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Viscosity solution of second order HJI equation and application to zero-sum stochastic diffrential game.

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Abstract. In this article, we study the existence and uniqueness of viscosity solution of the second order Hamilton-Jacobi-Isaacs (HJI) equation. Then we give an application to the zero-sum stochastic differential game of fixed duration where the state is given by controlled degenerate diffusions. We show that the unique solution of the HJI equation is the value function associated with the game in an appropriate sense. Finally, we prove the existence of saddle point strategies when the state dynamics and the cost are linear in the state variables.

1. Introduction

In this article, we are interested in studying the existence and uniqueness of degenerate second order Hamilton-Jacobi-Isaacs (HJI for short) equation of the form

$$\underline{F}(t,x,\phi_t,D_x\phi,D_x^2\phi) = \overline{F}(t,x,\phi_t,D_x\phi,D_x^2\phi) = 0 \quad \text{in} \quad [0,T]\times \mathbb{R}^d$$

where

$$\overline{F}(t,x,\phi_t,D_x\phi,D_x^2\phi) = \phi_t + \inf_{\mu_2\in\mathcal{M}_2}\sup_{\mu_1\in\mathcal{M}_1} H(t,x,D_x\phi,D_x^2\phi,\mu_1,\mu_2) \tag{1}$$

$$\underline{F}(t,x,\phi_t,D_x\phi,D_x^2\phi) = \phi_t + \sup_{\mu_1\in\mathcal{M}_1}\inf_{\mu_2\in\mathcal{M}_2} H(t,x,D_x\phi,D_x^2\phi,\mu_1,\mu_2). \tag{2}$$

Here $\mathcal{M}_i = \mathcal{P}(U_i)$, the space of probability measures on U_i, $i = 1,2$ and $U_i's$ are the given compact metric spaces and $D_x = (\frac{\partial}{\partial x_i})_{1\le i\le n}$, $D_x^2 = (\frac{\partial^2}{\partial x_i \partial x_j})_{1\le i,j\le n}$. The Hamiltonian H is given by

$$H(t,x,p,A,\mu_1,\mu_2) = \frac{1}{2}\mathrm{tr}(\mathrm{a}(\mathrm{t},\mathrm{x})\mathrm{A}) + \mathrm{b}(\mathrm{t},\mathrm{x},\mu_1,\mu_2)\cdot \mathrm{p} + \mathrm{r}(\mathrm{t},\mathrm{x},\mu_1,\mu_2)$$

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for all $(t, x, p, A) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ and $\mu_i \in \mathcal{M}_i, i = 1, 2$. Here, we define b and r as follows:

$$b(t, x, \mu_1, \mu_2) = \int_{U_2} \int_{U_1} \bar{b}(t, x, u_1, u_2)\ \mu_1(du_1)\ \mu_2(du_2) \tag{3}$$

$$r(t, x, \mu_1, \mu_2) = \int_{U_2} \int_{U_1} \bar{r}(t, x, u_1, u_2)\ \mu_1(du_1)\ \mu_2(du_2). \tag{4}$$

The assumptions on $a, b, \bar{b}, r, \bar{r}$ are given below.

The above problem (1) arises in the study of zero-sum stochastic differential game on the finite horizon in the setup of relaxed strategies. Note that it is a game of two players 1 and 2 and U_i are the action sets and $\mathcal{M}_i$ are sets of randomized actions. We will describe this in detail in Section 3. In this introductory Section 1, we specify the conditions on a, b and r, and we also introduce the concept of viscosity solutions. In Section 2, we state and prove the existence of a unique solution to (1). This is achieved by proving a related maximum principle. Finally, Section 3 is devoted to the study of application to the stochastic differential game.

To begin with, we consider the following two equations separately:

$$\underline{F}(t, x, \phi_t, D_x\phi, D_x^2\phi) = 0, \tag{5}$$

$$\overline{F}(t, x, \psi_t, D_x\psi, D_x^2\psi) = 0. \tag{6}$$

The equations (5) and (6) are, respectively, known as lower and upper HJI equations. We make the following assumptions.

