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Homogenization of eigenvalue problems of elasticity in perforated domains

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Abstract

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In this paper, we study the homogenization of eigenvalue problem associated with the elasticity system in a periodically (with period $\varepsilon > 0$, a small parameter) perforated domain with tiny holes. The critical size of the holes a_ε is given by $a_\varepsilon = C_0\varepsilon^{N/N-2}$ if $N \geqslant 3$ and $a_\varepsilon = \exp(-C_0/\varepsilon^2)$ if $N = 2$, where C_0 is a constant and N is the dimension. We will study the above eigenvalue problem as $\varepsilon \to 0$ and will obtain the homogenized system. We also study the correctors for the eigenvalues and eigenvectors.

1. Introduction and notations

In linear elasticity problems the displacement vector $u = (u_1, \ldots, u_N)$ of an elastic body under a force $f = (f_1, \ldots, f_N)$ can be described by a system of equations of the following form:

$$\begin{aligned} -\frac{\partial}{\partial x_j}\sigma_{ij}(u) = f_i \quad \text{in } \Omega,\ \forall i = 1, \ldots, N, \\ \sigma_{ij}(u) = a_{ijkl}(x)e_{kl}(u). \end{aligned} \tag{1.1}$$

Here

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

is the strain tensor and $\sigma_{ij}(u)$ is the stress tensor. The coefficients $a_{ijkl}(x)$ are given by the properties of the material of the body and $\Omega \subset \mathbb{R}^N$ is the region occupied by the body. Suitable boundary conditions can be associated with the above system, for instance $u_i = 0$ on Γ_1 (Dirichlet condition) and $\sigma_{ij}(u)n_j = 0$ on Γ_2 (Neumann condition), where $\Gamma = \Gamma_1 \cup \Gamma_2$ is the boundary of Ω

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and n is the unit exterior normal to the boundary of Ω. As a particular case if we take $a_{ijkl} = \delta_{ijkl}$ where $\delta_{ijkl} = 1$ if $i = k$, $j = l$ and $= 0$ otherwise, then $\sigma_{ij}(u) = e_{ij}(u)$.

A vast amount of literature is available on the elasticity problems in bounded and unbounded domains and for the derivation and the physical interpretation of the above, one can refer to [14]. In fact, the above system can be put in an elliptic variational form and one can study the existence and uniqueness results. For example see [9–12], and the references therein.

In this paper we consider the eigenvalue problems associated with the elasticity system in a domain perforated by periodic holes, the period being a small parameter. Our aim is to study the homogenization of the above problem. Before describing the problem, we first introduce some notations.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary Γ. Let $Y = (-1,1)^N$ and T be an open subset of Y containing the origin. Let $\varepsilon > 0$ be a small parameter and $0 < a_\varepsilon \leqslant \varepsilon$. Perforate the ε-periodic cell $Y_\varepsilon = \varepsilon Y$ by $T_{a_\varepsilon} = a_\varepsilon T$. The remaining part is $Y_\varepsilon^* = \varepsilon Y \setminus a_\varepsilon T$. Now cover the whole space $\mathbb{R}^N$ by ε-periodic cells which are translates of εY. Let I_ε be the index set so that

$$\Omega \subset \bigcup_{k \in I_\varepsilon} (\varepsilon Y^k) \quad \text{and} \quad \varepsilon Y^k \cap \Omega \neq \varnothing,$$

where $Y^k = Y + k$. Here $k \in Z^N$. Also put $T^k = T + k$. Now consider the perforated domain

$$\Omega_{a_\varepsilon} = \Omega \setminus \left(\overline{\bigcup_{k \in I_\varepsilon} T_{a_\varepsilon}^k} \right),$$

i.e., Ω_{a_ε} is the domain obtained from Ω by removing the holes $T_{a_\varepsilon}^k = T_{a_\varepsilon} + \varepsilon k$ from all cells $Y_\varepsilon^k = \varepsilon Y^k = \varepsilon Y + \varepsilon k$ which intersect Ω. Observe that a_ε is the size of the holes which are distributed periodically with period ε. The boundary of Ω_{a_ε} is given by

$$\partial \Omega_{a_\varepsilon} = \Gamma_\varepsilon \bigcup_{k \in I_\varepsilon} (\partial T_{a_\varepsilon}^k \cap \Omega),$$

where Γ_ε is the remaining part of the boundary Γ after removing the holes. Put $S_{a_\varepsilon}^k = \partial T_{a_\varepsilon}^k$. Let B_r be the ball of radius r with centre at the origin. We also use the standard Sobolev spaces. Let $V_{a_\varepsilon}^0 = H_0^1(\Omega_{a_\varepsilon})^N$, $V_{a_\varepsilon} = H_0^1(\Omega_{a_\varepsilon})^N$, $V^0 = H_0^1(\Omega)^N$, and $V = H^1(\Omega)^N$.

The aim of this work is to study the eigenvalue problem associated with (1.1) when ε and a_ε vary. We study the eigenvalue problem corresponding to the elasticity system in Ω_{a_ε} with Dirichlet condition on the holes. This paper is divided into various sections for obtaining the bounds on the eigenvalues, passage to the limit, construction of test functions, correctors etc.

The same problem with Neumann condition on the whole is quite standard as in the Laplacian case (see [18]). In this case, one can allow the size of the holes a_ε and the period ε to be of same size, i.e., $a_\varepsilon \simeq \varepsilon$. In this case the spectrum is bounded independent of ε and one can obtain the homogenized system using the standard techniques.

But as far as the Dirichlet case is concerned the situation is quite different. In fact, when $a_\varepsilon \simeq \varepsilon$ the spectrum is not bounded in ε and the limit analysis seems to be an open problem. The Dirichlet case with Laplacian operator was studied in [18]. In this paper we study the Dirichlet eigenvalue problem of elasticity when the size of the holes is much smaller than the period. More precisely, we take

$$a_\varepsilon = \begin{cases} C_0 \varepsilon^{N/N-2} & \text{if } N \geqslant 3, \\ \exp\left(\dfrac{-C_0}{\varepsilon^2}\right) & \text{if } N = 2 \end{cases} \tag{1.2}$$

where C_0 is a constant. In this case, we will prove that the spectrum is bounded in ε and we will obtain the homogenized system satisfied by the limit. The homogenized system is similar to the elasticity system, but with an extra term. We will also study the correctors of the eigenvalues and eigenvectors.

Now we will cite a few other references. Boundary value problems for Laplace operator in perforated domains where the size of the holes is given by (1.2) were studied in [4]. The homogenization of the elasticity system in a domain Ω with oscillating coefficients has been studied in [17] (see [7]). Duvaut [8] has studied the above system in perforated domains when the size of the holes a_ε is same as the period ε but with Neumann condition. For the homogenization of eigenvalue problems of Laplacian in perforated domains, one can see the references [18] (when $a_\varepsilon \simeq \varepsilon$) and [15] (when $a_\varepsilon \ll \varepsilon$). Stokes and Biharmonic eigenvalue problems in perforated domain when $a_\varepsilon \ll \varepsilon$ have also been studied in [15]. In [1, 2] Allaire studies the homogenization of the Stokes system, when the size of the holes a_ε is much smaller than the period ε and for the Laplacian and bi-Laplacian case one can see in Cioranescu and Murat [4]. We closely follow the same techniques as in [1, 2, 4, 15]. The test functions, however, are different in the present case and are presented in Section 6. Summation convention is adapted throughout the paper.

2. Problem description

We consider the following problem :
Find $u_\varepsilon \in V^0_{a_\varepsilon}$, $u_\varepsilon \not\equiv 0$, $\lambda_\varepsilon \in \mathbb{R}$ such that

$$\begin{aligned} &-\frac{\partial}{\partial x_j}\sigma_{ij}(u_\varepsilon) = \lambda_\varepsilon u_{\varepsilon i} \quad \text{in } \Omega_{a_\varepsilon}, \\ &\sigma_{ij}(u_\varepsilon) = \frac{1}{2}\left(\frac{\partial u_{\varepsilon i}}{\partial x_j} + \frac{\partial u_{\varepsilon j}}{\partial x_i}\right) \quad \text{in } \Omega_{a_\varepsilon}, \\ &u_\varepsilon = 0 \quad \text{on } \partial\Omega_{a_\varepsilon}. \end{aligned} \tag{2.1}$$

The above problem can be formulated in the variational form and is equivalent to
Find $u_\varepsilon \in V^0_{a_\varepsilon}$, $u_\varepsilon \not\equiv 0$, $\lambda_\varepsilon \in \mathbb{R}$, such that

$$b_\varepsilon(u_\varepsilon, v) = \lambda_\varepsilon (u_\varepsilon, v)_\varepsilon, \quad \forall v \in V^0_{a_\varepsilon} \tag{2.2}$$

where

$$\begin{aligned} &\text{i)} \quad b_\varepsilon(u, v) = \int_{\Omega_{a_\varepsilon}} \sigma_{ij}(u)\sigma_{ij}(v) = \frac{1}{2}\left[\int_{\Omega_{a_\varepsilon}} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} + \int_{\Omega_{a_\varepsilon}} \frac{\partial u_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\right], \\ &\text{ii)} \quad (u, v)_\varepsilon = \int_{\Omega_{a_\varepsilon}} u_i v_i. \end{aligned} \tag{2.3}$$

