section, we state and prove the crucial lemma which is used to transform the problem (1.1) to a problem with homogeneous condition on the boundary. Before that, we recall few lemmas by Tartar ([8], [7]) which we use to extend the pressure p_ε to all of Ω.

Lemma 3.1 (Tartar [8]). *The constant of the Poincaré-Friedrich's inequality in Ω_ε is of order ε^2; i.e., there exists constant C independent of ε such that*

$$\int_{\Omega_\varepsilon} |u|^2 \leq C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u|^2, \quad \forall u \in H_o^1(\Omega_\varepsilon). \tag{3.1}$$

Lemma 3.2 (Tartar [8]). *There exists an operator R: $H^1(Y) \to H^1(Y^*)$ such that*

$$\begin{aligned} &\text{i)} \quad Rw = w \quad \text{in a neighbourhood of } \partial Y, \\ &\text{ii)} \quad Rw = 0 \quad \text{on } S, \\ &\text{iii)} \quad w = 0 \quad \text{on } S \Rightarrow Rw = w \quad \text{in } Y^*, \\ &\text{iv)} \quad \operatorname{div} w = 0 \quad \text{in } Y \Rightarrow \operatorname{div} Rw = 0 \quad \text{in } Y^*, \\ &\text{v)} \quad \|Rw\|_{H^1(Y^*)} \leq C\|w\|_{H^1(Y)}, \quad \forall w \in H^1(Y). \end{aligned} \tag{3.2}$$

Now, for $w \in H^1(\Omega)$, define $w^\varepsilon(y) = w(\varepsilon y)$ if $y \in Y_k$, $k \in I_\varepsilon$ and define R_ε as follows:

$$(R_\varepsilon w)(x) = \begin{cases} (Rw^\varepsilon)\left(\frac{x}{\varepsilon}\right) & \text{if } x \in \varepsilon Y_k, \quad k \in I_\varepsilon \\ w, & \text{if } x \in \varepsilon Y_k, \quad k \in J_\varepsilon. \end{cases}$$

Then R_ε satisfies the following.

Lemma 3.3 (Tartar [8]). *There exists an operator $R_\varepsilon : H_o^1(\Omega) \to H_o^1(\Omega_\varepsilon)$ such that*

$$\begin{aligned} &\text{i)} \quad w = 0 \text{ on } S_\varepsilon \Rightarrow R_\varepsilon w = w \text{ in } \Omega_\varepsilon, \\ &\text{ii)} \quad \text{div } w = 0 \text{ in } \Omega \Rightarrow \text{div } R_\varepsilon w = 0 \text{ in } \Omega_\varepsilon, \\ &\text{iii)} \quad \|R_\varepsilon w\|_{L^2(\Omega_\varepsilon)} \le C\left(\|w\|_{L^2(\Omega)} + \varepsilon\|\nabla w\|_{L^2(\Omega)}\right), \forall w \in H_o^1(\Omega), \\ &\text{iv)} \quad \|\nabla(R_\varepsilon w)\|_{L^2(\Omega_\varepsilon)} \le C\big(\frac{1}{\varepsilon}\|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}\big), \ \forall w \in H_o^1(\Omega), \end{aligned} \tag{3.3}$$

where C is a constant independent of ε.

Now, we state and prove the crucial lemma. Using this result, we transfer our problem (1.1) to a problem with homogeneous boundary condition on the holes.

Lemma 3.4. *There exists an operator $Q : H_p^1(Y) \to H_o^1(Y)$ such that*

$$\begin{aligned} &\text{i)} \quad Qw = 0 \text{ in a neighbourhood of } \partial Y, \text{ neighbourhood being independent of } w, \\ &\text{ii)} \quad Qw = w \text{ in a neighbourhood of } T, \text{ neighbourhood being independent of } w, \\ &\text{iii)} \quad \text{div } w = 0 \text{ in } Y \Rightarrow \text{div } Qw = 0 \text{ in } Y, \\ &\text{iv)} \quad \|Qw\|_{H^1(Y)} \le C\|w\|_{H^1(Y)}, \forall w \in H_p^1(Y). \end{aligned} \tag{3.4}$$

Proof: Consider two smooth non intersecting hyper-surfaces γ_1 and γ_2 in Y^* such that γ_2 contains γ_1 which in turn contains S. Let A_1 be the region between γ_1 and S and A_2 be the region between γ_1 and γ_2. Let $Y^{**} = Y \setminus (\overline{T} \cup \overline{A}_1 \cup \overline{A}_2)$. Take any $w \in H_p^1(Y)$. Let v and q be the unique solution of the following problem:

$$\begin{aligned} &-\nabla v + \Delta q = -\Delta w \quad \text{in } A_2 \\ &\text{div } v = \text{ div } w + \frac{1}{|A_2|}\int_{Y^{**}} \text{div } w \quad \text{in } A_2 \\ &v|_{\gamma_1} = w|_{\gamma_1}, \quad v|_{\gamma_2} = 0. \end{aligned} \tag{3.5}$$

Here $|A_2|$ =volume of A_2. The problem (3.5) has a unique solution.

We express v in the form $v = \alpha + \beta + \tilde{v}$ where α, β, $\tilde{v}$ are defined as follows. First, we choose $\alpha \in H^1(A_2)^N$ such that $\|\alpha\|_{H^1(A_2)} \le C\|w\|_{H^1(Y)}$ and $\alpha|_{\gamma_1} = w|_{\gamma_1}$ and $\alpha|_{\gamma_2} = 0$ which exists by standard trace properties.

Now, define β as the solution of

$$\begin{aligned} &\text{i)} \quad \text{div } \beta = -\text{div } \alpha + \text{div } w + \frac{1}{|A_2|}\int_{Y^{**}} \text{div } w \equiv F(y) \text{ in } A_2 \\ &\text{ii)} \quad \beta \in H_o^1(A_2)^N \text{ and } \|\beta\|_{H^1(A_2)} \le C\|F\|_{L^2(A_2)}. \end{aligned} \tag{3.6}$$

The problem (3.6) has a solution since the compatibility condition $\int_{A_2} F(y)\,dy = 0$ is satisfied. For

$$\begin{aligned} \int_{A_2} F(y)\,dy &= -\int_{\gamma_1\cup\gamma_2} \alpha\cdot\nu + \int_{\gamma_1\cup\gamma_2} w\cdot\nu + \Big(\frac{1}{|A_2|}\int_{Y^{**}} \text{div } w\Big)|A_2| \\ &= -\int_{\gamma_1} w\cdot\nu + \int_{\gamma_1} w\cdot\nu + \int_{\gamma_2} w\cdot\nu + \int_{\gamma_2} w\cdot(-\nu) = 0. \end{aligned}$$

Now, $\tilde{v} = v - \alpha - \beta$ satisfies

$$\begin{aligned} &-\Delta\tilde{v} + \nabla q = -\Delta(w - \alpha - \beta) \quad \text{in } A_2, \\ &\text{div } \tilde{v} = 0 \quad \text{in } A_2, \\ &\tilde{v} \in H_o^1(A_2). \end{aligned} \tag{3.7}$$

This has a unique solution $\tilde{v}$, q and satisfies $\|\tilde{v}\|_{H^1(A_2)} \leq C\|w\|_{H^1(Y)}$ and, hence, there is a unique solution v for the problem (3.6). Obviously, if div $w = 0$ then div $v = 0$. Now, we define Q as follows:

$$(Qw)(y) = \begin{cases} w(y) & \text{if } y \in T \cup A_1 \\ v(y) & \text{if } y \in A_2 \\ 0 & \text{if } y \in Y^{**} \end{cases} \tag{3.8}$$

for all $w \in H_p^1(Y)$. Then Q satisfies (3.4) and the proof of Lemma 3.4 is complete.

