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Stackelberg's Optimization with Microstructure and Homogenization

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(In memory of my friend late Dr. Vivek Srinivas, Asst. Prof. ECE, IISc)

Abstract

In this article, we study the homogenization of a control problem, where the controls are acting on a subdomain which is periodically perforated. The periods are of order $\varepsilon > 0$, a small parameter, and we study the limiting behaviour of the controls and the solution as $\varepsilon \to 0$. We also prove the existence of a solution of a minimization problem by looking at the control problem in different way.

1 Introduction

In this short article, we prove a homogenization result of a control problem, when the controls are acting on a periodically perforated domain. We also prove the existence of a solution of a minimization problem. To make these assertions precise, first we have to describe the control problem which we will do it in this section.

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[0]

In [3], Lions has considered the following control problem:

$$\begin{aligned} &y' - \Delta y = v\chi_O \quad \text{in } \Omega_T = \Omega \times (0,T)\,, \\ &y = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T)\,, \\ &y(x,0) = 0 \quad \text{on } \Omega\,. \end{aligned} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary Γ. v is the control function which acts in an open set $O \subset \Omega$ and χ_O is the characteristic function of O. The problem is to find a control v so that the corresponding solution y achieves the following two objectives.

1. **(Approximate Controllability)** : Given a function $y^1 \in L^2(\Omega)$, we want to find a control v so that the solution y at time $T > 0$, is close enough to y^1. In other words, for given $\alpha > 0$, find a v so that the corresponding solution y of (1.1) satisfies

$$y(T) = y(\cdot,T) \in y^1 + \alpha B,$$

where B is the unit ball in $L^2(\Omega)$.

2. The second objective is that in the process of achieving the approximate controllability, we do not want the solution $y(x,t)$ "too far" from an apriorily fixed $y^2 = y^2(x,t) \in L^2(\Omega_T)$.

Physical Interpretation : We can think of $y = y(x,t)$ as being the concentration of some chemical product at time t, say, in a lake Ω and think of O as the region where one can apply a control. The first objective is to have, at time T, $y(t)$ "very close" of the optimal concentration y^1. We want to achieve this, in the course of action, without going "too far" from y^2.

A solution to the above problem can be obtained by minimizing the following two functionals simultaneously

$$J_1(v) = \frac{1}{2}\int_0^T \int_O v^2 \quad \text{subject to } y(T) \in y^1 + \alpha B \tag{1.2}$$

$$J_2(v) = \frac{1}{2}\int_0^T \int_\Omega (y - y^2)^2 + \frac{\beta}{2}\int_0^T \int_O v^2 \tag{1.3}$$

In general, it is not possible to achieve the two objectives at the same time with a single control. In [3], Lions has been tackled this problem using the method of Stackelberg's optimization (See [7]) by dividing the domain O of action of control into two disjoint domains O_1 and O_2 and taking controls separately on O_1 and O_2, which we briefly describe in the next section.

2 Stackelberg's Optimization

The main idea of Stackelberg's Optimization is to divide the domain O of action of control into two disjoint subdomains. i. e., write $O = O_1 \cup O_2$, where $O_1 \cap O_2 = \phi$. Then consider the problem

$$\begin{aligned} &y' - \Delta y = v_1\chi_{O_1} + v_2\chi_{O_2} \quad \text{in } \Omega_T, \\ &y(0) = 0, y(x,t) = 0 \quad \text{on } \textstyle\sum. \end{aligned} \tag{2.1}$$

That is the control v_1 is acting on O_1 and v_2 is acting on O_2. For a given $v_1 \in L^2(O_{1T}), v_2 \in L^2(O_{2T})$, the problem (2.1) is a heat equation and can be solved uniquely for y which depends on both v_1 and v_2. i.e., $y = y(v_1, v_2)$.
Now the method is as follows. Keeping v_1 fixed, solve the control problem (2.1) to obtain v_2, y so that the second objective is achieved.That is

$$\begin{aligned} &\text{Find } v_2 \in L^2(O_{2T}) \text{ such that} \\ &J_2(v_1, v_2) = \inf_{\tilde{v}_2} J_2(v_1, \tilde{v}_2) \ , \end{aligned} \tag{2.2}$$