Using Korn's inequality (see [17]), one can see that b_ε is an elliptic bilinear form on $V^0_{a_\varepsilon} \times V^0_{a_\varepsilon}$, so that the system (2.2) is an elliptic eigenvalue problem and hence, for each fixed ε, the system (2.2) has a sequence of eigenvalues $\{\lambda^l_\varepsilon\}_{l=1}^\infty$ and eigenvectors $\{u^l_\varepsilon\}_{l=1}^\infty$ such that

$$\begin{aligned} &\text{i)} \qquad 0 < \lambda^1_\varepsilon \leqslant \lambda^2_\varepsilon \leqslant \ldots \to \infty, \\ &\text{ii)} \qquad \{u^l_\varepsilon\} \text{ form an orthonormal basis for } L^2(\Omega_{a_\varepsilon})^N. \end{aligned} \tag{2.4}$$

Further, the eigenvalues λ^l_ε can be characterized as

$$\lambda^l_\varepsilon = \min\left\{ \max_{v \in S_l} R_\varepsilon(v)\colon\ S_l \subset V^0_{a_\varepsilon},\ \dim S_l = l \right\} \tag{2.5}$$

where

$$R_\varepsilon(v) = \frac{\int_{\Omega_{a_\varepsilon}} \sigma_{ij}(v)\sigma_{ij}(v)}{\int_{\Omega_{a_\varepsilon}} |v|^2}. \tag{2.6}$$

Our aim is to study the behaviour of u^l_ε and λ^l_ε as $\varepsilon \to 0$. In the next section we will obtain the estimates on the eigenvalues λ^l_ε.

3. Estimates on the eigenvalues

Introduce the following problem in the domain Ω.
Find $w \in V^0$, $w \not\equiv 0$, $\nu \in \mathbb{R}$, such that

$$\begin{aligned} &-\frac{\partial}{\partial x_j}\sigma_{ij}(w) = \nu w_i \quad \text{in } \Omega, \\ &\sigma_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \quad \text{in } \Omega, \\ &w = 0 \quad \text{on } \Gamma. \end{aligned} \tag{3.1}$$

This problem (3.1) is similar to the problem (2.1), but with domain Ω instead of the perforated domain Ω_{a_ε} and it has a point spectrum $\{\nu^l\}_{l=1}^\infty$ and eigenvectors $\{w^l\}_{l=1}^\infty$, such that

$$\begin{aligned} &\text{i)} \qquad 0 < \nu^1 \leqslant \nu^2 \leqslant \ldots \to \infty, \\ &\text{ii)} \qquad \{w^l\}_{l=1}^\infty \text{ form an orthonormal basis for } L^2(\Omega)^N. \end{aligned} \tag{3.2}$$

These eigenvalues ν^l have a characterization similar to (2.5). More precisely,

$$\nu^l = \min\left\{ \max_{v \in S_l} R(v)\colon\ S_l \subset V^0,\ \dim S_l = l \right\},$$

where

$$R(v) = \frac{\int_\Omega \sigma_{ij}(v)\sigma_{ij}(v)}{\int_\Omega |v|^2}.$$

We have the following theorem which provides the necessary bounds on the eigenvalues.

Theorem 3.1. *Let a_ε be as in* (1.2). *Then there are constants $C_1 = C_1(l) > 0$, $C_2 = C_2(l) > 0$, independent of ε such that*

$$C_1 \leqslant \lambda_\varepsilon^l \leqslant C_2. \tag{3.3}$$

Proof. The proof is similar to the one we used in [15] for studying the Laplacian case.

Let w_ε be a sequence of functions satisfying the following properties

$$\begin{aligned} &\text{i)} \qquad w_\varepsilon \in H^1(\Omega) \text{ and } w_\varepsilon = 0 \text{ in the holes } T_{a_\varepsilon}, \\ &\text{ii)} \qquad w_\varepsilon \to 1 \text{ in } H^1(\Omega) \text{ weak.} \end{aligned} \tag{3.4}$$

The existence of such functions w_ε has been proved in [4] (see also [15]). Let $w^1, \ldots, w^l$ be the first l eigenvectors of the problem (3.1) and consider the set

$$S_l' = \{w_\varepsilon w^1, \ldots, w_\varepsilon w^1\} \subset V_{a_\varepsilon}^0.$$

Notice $w_\varepsilon w^i \in V_{a_\varepsilon}^0$ because $w^i = 0$ on $\partial\Omega$, $w_\varepsilon = 0$ on the holes $T_{a_\varepsilon}^k$ and $w_\varepsilon \in H^1(\Omega)$ and $w^i \in C^\infty(\overline{\Omega})$.

Claim. *S_l' is an independent set (for sufficiently small ε).*

If

$$\sum_{i=1}^{l} c_{\varepsilon i} w_\varepsilon w^i = 0 \quad \text{in } \Omega_{a_\varepsilon},$$

where $c_{\varepsilon i}$'s are constants, and $c_{\varepsilon i} \neq 0$ for some i, then choose $k \in \{1, \ldots, l\}$, independent of ε, such that $|c_{\varepsilon k}| \geqslant |c_{\varepsilon i}|$ for all $i = 1, \ldots, l$, along a subsequence of ε. Then, if necessary, dividing $c_{\varepsilon i}$ by $c_{\varepsilon k}$, without loss of generality we can assume that $c_{\varepsilon k} = 1$ and $|c_{\varepsilon i}| \leqslant 1$ for all $i = 1, \ldots, l$, so that $c_{\varepsilon i} \to c_i$ as $\varepsilon \to 0$ and $c_k = 1$. Then by passing to the limit in the above relation as $\varepsilon \to 0$, we get

$$\sum_{i=1}^{l} c_i w^i = 0,$$

which in turn will imply that $c_i = 0$, $\forall i = 1, \ldots, l$ because $\{w^1, \ldots, w^l\}$ is independent. This is a contradiction and hence our claim.

Now orthogonalize S_l' using the Gram–Schmidt process in the following way.

$$z_\varepsilon^1 = w_\varepsilon w^1 \quad \text{and} \quad z_\varepsilon^k = w_\varepsilon w^k - \sum_{i=1}^{k-1} \left(w_\varepsilon w^k, w_\varepsilon w^i\right)_\varepsilon w_\varepsilon w^i.$$

Then $S_l'' = \{z_\varepsilon^1, \ldots, z_\varepsilon^l\}$ is an orthogonal set and it satisfies the following: There exist $C = C(l) > 0$, $C' = C'(l) > 0$, independent of ε, such that

$$\|z_\varepsilon^i\|_{L^2(\Omega)} \geqslant C \quad \text{and} \quad \|\nabla z_\varepsilon^i\|_{L^2(\Omega)} \leqslant C', \quad \forall i = 1, \ldots, l. \tag{3.5}$$

The estimates (3.5) can be proved using the convergence (3.4). Let S_l be the subspace spanned by z_ε^i, $i = 1, \ldots, l$. Then S_l is an l-dimensional subspace of $V^0_{a_\varepsilon}$ and using (2.5), we get

$$\lambda_\varepsilon^l \leqslant \max_{v \in S_l} R_\varepsilon(v).$$

Now for any $v \in S_l$, we can write

$$v = \sum_{i=1}^{l} c_{\varepsilon i} z_\varepsilon^i$$

and using (3.5) we see that

$$\int_{\Omega_{a_\varepsilon}} |v|^2 = \sum_{i=1}^{l} c_{\varepsilon i}^2 \int_{\Omega_{a_\varepsilon}} |z_\varepsilon^i|^2 \geqslant C \sum_{i=1}^{l} c_{\varepsilon i}^2,$$

$$\int_{\Omega_{a_\varepsilon}} \sigma_{ij}(v)\sigma_{ij}(v) \leqslant C' \sum_{i=1}^{l} c_{\varepsilon i}^2,$$

so that $\lambda_\varepsilon^l \leqslant C_2 = C_2(l)$ for some constant C_2.

The opposite inequality in (3.3) is a simple consequence of the fact that any $v \in V^0_{a_\varepsilon}$ can be extended by zero in the holes and hence $v \in V^0$, so that $\lambda_\varepsilon^l > \nu^l \geqslant \nu^1 > 0$. This completes the proof of Theorem 3.1. □

4. Homogenization

In this section we pass to the limit in the system (2.1) and we will obtain the homogenized system. To pass to the limit we need some test functions. In this section we will only state the required properties of the test functions (Lemma 4.1), and the construction and the proof will be given in Section 6. Of course we will not explicitly construct the test functions, but we do construct certain explicit approximate functions using the fundamental solution of the elasticity system.