Suppose $\{\psi_\varepsilon\}$ is a sequence from $H_o^1(\Omega)$ and $\psi_\varepsilon \to \psi$ in $L^2(\Omega)$ weak, then, in general, we cannot conclude anything about the value of ψ on the boundary Γ of Ω. But if div $\psi_\varepsilon = 0$, then we have the following result which is trivial.

Lemma 3.5. *Let $\{\psi_\varepsilon\}$ be a family from $H_o^1(\Omega)$ such that* div $\psi_\varepsilon = 0$ *in Ω and suppose that $\psi_\varepsilon \to \psi$ in $L^2(\Omega)$ weak. Then $\psi \cdot \nu = 0$ on Γ.*

We state another lemma from a recent paper by G. Nguetseng (see Theorem 2 in [6]) which we use to obtain the limit equation in both variables x and y. Roughly speaking, it says that weak limit in $L^2(\Omega)$ of any sequence u_ε is the weak limit of a sequence of the form $u_o\left(x, \frac{x}{\varepsilon}\right)$ for some $u_o = u_o(x, y)$.

Lemma 3.6 (Nguetseng [6]). *Let $\{u_\varepsilon\}$ be a sequence in $L^2(\Omega)$. Suppose that there exists a constant $C > 0$ such that*

$$\|u_\varepsilon\|_{L^2(\Omega)} \leq C, \quad \forall \varepsilon.$$

Then there is a subsequence of ε, denoted again by ε, and

$$u_o = u_o(x, y) \in L^2\left(\Omega, L_p^2(Y)\right)$$

such that

$$\int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_{\Omega\times Y} u_o(x, y)\psi(x, y)\, dx\, dy \tag{3.9}$$

as $\varepsilon \to 0$, for all $\psi \in C_c\left(\overline{\Omega}, C_p(Y)\right)$. Moreover,

$$\int_\Omega u_\varepsilon(x)v(x)w\left(\frac{x}{\varepsilon}\right) dx \to \int_{\Omega\times Y} u_o(x, y)v(x)w(y)\, dxdy \tag{3.10}$$

as $\varepsilon \to 0$, for all $v \in C_c(\overline{\Omega})$ and $w \in L_p^2(Y)$. Further, if u is the L^2-weak limit of u_ε, then by taking $w = 1$ in (3.10) *we get*

$$u(x) = \int_Y u_o(x, y)\, dy. \tag{3.11}$$

We close this section by proving the following lemma.

Lemma 3.7. *Let $w \in H^1_p(Y)^N$ such that $\int_S w \cdot \nu = 0$; then there exists $\overline{w} \in H^1_p(Y)^N$ such that*

$$\begin{aligned} &\text{i)} \quad \operatorname{div} \overline{w} = 0 \quad \text{in } Y, \\ &\text{ii)} \quad \overline{w} = w \quad \text{on } S, \\ &\text{iii)} \quad \|\overline{w}\|_{H^1(Y)^N} \leq C|w|_{H^1(Y)^N}, \end{aligned} \tag{3.12}$$

where C is a constant independent of w.

Proof: Consider the following problems. Look for w_1, q_1 such that

$$\begin{aligned} &-\Delta w_1 + \nabla q_1 = -\Delta w \quad \text{in } Y^*, \\ &\operatorname{div} w_1 = \operatorname{div} w \quad \text{in } Y^*, \\ &w_1 \in H^1_o(Y^*)^N \end{aligned} \tag{3.13}$$

and look for w_2, q_2 satisfying

$$\begin{aligned} &-\Delta w_2 + \nabla q_2 = -\Delta w \quad \text{in } T, \\ &\operatorname{div} w_2 = \operatorname{div} w \quad \text{in } T, \\ &w_2 \in H^1_o(T)^N. \end{aligned} \tag{3.14}$$

The problems (3.13) and (3.14) have unique solutions because the compatibility conditions

$$\int_{Y^*} \operatorname{div} w = \int_{\partial Y^*} w \cdot \nu = \int_S w \cdot \nu = 0$$

and

$$\int_T \operatorname{div} w = \int_S w \cdot (-\nu) = 0$$

are satisfied. Now, define $\overline{w}$ as follows:

$$\overline{w} = \begin{cases} w - w_2 & \text{in } T \\ w - w_1 & \text{in } Y^*. \end{cases}$$

Then it is easy to see that $\overline{w}$ satisfies (3.12) and the proof of Lemma 3.7 is complete.

4. Transformation, estimates and extension. In this section, we transform our problem to another problem with homogeneous condition and then estimate the solution. Finally, we obtain an extension of the pressure p_ε using the technique developed by Tartar [8]. First, define

$$b_\varepsilon(x) = \begin{cases} (Q\overline{g})^\varepsilon\left(\frac{x}{\varepsilon}\right) & \text{if } x \in \varepsilon Y_k, \quad k \in I_\varepsilon \\ 0 & \text{if } x \in \varepsilon Y_k, \quad k \in J_\varepsilon, \end{cases} \tag{4.1}$$

where $\overline{g}$ is given by Lemma 3.1 corresponding to g, which is the non-homogeneous boundary term in the problem (1.1). Recall the operator Q constructed in the previous section. Then one can easily verify that b_ε satisfies the following.

Lemma 4.1. *We have*

$$\begin{aligned}
&\text{i)}\quad b_\varepsilon \in H_o^1(\Omega)^N \text{ and } b_\varepsilon = g^\varepsilon \text{ on } \varepsilon S_k, \quad k \in I_\varepsilon,\\
&\text{ii)}\quad \operatorname{div} b_\varepsilon = 0 \text{ in } \Omega,\\
&\text{iii)}\quad \|b_\varepsilon\|_{L^2(\Omega)} \le C\|g\|_{H^1(Y)} \text{ and } \|\nabla b_\varepsilon\|_{L^2(\Omega)^N} \le \frac{C}{\varepsilon}\|g\|_{H^1(Y)},\\
&\text{iv)}\quad b_\varepsilon \to 0 \text{ in } L^2(\Omega) \text{ weak,}
\end{aligned} \tag{4.2}$$

where C is independent of ε.

Note: Observe that $b_\varepsilon \to m(Q\overline{g})$ in $L^2(\Omega)$ weak. But, $m(Q\overline{g}) = \int_Y (Q\overline{g})\,dy = 0$ because the average of a divergence-free, compactly supported vector is zero.