where

$$J_2(v_1, \tilde{v}_2) = \frac{1}{2}\int_0^T \int_\Omega (\tilde{y} - y^2)^2 + \frac{\beta}{2}\int_0^T \int_{O_2} \tilde{v}_2^2 \ , \tag{2.3}$$

and $\tilde{y}$ is the unique solution of the equation (2.1) corresponding to $v_1, \tilde{v}_2$. The problem (2.2) has a unique solution v_2. In fact, the control v_2 is given by

$$v_2 = -\frac{1}{\beta} p\chi_{O_2} \equiv F(v_1) \tag{2.4}$$

The corresponding solution y and the function p are the unique solution of the following optimality system

$$\begin{aligned} &y' - \Delta y = v_1\chi_{O_1} - \tfrac{1}{\beta} p\chi_{O_2} \quad \text{in } \Omega_T \\ &-p' - \Delta p = y - y^2 \quad \text{in } \Omega_T \\ &y(0) = p(T) \ = 0, \ y = p = 0 \quad \text{on } \textstyle\sum \end{aligned} \tag{2.5}$$

Now consider the family of solutions $\{(v_2, y)\}$ as v_1 varies in $L^2(O_{1T})$. Then search for a suitable v_1 with minimal L^2 norm and the corresponding solution (v_2, y) so that the approximate controllability is achieved. The existence of the unique solution (v_1, v_2, y) is proved in [3]. That is, we have to solve the following problem

$$\begin{aligned} &\text{Find } v_1 \in L^2(O_{1T}) \text{ such that} \\ &J_1(v_1) = \inf_{\tilde{v}_1} J_1(\tilde{v}_1) \text{ subject to } \tilde{y}(T, \tilde{v}_1, F(\tilde{v}_1)) \in y^1 + \alpha B, \end{aligned} \tag{2.6}$$

where $J_1(\tilde{v}_1) = \frac{1}{2}\int_0^T \int_{O_1} \tilde{v}_1^2$. Here $\tilde{v}_2 = F(\tilde{v}_1)$ is the solution of (2.2) for the given $\tilde{v}_1$ and $\tilde{y}$ is the solution of (2.1) corresponding to $\tilde{v}_1, \tilde{v}_2$.

The unique solution v_1, given by (2.6) is called the 'leader' and for this unique v_1, find the unique v_2, the 'follower' by solving (2.2) which is given explicitely by (2.4).

One can also obtain the optimality system for the final solution (v_1, v_2, y) and is given as follows. The best leader v_1 and the follower v_2 are given by

$$v_1 = \phi\chi_{O_1}\ ,\ v_2 = -\frac{1}{\beta}p\chi_{O_2}, \tag{2.7}$$

where $\{y, \phi, p, \theta\}$ is the unique solution of

$$\begin{aligned}
&y' - \Delta y = \phi\chi_{O_1} - \tfrac{1}{\beta}p\chi_{O_2}, \quad -p' - \Delta p = y - y^2 \quad \text{in } \Omega_T,\\
&-\phi' - \Delta\phi = \theta, \quad \theta' - \Delta\theta = -\tfrac{1}{\beta}\phi\chi_{O_2} \quad \text{in } \Omega_T,\\
&y(0) = p(T) = \theta(0) = 0, \quad \phi(T) = f \quad \text{in } \Omega,\\
&y = p = 0, \theta = 0 \quad \text{on } \textstyle\sum,
\end{aligned} \tag{2.8}$$

where $f \in L^2(\Omega)$ is uniquely determined by

$$(y(T) - y^1, \hat{f} - f) + \alpha\|\hat{f}\| - \alpha\|f\| \geq 0 \text{ for all } \hat{f} \in L^2(\Omega)$$

The details are given in Lions [3].

Our main objective of this paper is to prove a homogenization result. A large amount of literature is available regarding the theory of homogenization. It is beyond the scope of this article to give an exhaustive list of all the references. The interested readers can refer to Bensoussan et. al. [1], G. Dal Maso [2], Sanchez-Palencia [5] to name a few. Some references regarding the homogenization in perforated domains, one can refer to Nandakumar [4], Vanninathan [6]. More references are available in Dal Maso [2].

