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PARTIAL EXACT CONTROLLABILITY OF A LINEAR THERMOELASTIC SYSTEM

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In this article, we prove the partial exact controllability of a one dimensional linear thermoelastic system. We use RHUM method which is a variation of HUM method to study the present system.

1. Introduction

In this short note, we study the partial exact controllability, which we will make precise later, of the following thermoelastic linear system (see [**6**])

$$u_{tt} - u_{xx} + \alpha\theta_x = 0, \quad 0 < x < L, \quad 0 < t < T \tag{1}$$

$$\theta_t - \theta_{xx} + \beta u_{xt} 0, \quad 0 < x < L, 0 < t < T \tag{2}$$

with initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \tag{3}$$

$$\theta(0, x) = \theta_0(x) \tag{4}$$

and boundary conditions

$$u(t, x) = v \text{ in } \Sigma \tag{5}$$

$$\theta(t, x) = 0 \text{ in } \Sigma. \tag{6}$$

Here u, θ are the unknowns and v is the control function; $Q = (0, L) \times (0, T)$ and $\Sigma = \{0\} \times (0, T) \cup \{L\} \times (0, T)$.

DEFINITION (Partial Exact Controllability) We say the system is partially

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exactly controllable if there exists $T>0$ such that for any given initial data (u_0, u_1, θ_0) in a suitable space, there exists a control function v such that the corresponding solution of system satisfies

$$u(T,\cdot)=u_t(T,\cdot)=0.$$

There is enormous literature in the field of exact controllability. For a good survey in this field using a new technique, the so called HUM method introduced by Lions, one can refer to Lions [4]-[5] and the references therein. We use the RHUM method (see [3]) to study our system. We would like to mention that RHUM is a variation of HUM method to deal with irreversible systems. We will not go into the details of the literature and one can see the above cited references.

2. Main Result

In this section, we study the following thermoelastic system described as in our introduction:

$$u_{tt}-u_{xx}+\alpha\theta_x=0 \text{ in } Q$$

$$\theta_t-\theta_{xx}+\beta u_{xt}=0 \text{ in } Q \tag{7}$$

where $Q=(0,L)\times(0,T)$, with the initial conditions

$$u(x,0)=u^0(x),\ u_t(x,0)=u^1(x),\ \theta(x,0)=\theta^0(x) \text{ in } (0,L) \tag{8}$$

and boundary control on u,

$$u(0,t)=g_1(t);\quad u(L,t)=g_2(t) \text{ in } (0,T)$$

$$\theta(0,t)=\theta(L,t)=0 \text{ in } (0,T). \tag{9}$$

As mentioned earlier our main aim is to obtain partial exact controllability of the above problem. Here we use the RHUM method introduced by Lions (see [3]). We have the following main theorem.

THEOREM 2.1 *If $\alpha\beta$ is sufficiently small and $T>L$, the system* (7)-(9) *is partially exactly controllable.*

To apply the RHUM method, we transform the problem in the following fashion. In other words, we convert the question of 'controllability' to 'reachability'.

Write $u=u_0+u_1$ and $\theta=\theta_0+\theta_1$, where (u_0,θ_0) is the solution of the system (7) with the conditions

$$u_0(x,0)=u^0(x),\quad u_{0t}(x,0)=u^1(x),\quad \theta_0(x,0)=\theta^0(x) \text{ in } (0,L) \tag{10}$$

and boundary conditions

$$u_0(0,t)=0=u_0(L,t),\quad \theta_0(0,t)=0=\theta_0(L,t) \text{ in } (0,T). \tag{11}$$