Lemma 4.1. (Test functions.) *Let a_ε be as in (1.2). Then there exist test functions w_ε^k and μ_k, $1 \leqslant k \leqslant N$, such that*

i) $w_\varepsilon^k \in V$ *and* $w_\varepsilon^k = 0$ *in* $T^k_{a_\varepsilon}$, $\forall k \in I_\varepsilon$,

ii) $\mu_k \in W^{-1,\infty}(\Omega)^N$,

iii) $w_\varepsilon^k \rightharpoonup e_k$ *in* V *weak, where* e_k *is the* k*-th unit vector in the canonical basis of* $\mathbb{R}^N$, (4.1)

iv) *whenever* $v_\varepsilon \in V$, $v_\varepsilon = 0$ *in* $T^k_{a_\varepsilon}$, $\forall k \in I_\varepsilon$ *such that* $v_\varepsilon \rightharpoonup v$ *in* V *weak then we have,* $\forall \phi \in \mathcal{D}(\Omega)^N$,

$$_{V'}\left\langle -\frac{\partial}{\partial x_j}\sigma_{ij}(w_\varepsilon^k),\ \phi v_\varepsilon^i \right\rangle_V \to {}_{V'}\langle \mu_k, \phi v\rangle_V.$$

Corollary 4.1. *Let a_ε be of non-critical size, i.e., $a_\varepsilon = C_0\varepsilon^\alpha$, $\alpha > N/(N-2)$ if $N \geqslant 3$, and $a_\varepsilon = \exp(-C_0/\varepsilon^\alpha)$, $\alpha > 2$ if $N = 2$. Then the test functions w_ε^k defined above will satisfy the strong convergence in* (4.1, iii). *Moreover, in this case, $\mu_k = 0$.*

Corollary 4.2.. *By taking $v_\varepsilon = w_\varepsilon^l$ and $v = e_l$ in* (4.1, iv) *we get*

$$ {}_{\mathcal{D}'(\Omega)}\langle \mu_{kl}, \phi\rangle_{\mathcal{D}(\Omega)} = \lim_{\varepsilon\to 0}\int_{\Omega_{a_\varepsilon}} \phi\sigma_{ij}\left(w_\varepsilon^k\right)\sigma_{ij}\left(w_\varepsilon^l\right) \tag{4.2} $$

for every $\phi \in \mathcal{D}(\Omega)$. Hence, if M is the matrix with elements $M_{kl} = \mu_{kl}$, then M is a symmetric, non-negative definite matrix, i.e., for every $\phi \in \mathcal{D}(\Omega)$, we have

$$ \langle M\phi, \phi\rangle = \lim_{\varepsilon\to 0}\int_\Omega \left|\sum_k \sigma_{ij}\left(w_\varepsilon^k\right)\phi_k\right|^2 \geqslant 0. \tag{4.3} $$

We are now in a position to pass to the limit in (2.1) and we have the following theorem.

Theorem 4.1. *Let a_ε be as in* (1.2). *Let $\{u_\varepsilon^l\}$, $\{\lambda_\varepsilon^l\}$ be the solution of the problem* (2.2) *and $\widetilde{u}_\varepsilon^l$ be the extension of u_ε^l by zero in the holes. Then there is a subsequence of ε, denoted again by ε, such that*

$$ \begin{aligned} &\text{i)} \quad \lambda_\varepsilon^l \to \lambda^l, \\ &\text{ii)} \quad \widetilde{u}_\varepsilon^l \to u^l \text{ in } V^0 \text{ weak as } \varepsilon \to 0, \end{aligned} \tag{4.4} $$

where λ^l is the l-th eigenvalue and u^l is the corresponding eigenvector of the following problem:

$$ \begin{aligned} &-\frac{\partial}{\partial x_j}\sigma_{ij}(u) + (Mu)_i = \lambda u_i \quad \text{in } \Omega, \\ &\sigma_{ij}(u) = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \Gamma. \end{aligned} \tag{4.5} $$

Moreover,

$$ \int_\Omega \left|\sigma_{ij}\left(\widetilde{u}_\varepsilon^l\right)\right|^2 \to \int_\Omega \left|\sigma_{ij}\left(u^l\right)\right|^2 + \langle Mu^l, u^l\rangle, \tag{4.6} $$

where M is defined in Corollary 4.2.

Proof. Taking $v = u_\varepsilon^l$ in (2.2) and using the estimate on the eigenvalues we get (after normalizing u_ε by $\|u_\varepsilon^l\|_{L^2(\Omega_{a_\varepsilon})} = 1$):

$$ \left\|\sigma_{ij}\left(u_\varepsilon^l\right)\right\|_{L^2(\Omega_{a_\varepsilon})} \leqslant \text{constant.} $$

So using Korn's inequality (see [17]), we have

$$\|\widetilde{u}_\varepsilon^l\|_{H^1(\Omega)} \leqslant \text{constant.} \tag{4.7}$$

Hence there is a subsequence of ε, satisfying the convergence (4.4, i and ii) using the estimates (3.3) and (4.7).

Now we will show that λ^l, u^l satisfies the system (4.5). Let $\phi \in \mathcal{D}(\Omega)$ and multiplying the equation (2.1) by ϕw_ε^k and integrating by parts, we get

$$\int_{\Omega_{a_\varepsilon}} \sigma_{ij}(u_\varepsilon^l)\sigma_{ij}(\phi w_\varepsilon^k) = \lambda_\varepsilon^l \int_{\Omega_{a_\varepsilon}} u_{\varepsilon i}^l \phi w_{\varepsilon i}^k \to \lambda^l \int_\Omega u_i^l \phi e_{ki} = \lambda^l \int_\Omega u_k^l \phi.$$

On the other hand,

$$\text{L.H.S.} = \int_{\Omega_{a_\varepsilon}} \sigma_{ij}(u_\varepsilon^l)\,\frac{1}{2}\left(\frac{\partial\phi}{\partial x_j} w_{\varepsilon i}^k + \frac{\partial\phi}{\partial x_j} w_{\varepsilon j}^k\right) + \int_{\Omega_{a_\varepsilon}} \sigma_{ij}(u_\varepsilon^l)\phi\sigma_{ij}(w_\varepsilon^k).$$

But the first term on the right hand side converges to

$$\begin{aligned}
&\int_\Omega \sigma_{ij}(u^l)\,\frac{1}{2}\left(\frac{\partial\phi}{\partial x_j} e_{ki} + \frac{\partial\phi}{\partial x_i} e_{kj}\right)\\
&= \int_\Omega \sigma_{ij}(u^l)\,\frac{\partial\phi}{\partial x_j} e_{ki} \quad (\text{since } \sigma_{ij} = \sigma_{ji})\\
&= -\int_\Omega \frac{\partial\sigma_{ij}(u^l)}{\partial x_j}\,\phi e_{ki}.
\end{aligned}$$

Second term is equal to

$$\begin{aligned}
&-\int_{\Omega_{a_\varepsilon}} u_{\varepsilon i}^l \frac{\partial}{\partial x_j}\left(\phi\sigma_{ij}(w_\varepsilon^k)\right) = {}_{V'}\left\langle -\frac{\partial}{\partial x_j}\sigma_{ij}(w_\varepsilon^k),\ \phi\widetilde{u}_{\varepsilon i}^l\right\rangle_V - \int_\Omega \widetilde{u}_{\varepsilon i}^l \frac{\partial\phi}{\partial x_j}\sigma_{ij}(w_\varepsilon^k)\\
&\to {}_{V'}\langle \mu_k, \phi u^l\rangle_V \quad \text{by Lemma 4.1.}
\end{aligned}$$

So we get, $\forall\phi \in \mathcal{D}(\Omega)$,

$$-\int_\Omega \frac{\partial\sigma_{ij}(u^l)}{\partial x_j} e_{ki}\phi + {}_{V'}\langle \mu_k, \phi u^l\rangle_V = \lambda^l \int_\Omega u_i^l e_{ki}\phi.$$

This shows that λ^l, u^l is a solution of the system (4.5). In fact, λ^l is the l-th eigenvalue and u^l is the corresponding eigenvector and the proof of which is quite standard and hence we will omit the proof here (see [15]).

The convergence (4.6) follows by taking $v = u_\varepsilon$ in (2.2) and then, after passing to the limit, use the system (4.5). This completes the proof of Theorem 4.1.