We are now ready to transform the problem (1.1) to a problem with homogeneous condition on the boundary of the holes. Let

$$u_\varepsilon = v_\varepsilon - b_\varepsilon \quad \text{in } \Omega_\varepsilon. \tag{4.3}$$

Then u_ε will satisfy the system of equations:

$$\begin{aligned}
&\text{i)}\quad -\Delta u_\varepsilon - \Delta b_\varepsilon + \nabla p_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon\\
&\text{ii)}\quad \operatorname{div} u_\varepsilon = 0 \quad \text{on } \Omega_\varepsilon\\
&\text{iii)}\quad u_\varepsilon \in H_o^1(\Omega_\varepsilon)^N, \quad p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbf{R}.
\end{aligned} \tag{4.4}$$

We want to study the behaviour of the problem (4.4) as $\varepsilon \to 0$. Further, we need to extend $v_\varepsilon, p_\varepsilon$ to Ω. Since $u_\varepsilon = 0$ on the boundary S_ε of the holes, one can extend u_ε to $\tilde{u}_\varepsilon$ by zero inside the holes and so define

$$\tilde{v}_\varepsilon = \tilde{u}_\varepsilon + b_\varepsilon \quad \text{in } \Omega. \tag{4.5}$$

Then div $\tilde{v}_\varepsilon = 0$ in Ω automatically.

Next, we extend the pressure p_ε. We have the following result, which achieves this apart from providing basic estimates on the solutions.

Theorem 4.1. *There exists an extension $\tilde{p}_\varepsilon$ of p_ε such that*

$$\left\|\varepsilon^2 \tilde{p}_\varepsilon\right\|_{L^2(\Omega)/\mathbf{R}} \le C. \tag{4.6}$$

Also, $\tilde{u}_\varepsilon$, the extension by zero of the solution u_ε of the problem (4.4), satisfies

$$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)} \le C, \quad \text{and} \tag{4.7}$$

$$\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)} \le \frac{C}{\varepsilon}, \text{ where } C \text{ is independent of } \varepsilon. \tag{4.8}$$

Proof: Multiplying (4.4) by u_ε and integrating by parts, we get

$$\begin{aligned}
\|\nabla u_\varepsilon\|_{L^2}^2 &\le \|\nabla b_\varepsilon\|_{L^2} \|\nabla u_\varepsilon\|_{L^2} + \|f_\varepsilon\|_{L^2} \|u_\varepsilon\|_{L^2}\\
&\le \frac{C}{\varepsilon} \|\nabla u_\varepsilon\| + \frac{C}{\varepsilon^2}\varepsilon \|\nabla u_\varepsilon\| \le \frac{C}{\varepsilon} \|\nabla u_\varepsilon\|
\end{aligned}$$

which gives (4.8). Again by (3.1), we have

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C.$$

Now, we extend the pressure p_ε using the same technique as in Tartar [8]. We will roughly sketch the proof. Define an element $F_\varepsilon \in H^{-1}(\Omega)^N$ as follows. For any $w \in H_o^1(\Omega)^N$, define

$$\begin{aligned}(F_\varepsilon, w)_\Omega &= (\nabla p_\varepsilon, R_\varepsilon w)_{\Omega_\varepsilon}, \text{ where } R_\varepsilon \text{ is given by Lemma 2.3}\\ &= -\int_{\Omega_\varepsilon} \frac{\partial u_{\varepsilon i}}{\partial x_j}\frac{\partial (R_\varepsilon w)_i}{\partial x_j} - \int_{\Omega_\varepsilon} \frac{\partial b_{\varepsilon i}}{\partial x_j}\frac{\partial (R_\varepsilon w)_i}{\partial x_j} + \int_{\Omega_\varepsilon} f_{\varepsilon i}\,(R_\varepsilon w)_i .\end{aligned} \tag{4.9}$$

One can easily check that, in fact, $F_\varepsilon \in H^{-1}(\Omega)^N$. Further, if div $w = 0$ in Ω then $(F_\varepsilon, w)_\Omega = 0$ which shows that F_ε is a gradient in Ω. However, we know $F_\varepsilon = \nabla p_\varepsilon$ in Ω_ε because if $w \in H_o^1(\Omega_\varepsilon)^N$, then $R_\varepsilon w = w$. Hence, there exists an extension $\tilde{p}_\varepsilon$ of p_ε such that

$$F_\varepsilon = \nabla \tilde{p}_\varepsilon \quad \text{in } \Omega.$$

Again from (4.9), by using Lemma 2.3, one can obtain

$$\varepsilon^2 \left|(\nabla \tilde{p}_\varepsilon, w)\right| \leq C\left(\|w\|_{L^2(\Omega)} + \varepsilon\|\nabla w\|_{L^2(\Omega)}\right), \quad \forall w \in H_o^1(\Omega)^N, \tag{4.10}$$

which gives

$$\left\|\varepsilon^2 \nabla \tilde{p}_\varepsilon\right\|_{H^{-1}(\Omega)} \leq C \tag{4.11}$$

and

$$\left\|\varepsilon^2 \tilde{p}_\varepsilon\right\|_{L^2(\Omega)/\mathbf{R}} \leq C. \tag{4.12}$$

This completes the proof.

Remark 4.1. Using the estimate (4.10), one can see that the extension $\tilde{p}_\varepsilon$ of p_ε satisfies

$$\varepsilon^2 \nabla \tilde{p}_\varepsilon \to \nabla p \quad \text{in } H^{-1}(\Omega) \text{ strong}, \tag{4.13}$$

$$\varepsilon^2 \tilde{p}_\varepsilon \to p \quad \text{in } L^2(\Omega)/\mathbf{R} \text{ strong}. \tag{4.14}$$

Also, from the Lemma 4.1 and Theorem 4.1, it follows that

$$\|\tilde{v}_\varepsilon\|_{L^2(\Omega)} \leq C, \tag{4.15}$$

and

$$\|\nabla \tilde{v}_\varepsilon\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon}. \tag{4.16}$$

5. Convergence results. Asymptotic expansion. Applying a two-scale asymptotic expansion for u_ε and p_ε, namely,

i) $u_\varepsilon(x) = u_o(x, y) + \varepsilon u_1(x, y) + \cdots$,

ii) $\varepsilon^2 p_\varepsilon(x) = p(x) + \varepsilon p_1(x, y) + \cdots$,

iii) $u_i(x, y) = 0$ for $x \in \Omega$, $y \in S$ and u_i, p_i are Y-periodic, $\forall i = 1, 2, 3, \cdots$.

We can see that u_o, p and p_1 satisfy the system:

$$\begin{aligned} &\text{i)} \quad -\Delta_y u_o + \nabla_y p_1 + \nabla_x p = f + \Delta_y(Q\overline{g}) \text{ in } \Omega \times Y^* \\ &\text{ii)} \quad u_o, p_1 \text{ are } Y\text{-periodic}, u_o = 0 \text{ on } S, \forall x \in \Omega \\ &\text{iii)} \quad \operatorname{div}_y u_o = 0 \text{ in } \Omega \times Y^* \\ &\text{iv)} \quad \operatorname{div}_x \int_{Y^*} u_o(x,y)\,dy = 0 \text{ in } \Omega, \text{ and } \nu \cdot \int_{Y^*} u_o(x,y)\,dy = 0 \text{ on } \Gamma. \end{aligned} \tag{5.1}$$

The above system (5.1) has been studied in Lions [3] (of course, without the term $\Delta_y(Q\overline{g})$) and there exists a unique solution u_o, p, p_1 (p unique up to an additive constant and p_1 up to an additive function of x).

First, we state and prove the homogenization of u_ε and p_ε.