Then (u_1, θ_1) will satisfy the system (7) with zero initial conditions as

$$u_1(x,0)=0=u_{1t}(x,0),\quad \theta_1(x,0)=0 \text{ in } (0,L) \tag{12}$$

and the boundary conditions

$$u_1(0,t)=g_1(t),\quad u_1(L,t)=g_2(t) \text{ in } (0,T)$$

$$\theta_1(0,t)=0=\theta_1(L,t) \text{ in } (0,T). \tag{13}$$

So the problem reduces to the following: We want to find g_1, g_2 such that

$$u_1(T)=-u_0(T) \text{ and } u_{1t}(T)=-u_{0t}(T).$$

Now onwards, we consider the problem (7), (9) with the zero initial conditions

$$u(x,0)=0=u_t(x,0), \text{ and } \theta(x,0)=0 \text{ in } (0,L). \tag{14}$$

Let $z^0, z^1 \in X=H_0^1(\Omega)\times L^2(\Omega)$. (This is the space we use to obtain the controllability.) We want to find g_1, g_2 and $T_0>0$ such that the solution of (7), (9), (14) satisfies

$$u(\cdot,T_0)=z^0,\quad u_t(\cdot,T_0)=z^1.$$

We now use the RHUM method. Consider the adjoint system, for given $\phi^0, \phi^1 \in X$

$$\phi_{tt}-\phi_{xx}+\beta\eta_{xt}=0 \text{ in } Q$$

$$\eta_t+\eta_{xx}+\alpha\phi_x=0 \text{ in } Q$$

$$\phi(T)=\phi^0,\ \phi_t(T)=\phi^1,\ \eta(T)=0 \text{ in } (0,L)$$

$$\phi(0,t)=0=\phi(L,t),\ \eta(0,t)=0=\eta(L,t) \text{ in } (0,T). \tag{15}$$

Then consider the following system for (ψ, δ):

$$\psi_{tt} - \psi_{xx} + \alpha\delta_x = 0 \text{ in } Q$$

$$\delta_t - \delta_{xx} + \beta\psi_{xt} = 0 \text{ in } Q$$

$$\psi(x, 0) = 0 = \psi_t(x, 0),\ \delta(x, 0) = 0 \text{ in } (0, L)$$

$$\psi(0, t) = -\phi_x(0, t),\ \psi(L, t) = \phi_x(L, t) \text{ in } (0, T)$$

$$\delta(0, t) = 0 = \delta(L, t) \text{ in } (0, T). \tag{16}$$

Now introduce the operator Λ by

$$\Lambda\{\phi^0, \phi^1\} = \{-\psi'(T), \psi(T)\}.$$

Suppose the operator Λ is invertible, then solve

$$\Lambda\{\phi^0, \phi^1\} = \{-z^1, z^0\}.$$

Using this $\{\phi^0, \phi^1\}$ solve for $\{\phi, \eta\}$ and then for $\{\psi, \delta\}$. Then it is clear that our problem is solved by taking $u = \psi$ and $\theta = \delta$; and the control $\{g_1, g_2\}$ is given by

$$g_1(t) = -\phi_x(0, t), \quad g_2(t) = \phi_x(L, t) \text{ in } (0, T).$$

Also

$$u(T) = \psi(T) = z^0, \quad u_t(T) = \psi_t(T) = z^1.$$

We now calculate $\langle \Lambda\{\phi^0, \phi^1\}, \{\phi^0, \phi^1\}\rangle$. Multiply the equations (16) by ϕ, η respectively and (15) by ψ, δ respectively, integrate by parts, add and subtract the resulting equations to obtain the following equations:

$$\int_0^T (\phi_x^2(0, t) + \phi_x^2(L, t)) = \int_0^L (\phi^1\psi(T) - \phi^0\psi'(T)) + \beta\int_0^T\int_0^L \eta_{xt}\psi - \alpha\int_0^T\int_0^L \delta_x\phi \tag{17}$$

$$\beta\int_0^T\int_0^L \psi_{xt}\eta + \alpha\int_0^T\int_0^L \phi_x\delta = 0. \tag{18}$$