5. Correctors

Here we introduce the correctors for the eigenvalues and eigenvectors. Let $W_\varepsilon = (w_\varepsilon^k)_{1\leqslant k\leqslant N}$ be the matrix with columns w_ε^k given by Lemma 4.1. Let u_ε and u be eigenvectors of the system (2.1) and (4.5), respectively, and $\widetilde{u}_\varepsilon \to u$ in V weak. Let

$$r_\varepsilon = \widetilde{u}_\varepsilon - W_\varepsilon u. \tag{5.1}$$

Then we have the following theorem

Theorem 5.1. *Let r_ε be given by* (5.1). *Then*

$$r_\varepsilon \to 0 \quad \text{in } V^0 \text{ strong.} \tag{5.2}$$

Proof. Since u is an eigenvector, $u \in C^\infty(\overline{\Omega})^N \cap V^0$ and hence given a small number $\eta > 0$, for any $1 \leqslant p < \infty$ choose $\phi \in \mathcal{D}(\Omega)^N$ such that

$$\|u-\phi\|^2_{L^2(\Omega)} + \|u-\phi\|_{W_0^{1,p}(\Omega)} \leqslant \eta. \tag{5.3}$$

Now

$$\sigma_{ij}(r_\varepsilon) = \sigma_{ij}\big(\widetilde{u}_\varepsilon - W_\varepsilon\phi\big) + \sigma_{ij}\big(W_\varepsilon(\phi-u)\big).$$

But $\|\sigma_{ij}(W_\varepsilon(\phi-u))\|_{L^2(\Omega)}$ can be make arbitrarily small enough using the inequality (5.3) and the fact that w_ε^k is bounded in V independent of ε. (Also use the fact that $H^1(\Omega) \hookrightarrow L^q(\Omega)$, where $1 \leqslant q \leqslant 2N/(N-2)$ if $N \geqslant 3$ and $1 \leqslant q < \infty$ if $N = 2$). Now without much difficulty, by expanding $\sigma_{ij}(\widetilde{u}_\varepsilon - W_\varepsilon\phi)$ and using (4.1, iv), (4.2) and (4.6), one can see that

$$\big\|\sigma_{ij}\big(\widetilde{u}_\varepsilon - W_\varepsilon\phi\big)\big\|_{L^2(\Omega)} \to \big\|\sigma_{ij}(u-\phi)\big\|_{L^2(\Omega)} + \big\langle M(u-\phi), u-\phi\big\rangle.$$

The right-hand side of the above equation can be made small because of (5.3) and

$$M \in W^{-1,\infty}(\Omega)^{N^2}.$$

Hence it follows that $\sigma_{ij}(r_\varepsilon) \to 0$ in $L^2(\Omega)$ strong. Now using Korn's inequality we get $\nabla r_\varepsilon \to 0$ in $L^2(\Omega)$ strong, which completes the proof of Theorem 5.1. □

The above result can be interpreted as a corrector result for eigenvectors. Next, we will give the corrector result for the eigenvalues λ_ε. Let U_ε be the unique solution of the following problem:

$$b_\varepsilon(U_\varepsilon, v) = \lambda(u, v)_\varepsilon, \quad \forall v \in V^0_{a_\varepsilon},\ U_\varepsilon \in V^0_{a_\varepsilon}, \tag{5.4}$$

where λ is a simple eigenvalue and u is the normalized eigenvector corresponding to λ of the homogenized problem (4.5). Here b_ε is the bilinear form given by (2.3, i). Then one can see that (following the proof of Theorem 4.1)

$$\widetilde{U}_\varepsilon \to u \quad \text{in } V^0 \text{ weakly.} \tag{5.5}$$

Moreover, using the same proof as in Theorem 5.1, we get

$$\widetilde{U}_\varepsilon - W_\varepsilon u \to 0 \quad \text{in } V^0 \text{ strongly.}$$

Let λ_ε be the eigenvalue of the problem (2.1) converging to λ and u_ε be the corresponding (normalized) eigenvector. Then λ_ε will be simple for ε small. It follows that (using the above strong convergence)

$$\widetilde{U}_\varepsilon - \widetilde{u}_\varepsilon \to 0 \quad \text{in } V^0 \text{ strongly.}$$

Now taking $v = u_\varepsilon$ in (5.4) and $v = U_\varepsilon$ in (2.2), we get

$$\lambda_\varepsilon = \frac{(u, u_\varepsilon)_\varepsilon}{(u_\varepsilon, U_\varepsilon)_\varepsilon}\lambda. \tag{5.6}$$

Then it is easy to see that

$$|\lambda_\varepsilon - \lambda| \leqslant C\|u - \widetilde{U}_\varepsilon\|_{L^2(\Omega)}. \tag{5.7}$$

Since $\widetilde{U}_\varepsilon - W_\varepsilon u \to 0$ strongly in V^0, the convergence of $\|\widetilde{U}_\varepsilon - W_\varepsilon u\|_{L^2(\Omega)}$ is better than the convergence of $\|\widetilde{U}_\varepsilon - u\|_{L^2(\Omega)}$. Hence our aim, in the remaining part of this section, is to obtain an estimate of $\lambda_\varepsilon - \lambda$ in terms of $\|\widetilde{U}_\varepsilon - W_\varepsilon u\|_{L^2(\Omega)}$. The estimate of $\lambda_\varepsilon - \lambda$ (see (5.8) below) also contains terms of the form $\|u - \widetilde{U}_\varepsilon\|^2_{L^2(\Omega)}$ and $\|Id - W_\varepsilon\|_{L^2(\Omega)}\,\|u - \widetilde{U}_\varepsilon\|_{L^2(\Omega)}$ and observe that these two terms converge to zero much faster than $\|u - \widetilde{U}_\varepsilon\|_{L^2(\Omega)}$ and we have the following result.

Theorem 5.2. *Let λ_ε and λ be simple eigenvalues of the problem* (2.1) *and* (5.4), *respectively, and $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$. Let u_ε and u be the normalized eigenvectors corresponding to λ_ε and λ, respectively, then $(u, (2Id - W_\varepsilon)u)$ acts as a corrector in the following sense*:

$$\begin{aligned}\left|\lambda_\varepsilon - \big(u, (2Id - W_\varepsilon)u\big)\lambda\right| \leqslant C\Big[&\|\widetilde{U} - W_\varepsilon u\|_{L^2(\Omega)} + \|\widetilde{U}_\varepsilon - u\|^2_{L^2(\Omega)} \\ &+ \|Id - W_\varepsilon\|_{L^2(\Omega)}\|\widetilde{U}_\varepsilon - u\|_{L^2(\Omega)}\Big].\end{aligned} \tag{5.8}$$

Proof. We roughly sketch the proof. For a detailed proof in other cases see [15]. Using the following equality

$$\lambda_\varepsilon\big(u - u_\varepsilon, \widetilde{U}_\varepsilon - u_\varepsilon\big)_\varepsilon = \big[\lambda_\varepsilon(u, U_\varepsilon)_\varepsilon - \lambda\big] + (\lambda_\varepsilon - \lambda)\big(u, (Id - W_\varepsilon u)\big)_\varepsilon$$

we get

$$\begin{aligned}\lambda_\varepsilon - \big(u, (2Id - W_\varepsilon)u\big)\lambda = {}&\lambda_\varepsilon\big(u, W_\varepsilon u - \widetilde{U}_\varepsilon\big) + (\lambda_\varepsilon - \lambda)\big(u, (Id - W_\varepsilon)u\big) \\ &+ \lambda_\varepsilon\big(u - \widetilde{u}_\varepsilon,\ \widetilde{U}_\varepsilon - u\big) - (\lambda - \lambda_\varepsilon)\big(u, u - \widetilde{u}_\varepsilon\big).\end{aligned}$$

Then the result (5.8) will follow from (5.7) and if we assume that the following estimate (5.9) is true

$$\|u - \widetilde{u}_\varepsilon\|_{L^2(\Omega)} \leqslant C\|u - \widetilde{U}_\varepsilon\|_{L^2(\Omega)}. \tag{5.9}$$

To prove this, introduce z_ε as the solution of

$$\begin{aligned} &b_\varepsilon(z_\varepsilon, v) - \lambda_\varepsilon(z_\varepsilon, v)_\varepsilon = \lambda_\varepsilon(U_\varepsilon, v)_\varepsilon - \lambda(u, v)_\varepsilon, \\ &\quad \forall v \in V^0_{a_\varepsilon},\ z_\varepsilon \in V^0_{a_\varepsilon} \text{ and } (z_\varepsilon, u_\varepsilon)_\varepsilon = 0. \end{aligned} \tag{5.10}$$