Theorem 5.1. *Let $\tilde{u}_\varepsilon$ be the extension by zero of u_ε and $\tilde{p}_\varepsilon$ be the extension of p_ε given by Theorem 4.1, where u_ε, p_ε are given by the problem (4.4). Then*

$$\tilde{u}_\varepsilon \to u(x) \textit{ in } L^2(\Omega) \textit{ weak}, \tag{5.2}$$

$$\varepsilon^2 \tilde{p}_\varepsilon \to p(x) \textit{ in } L^2(\Omega)/\mathbf{R} \textit{ strong, and} \tag{5.3}$$

$$u(x) = \int_{Y^*} u_o(x,y)\,dy, \tag{5.4}$$

where u_o, p are the unique solution of the problem (5.1).

Theorem 5.2. *Let u and p be as in Theorem 5.1. Then u and p are given by the unique solution of the system* (2.8).

Proof of Theorem 5.1: The convergence (5.1) and (5.2) follow from (4.7) and (4.14), respectively. Now, put

$$\xi_{\varepsilon ij} = \varepsilon \frac{\partial u_{\varepsilon i}}{\partial x_j} \quad \text{in } \Omega_\varepsilon$$

and let $\tilde{\xi}_{\varepsilon ij}$ be the extension by zero inside the holes. Then, by (4.8), we have

$$\left\| \tilde{\xi}_{\varepsilon ij} \right\|_{L^2(\Omega)} \le C. \tag{5.5}$$

So, applying Lemma 3.6 to $\tilde{u}_{\varepsilon i}$ and $\tilde{\xi}_{\varepsilon ij}$, there exist

$$u_{oi}(x,y) \quad \text{and} \quad \xi_{oij}(x,y) \in L^2(\Omega, L^2_p(Y))$$

corresponding to $\tilde{u}_{\varepsilon i}$ and $\tilde{\xi}_{\varepsilon ij}$, respectively, satisfying (3.9), (3.10) and (3.11). So that we have

$$u(x) = \int_Y u_o(x,y)\,dy.$$

Now, we prove that u_o will satisfy the system (5.1).

Step 1. In this step, we derive a relationship between ξ_{oij}, p and p_1. Let $\phi \in \mathcal{D}(\Omega)$ and $w \in (\mathcal{D}(Y^*))^N$ with div $w = 0$ and define $w^\varepsilon(x) = w\left(\frac{x}{\varepsilon}\right)$ (by extending w to all of $\mathbf{R}^N$). Multiplying the equation (4.4, i) by $\varepsilon^2\phi w^\varepsilon$ and integrating by parts and passing to the limit as $\varepsilon \to 0$ (which can be achieved using Lemma 3.6 and the results from §4), we get

$$\int_\Omega \Big[\int_{Y^*} (\xi_{oij}\frac{\partial w_i}{\partial y_j} + \frac{\partial (Q\overline{g})_i}{\partial y_j}\frac{\upsilon w_i^\varepsilon}{\partial x_j} - f_i w_i)\, dy\Big]\phi(x)\, dx$$
$$= \int_\Omega \Big[\int_{Y^*} (w_i(y)\, dy)\, p(x)\frac{\partial \phi}{\partial x_i}\Big]\, dx,$$

which holds for all $\phi \in \mathcal{D}(\Omega)$, so that

$$\int_{Y^*}\left(\xi_{oij}\frac{\partial w_i}{\partial y_j} + \frac{\partial (Q\overline{g})_i}{\partial y_j}\frac{\partial w_i}{\partial y_j} - f_i w_i\right) = \int_{Y^*}\frac{\partial p}{\partial x_i} w_i(y)\, dy. \tag{5.6}$$

This holds for all $w \in \mathcal{D}(Y^*)^N$ with div $w = 0$. In fact, (5.6) is true for all $w \in C_p^\infty(Y^*)^N$ (set of all C^∞ functions which are Y-periodic) with div $w = 0$ and $w = 0$ on S. So, there exists a function $p_1(x, y)$ (see Temam [9]), Y-periodic, and $p_1(x, \cdot) \in L^2(Y^*)$ such that

$$-\frac{\partial \xi_{oij}}{\partial y_j} - \frac{\partial^2 (Q\overline{g})_i}{\partial y_j^2} + \frac{\partial p_1}{\partial y_i} = \frac{\partial p}{\partial x_i} + f_i. \tag{5.7}$$

The existence of p_1 is the standard problem of the solvability of the equation div $\eta = f$ for $f \in L^2(Y^*)$ and one can see the references [9], [6].

Calculation of ξ_{oij}: For any $\phi \in \mathcal{D}(\Omega)$ and $w \in \mathcal{D}(Y^*)$, we have

$$\int_{\Omega_\varepsilon} \xi_{\varepsilon ij}\phi(x) w^\varepsilon \to \int_{\Omega\times Y^*} \xi_{oij}\phi(x) w(y)\, dx dy.$$

But on the other hand,

$$\int_\Omega \tilde{\xi}_{\varepsilon ij}\phi(x) w^\varepsilon(x)\, dx = -\varepsilon\int_{\Omega_\varepsilon} u_{\varepsilon i}\left(\frac{\partial \phi}{\partial x_j} w^\varepsilon + \phi\frac{\partial w^\varepsilon}{\partial x_j}\right).$$

The first term on the right hand side goes to zero as $\varepsilon \to 0$ and the second term is equal to

$$-\int_\Omega \tilde{u}_{\varepsilon i}\phi(x)\left(\frac{\partial w}{\partial y_j}\right)^\varepsilon (x)\, dx \to -\int_{\Omega\times Y^*} u_{oi}(x, y)\phi(x)\frac{\partial w}{\partial y_j}.$$

Hence, it follows that

$$\int_{Y^*}\xi_{oij} w(y)\, dy = -\int_{Y^*} u_{oi}\frac{\partial w}{\partial y_j}\, dy, \text{ a.e. } x \in \Omega, \forall w \in \mathcal{D}(Y^*),$$

which implies that

$$\xi_{oij} = \frac{\partial u_{oi}}{\partial y_j}. \tag{5.8}$$

The equation (5.1,i) follows from (5.7) and (5.8).

Step 2. (Conditions (5.1, ii, iii, and iv)): u_o and p_1 are Y-periodic follows from the existence of u_o and p_1.

Claim: $u_o(x,y) = 0$ on S for $x \in \Omega$. In fact, we prove $u_o(x,y) = 0$ for $y \in T$, $x \in \Omega$. Let $\phi \in \mathcal{D}(\Omega)$ and $w \in \mathcal{D}(Y)$. Let χ_{Y^*} be the characteristic function of Y^*. Then

$$\int_{\Omega_\varepsilon} u_{\varepsilon i}\phi w^\varepsilon = \int_\Omega \tilde{u}_{\varepsilon i}\phi w^\varepsilon \to \int_{\Omega\times Y} u_{oi}\phi w \, dxdy.$$

Now, observe that $\chi_{Y^*}w \in L^2_p(Y)$ and we have

$$\int_{\Omega_\varepsilon} u_{\varepsilon i}\phi w^\varepsilon = \int_\Omega \tilde{u}_{\varepsilon i}\phi \left(\chi_{Y^*}w\right)^\varepsilon(x)\, dx \to \int_{\Omega\times Y} u_{oi}\phi\chi_{Y^*}(y)w(y)\, dxdy,$$

so that we get

$$\int_\Omega \left(\int_Y u_{oi}w(y)\, dy\right)\phi(x)\, dx = \int_\Omega \left(\int_Y u_{oi}\chi_{Y^*}(y)w(y)\, dy\right)\phi(x)\, dx$$

which holds for all $\phi \in \mathcal{D}(\Omega)$ and $w \in \mathcal{D}(Y)$ and, hence, we have

$$u_{oi}(x,y) = \chi_{Y^*}(y)u_{oi}(x,y).$$

Therefore, we get $u_o(x,y) = 0$ in T.