But

$$-\alpha \int\int \delta_x \phi = \alpha \int\int \delta \phi_x - \alpha \int_0^T \delta \phi |_0^L = \alpha \int\int \delta \phi_x,$$

and

$$\beta \int\int \eta_{xt} \phi = -\beta \int\int \eta_x \psi_t + \beta \int_0^L \eta_x \psi |_0^T$$

$$= \beta \int\int \eta \psi_{xt} - \beta \int_0^T \eta \psi_t |_0^L$$

$$= \beta \int\int \eta \psi_{xt},$$

because $\psi(x, 0) = 0$ and $\eta_x(x, T) = 0$ since $\eta(x, T) = 0$. So from (17) and (18), it follows that

$$\int_0^T (\phi_x^2(0, t) + \phi_x^2(L, t)) = \int_0^L (\phi^1 \psi(T) - \phi^0 \psi'(T)).$$

Now

$$\langle \Lambda \{\phi^0, \phi^1\}, \{\phi^0, \phi^1\} \rangle = \int_0^L (-\psi'(T)\phi^0 + \psi(T)\phi^1)$$

$$= \int_0^T (\phi_x^2(0, t) + \phi_x^2(L, t)). \tag{19}$$

We denote the right hand side of (19) as $\| \{\phi^0, \phi^1\} \|^2$. Now to prove Λ is invertible, it is enough to prove that $\| \{\phi^0, \phi^1\} \|^2$ is a norm equivalent to the standard norm in $H_0^1(\Omega) \times L^2(\Omega)$ and that will complete the proof of Theorem 4.1. We have to prove

$$\|\{\phi^0, \phi^1\}\|^2 \leq C_1 \left[\| \phi^0 \|_{H^1(\Omega)}^2 + \| \phi^1 \|_{L^2(\Omega)}^2 \right] \tag{20}$$

and

$$\| \{\phi^0, \phi^1\} \|^2 \geq C_2 \left[\| \phi^0 \|_{H^1(\Omega)}^2 + \| \phi^1 \|_{L^2(\Omega)}^2 \right] \tag{21}$$

for some constants C_1 and $C_2 > 0$.

PROOF OF (20). Choose $h(x)=\frac{2x-L}{L}$. Note that $h(0)=-1$ and $h(L)=+1$. Now multiply the equation $\phi_{tt}-\phi_{xx}+\beta\eta_{xt}$ by $h\,\phi_x$ and integrate by parts to get

$$\|\{\phi^0,\phi^1\}\|^2=\iint h_x\left(\phi_t^2+\phi_x^2\right)+2\int_0^L h\,\phi_t\,\phi_x\,|_0^T+2\beta\iint \eta_{xt}\,h\,\phi_x.$$

Now differentiate the equation (15) with respect to t, to obtain

$$\eta_{tt}+\eta_{xxt}+\alpha\phi_{xt}=0 \text{ in } Q. \tag{23}$$

Multiply (15) by ϕ_t and (23) by $\frac{\beta}{\alpha}\eta_t$, integrate by parts, in $(0,L)\times(t,T)$, add the equations to obtain

$$\frac{1}{2}\iint\frac{d}{dt}\left[\phi_t^2+\phi_x^2+\frac{\beta}{\alpha}\eta_t^2\right]=\frac{\beta}{\alpha}\iint\eta_{xt}^2 .$$

Using the boundary condition $\eta(x,T)=0$ and the equation, we can obtain $\eta_t(x,T)=-\alpha\frac{d\phi^0}{dx}$. Hence, we have

$$\frac{1}{2}\int_0^L\left(\phi_t^2+\phi_x^2+\frac{\beta}{\alpha}\eta_t^2\right)(x,t)+\frac{\beta}{\alpha}\iint\eta_{xt}^2=\frac{1}{2}\int_0^L\left[|\phi^1|^2+\left|\frac{d\phi^0}{dx}\right|^2+\alpha\beta\left|\frac{d\phi^0}{dx}\right|^2\right]$$