Using a contradictory argument and using the fact that u is a simple eigenvector, one proves that

$$\|\sigma_{ij}(z_\varepsilon)\|_{L^2(\Omega_{a_\varepsilon})} \leqslant C\|\lambda_\varepsilon U_\varepsilon - \lambda u\|_{L^2(\Omega_{a_\varepsilon})} \tag{5.11}$$

where C is independent of ε. Moreover, from (5.10), we can see that $z_\varepsilon + U_\varepsilon$ is an eigenvector corresponding to λ_ε and since λ_ε is simple, after normalizing $z_\varepsilon + U_\varepsilon$, we get

$$u_\varepsilon = z_\varepsilon + U_\varepsilon.$$

Thus

$$\|u - \widetilde{u}_\varepsilon\|_{L^2(\Omega)} \leqslant \|u - \widetilde{U}_\varepsilon\|_{L^2(\Omega)} + \|\widetilde{z}_\varepsilon\|_{L^2(\Omega)},$$

so that the estimate (5.9) follows, because by using (5.11), we have

$$\begin{aligned} \|\widetilde{z}_\varepsilon\|_{L^2(\Omega)} &\leqslant C\|\lambda_\varepsilon(U_\varepsilon - u)\|_{L^2(\Omega_{a_\varepsilon})} + \|(\lambda_\varepsilon - \lambda)u\|_{L^2(\Omega_{a_\varepsilon})} \\ &\leqslant C\|u - \widetilde{U}_\varepsilon\|_{L^2(\Omega)}. \end{aligned}$$

This completes the proof of Theorem 5.2. □

6. Test functions w^k_ε and the proof of Lemma 4.1

First define w^k_ε in the ε-periodic cell Y_ε in the following way. Let e_k be the k-th unit vector in the canonical basis of $\mathbb{R}^N$. Define w^K_ε as the unique solution of the following problem:

$$\begin{aligned} &-\frac{\partial}{\partial x_j}\sigma_{ij}(w^k_\varepsilon) = 0 \quad \text{in } B_{\varepsilon/2} \setminus \overline{T}_{a_\varepsilon}, \\ &w^k_\varepsilon = 0 \quad \text{in } T_{a_\varepsilon}, \\ &w^k_\varepsilon = e_k \quad \text{in } Y_\varepsilon \setminus \overline{B}_{\varepsilon/2}. \end{aligned} \tag{6.1}$$

Then extend w^k_ε to all of $\mathbb{R}^N$ using ε-periodicity. It will be shown that w^k_ε defined in this way will satisfy Lemma 4.1. First, let us prove the following

Claim. *Let a_ε be as in* (1.2). *Then*

$$\|w^k_\varepsilon\|_{H^1(\Omega)} \leqslant \text{constant}, \quad \textit{independent of } \varepsilon. \tag{6.2}$$

Proof. Choose a sequence of test functions w_ε satisfying the following properties

i) w_ε is bounded in $H^1(\Omega)$, independent of ε,

ii) $w_\varepsilon = 0$ in $T^i_{a_\varepsilon}$, $\forall i \in I_\varepsilon$, $w_\varepsilon = 1$ in $\Omega \setminus \bigcup_{i \in I_\varepsilon} B^i_{\varepsilon/2}$ and $0 \leqslant w_\varepsilon \leqslant 1$.

For the existence of such functions, for the above choice of a_ε one can refer to [4]. Let $\overline{w}^k_\varepsilon = w^k_\varepsilon - w_\varepsilon e_k$. Then $\overline{w}^k_\varepsilon$ satisfies

$$-\frac{\partial}{\partial x_j}\sigma_{ij}(\overline{w}^k_\varepsilon) = \frac{\partial}{\partial x_j}\sigma_{ij}(w_\varepsilon e_k) \quad \text{in } B_{\varepsilon/2} \setminus \overline{T}_{a_\varepsilon},$$

$$\overline{w}^k_\varepsilon = 0 \quad \text{on } \partial B_{\varepsilon/2} \cup \partial T_{a_\varepsilon}.$$

Multiplying the equation by $\overline{w}^k_\varepsilon$ and integrating by parts, we get

$$\left\|\sigma_{ij}(\overline{w}^k_\varepsilon)\right\|_{L^2(B_{\varepsilon/2}\setminus T_{a_\varepsilon})} \leqslant \left\|\sigma_{ij}(w_\varepsilon e_k)\right\|_{L^2(B_{\varepsilon/2}\setminus T_{a_\varepsilon})} \leqslant C\|\nabla w_\varepsilon\|_{L^2(B_{\varepsilon/2}\setminus T_{a_\varepsilon})}.$$

Hence we have

$$\left\|\sigma_{ij}(\overline{w}^k_\varepsilon)\right\|^2_{L^2(\Omega)} \leqslant \|\nabla w_\varepsilon\|^2_{L^2(\Omega)} \leqslant C.$$

Since $\overline{w}^k_\varepsilon = 0$ on $\partial B_{\varepsilon/2}$, using the transformation

$$x \in B_{\varepsilon/2} \mapsto y = \frac{x}{\varepsilon} \in B_{1/2}$$

and applying Korn's inequality, we get

$$\left\|\nabla \overline{w}^k_\varepsilon\right\|^2_{L^2(B_{\varepsilon/2})} \leqslant C\left\|\sigma_{ij}(\overline{w}^k_\varepsilon)\right\|^2_{L^2(B_{\varepsilon/2})}.$$

Summing overall cells and using the estimate obtained above, we get

$$\left\|\nabla \overline{w}^k_\varepsilon\right\|_{L^2(\Omega)} \leqslant C.$$

Hence it follows that

$$\left\|\nabla w^k_\varepsilon\right\|_{L^2(\Omega)} \leqslant C.$$

This completes the proof of the claim (6.2). □

By the above claim, it follows that $w^k_\varepsilon \to w^k$ in V weak and hence in $L^2(\Omega)^N$ strong. In fact, we have $w^k = e_k$. To see this, let $\chi_\varepsilon(x)$ be the characteristic function of

$$\Omega \setminus \bigcup_{i \in I_\varepsilon} B^i_{\varepsilon/2}.$$

Then, we have $\chi_\varepsilon w_\varepsilon^k = \chi_\varepsilon e_k$. Observe that if χ is the characteristic function of $Y \setminus B_{1/2}$, then $\chi_\varepsilon(x) = \chi(x/\varepsilon)$, so that χ_ε converges to a positive quantity in $L^\infty(\Omega)$ weak-star namely the average of χ. Therefore by passing to the limit we get $w^k = e_k$. So it remains to prove w_ε^k defined by (6.1) satisfies (4.1, iv). To do this we construct certain approximation to w_ε^k. In the case of dimension 2, such approximate solution will be obtained directly from the fundamental solution. This procedure will not work, in the case of dimension $N \geqslant 3$ (see Remark 6.1). So, for $N \geqslant 3$, we proceed differently. The method we employ for $N \geqslant 3$ does not seem to work for $N = 2$. So we need to treat the two cases $N = 2$, $N \geqslant 3$ separately. □

Now let us consider fundamental solutions of elasticity system. Define a solution u^k of the following equation:

$$-\frac{\partial}{\partial x_j}\sigma_{ij}(u^k) = \delta(x-y)e_{ki} \quad \text{in } \mathbf{R}^N.$$

Then a solution u^k with singularity at y is given by (see [3])

$$u_i^k = \begin{cases} \dfrac{1}{4\pi}\left(-3\log r\cdot e_{ki} + \dfrac{x_i x_k}{r^2}\right), & \text{if } N = 2,\\[2mm] \dfrac{1}{2S_N}\left(\dfrac{3}{N-2}\dfrac{e_{ki}}{r^{N-2}} + \dfrac{x_i x_k}{r^N}\right), & \text{if } N \geqslant 3. \end{cases} \tag{6.3}$$

Here $r = |x-y|$ and S_N is the area of the unit sphere in $\mathbf{R}^N$.

Case $N = 2$. To simplify matters, we assume T is a ball with centre at the origin and radius 1. Therefore $a_\varepsilon T = B_{a_\varepsilon}$. This is not a serious restriction because for any T, let B be some ball such that $T \subset B$ and then the approximate solution in $B_{\varepsilon/2} \setminus a_\varepsilon B$ will be an approximate solution in $B_{\varepsilon/2} \setminus a_\varepsilon T$ and we can work out the details in a similar way.