Claim: $\text{div}_y u_o = 0$. Multiplying div $\tilde{u}_\varepsilon = 0$ by $\varepsilon\phi w^\varepsilon$, where $\phi \in \mathcal{D}(\Omega)$, $w \in \mathcal{D}(Y^*)$, and integrating by parts and passing to the limit, we get

$$\begin{aligned} 0 = \varepsilon \int_\Omega \text{div}_x\tilde{u}_\varepsilon \cdot \phi w^\varepsilon &= -\varepsilon \int_\Omega \tilde{u}_\varepsilon \left(\nabla\phi \cdot w^\varepsilon + \phi\nabla w^\varepsilon\right) \\ &= -\varepsilon \int_\Omega \tilde{u}_\varepsilon \nabla\phi \cdot w^\varepsilon - \int_\Omega \tilde{u}_\varepsilon\phi \left(\nabla_y w\right)^\varepsilon dx. \end{aligned}$$

The first term goes to 0 as $\varepsilon \to 0$ and the second term is equal to

$$-\int_\Omega \tilde{u}_\varepsilon\phi\left(\nabla_y w\right)^\varepsilon dx \to -\int_{\Omega\times Y^*} u_o(x,y)\phi(x)\nabla_y w(y)\, dxdy;$$

i.e.,

$$\int_\Omega \left(\int_{Y^*} u_{oi}\frac{\partial w}{\partial y_i}\, dy\right)\phi(x) = 0, \forall\phi \in \mathcal{D}(\Omega), w \in \mathcal{D}(Y^*),$$

which gives $\text{div}_y\ u_o = 0$. Since div $u_\varepsilon = 0$, it follows that $\text{div}_x \int_{Y^*} u_o(x,y)\, dy = \text{div}_x u(x) = 0$ and the condition $\nu \cdot \int_{Y^*} u_o(x,y)\, dy = \nu \cdot u = 0$ on Γ is an easy consequence of the Lemma 3.5. This completes the proof of Theorem 5.1.

Proof of Theorem 5.2: It suffices to show that u and p, given by the above Theorem 5.1, satisfy the equation (2.8,ii).

Multiplying the equation (5.1,i) by v^k, where v^k is the solution of (2.1), and integrating with respect to y and observing that $\text{div}_y v^k = 0$ in Y^*, $v^k = 0$ on S, we get

$$\int_{Y^*} \frac{\partial u_{0i}}{\partial y_j}\frac{\partial v_i^k}{\partial y_j} + \frac{\partial p}{\partial x_i}\int_{Y^*} v_i^k(y)\, dy = f_i(x)\int_{Y^*} v_i^k\, dy - \int_{Y^*} \frac{\partial(Q\overline{g})_i}{\partial y_j}\frac{\partial v_i^k}{\partial y_j}\, dy;$$

i.e.,

$$\int_{Y^*} \frac{\partial u_{0i}}{\partial y_j}\frac{\partial v_i^k}{\partial y_j}\,dy = K_{ik}\left(f_i - \frac{\partial p}{\partial x_i}\right) - \int_{Y^*} \frac{\partial (Q\overline{g})_i}{\partial y_j}\frac{\partial v_i^k}{\partial y_j}\,dy. \tag{5.9}$$

On the other hand, if we multiply the equation (2.1,i) by u_o and integrate by parts, we get

$$\int_{Y^*} \frac{\partial u_{oi}}{\partial y_j}\frac{\partial v_i^k}{\partial y_j}\,dy = \int_{Y^*} e_{ki}u_{oi}\,dy = u_k. \tag{5.10}$$

Therefore, the proof of Theorem 5.2 is complete if we show that

$$C_k = -\int_{Y^*} \frac{\partial (Q\overline{g})_i}{\partial y_j}\frac{\partial v_i^k}{\partial y_j}\,dy, \tag{5.11}$$

where C_k is given by (2.9).

Proof of (5.11): Multiplying the equation (2.1,i) by $Q\overline{g}$ and integrating by parts, we get

$$\int_{Y^*} \frac{\partial v_i^k}{\partial y_j}\frac{\partial (Q\overline{g})_i}{\partial y_j} + \int_S \frac{\partial v_i^k}{\partial \nu_y}(Q\overline{g})_i - \int_S q^k Q\overline{g}\cdot\nu_y = \int_{Y^*} e_{ki}(Q\overline{g})_i;$$

i.e.,

$$\int_{Y^*} \frac{\partial v_i^k}{\partial y_j}\frac{\partial (Q\overline{g})_i}{\partial y_j} = \int_S \left(q^k\,(g\cdot\nu_y) - g\cdot\frac{\partial v_i^k}{\partial \nu_y}\right) + \int_{Y^*} (Q\overline{g})_k. \tag{5.12}$$

Note that ν_y is the exterior unit normal at S (i.e., exterior to T).

Now, since $\operatorname{div}(Q\overline{g}) = 0$ in Y^*, multiplying this equation by y_k and integrating by parts, we get

$$0 = -\int_{Y^*} (Q\overline{g})\cdot\nabla y_k - \int_S (Q\overline{g}\cdot\nu_y)\,y_k,$$

so that

$$\int_{Y^*} (Q\overline{g})_k = -\int_S (g\cdot\nu_y)\,y_k. \tag{5.13}$$

Substituting this in (5.12), we get (5.11) and, hence, the proof of Theorem 5.2 is complete.

Proof of the main results (Theorems 2.1 and 2.2): Theorems 2.1 and 2.2 follow directly from Theorems 5.1 and 5.2, respectively. Define

$$v_o(x,y) = u_o(x,y) + (Q\overline{g})(y), \quad x\in\Omega, \quad y\in Y. \tag{5.13}$$

Then v_o satisfies all the equations in (2.4) trivially, except the boundary condition

$$\nu_x\cdot\int_{Y^*} v_o\,dy = -\nu_x\cdot\int_S (g\cdot\nu_y)y\,ds \quad \text{on } \Gamma. \tag{5.14}$$

Since $\nu_x\cdot\int_{Y^*} u_o(x,y)\,dy = 0$ on Γ, it follows that

$$\nu_x\cdot\int_{Y^*} v_o(x,y) = \nu_x\cdot\int_{Y^*} Q\overline{g}\,dy.$$

Then (5.14) follows from the equation (5.13) and, hence, v_o is a solution of system (2.14). Now, since $\tilde{v}_\varepsilon = u_\varepsilon + b_\varepsilon$ and $b_\varepsilon \to m(Q\overline{g}) = 0$ in $L^2(\Omega)$ weak, we have

$$v(x) = \int_Y v_o(x,y)\,dy = \int_{Y^*} u_o(x,y) = u(x).$$

This completes the proof of Theorems 2.1 and 2.2.