$$\leq C\left[\|\phi^0\|_{H^1}^2+\|\phi^1\|_{L^2}^2\right]. \tag{24}$$

This shows that

$$\{\phi,\phi_t\}\in L^\infty(0,T;H^1(\Omega)\times L^2(\Omega))$$

$$\eta_t\in L^\infty(0,T;H^1(Q))$$

$$\eta_x\in L^\infty(0,T;L^2(\Omega)),\quad \Omega=(0,L). \tag{25}$$

Moreover,

$$\frac{\beta}{\alpha}\|\eta_{xt}\|_{L^2(Q)}^2\leq C\left(\|\phi^0\|_{H^1}^2+\|\phi^1\|_{L^2}^2\right). \tag{26}$$

By applying these estimates in (22) we get (20).

LEMMA 2.1 *If $T > L$ and $\alpha\beta$ is small enough, then* (21) *holds.*

PROOF. Let $\hat{\phi}$ be the solution of the following problem

$$\hat{\phi}_{tt} - \hat{\phi}_{xx} = 0 \text{ in } Q$$

$$\hat{\phi}(T) = \phi^0, \ \ \hat{\phi}_t(T) = \phi_1 \text{ in } (0, L)$$

$$\hat{\phi}(0, t) = 0 = \hat{\phi}(L, t) \text{ in } (0, T). \tag{27}$$

Now write $\phi = \hat{\phi} + \xi$, then ξ satisfies

$$\xi_{tt} - \xi_{xx} = -\beta\eta_{xt} \text{ in } Q$$

$$\xi(T) = 0 = \xi_t(T) \text{ in } (0, L)$$

$$\xi(0, t) = 0 = \xi(L, t) \text{ in } (0, T). \tag{28}$$

CLAIM :

$$\int_0^T (\xi_x^2(0, t) + \xi_x^2(L, t)) \leq C\alpha\beta\left(\| \phi^0 \|_{H^1}^2 + \| \phi^1 \|_{L^2}^2 \right). \tag{29}$$

Define the energy of (28) as

$$E(t) = \int_0^L (\xi_t^2 + \xi_x^2).$$

Then $\dfrac{dE}{dt} = -2\beta\displaystyle\int_0^L \eta_{xt}\xi_t$ and since $E(T) = 0$, we have

$$E(t) = -\int_t^T \frac{dE}{dt}(\tau)\, d\tau = 2\beta \int_0^L \int_t^T \eta_{xt}\xi_t$$

$$\leq 2\beta \| \eta_{xt} \|_{L^2(Q)} \left[\int_0^L \int_t^T |\xi_t|^2 \right]^{1/2}$$

$$\leq 2\beta T^{1/2} \| \eta_{xt} \|_{L^2(Q)} \| \xi_t \|_{L^\infty(0, T; L^2)}.$$

So we get

$$\| \xi_t \|_{L^\infty(0, T; L^2)} \leq 2\beta T^{1/2} \| \eta_{xt} \|_{L^2(Q)}$$

and we have

$$\| \xi_t \|^2_{L^\infty(0,T,L^2)} + \| \xi_x \|^2_{L^\infty(0,T,L^2)} \le C\beta^2 \| \eta_{xt} \|^2_{L^2(Q)}$$

and

$$\| \xi_t \|^2_{L^2(Q)} + \| \xi_x \|^2_{L^2(Q)} \le C\beta^2 \| \eta_{xt} \|^2_{L^2(Q)}.$$

Now using (26) we get

$$\| \xi_t \|^2_{L^2(Q)} + \| \xi_x \|^2_{L^2(Q)} \le C\alpha\beta \left(\| \phi^0 \|^2_{H^1} + \| \phi^1 \|^2_{L^2} \right). \tag{30}$$

Again apply the multiplier technique, that is, multiply the equation (28) by $h\,\xi_x$, where $h(x) = \frac{2x-1}{L}$ to obtain

$$\int_0^T (\xi_x^2(0,t) + \xi_x^2(L,t)) = \iint h_x(\xi_x^2 + \xi_t^2) + \int_0^L h\,\xi_t\,\xi_x \Big|_0^T - \beta \iint h\,\eta_{xt}\,\xi_x.$$

Use (30) to obtain the proof of the claim.