Define g_ε^k in $B_{\varepsilon/2} \setminus B_{a_\varepsilon}$ as follows

$$\begin{aligned} g_{\varepsilon i}^k &= e_{ki} - \frac{3}{3C_0}4\pi u_i^k \quad \text{in } B_{\varepsilon/2} \setminus B_{a_\varepsilon} \\ &= e_{ki} - \frac{\varepsilon^2}{3C_0}\left(-3\log r\cdot e_{ki} + \frac{x_k x_i}{r^2}\right) \end{aligned} \tag{6.4}$$

where C_0 is the constant given in the definition of a_ε and $r = |x|$. The $g_{\varepsilon i}^k$ defined by (6.4) satisfies

$$\frac{\partial}{\partial x_j}\sigma_{ij}(g_\varepsilon^k) = 0 \quad \text{in } B_{\varepsilon/2} \setminus B_{a_\varepsilon},$$

$$g_{\varepsilon i}^k\big|_{\partial B_{\varepsilon/2}} = e_{ki} - \frac{\varepsilon^2}{3C_0}\left(-3\log\frac{\varepsilon}{2}\cdot e_{ki} + \frac{(x_i x_k)|_{\partial B_{\varepsilon/2}}}{(\varepsilon/2)^2}\right),$$

$$g_{\varepsilon i}^k\big|_{\partial B_{a_\varepsilon}} = \frac{-\varepsilon^2}{3C_0 a_\varepsilon^2}(x_i x_k)\Big|_{\partial B_{a_\varepsilon}}$$

Since

$$(x_ix_k)\big|_{\partial B_{\varepsilon/2}} \leqslant \left(\frac{\varepsilon}{2}\right)^2 \quad \text{and} \quad (x_ix_k)\big|_{\partial B_{a_\varepsilon}} \leqslant a_\varepsilon^2,$$

it follows that

$$g_\varepsilon^k\big|_{\partial B_{\varepsilon/2}} \sim e_k + \left|\varepsilon^2 \log \varepsilon\right| \quad \text{and} \quad g_\varepsilon^k\big|_{\partial B_{a_\varepsilon}} \sim \varepsilon^2. \tag{6.6}$$

This shows that g_ε^k is an approximation to w_ε^k in $B_{\varepsilon/2} \setminus B_{a_\varepsilon}$ defined by (6.1). Define the error:

$$z_\varepsilon^k = w_\varepsilon^k - g_\varepsilon^k \quad \text{in } B_{\varepsilon/2} \setminus B_{a_\varepsilon}.$$

Then z_ε^k satisfies

$$\begin{aligned} &\text{i)} && \frac{\partial}{\partial x_j}\sigma_{ij}\left(z_\varepsilon^k\right) = 0 \quad \text{in } B_{\varepsilon/2} \setminus \overline{B}_{a_\varepsilon}, \\ &\text{ii)} && z_\varepsilon^k \sim \left|\varepsilon^2 \log \varepsilon\right| \quad \text{on } \partial B_{\varepsilon/2}, \\ &\text{iii)} && z_\varepsilon^k \sim \varepsilon^2 \quad \text{on } \partial B_{a_\varepsilon}. \end{aligned} \tag{6.7}$$

Note that w_ε^k is defined in all of Y_ε. Now extend z_ε^k to all of Y_ε in some way for example, define

$$\frac{\partial}{\partial x_j}\sigma_{ij}\left(z_\varepsilon^k\right) = 0 \quad \text{in } Y_\varepsilon \setminus \overline{B}_{\varepsilon/2},$$

z_ε^k is Y-periodic and continuous across $\partial B_{\varepsilon/2}$

and in B_{a_ε}

$$\frac{\partial}{\partial x_j}\sigma_{ij}\left(z_\varepsilon^k\right) = 0 \quad \text{in } B_{a_\varepsilon},$$

z_ε^k is continuous across $\partial B_{a_\varepsilon}$.

We can then extend z_ε^k periodically to all of $\mathbb{R}^2$. We denote the extension also by z_ε^k. From (6.7, ii and iii) it follows that on $\partial B_{\varepsilon/2}$,

$$\frac{z_\varepsilon^k}{|\varepsilon^2 \log \varepsilon|} \sim \mathrm{O}(1)$$

and on $\partial B_{a_\varepsilon}$,

$$\frac{z_\varepsilon^k}{|\varepsilon^2 \log \varepsilon|} \sim \frac{\varepsilon^2}{|\varepsilon^2 \log \varepsilon|} \to 0 \quad \text{as } \varepsilon \to 0,$$

so that in a similar fashion as in (6.2), we get

$$\left\|\nabla\left(\frac{z_\varepsilon^k}{|\varepsilon^2 \log \varepsilon|}\right)\right\|_{L^2(\Omega)} \leqslant C$$

and hence

$$\|\nabla z_\varepsilon^k\|_{L^2(\Omega)} \leqslant C|\varepsilon^2 \log \varepsilon|.$$

Note that g_ε^k was defined only in the domain

$$\bigcup_{l\in I_\varepsilon} (B_{\varepsilon/2}^l \setminus B_{a_\varepsilon}^l).$$

Now, since z_ε^k, w_ε^k were defined in all of Ω, one can define g_ε^k in all of Ω satisfying the equation $g_\varepsilon^k = w_\varepsilon^k - z_\varepsilon^k$ in Ω.

Now observe that if we consider

$$\frac{\partial}{\partial x_j}\sigma_{ij}(w_\varepsilon^k) \quad \text{in } \Omega,$$

it is concentrated only on the boundary of the balls $B_{\varepsilon/2}^l$ and $B_{a_\varepsilon}^l$, for $l \in I_\varepsilon$. More precisely,

$$_{V'}\left\langle -\frac{\partial}{\partial x_j}\sigma_{ij}(w_\varepsilon^k), \phi\right\rangle_V = \sum_{l\in I_\varepsilon}\int_{\partial B_{\varepsilon/2}^l \cup \partial B_{a_\varepsilon^l}} \sigma_{ij}(w_\varepsilon^k) n_j \phi_i \,\mathrm{d}s.$$

In terms of g_ε^k and z_ε^k, we can write,

$$-\frac{\partial}{\partial x_j}\sigma_{ij}(w_\varepsilon^k) = -\frac{\partial}{\partial x_j}\sigma_{ij}(z_\varepsilon^k) - \frac{\partial}{\partial x_j}\sigma_{ij}(g_\varepsilon^k) = (\overline{\mu}_\varepsilon^k - \overline{\gamma}_\varepsilon^k) + (\mu_\varepsilon^k - \gamma_\varepsilon^k),$$

where $\overline{\mu}_\varepsilon^k$, $\overline{\gamma}_\varepsilon^k$, μ_ε^k, γ_ε^k all belong to V' and are given by

i) $$_{V'}\langle \overline{\mu}_\varepsilon^k, \phi\rangle_V = \sum_{l\in I_\varepsilon}\int_{\partial B_{\varepsilon/2}^l} \sigma_{ij}(z_\varepsilon^k) n_j \phi_i \,\mathrm{d}s,$$

ii) $$_{V'}\langle \overline{\gamma}_\varepsilon^k, \phi\rangle_V = \sum_{l\in I_\varepsilon}\int_{\partial B_{a_\varepsilon^l}} \sigma_{ij}(z_\varepsilon^k) n_j \phi_i \,\mathrm{d}s,$$

iii) $$_{V'}\langle \mu_\varepsilon^k, \phi\rangle_V = \sum_{l\in I_\varepsilon}\int_{\partial B_{\varepsilon/2}^l} \sigma_{ij}(g_\varepsilon^k) n_j \phi_i \,\mathrm{d}s,$$

iv) $$_{V'}\langle \gamma_\varepsilon^k, \phi\rangle_V = \sum_{l\in I_\varepsilon}\int_{\partial B_{a_\varepsilon^l}} \sigma_{ij}(g_\varepsilon^k) n_j \phi_i \,\mathrm{d}s,$$

for all $\phi \in V$.

Hence, for any v_ε, v, and ϕ as in Lemma 4.1, i.e., $v_\varepsilon \in V$, $v_\varepsilon = 0$ in $T^l_{a_\varepsilon}$, $l \in I_\varepsilon$, and $v_\varepsilon \to v$ in V weak and $\phi \in \mathcal{D}(\Omega)$, we have

$$_{V'}\left\langle -\frac{\partial}{\partial x_j}\sigma_{ij}(w^k_\varepsilon), \phi v_\varepsilon \right\rangle_V = {}_{V'}\langle \overline{\mu}^k_\varepsilon, \phi v_\varepsilon\rangle_V + {}_{V'}\langle \mu^k_\varepsilon, \phi v_\varepsilon\rangle_V.$$

Multiplying (6.7, i) by ϕv_ε and integrating by parts, we get

$$\int_{\partial B_{\varepsilon/2}} \sigma_{ij}(z^k_\varepsilon) n_j \phi v_\varepsilon = \int_{B_{\varepsilon/2}\setminus B_{a_\varepsilon}} \sigma_{ij}(z^k_\varepsilon)\sigma_{ij}(\phi v_\varepsilon).$$

So that, by summing over all cells and using (6.8) we get

$$\left|{}_{V'}\langle \overline{\mu}^k_\varepsilon, \phi v_\varepsilon\rangle_V\right| \leqslant \int_\Omega |\sigma_{ij}(z^k_\varepsilon)|\,|\sigma_{ij}(\phi v_\varepsilon)| \leqslant C|\varepsilon^2 \log \varepsilon| \to 0 \quad \text{as } \varepsilon \to 0.$$

So it suffices to prove that

$$_{V'}\langle \mu^k_\varepsilon, \phi v_\varepsilon\rangle_V \to {}_{V'}\langle \mu_k, \phi v\rangle_V \quad \text{as } \varepsilon \to 0, \tag{6.10}$$

for some $\mu_k \in (W^{-1,\infty}(\Omega))^N$.

Now the technique is similar to the one in Laplacian and Stokes case. We prove that

$$\mu^k_\varepsilon \to \mu_k \quad \text{in } V' \text{ strongly},$$

which in turn will imply (6.10).