Observe that in the Stackelberg's optimization technique, the partition of domain of action O into O_1 and O_2 seems to be quite artificial. It does not indicate any method to find the best partition. One can ask several questions in this direction. We do not want to give more details as we are not planning to prove any results regarding the optimal partition. While attempting to answer some of the questions, we arrived at a homogenization problem, where we consider the domain of action O with periodic perforations (with period $\varepsilon > 0$, a small parameter). That is the controls will act on a periodically perforated domain and the purpose of this paper is to study the asymptotic behaviour of the solution and the controls (which now depends on ε) as $\varepsilon \to 0$. This is done in section 4. This homogenization process shows some hints exhibiting that an optimal partition might involve certain microstuctures and there are some partial results which are at a premature level and we do not discuss it here.

Before coming to the homogenization results, let us look at the problem considered by Lions in a slightly different way. Consider the following space

$$\begin{aligned} X &= \Big\{ (\tilde{v}, \tilde{y}) = (\tilde{v}_1, \tilde{v}_2, \tilde{y}) : \tilde{v}_1 \in L^2(O_{1T}) \text{ and} \\ &\quad (\tilde{v}_2, \tilde{y}) \text{ is the solution of (2.2) such that} \\ &\quad \tilde{y}(T, \tilde{v}_1, \tilde{v}_2) \in y^1 + \alpha B \Big\}. \end{aligned}$$

The space X is non empty because the unique solution (v_1, v_2, y) given by (2.2), (2.6) is in X. Now we would like to study the following problem

$$\begin{aligned} &\text{Find } (v_1, v_2, y) \in X \text{ such that} \\ &J_1(v_1) + J_2(v_1, v_2) = \inf{}_X (J_1(\tilde{v}_1) + J_2(\tilde{v}_1, \tilde{v}_2)) \end{aligned} \tag{2.9}$$

Theorem 2.1. The problem (2.9) has a solution.

In section 3 we derive certain estmates and also we prove Theorem 2.1.

Open problem. Find an optimality system for the above solution.

3 Estimates and proof of Theorem 2.1

Let $g \in L^2(\Omega_T)$ and y be the unique solution of $y' - \Delta y = g$ in Ω_T, $y(0) = 0, y = 0$ on $\sum$. Multiplying the equation by y and integrating, we get

$$\begin{aligned} \frac{1}{2}\int_\Omega y^2(t) + \int_0^t \int_\Omega |\nabla y|^2 &= \int_0^t \int_\Omega gy \\ &\leq \frac{1}{2\delta}\int_0^t \int_\Omega |g|^2 + \frac{\delta}{2}\int_0^t \int_\Omega |y|^2 \\ &\leq \frac{1}{2\delta}\int\int |g|^2 + \frac{\delta}{2\lambda_1}\int\int |\nabla y|^2, \end{aligned} \tag{3.1}$$

where λ_1 is the first eigenvalue of the Laplacian and $\delta > 0$. Choosing $\delta = 2\lambda_1$, we see that

$$\|y\|^2_{L^\infty(0,T,L^2(\Omega_T))} \leq \frac{c^2}{2\lambda_1}, \text{ with } c = \|g\|_{L^2(\Omega_T)}. \tag{3.2}$$

Again from (3.1), we see that

$$\|\nabla y\|^2_{L^2(\Omega_T)} \leq c\|y\|_{L^2(\Omega_T)},$$

and since $\|y\|_{L^2(\Omega_T)} \leq \frac{1}{\lambda_1}\|\nabla y\|_{L^2(\Omega_T)}$, we get

$$\|\nabla y\|_{L^2(\Omega_T)} \leq \frac{c}{\lambda_1} \tag{3.3}$$

In fact, we can get better estimates on ∇y than (3.3). Since $y = 0$ on $\sum, y' = 0$ on $\sum$ as well. Moreover $y(x,0) = 0$ on Ω implies $\nabla y(x,0) = 0$ on Ω. Now multiplying the equation by y', we get

$$\int_0^t \int_\Omega |y'|^2 + \frac{1}{2}\int_\Omega |\nabla y(t)|^2 = \int_0^t \int_\Omega gy' \leq \|g\|_{L^2(\Omega_T)}\|y'\|_{L^2(\Omega_T)}. \tag{3.4}$$

Hence, we have

$$\|y'\|_{L^2(\Omega_T)} \leq c, \tag{3.5}$$

$$\|\nabla y\|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{2}c. \tag{3.6}$$

Suppose now that g_n is a bounded sequence in $L^2(\Omega_T)$ and $g_n \rightharpoonup g$ in $L^2(\Omega_T)$ weak. If y_n, y respectively represents the solutions corresponding to g_n, g, then the following convergences follows from the above estimates.