Now

$$\| \{\phi^0, \phi^1\} \|^2 \ge \int_0^T [\hat{\phi}_x^2(0,t) + \hat{\phi}_x^2(L,t)] - \int_0^T [\xi_x^2(0,t) + \xi_x^2(L,t)].$$

Then the proof of Lemma 4.1, and hence the proof of theorem, will follow if we can show that

$$\int_0^T [\hat{\phi}_x^2(0,t) + \hat{\phi}_x^2(L,t)] \ge \gamma \left[\| \phi^0 \|^2_{H^1} + \| \phi^1 \|^2_{L^2} \right] \tag{32}$$

for some $\gamma > 0$.

CLAIM: (32) is true if $T > L$.

PROOF. Multiplying the equation (28) by $h\,\hat{\phi}_x$, where $h(x) = \frac{2x-L}{L}$ and integrating by parts, we get

$$\| \{\phi^0, \phi^1\} \|^2 = \iint (\phi_t^2 + \phi_x^2)\, h_x + 2\int_0^L h\,\phi_t\,\phi_x \Big|_0^T$$

$$= \frac{2}{L} \iint (\phi_t^2 + \phi_x^2) + 2\int_0^L h\,\phi_t\,\phi_x \Big|_0^T.$$

Define the energy $E(t) = \frac{1}{2}\int_0^L (\phi_t^2 + \phi_x^2)$, then it is easy to see that $E(t) = a$ constant $= E(T) = \frac{1}{2}\left[\| \frac{d\phi^0}{dx} \|_{L^2}^2 + \| \phi^1 \|_{L^2}^2\right]$.

Also, since $|h| \leq 1$, we have

$$\int_0^L |h\, \phi_t \phi_x| \leq \frac{1}{2}\int_0^L (\phi_t^2 + \phi_x^2) = E(T)$$

Combining these estimates in the previous identity, we can see that

$$\int_0^T \left[\hat{\phi}_x^2(0, t) + \hat{\phi}_x^2(L, t)\right] \geq 2\left(\frac{T}{L} - 1\right)\left(\| \frac{d\phi^0}{dx} \|_{L^2}^2 + \| \phi^1 \|_{L^2}^2\right)$$

Hence the inequality (32) is true if $T > L$, by the Poincaré inequality.

This completes the proof of partial exact controllability.

REMARK 2.1 After the completion of this work, it is learnt from Prof. Zuazua regarding the work of Prof. Hansen [1] in which he has investigated the following problem:

$$u_{tt} - c^2 u_{xx} + c^2 \gamma \theta_x = 0$$

$$\theta_t - \theta_{xx} + \gamma u_{xt} = 0,$$

with the following boundary conditions

$$\sigma(t, 0) = \theta(t, 0) = v(t, 1) = q(t, 1) = 0$$

where $v = u_t$, $q = -\theta_x$, $\sigma = u_x - \gamma\theta$.

He has studied the controllability of the vector $y(t) = (u_x\, u_t, \theta)$ by applying the boundary control either at $\theta(t, 0)$ or at $q(t, 1)$. He assumes that $0 < \gamma \leq 1$. On the other hand, we assumed that $\alpha\beta << 1$ where α is the coefficient of θ_x and β is that of u_{xt}. The main feature of his work is the obtension of the controllability of the temperature component θ.

In our paper, the boundary conditions are different and we obtain the partial exact controllability of u, u_t. Our techniques are based on the HUM method. For a recent work on thermoelasticity system, see Zuazua [7].

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