Using formulae (6.4) and $n_j = x_j/r$, we get

$$\sigma_{ij}(g^k_\varepsilon) n_j\Big|_{\partial B_{\varepsilon/2}} = \frac{2\varepsilon}{3C_0} e_{ki} + \frac{4\varepsilon}{3C_0}\left(\frac{x_i x_k}{r^2}\right)\Big|_{\partial B_{\varepsilon/2}},$$

so that if x_l is the centre of the ball $B^l_{\varepsilon/2}$, $l \in I_\varepsilon$, then for any $\phi \in H^1_0(\Omega)$, we have

$$\begin{aligned}\langle \mu^k_{\varepsilon i}, \phi\rangle &= \sum_l \int_{\partial B^l_{\varepsilon/2}} \left[\frac{2\varepsilon}{3C_0} e_{ki} + \frac{4\varepsilon}{3C_0}\frac{(x_i - x_{li})(x_k - x_{lk})}{|x - x_l|^2}\right]\phi(s)\,\mathrm{d}s \\ &= \frac{2e_{ki}}{3C_0}\sum_l \varepsilon\langle \delta^l_{\varepsilon/2}, \phi\rangle + \frac{4}{3C_0}\sum_l \varepsilon\langle g^l_{ik}\delta^l_{\varepsilon/2}, \phi\rangle\end{aligned} \tag{6.11}$$

where

$$g^l_{ik}(x) = \frac{(x_i - x_{li})(x_k - x_{lk})}{|x - x_l|^2}$$

and $\delta^l_{\varepsilon/2}$ is the Dirac mass concentrated on the sphere $\partial B^l_{\varepsilon/2}$ given by

$$\langle \delta^l_{\varepsilon/2}, \phi\rangle = \int_{\partial B^l_{\varepsilon/2}} \phi(s)\,\mathrm{d}s \quad \text{and} \quad \langle g^l_{ik}\delta^l_{\varepsilon/2}, \phi\rangle = \int_{\partial B^l_{\varepsilon/2}} g^l_{ik}(x)\phi(s)\,\mathrm{d}s.$$

Now to find the limit of the right-hand side of (6.11), we use the following lemma, whose proof may be found in [4] (see also [1]).

Lemma 6.1. *Let $\delta^l_{\varepsilon/2}$ be the Dirac mass concentrated on the sphere $\partial B^l_{\varepsilon/2}$ and let*

$$e^l_r = (x - x_l)/|x - x_l|,$$

where $x \in \partial B^l_{\varepsilon/2}$. Let S_N denote the area of the unit sphere in $\mathbb{R}^N$. Then for $N \geqslant 2$, we have

$$\sum_{l\in I_\varepsilon} \varepsilon\delta^l_{\varepsilon/2} \to \frac{S_N}{2^{2N-1}} \quad \text{in } H^{-1}(\Omega) \text{ strongly,} \tag{6.12}$$

$$\sum_{l\in I_\varepsilon} \varepsilon\delta^l_{\varepsilon/2}(e_k, e^l_r)e^l_r \to \frac{S_N}{N2^{2N-1}}e_k \quad \text{in } H^{-1}(\Omega)^N \text{ strongly,} \tag{6.13}$$

where (e_k, e^l_r) is the standard product in $\mathbb{R}^N$.

Now from the strong convergence (6.12) and (6.13), it follows that, for $N = 2$, we have

$$\sum_{l\in I_\varepsilon} \varepsilon\langle\delta^l_{\varepsilon/2}, \phi v_{\varepsilon i}\rangle \to \left\langle\frac{\pi}{4}, \phi v_i\right\rangle,$$

and

$$\sum_{l\in I_\varepsilon} \varepsilon\langle g^l_{ik}\delta^l_{\varepsilon/2}, \phi v_{\varepsilon i}\rangle \to \left\langle\frac{\pi}{8}e_{ki}, \phi v_i\right\rangle.$$

Hence it follows that (from (6.11))

$$\langle\mu_{\varepsilon i}, \phi v_{\varepsilon i}\rangle \to \left\langle\frac{\pi}{3C_0}e_{ki}, \phi v_i\right\rangle,$$

so that $\mu_k = \pi/(3C_0)e_k$, is a constant and belongs to $W^{-1,\infty}(\Omega)^N$. This completes the proof of the convergence (6.10) and hence Lemma 4.1, in the case $N = 2$.

Remark 6.1. The above method does not work for $N \geqslant 3$ for the following reasons. Observe that in $N = 2$, we defined $g^k_\varepsilon = e_k - cu^k$ and the constant $C = 4\pi/(3C_0)\varepsilon^2$ is determined in such a way that g^k_ε is an approximate solution. More precisely, let us observe that

$$g^k_{\varepsilon i}|_{\partial B_{a_\varepsilon}} = e_{ki} - \frac{C}{4\pi}\left(-3\log a_\varepsilon e_{ki} + \frac{x_i x_k|_{\partial B_{a_\varepsilon}}}{a^2_\varepsilon}\right)$$

and our choice of C implies that $3C/(4\pi)\log a_\varepsilon = 1$, so that the first two terms get cancelled and hence

$$g^k_{\varepsilon i}|_{\partial B_{a_\varepsilon}} = \mathrm{O}(\varepsilon^2).$$

For $N \geqslant 3$, the same procedure does not yield an approximation to w^k_ε. This can be seen as follows. Let us take $g^k_\varepsilon = e_k - Cu^k$, then

$$g^k_{\varepsilon i}|_{\partial B_{a_\varepsilon}} = e_{ki} - \frac{C}{2S_N}\left(\frac{3}{N-2}\frac{e_{ki}}{a^{N-2}_\varepsilon} + \frac{x_i x_k|_{\partial B_{a_\varepsilon}}}{a^N_\varepsilon}\right).$$

So if we choose C in such a manner that

$$\frac{C}{2S_N}\frac{3}{(N-2)a_\varepsilon^{N-2}} = 1.$$

Then

$$g_{\varepsilon i}^k|_{\partial B_{a_\varepsilon}} = -\frac{N-2}{3}\left(\frac{x_i x_k|_{\partial B_{a_\varepsilon}}}{a_\varepsilon^2}\right) = O(1),$$

i.e., $g_{\varepsilon i}^k|_{\partial B_{a_\varepsilon}}$ is not small. The reason lies in the fact that the two terms in the fundamental solution (6.3) behave differently depending on $N = 2$ and $N \geqslant 3$. Indeed the two terms in u^k on $\partial B_{a_\varepsilon}$, namely,

$$\frac{1}{a_\varepsilon^{N-2}} \quad \text{and} \quad \frac{x_i x_k|_{\partial B_{a_\varepsilon}}}{a_\varepsilon^N}$$

are of the same order for $N \geqslant 3$, but for $N = 2$ the first term is $\log a_\varepsilon$ which is of order ε^{-2} and the second term

$$\frac{x_i x_k|_{\partial B_{a_\varepsilon}}}{a_\varepsilon^2}$$

is of order 1.

Case $N \geqslant 3$. Here we adopt the same method as in the case of Stokes system (see [1]). Define w^k as follows

$$\begin{aligned}
&-\frac{\partial}{\partial x_j}\sigma_{ij}(w^k) = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{T},\\
&w^k = 0 \quad \text{on } \partial T,\\
&w^k \to e_k \quad \text{as } |x| \to \infty,\\
&\|\sigma_{ij}(w^k)\|_{L^2(\mathbb{R}^N\setminus T)} < \infty.
\end{aligned} \tag{6.14}$$

This problem can be solved using Beppo Levi spaces [6]. Also see [13].

Now we define the approximate solution g_ε^k in $B_{\varepsilon/2}$ as:

$$g_\varepsilon^k(x) = \begin{cases} w^k\left(\dfrac{x}{a_\varepsilon}\right) & \text{in } B_{\varepsilon/2} \setminus \overline{B}_{a_\varepsilon},\\ 0 & \text{in } B_{a_\varepsilon}. \end{cases} \tag{6.15}$$

Observe that g_ε^k is an approximation to w_ε^k because

$$g_\varepsilon^k|_{\partial B_{a_\varepsilon}} = 0, \qquad g_\varepsilon^k|_{\partial B_{\varepsilon/2}} = w^k|_{\partial B_{\varepsilon/2a_\varepsilon}}$$

which converges to e_k as $\varepsilon \to 0$ since $\varepsilon/(2a_\varepsilon) \to \infty$ as $\varepsilon \to 0$ and g_ε^k satisfies the elasticity system.

Now using (6.14), it follows that

$$\|\sigma_{ij}(g_\varepsilon^k)\|^2_{L^2(B_{\varepsilon/2})} \leqslant C a_\varepsilon^{N-2},$$

so that

$$\|\sigma_{ij}(g_\varepsilon^k)\|^2_{L^2\left(\bigcup_{l\in I_\varepsilon} B^l_{\varepsilon/2}\right)} \leqslant C\,\frac{a_\varepsilon^{N-2}}{\varepsilon^N} = C,$$

for our choice of a_ε.