$$y_n \rightharpoonup y \text{ in } L^\infty(0,T,L^2(\Omega)) \text{ weak } *, \tag{3.7a}$$

$$\nabla y_n \rightharpoonup \nabla y \text{ in } L^2(\Omega_T) \text{ weak}, \tag{3.7b}$$

$$y_n' \rightharpoonup y' \text{ in } L^2(\Omega_T) \text{ weak}. \tag{3.7c}$$

Moreover from (3.4) and (3.5), we see that for each t, $\nabla y_n(t)$ is a bounded sequence in $L^2(\Omega)$ so that

$$\nabla y_n(t) \rightharpoonup \nabla y(t) \text{ in } L^2(\Omega) \text{ weakly}. \tag{3.7d}$$

Finally using the compact imbedding of L^2 into H_0^1, we get

$$y_n(t) \longrightarrow y(t) \text{ in } L^2(\Omega) \text{ strong}. \tag{3.7e}$$

In particular,

$$y_n(0) \to y(0) \text{ and } y_n(T) \to y(T) \text{ in } L^2(\Omega) \text{ strong}$$

Proof of Theorem 2.1. Let (v_1^n, v_2^n, y^n) be a minimizing sequence i.e.,

$$J_1(v_1^n) + J_2(v_1^n, v_2^n) \to \inf_X [J_1(\bar{v}_1) + J_2(\bar{v}_1, \bar{v}_2)]$$

Then $v_1^n \rightharpoonup v_1$ in $L^2(O_{1T})$ weak and $v_2^n \rightharpoonup v_2$ in $L^2(O_{2T})$.
Moreover,

$$y^n \rightharpoonup y, \nabla y^n \rightharpoonup \nabla y \text{ in } L^\infty(0,T,L^2(\Omega)) \text{ weak } *,$$
$$y^n(\cdot,t) \to y(\cdot,t) \text{ in } L^2(\Omega) \text{ strong, for all } t.$$

Since $y^n(T) \in y^1 + \alpha B$ and $y^n(T) \to y(T)$, we have $y(T) \in y^1 + \alpha B$.
Claim. $(v_1, v_2, y) \in X$.
If the claim is true, then by lower semi continuity, we have

$$\begin{aligned} J_1(v_1) + J_2(v_1, v_2) &\leq \underline{\lim}\,[J_1(v_1^n) + J_2(v_1^n, v_2^n)] \\ &= \inf_X [J_1 + J_2] , \end{aligned}$$

and hence the problem is solved.

To prove the claim, one has to show that (v_2, y) solves (2.2). Let $\tilde{v}_2$ be any element in $L^2(O_{2T})$ and $\tilde{y}$ be the solution of

$$\tilde{y}' - \Delta\tilde{y} = v_1\chi_{O_1} + \tilde{v}_2\chi_{O_2} , \tag{3.8}$$

with zero boundary and initial conditions. We have to show that

$$J_2(v_1, v_2) \leq J_2(v_1, \tilde{v}_2).$$

Let y_n be the solution of

$$y_n' - \Delta y_n = v_1^n\chi_{O_1} + \tilde{v}_2\chi_{O_2}, \tag{3.9}$$

with zero conditions. Since (v_2^n, y^n) is the minimizing solution for v_1^n, we get

$$J_2(v_2^n, y^n) \leq J_2(\tilde{v}_2, y_n) . \tag{3.10}$$

But $y_n \rightharpoonup \tilde{y}$ as $v_1^n \rightharpoonup v_1$. Hence

$$\begin{aligned} J_2(v_1, v_2, y) &\leq \underline{\lim} J_2(v_1^n, v_2^n, y^n) \text{ by l.s.c.} \\ &\leq \underline{\lim} J_2(v_1^n, \tilde{v}_2, y^n) \text{ by (3.8)} \\ &= \frac{1}{2}\int\int(\tilde{y} - y^2)^2 + \frac{\beta}{2}\int\int\tilde{v}_2^2 \\ &= J_2(v_1, \tilde{v}_2, \tilde{y}). \end{aligned}$$

Hence the claim.

4 A homogenization result

Let $O_{1\varepsilon}$ and $O_{2\varepsilon}$ be the domains obtained from O_1 and O_2 respectively, by removing holes from all the periodic cells of small period $\varepsilon > 0$. This is a case of homogenization with periodically perforated domains. We briefly describe how to obtain such a perforated domain and for more details one can refer to Nandakumar [4].