Remark 5.2. $v_o = u_o + Q\overline{g}$ is the unique solution of the system (2.4). Also, what we have observed is that the weak limits of $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ are the same but the equation satisfied by u_o and v_o are different. Further, from the uniqueness of the system (2.4), v_o is independent of the operator Q and the construction of $\overline{g}$ in $\Omega \times Y^*$ and $v_o = u_o + Q\overline{g}$ provides an extension to all of $\Omega \times Y$. The weak limit $v = \int_Y v_o(x,y)$. Even though the extension of v_o outside $\Omega \times Y^*$ depends on the construction of $\overline{g}$, v is independent of this because it is the unique solution of the system (2.8).

B. Evolution Stokes Equation. Now we proceed to study the behaviour of v_ε, p_ε as $\varepsilon \to 0$ for evolution Stokes equation given by the system (1.3) with the conditions (1.4) and (1.5). Here also we transform the problem to another problem with homogeneous boundary condition on the holes as in the case of Stokes equation. For this, we have to modify the Lemma 3.4 in a different form. Since the method is same as in part A, we do not present all the details.

6. Transformation, estimates and extensions. Because of the Lemma 3.7 and the compatibility condition (1.4), without loss of generality we can assume, in addition to the hypothesis on g given in §1, that

$$\mathrm{div}_y g(\cdot,t) = 0. \tag{6.1}$$

We will state the Lemma 3.4 in the following form.

Lemma 6.1. *There exists an operator* $Q_T : L^\infty(0,T : H^1_p(Y)) \to L^\infty(0,T : H^1_o(Y))$ *such that*

$$\begin{aligned}
&\text{i)} \quad Q_T w = 0 \quad \textit{in a neighbourhood of } \partial Y, \quad \forall t \in [0,T],\\
&\text{ii)} \quad Q_T w = w \quad \textit{in a neighbourhood of } T, \quad \forall t \in [0,T],\\
&\text{iii)} \quad \mathrm{div}_y w = 0 \Rightarrow \mathrm{div}_y Q_T w = 0,\\
&\text{iv)} \quad \|Q_T w\|_{\infty,2,Y} + \|\nabla(Q_T w)\|_{\infty,2,Y} \le C\left(\|w\|_{\infty,2,Y} + \|\nabla w\|_{\infty,2,Y}\right),\\
&\text{v)} \quad \|\frac{\partial}{\partial t} Q_T w\|_{\infty,2,Y} \le C(\|\frac{\partial w}{\partial t}\|_{\infty,2,Y} + \|\nabla(\frac{\partial w}{\partial t})\|_{\infty,2,Y})\\
&\qquad \textit{for all } w \in L^\infty\left(0,T : H^1_p(Y)\right).
\end{aligned} \tag{6.2}$$

Proof: For any $w \in L^\infty\left(0,T : H^1_p(Y)\right)$, let $w_t(x) = w(x,t)$, then $w_t \in H^1_p(Y)$. Then define Q_T as follows

$$\left(Q_T w\right)(x,t) = \left(Q w_t\right)(x),$$

where Q is given by Lemma 3.4. This Q_T satisfies (6.2) which completes the proof.

Now define $d_\varepsilon = d_\varepsilon(x,t)$ as follows:

$$d_\varepsilon(x,t) = \begin{cases} (Q_T g)^\varepsilon (x,t) = (Q_T g)\left(\frac{x}{\varepsilon}, t\right) & \text{if } x \in \varepsilon Y_k, \quad k \in I_\varepsilon \\ 0, \quad x \in \varepsilon Y_k, \quad k \in J_\varepsilon. \end{cases} \tag{6.3}$$

Here g is given by (1.4) and (6.1). This d_ε defined by (6.3) satisfies the following lemma which can be verified easily using Lemma 6.1.

Lemma 6.2. *We have*

$$\begin{aligned} &\text{i)} \quad d_\varepsilon(x,t) = 0 \quad \text{on } \Gamma_T \text{ and } d_\varepsilon = g^\varepsilon \quad \text{on } S_{\varepsilon T}, \\ &\text{ii)} \quad \operatorname{div}_y d_\varepsilon = 0 \quad \text{in } \Omega_T, \\ &\text{iii)} \quad \|d_\varepsilon\|_{\infty,2,\Omega} \leq C, \\ &\text{iv)} \quad \|\nabla d_\varepsilon\|_{\infty,2,\Omega} \leq \frac{C}{\varepsilon}, \\ &\text{v)} \quad \left\|\frac{\partial d_\varepsilon}{\partial t}\right\|_{\infty,2,\Omega} \leq C, \text{ where } C \text{ is independent of } \varepsilon, \\ &\text{vi)} \quad d_\varepsilon \to 0 \text{ in } L^2(\Omega) \text{ weak, uniformly in } t. \end{aligned} \tag{6.4}$$

Now, we transform the problem (1.3) to a problem with homogeneous boundary condition. Put

$$u_\varepsilon = v_\varepsilon - d_\varepsilon. \tag{6.5}$$

Then u_ε is the solution of the following equation:

$$\begin{aligned} &\text{i)} \quad \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon - \frac{\partial d_\varepsilon}{\partial t} + \Delta d_\varepsilon \text{ in } \Omega_{\varepsilon T}, \\ &\text{ii)} \quad \operatorname{div} u_\varepsilon = 0 \text{ in } \Omega_{\varepsilon T}, \\ &\text{iii)} \quad u_\varepsilon = 0 \text{ on } \Gamma_T \cup S_{\varepsilon T}, \\ &\text{iv)} \quad u_\varepsilon(x,0) = v_o(x) - b_\varepsilon(x,0) \text{ in } \Omega_\varepsilon. \end{aligned} \tag{6.6}$$

Now, extend u_ε to $\tilde{u}_\varepsilon$ by zero inside the holes and define

$$\tilde{v}_\varepsilon = \tilde{u}_\varepsilon + d_\varepsilon \text{ in } \Omega_T \text{ and we have } \operatorname{div}_y \tilde{v}_\varepsilon = 0. \tag{6.7}$$

Estimates on u_ε and v_ε: Multiplying the equation (6.6,i) by u_ε and integrating, it is easy to see that

$$\int_{\Omega_\varepsilon} u_\varepsilon^2(x,t) + \int_0^t \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C + \frac{C}{\varepsilon}\int_0^t \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\, ds, \tag{6.8}$$

so that we have

$$\left[\int_0^t \|\nabla u_\varepsilon\|\right]^2 \leq \int_0^t \|\nabla u_\varepsilon\|^2 \leq C + \frac{C}{\varepsilon}\int_0^t \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\, ds. \tag{6.9}$$

Hence, it follows that

$$\int_{\Omega_\varepsilon} u_\varepsilon^2(x,t)\,dx + \int_0^t \|\nabla_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\,ds \le \frac{C}{\varepsilon^2}, \tag{6.10}$$

C is independent of ε, $\forall\ 0 \le t \le T$. In terms of $\tilde{v}_\varepsilon$, we have

$$\int_{\Omega} \tilde{v}_\varepsilon^2(x,t)\,dx + \int_0^t \|\nabla \tilde{v}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\,ds \le \frac{C}{\varepsilon^2}. \tag{6.11}$$