(This method does not work for $N = 2$ because in this case

$$\|\sigma_{ij}(g_\varepsilon^k)\|^2_{L^2\left(\bigcup_{l\in I_\varepsilon} B^l_{\varepsilon/2}\right)} \leqslant \frac{C}{\varepsilon^2}$$

which is unbounded as $\varepsilon \to 0$.)

Now define $z_\varepsilon^k = w_\varepsilon^k - g_\varepsilon^k$ in $B_{\varepsilon/2} \setminus B_{a_\varepsilon}$. Then one can extend z_ε^k in $Y_\varepsilon \setminus B_{\varepsilon/2}$ (also g_ε^k) as in the case of $N = 2$.

Again one has to prove (3.6, iv) of Lemma 4.1. We have

$$-\frac{\partial}{\partial x_j}\sigma_{ij}(w_\varepsilon^k) = -\frac{\partial}{\partial x_j}\sigma_{ij}(z_\varepsilon^k) - \frac{\partial}{\partial x_j}\sigma_{ij}(g_\varepsilon^k) = (\overline{\mu}_\varepsilon^k - \overline{\gamma}_\varepsilon^k) + (\mu_\varepsilon^k - \gamma_\varepsilon^k)$$

where $\overline{\mu}_\varepsilon^k$, $\overline{\gamma}_\varepsilon^k$, μ_ε^k, γ_ε^k all belong to V' with similar definitions as in (6.9). Hence for given v_ε, v, and ϕ as in Lemma 4.1, we get

$$_{V'}\left\langle -\frac{\partial}{\partial x_j}\sigma_{ij}(w_\varepsilon^k), \phi v_\varepsilon\right\rangle_V = {_{V'}}\langle \overline{\mu}_\varepsilon^k, \phi v_\varepsilon\rangle_V + {_{V'}}\langle \mu_\varepsilon^k, \phi v_\varepsilon\rangle_V. \tag{6.16}$$

So it remains to find the limit of the right hand side of (6.16). We express w^k using the fundamental solution (6.3). Assume T be the open set such that $T \subset B_1$, where B_1 is the ball with centre at the origin and radius 1. Let $\theta \in \mathcal{D}(B_1)$ such that $\theta \equiv 1$ in a neighbourhood of T and put

$$\widetilde{w}^k = (1-\theta)w^k, \quad \text{and} \quad f_i^k = -\frac{\partial}{\partial x_j}\sigma_{ij}(\widetilde{w}^k) \quad \text{in } \mathbb{R}^N. \tag{6.17}$$

Observe that $\widetilde{w}^k = w^k$ outside B_1, i.e., in $\mathbb{R}^N \setminus \overline{B}_1$ and $f_i^k \in \mathcal{D}(B_1)$. Now consider $f_m^k * u_i^m$, where u^m is the fundamental solution given by (6.3), which is smooth and satisfies the elasticity system in $\mathbb{R}^N$ and

$$\begin{aligned} f_m^k * u_i^m &= \frac{1}{2S_N}\left[\int_{\mathbb{R}^N} \frac{3}{N-2}\,\frac{e_{km}}{|x-y|^{N-2}}\,f_m^k(y)\,\mathrm{d}y + \int_{\mathbb{R}^N} \frac{(x_i-y_i)(x_m-y_m)}{|x-y|^N}\,f_m^k(y)\,\mathrm{d}y\right] \\ &= \frac{1}{2S_N}\left[\int_{B_1} \frac{3}{N-2}\,\frac{e_{km}}{|x-y|^{N-2}}\,f_m^k(y)\,\mathrm{d}y + \int_{B_1} \frac{(x_i-y_i)(x_m-y_m)}{|x-y|^N}\,f_m^k(y)\,\mathrm{d}y\right]. \end{aligned}$$

So if $|x|$ is large enough,

$$f_m^k * u_i^m \sim \frac{3}{2S_N(N-2)} \frac{e_{km}}{|x|^{N-2}} F_m^k + \frac{1}{2S_N} \frac{x_i x_m}{|x|^N} F_m^k$$

where

$$\begin{aligned} F_i^k &= \int_{B_1} f_i^k(y)\,\mathrm{d}y = -\int_{B_1} \frac{\partial}{\partial y_j} \sigma_{ij}(\tilde{w}^k)\,\mathrm{d}y = -\int_{\partial B_1} \sigma_{ij}(w^k) n_j\,\mathrm{d}s \\ &= -\int_{\partial T} \sigma_{ij}(w^k) n_j\,\mathrm{d}s. \end{aligned} \tag{6.18}$$

Here n_j denotes the unit normal to the ball B_1 or the hole T. Also

$$\nabla(f_m^k * u_i^m) \sim \mathrm{O}\left(\frac{1}{|x|^{N-1}}\right) \quad \text{as } |x| \to \infty,$$

and hence

$$\|\sigma_{ij}(f_m^k * u_i^m)\|_{L^2(\mathbb{R}^N)} < \infty.$$

Further,

$$w_i^k - (e_{ki} + f_m^k * u_i^m) \to 0 \quad \text{as } |x| \to \infty$$

and using the uniqueness of the elasticity system in $\mathbb{R}^N$, it follows that

$$\tilde{w}_i^k = e_{ki} + f_m^k * u_i^m$$

and hence for $|x|$ large enough,

$$w_i^k(x) \sim e_{ki} + \frac{3}{2S_N(N-2)} \frac{e_{km}}{|x|^{N-2}} F_m^k + \frac{1}{2S_N} \frac{x_i x_m}{|x|^N} F_m^k, \tag{6.19}$$

where F_m^k is given by (6.18).

So from (6.19) and from the definition of g_ε^k and z_ε^k, it follows that

$$z_\varepsilon^k|_{\partial T_{a_\varepsilon}} = 0$$

and

$$z_\varepsilon^k|_{\partial B_{\varepsilon/2}} \sim \left(\frac{a_\varepsilon}{\varepsilon}\right)^{N-2} \left[\frac{3}{2S_N(N-2)} 2^{N-2} F_i^k e_{ki} + \frac{2^{N-2}}{2S_N} F_m^k \frac{x_i x_m}{|x|^2}\bigg|_{\partial B_{\varepsilon/2}}\right]$$

and the right-hand side is of order $(a_\varepsilon/\varepsilon)^{N-2} = \varepsilon^2$ and in a similar fashion as in the case of dimension 2, one can show that

$$\|\sigma_{ij}(z_\varepsilon^k)\|_{L^2(\Omega)} \leqslant C\varepsilon^2,$$

and

$$\bar{\mu}_\varepsilon^k \to 0 \quad \text{in } V' \text{ strongly.} \tag{6.21}$$

Claim. *$\mu_\varepsilon^k \to \mu_k$ in V' strongly, where*

$$\mu_k = \frac{C_0^{N-2}}{2^N} F^k, \quad \text{and} \quad F^k = -\int_{\partial T} \sigma_{ij}(w^k) n_j \, ds, \tag{6.22}$$

where n_j is the exterior unit normal to the hole T.

Proof. Using (6.15), (6.19) and $n_j = x_j/|x|$ on $\partial B_{\varepsilon/2}$, we have

$$\sigma_{ij}(g_\varepsilon^k) n_j \sim \frac{1}{2S_N} \frac{a_\varepsilon^{N-2}}{|x|^{N-1}} \left[F_i^k + \sum_m \frac{N(F_m^k x_m x_i)}{|x|^2} \right]$$

so that

$$\sigma_{ij}(g_\varepsilon^k) n_j \Big|_{\partial B_{\varepsilon/2}} \sim \frac{C_0^{N-2} 2^{N-1}}{2S_N} \varepsilon \left[F_i^k + \frac{N(F_m^k x_m x_i)}{|x|^2} \Big|_{\partial B_{\varepsilon/2}} \right].$$

Hence we have

$$\begin{aligned} \mu_\varepsilon^k &= \frac{C_0^{N-2} 2^{N-1}}{2S_N} \left[\sum_{l \in I_\varepsilon} F_i^k \varepsilon \delta_{\varepsilon/2}^l + \sum_{l \in I_\varepsilon} N \varepsilon \delta_{\varepsilon/2}^l \left(\frac{F_m^k x_m x_i}{|x|^2} \right) \Big|_{\partial B_{\varepsilon/2}} \right] \\ &\to \frac{C_0^{N-2} 2^{N-1}}{2S_N} \left[F_i^k \frac{S_N}{2^{2N-1}} + \frac{S_N}{2^{2N-2}} F_i^k \right] \quad \text{in } V' \text{ strongly} \end{aligned}$$

by using Lemma 6.1. So it follows that

$$\mu_\varepsilon^k \to \mu_k,$$

where μ is given by (6.22) and this completes the proof of the claim and hence Lemma 4.1.

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