Let $Y = (0,1)^N$ be the unit cell and $T \subset Y$ be an open set contained in Y. Let $Y^* = Y \setminus T$ and put $\theta = |Y^*|, 0 < \theta < 1$. Consider the ε-periodic ($\varepsilon > 0$, given) cells εY and εY^* and one can divide the whole space $\mathbb{R}^N$ into ε-periodic cells and its translations i.e., one can write $\mathbb{R}^N = \bigcup_{k \in \mathbb{Z}^N} \varepsilon Y_k$ where $Y_k = Y + k$. The domain $\mathbb{R}^N_{\text{per}} = \bigcup_{k \in \mathbb{Z}^N} \varepsilon Y_k^*, Y_k^* = Y^* + k$ is the periodically perforated domain which is obtained from $\mathbb{R}^N$ by removing holes from all the periodic cells εY_k. Then the perforated subdomains $O_{1\varepsilon}$ and $O_{2\varepsilon}$ are given by

$$O_{1\varepsilon} = O_1 \cap \mathbb{R}^N_{\text{per}} \text{ and } O_{2\varepsilon} = O_2 \cap \mathbb{R}^N_{\text{per}}.$$

In this section we would like to consider the problem (2.1), where the domain of action of the control is $O_{1\varepsilon} \cup O_{2\varepsilon}$ instead of $O_1 \cup O_2$. So, naturally the solution will depend on ε, and we would like to study the behaviour of the solution as $\varepsilon \to 0$ and obtain the limiting problem. This problem has not been studied completely. What we do in this section is that we consider the perforations only in O_2. More precisely, we consider the problem (2.1) with controls acting on $O_1 \cup O_{2\varepsilon}$. We denote $P = \{O_1, O_{2\varepsilon}\}$ a partition and look at the problem

$$\begin{aligned} &y_\varepsilon' - \Delta y_\varepsilon = v_{1\varepsilon}\chi_{O_1} + v_{2\varepsilon}\chi_{O_{2\varepsilon}} \text{ in } \Omega_T, \\ &y_\varepsilon(0) = 0 \text{ and } y_\varepsilon = 0 \text{ on } \textstyle\sum \end{aligned} \tag{4.1}$$

Let $\{v_{1\varepsilon}, v_{2\varepsilon}, y_\varepsilon\}$ be the solution given by the Stackelberg's optimization method with the functionals given by

$$\begin{aligned} J_{2\varepsilon} &= \frac{1}{2}\int_0^T \int_\Omega (y_\varepsilon - y^2)^2 + \frac{\beta}{2}\int_0^T \int_{O_{2\varepsilon}} v_{2\varepsilon}^2 \\ J_1(v_1) &= \frac{1}{2}\int_0^T \int_{O_1} v_1^2 \end{aligned}$$

i.e., we have

$$\min_{\tilde{v}_{2\varepsilon}} J_{2\varepsilon}(v_{1\varepsilon}, \tilde{v}_{2\varepsilon}, \tilde{y}_\varepsilon) = J_{2\varepsilon}(v_{1\varepsilon}, v_{2\varepsilon}, y_\varepsilon), \tag{4.2}$$

and

$$\begin{aligned} &\min J_1(v_1) = J_1(v_{1\varepsilon}) \\ &\text{subject to } y_\varepsilon(T, v_1, v_{2\varepsilon}) \in y^1 + \alpha B \end{aligned} \tag{4.3}$$

We have the following main theorem regarding the homogenization.

Theorem 4.1. Let $\{v_{1\varepsilon}, v_{2\varepsilon}, y_\varepsilon\}$ be given as above. Then

$$\{v_{1\varepsilon}, v_{2\varepsilon}\chi_{O_{2\varepsilon}}\} \rightharpoonup \{v_1, v_2\} \text{ in } L^2(O_{1T}) \times L^2(O_{2T}) \text{ weak }. \tag{4.4}$$

$$y_\varepsilon \rightharpoonup y \text{ as in (3.7)}, \tag{4.5}$$

where $\{v_1, v_2, y\}$ is the solution for the partition $P = \{O_1, O_2\}$ with the functionals

$$J_1 = \frac{1}{2}\int_0^T \int_{O_1} v_1^2, \tag{4.6}$$

$$J_2 = \frac{1}{2}\int_0^T \int_\Omega (y - y^2)^2 + \frac{\beta}{2\theta}\int_0^T \int_{O_2} v_2^2. \tag{4.7}$$

Here $\theta = |Y^*|$.