Let

$$V_\varepsilon(t) = V_\varepsilon(x,t) = \int_0^t v_\varepsilon(x,s)\,ds, \quad \widetilde{V}_\varepsilon(t) = \int_0^t \tilde{v}_\varepsilon(x,s)\,ds \tag{6.12}$$

$$U_\varepsilon(x,t) = \int_0^t u_\varepsilon(x,s)\,ds, \quad \widetilde{U}_\varepsilon(t) = \int_0^t \tilde{u}_\varepsilon(x,s)\,ds, \text{ and} \tag{6.13}$$

$$F_\varepsilon(x,t) = \int_0^t f_\varepsilon(x,s)\,ds, \quad D_\varepsilon(t) = \int_0^t d_\varepsilon(x,s)\,ds. \tag{6.14}$$

Then if v_ε and u_ε are the weak solutions of the problems (1.3) and (6.6), respectively, then

$$V_\varepsilon, U_\varepsilon \in C([0,T], H^1(\Omega_\varepsilon)^N),\ \text{div}\ V_\varepsilon = 0,\ \text{div}\ U_\varepsilon = 0 \tag{6.15}$$

and, due to the theory developed by Temam [9], there exist $P_\varepsilon \in C([0,T], L_2(\Omega))$, $\nabla P_\varepsilon \in C([0,T], H^{-1}(\Omega_\varepsilon)^N)$ such that

$$v_\varepsilon(t) - v_\varepsilon(0) - \Delta V_\varepsilon + \nabla P_\varepsilon = F_\varepsilon \text{ in } \Omega_{\varepsilon T}, \tag{6.16}$$

$$u_\varepsilon(t) - u_\varepsilon(0) - \Delta U_\varepsilon + \nabla P_\varepsilon = F_\varepsilon - (d_\varepsilon(t) - d_\varepsilon(0)) + \Delta D_\varepsilon \text{ in } \Omega_{\varepsilon T}. \tag{6.17}$$

We have the following result and the proof follows as in §4 of Part A.

Lemma 6.3. *There exists an extension $\tilde{P}_\varepsilon$ of P_ε such that*

$$\begin{aligned} &\text{i)} \quad \|\nabla \tilde{P}_\varepsilon\|_{C([0,T],H^{-1}(\Omega))} \le \frac{C}{\varepsilon^2}, \\ &\text{ii)} \quad \|\tilde{P}_\varepsilon\|_{C([0,T],L^2(\Omega)/\mathbf{R})} \le \frac{C}{\varepsilon^2}, \quad C \text{ is independent of } \varepsilon. \end{aligned} \tag{6.18}$$

7. Convergence theorems.

Now we state and prove the homogenization results.

Theorem 7.1. *Let U_ε, $\widetilde{U}_\varepsilon$ be given by (6.13) and $\tilde{P}_\varepsilon$ be as in Lemma 6.3. Then*

$$\widetilde{U}_\varepsilon \to U \quad \text{in } L^\infty(0,T,L^2(\Omega)/\mathbf{R}) \quad \text{weak}^*, \tag{7.1}$$

$$\varepsilon^2 \tilde{P}_\varepsilon \to P \quad \text{in } L^\infty(0,T,L^2(\Omega)/\mathbf{R}) \quad \text{weak}^*, \tag{7.2}$$

where U and P satisfy the elliptic system:

$$\begin{aligned}
&\text{i)}\quad \operatorname{div} U = 0 \quad \text{in } \Omega_T \\
&\text{ii)}\quad U = \alpha + K(F - \nabla P) \quad \text{in } \Omega_T \\
&\text{iii)}\quad U \cdot \nu = 0 \quad \text{on } \Gamma, \quad \text{a.e. } t \in [0,T].
\end{aligned} \tag{7.3}$$

Here $K = [K_{ij}]$ is given by (2.2) and $\alpha = \alpha(t) = (\alpha^k(t))$, where

$$\text{iv)}\quad \alpha^k(t) = \int_0^t \int_S \left(g \cdot \frac{\partial v^k}{\partial \nu_y} - g \cdot \nu_y q^k\right) + \int_0^t \int_S (g \cdot \nu_y)\, y_k$$

and

$$\text{v)}\quad F(x,t) = \int_0^t f(x,s)\,ds.$$

Theorem 7.2. *Let u_ε, p_ε be the solution of (6.6). Then there exist extensions $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ of u_ε, p_ε, respectively, such that*

$$\begin{aligned}
&\text{i)}\quad \tilde{u}_\varepsilon \to u \quad \text{in } L^2(\Omega_T) \text{ weak,} \\
&\text{ii)}\quad \varepsilon^2 \tilde{p}_\varepsilon \to p = \frac{\partial P}{\partial t} \quad \text{in } L^2(0,T,L^2(\Omega)) \text{ weak,}
\end{aligned} \tag{7.4}$$

where u and p are given by the unique solution of the system (2.11). Moreover,

$$U(x,t) = \int_0^t u(x,\sigma)\,d\sigma \text{ and } p = \frac{\partial P}{\partial t}. \tag{7.5}$$

Proof of Theorem 7.1: We briefly sketch the proof. It is easy to see that

$$\|\tilde{u}_\varepsilon\|_{L^2(\Omega_T)} \leq C \text{ and } \|\widetilde{U}_\varepsilon\|_{L^\infty(0,T,L^2(\Omega))} \leq \text{ constant.} \tag{7.6}$$

The convergence (7.1)-(7.2) and the equation (7.3,i,iii) can be verified without much difficulty. So, it remains to prove the equation (7.3,ii). Let $\phi \in \mathcal{D}(\Omega)$ and $v_\varepsilon^k(x) = v^k\left(\frac{x}{\varepsilon}\right)$, where v^k is the solution of (2.1). Then multiplying the equation (6.17) by $\varepsilon^2 \phi v_\varepsilon^k$ and integrating by parts we get,

$$\begin{aligned}
&\varepsilon^2 \int_{\Omega_{\varepsilon t}} u_{\varepsilon i}(x,t)\phi(x) v_{\varepsilon i}^k(x) - \varepsilon^2 \int_{\Omega_{\varepsilon T}} u_{\varepsilon i}(x,0)\phi v_{\varepsilon i}^k \\
&+ \varepsilon^2 \int_{\Omega_{\varepsilon T}} \frac{\partial U_{\varepsilon i}}{\partial x_j} \phi \frac{\partial v_{\varepsilon i}^k}{\partial x_j} + \varepsilon^2 \int_{\Omega_{\varepsilon T}} \frac{\partial U_{\varepsilon i}}{\partial x_j} \frac{\partial \phi}{\partial x_j} v_{\varepsilon i}^k - \varepsilon^2 \int_{\Omega_{\varepsilon T}} P_\varepsilon \frac{\partial \phi}{\partial x_i} v_{\varepsilon i}^k \\
&= \varepsilon^2 \int_{\Omega_{\varepsilon T}} F_{\varepsilon i} \phi v_{\varepsilon i}^k - \varepsilon^2 \int_{\Omega_{\varepsilon T}} d_{\varepsilon i}(x,t)\phi v_{\varepsilon i}^k + \varepsilon^2 \int_{\Omega_{\varepsilon T}} d_{\varepsilon i}(x,0)\phi v_{\varepsilon i}^k \\
&- \varepsilon^2 \int_{\Omega_{\varepsilon T}} \frac{\partial D_{\varepsilon i}}{\partial x_j} \phi \frac{\partial v_{\varepsilon i}^k}{\partial x_j} - \varepsilon^2 \int_{\Omega_{\varepsilon T}} \frac{\partial D_{\varepsilon i}}{\partial x_j} \frac{\partial \phi}{\partial x_j} v_{\varepsilon i}^k.
\end{aligned} \tag{7.7}$$