Proof. We only give a sketch of the proof. The details and more results will appear elsewhere.

Let $\{v_1, v_2, y\}$ be the solution for $P = \{O_1, O_2\}$ with the functionals J_2 and J_1 as in the theorem which exists uniquely. Then one has to prove the convergence (4.4) and (4.5).

Step 1. Let $v_2^\varepsilon = v_2\chi_{O_{2\varepsilon}}$ and $\overline{y}_\varepsilon$ be the solution of $\overline{y}_\varepsilon' - \Delta\overline{y}_\varepsilon = v_1\chi_{O_1} + \frac{1}{\theta}v_2^\varepsilon\chi_{O_{2\varepsilon}}$. We have $v_2^\varepsilon\chi_{O_{2\varepsilon}} \rightharpoonup \theta v_2\chi_{O_2}$ in $L^2(O_{2T})$ weak. From the convergence results of section 3, it follows that $\overline{y}_\varepsilon \to \overline{y}$ to a solution of $\overline{y}' - \Delta\overline{y}' = v_1\chi_1 + v_2\chi_2$. By uniqueness $\overline{y} = y$.

Step 2. Given v_1 as in Theorem, let $\{\tilde{v}_{2\varepsilon}, \tilde{y}_\varepsilon\}$ be the control solution with $P = \{O_1, O_{2\varepsilon}\}$ of

$$\min_{\hat{v}_{2\varepsilon}} J_{2\varepsilon}(v_1, \hat{v}_{2\varepsilon}, \hat{y}_\varepsilon) = J_{2\varepsilon}(v_1, \tilde{v}_{2\varepsilon}, \tilde{y}_\varepsilon).$$

Since $\hat{v}_{2\varepsilon} = \frac{1}{\theta}v_2^\varepsilon, \hat{y}_\varepsilon = \overline{y}_\varepsilon$ is a candidate for the above minimization we see that

$$\begin{aligned} J_{2\varepsilon} &\le \frac{1}{2}\int\int(\hat{y}_\varepsilon - y^2)^2 + \frac{\beta}{2}\int\int_{O_{2\varepsilon}} \hat{v}_{2\varepsilon}^2 \\ &\le \frac{1}{2}\int\int(\overline{y}_\varepsilon - y^2)^2 + \frac{\beta}{2}\int\int_{O_2}\left(\frac{1}{\theta}v_2^\varepsilon\chi_{O_{2\varepsilon}}\right)^2 \\ &= \frac{1}{2}\int\int(\overline{y}_\varepsilon - y^2)^2 + \frac{\beta}{2\theta^2}\int\int v_2^2\chi_{O_{2\varepsilon}} \end{aligned}$$

Thus we get

$$\lim_{\varepsilon\to 0} J_{2\varepsilon}(v_1, \tilde{v}_{2\varepsilon}, \tilde{y}_\varepsilon) \le \frac{1}{2}\int\int(y - y^2)^2 + \frac{\beta}{2\theta}\int\int_{O_2} v_2^2.$$

From this we conclude that $\{\tilde{v}_{2\varepsilon}\chi_{O_{2\varepsilon}}\}$ is bounded in $L^2(O_{2T})$ and hence it converges weakly to, say, $\tilde{v}_2$ and $\tilde{y}_\varepsilon \longrightarrow \tilde{y}$. Moreover $\tilde{y}' - \Delta\tilde{y} = v_1\chi_{O_1} + \tilde{v}_2\chi_{O_2}$. In fact, we will show that $\tilde{y} = y$ and $\tilde{v}_2 = v_2$. By lower semi continuity,

$$\frac{1}{2}\int\int(\tilde{y} - y^2)^2 + \frac{\beta}{2}\int\int\tilde{v}_2^2 \quad\le\quad \lim_{\varepsilon\to 0} J_{2\varepsilon} \quad\le\quad \frac{1}{2}\int\int(y - y^2)^2 + \frac{\beta}{2\theta}\int\int_{O_2} v_2^2.$$

From this we cannot conclude that $\tilde{v}_2$ minimizes the functionals on the right hand side. But this can be proved using the optimality system. It is not very difficult and we omit the details here.

Thus we conclude that for v_1, if we consider only the minimization (4.2), then the solution of the inhomogenized problem converges to the solution of the homogenized problem with functional J_2 as in (4.7). In fact, the same proof shows that if we take any sequence $\bar{v}_{1\varepsilon} \in L^2(O_{1T})$ that converges to v_1 weakly and $\{\bar{v}_{2\varepsilon}, \bar{y}_\varepsilon\}$ be the solution of (4.2) with $\bar{v}_{1\varepsilon}$ then $\bar{v}_{2\varepsilon} \rightharpoonup v_2$ and $\bar{y}_\varepsilon \to y$ as above and $\{v_2, y\}$ solves the minimization problem.

This is the case in our Theorem. So first of all we should get a uniform L^2 estimate for $v_{1\varepsilon}$, where $\{v_{1\varepsilon}, v_{2\varepsilon}, y\}$ is the solution of the combined problem (4.2) and (4.3).

Fix $u_1 \in L^2(O_{1T})$, let $\{\tilde{z}, \tilde{u}_2\}$ and $\{\tilde{z}_\varepsilon, \tilde{u}_{2\varepsilon}\}$ be respectively, the solution with partition $P = \{O_1, O_2\}$ and $P_\varepsilon = \{O_1, O_{2\varepsilon}\}$. Set

$$\begin{aligned} X_0 &= \left\{u_1 \in L^2(O_{1T}) : \tilde{z}(T, u_1, \tilde{u}_2) \in y^1 + \alpha B\right\}, \\ X_\varepsilon &= \left\{u_1 \in L^2(O_{1T}) : \tilde{z}_\varepsilon(T, u_1, \tilde{u}_{2\varepsilon}) \in y^1 + \alpha B\right\} \end{aligned}$$

Then

$$\int\!\!\int v_1^2 = \min_{X_0} \int\!\!\int u_1^2 \quad \text{and} \quad \int\!\!\int v_{1\varepsilon}^2 = \min_{X_\varepsilon} \int\!\!\int u_1^2.$$

Let $u_1 \in X_0$, then by step 2, $\tilde{z}_\varepsilon \to \tilde{z}$ and $\tilde{z}_\varepsilon(T) \to \tilde{z}(T)$. So there exists $\varepsilon_0 = \varepsilon_0(u_1)$ such that $\tilde{z}_\varepsilon(T) \in y^1 + \alpha B$ because $\tilde{z}(T) \in y^1 + \alpha B$, which shows that $u_1 \in X_\varepsilon$ for all $\varepsilon \leq \varepsilon_0(u_1)$. Therefore

$$\int v_{1\varepsilon}^2 \leq \int u_1^2 \quad \forall \quad \varepsilon \leq \varepsilon_0(u_1).$$

i.e., $\{v_{1\varepsilon}\}$ is bounded in $L^2(O_{1T})$ and hence coverges weakly, say, to $\bar{v}_1$ and

$$\int\!\!\int \bar{v}_1^2 \leq_{by\ l.s.c} \varliminf_{\varepsilon\to 0} \int v_{1\varepsilon}^2 \leq \min_{X_0} \int u_1^2 = \int v_1^2.$$

Then, we conclude that $v_{2\varepsilon} \rightharpoonup \bar{v}_2$ and $y_\varepsilon \to \bar{y}$ and $\{\bar{v}_2, \bar{y}\}$ minimizes $\frac{1}{2}\int\int(\bar{y} - y^2)^2 + \frac{\beta}{2\theta}\int\int \bar{v}_2^2$.

Since $y_\varepsilon(T) \in y^1 + \alpha B$, we have $\bar{y}(T) \in y^1 + \alpha B$ and hence $\bar{v}_1 \in X_0$. Since $\int\int \bar{v}_1^2 \leq \int\int v_1^2$, we get $\bar{v}_1 = v_1$ by uniqueness which in turn will imply that $\bar{v}_2 = v_2$ and $\bar{y} = y$. This completes the proof of partial homogenization result.

Remark. As remarked earlier we have considered the perforations only in O_2. But when we consider perforations both in O_1 and O_2, then to study the homogenization result, one should obtain estimates on the solution of the joint optimality system. We have not yet succeeded in this goal.

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