$$I_1 + I_2 + I_3 + I_4 + I_5 = I_6 + I_7 + I_8 + I_9 + I_{10}.$$

Note that $v_{\varepsilon i}^k \to K_{ki}$ in $L^2(\Omega)$ weak and $\|\nabla v_{\varepsilon i}^k\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon}$. Using this and the estimates on u_ε and U_ε, it is easy to pass to the limit in all the terms, except possibly on I_3 and I_5. In I_5, one cannot pass to limit immediately because we do not have the strong convergence of $\tilde{P}_\varepsilon$ in $L^2(\Omega_T)$. We have

$$I_5 = -\int_{\Omega_T} (\varepsilon^2 \tilde{P}_\varepsilon) \frac{\partial \phi}{\partial x_i} v_{\varepsilon i}^k = -\int_{\Omega_T} (\varepsilon^2 \tilde{P}_\varepsilon) \frac{\partial \phi}{\partial x_i} \overline{v}_{\varepsilon i}^k - m(v_{\varepsilon i}^k) \int_{\Omega_T} (\varepsilon^2 \tilde{P}_\varepsilon) \frac{\partial \phi}{\partial x_i}, \tag{7.8}$$

where

$$\overline{v}_{\varepsilon i}^k = v_{\varepsilon i}^k - m\left(v_{\varepsilon i}^k\right) \text{ and } m\left(v_{\varepsilon i}^k\right) = \frac{1}{|\Omega|} \int_\Omega v_{\varepsilon i}^k \, dx; \tag{7.9}$$

then

$$\int_\Omega \overline{v}_\varepsilon^k = 0 \text{ and } \overline{v}_\varepsilon^k \to 0 \text{ in } L^2(\Omega) \text{ weak} \tag{7.10}$$

and

$$\frac{1}{|\Omega|} \int_\Omega v_\varepsilon^k \to \int_{Y^*} v^k(y) \, dy \text{ in } \mathbf{R} \tag{7.11}$$

because

$$v_\varepsilon^k \to \int_{Y^*} v^k(y) \, dy \text{ in } L^2(\Omega) \text{ weak.}$$

Claim: $\int_{\Omega_T} (\varepsilon^2 \tilde{P}_\varepsilon) \frac{\partial \phi}{\partial x_i} \overline{v}_{\varepsilon i}^k \to 0$ as $\varepsilon \to 0$. Once the claim is proved, it is easy to see that from (7.8):

$$I_5 \to K_{ki} \int_{\Omega_T} P \frac{\partial \phi}{\partial x_i}.$$

So, from (7.7), it follows that

$$I_3 = \varepsilon^2 \int_\Omega \frac{\partial U_{\varepsilon i}}{\partial x_j} \frac{\partial v_{\varepsilon i}^k}{\partial x_j} \to \int_{\Omega_T} K_{ki} F_i \phi + \int_{\Omega_T} K_{ki} P \frac{\partial \phi}{\partial x_i} - \int_{\Omega_T} \alpha^k(t) \phi(x) \, dx, \tag{7.12}$$

where $\alpha^k(t) = \int_{Y^*} \left(\frac{\partial D_i}{\partial y_j} \frac{\partial v_i^k}{\partial y_j} \right) (y,t) \, dy$, where $D(y,t) = \int_o^t (Q_T g)(y,\sigma) \, d\sigma$. But using the same argument as in part A, one can prove that, in fact, $\alpha^k(t) = \alpha_k(t)$, where $\alpha^k(t)$ is given by (7.3,iv).

On the other hand, by multiplying the equation (2.1,i) by ϕU_ε and passing to the limit, we get

$$\varepsilon^2 \int_{\Omega_{\varepsilon T}} \frac{\partial v_{\varepsilon i}^k}{\partial x_j} \frac{\partial U_{\varepsilon i}}{\partial x_j} \phi \to \int_{\Omega_{\varepsilon T}} e_{ki} \phi U_i. \tag{7.13}$$

So, from (7.12) and (7.13), it follows that U satisfies the equation (7.3,ii). Hence, the proof of Theorem 7.1 is complete if we prove the claim.

Proof of the claim: Because of (7.10), for each k, i, there exist $\psi_\varepsilon^{k,i} \in H_o^1(\Omega)^N$ such that

$$\operatorname{div} \psi_\varepsilon^{k,i} = \overline{v}_{\varepsilon i}^k \text{ and } \psi_\varepsilon^{k,i} \to 0 \text{ in } H_o^1(\Omega) \text{ weak and, hence, in } L^2(\Omega) \text{ strong.} \tag{7.14}$$

Now,

$$\begin{aligned}
\left| \int_{\Omega_T} \varepsilon^2 \tilde{P}_\varepsilon \frac{\partial \phi}{\partial x_i} \overline{v}_{\varepsilon i}^k \right| &\le \left| \int_0^t \langle \varepsilon^2 \frac{\partial \tilde{P}_\varepsilon}{\partial x_j}, \frac{\partial \phi}{\partial x_i} \psi_{\varepsilon j}^{k,i} \rangle_\Omega \right| + \left| \int_0^t \int_\Omega \varepsilon^2 \tilde{P}_\varepsilon \psi_{\varepsilon j}^{k,i} \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_i} \right| \\
&\le C \left\| \varepsilon^2 \nabla \tilde{P}_\varepsilon \right\|_{L^\infty(0,T,H^{-1}(\Omega))} \left\| R_\varepsilon \psi_\varepsilon^{k,i} \right\|_{H_o^1(\Omega_\varepsilon)} + C \left\| \varepsilon^2 \tilde{P}_\varepsilon \right\|_{\infty,2,\Omega} \left\| \psi_\varepsilon^{k,i} \right\|_{L^2(\Omega)} \\
&\le C \left(\left\| \psi_\varepsilon^{k,i} \right\|_{L^2(\Omega)} + \varepsilon \left\| \nabla \psi_\varepsilon^{k,i} \right\|_{L^2(\Omega)} \right) \to 0 \text{ as } \varepsilon \to 0,
\end{aligned}$$

where R_ε is as in Lemma 3.3. This completes the proof of the claim and, hence, the proof of Theorem 7.1.

Proof of Theorem 7.2: This theorem follows from the above Theorem 7.1 by observing that $p_\varepsilon = \frac{\partial P_\varepsilon}{\partial t}$ and p_ε has an extension $\tilde{p}_\varepsilon$ given by $\tilde{p}_\varepsilon = \frac{\partial \tilde{P}_\varepsilon}{\partial t}$. Moreover, $\tilde{p}_\varepsilon \in L^2(0,T,L^2(\Omega))$ and $\nabla \tilde{p}_\varepsilon \in H^{-1}(0,T,H^{-1}(\Omega))$.

Proof of the main result (Theorem 2.3): Follows from Theorem 7.2 and the fact that $d_\varepsilon \to 0$ in $L^2(\Omega_T)$.

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