Assumptions:

(A1) Let the matrix $a = \sigma\sigma^* \geq 0$, where $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and there exists a constant $C_1 > 0$ such that

$$|\sigma(t, x) - \sigma(s, y)| \leq C_1(|t - s| + |x - y|).$$

(A2) The function $\bar{b} : [0,T] \times \mathbb{R}^d \times \mathrm{U}_1 \times \mathrm{U}_2 \to \mathbb{R}^d$ is continuous and there exists a constant $C_2 > 0$ such that

$$|\bar{b}(t, x, u_1, u_2) - \bar{b}(s, y, u_1, u_2)| \leq C_2(|t - s| + |x - y|).$$

(A3) The function $\bar{r} : [0,T] \times \mathbb{R}^d \times \mathrm{U}_1 \times \mathrm{U}_2 \to \mathbb{R}^d$ is continuous and there exists a constant $C_3 > 0$ such that

$$|\bar{r}(t, x, u_1, u_2) - \bar{r}(s, y, u_1, u_2)| \leq C_3(|t - s| + |x - y|).$$

We do not assume the uniform ellipticity of a and thus we are considering the degenerate case. In the non-degenerate case, there exists a unique classical $C^{1,2}$ solution for the equations (5), (6) (See [2]). In the degenerate case, the existence of classical solution is more of an exception than a rule and it is far more involved. We study these equations in the framework of viscosity solutions. There is enormous literature in the field of viscosity solutions which was developed in the last two decades and is beyond the scope of this short article to present a complete survey. However, the notion of viscosity solution for general non-linear partial differential equations was introduced by Crandall and Lions [5] in the early eighties even though some basic ideas were available prior to it in the works of Evans [6]. Further significant contributions have been made by various authors, especially in the case of Hamilton-Jacobi equation. For example, see Crandall, Evans and Lions [3], Crandall, Ishii and Lions [4], Crandall and Lions [5], Ishii [18], Ishii and Lions [19], Jensen [20] and the references therein.

We now introduce the definitions of viscosity solutions.

Definition 1. *An upper semi-continuous function $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is said to be a viscosity subsolution of (5) if whenever $(t_0, x_0) \in [0,T] \times \mathbb{R}^d, \eta \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that $\phi - \eta$ has a local maximum at (t_0, x_0) then*

$$\underline{F}(t_0, x_0, \eta_t(t_0, x_0), D_x\eta(t_0, x_0), D_x^2\eta(t_0, x_0)) \leq 0.$$

Definition 2. *A lower semi-continuous function $\psi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is said to be a viscosity supersolution of (5) if whenever $(t_0, x_0) \in [0,T] \times \mathbb{R}^d, \eta \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that $\psi - \eta$ has a local minimum at (t_0, x_0) then*

$$\underline{F}(t_0, x_0, \eta_t(t_0, x_0), D_x\eta(t_0, x_0), D_x^2\eta(t_0, x_0)) \geq 0.$$

Definition 3. *A continuous function $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is said to be a viscosity solution of (5) if it is both viscosity sub and super solution of (5).*

Remark 1. *One need not have to include the assumption upper semicontinuous in definition (1) (similarly l.s.c in definition (2)) as is done in* [18]. *If ϕ is not u.s.c or l.s.c, one can define the u.s.c and l.s.c envelopes of ϕ and can give the definitions in terms of these envelopes. We do not go into such technicalities.*

Remark 2. *The concept of viscosity solutions has many interesting properties like compactness which are not preserved by other types weak solutions (for example: a.e. solution).*

2. Existence and uniqueness

Let

$$CL([0,T] \times \mathbb{R}^d) := \{\phi \in C([0,T] \times \mathbb{R}^d) \to \mathbb{R}^d : |\phi(t,x)| \leq C(1 + |x|) \text{ for some } C > 0\}.$$

We first prove the following maximum principle.

Theorem 1. *Let $\phi, \psi \in CL([0,T] \times \mathbb{R}^d)$ be, respectively, the subsolution and supersolution of (1). Assume that*

$$\phi(T,x) \leq \psi(T,x), \forall x \in \mathbb{R}^d.$$

Then

$$\phi \leq \psi \text{ in } [0,T] \times \mathbb{R}^d.$$

Proof. Let $\varepsilon > 0$. Define

$$\phi_1(t,x) := \phi(t,x) - \varepsilon(1+|x|^2) - e^{\alpha t}\delta(x) \quad (1)$$

$$\psi_1(t,x) := \psi(t,x) + \varepsilon(1+|x|^2) + e^{\alpha t}\delta(x), \quad (2)$$

where $\delta(x) = \beta|x|^2 + \gamma$ and α, β, γ are positive constants to be chosen later.

Let $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$ and $\eta \in C^{1,2}([0,T) \times \mathbb{R}^d)$ be such that $\phi_1 - \eta$ has a local maximum at (t_0, x_0). Then $\phi - \bar{\eta}$ has a local maximum at (t_0, x_0), where

$$\bar{\eta}(t,x) = \eta(t,x) + \varepsilon(1+|x|^2) + e^{\alpha t}\delta(x).$$

Further,

$$\begin{aligned} \bar{\eta}_t(t_0,x_0) &= \eta_t(t_0,x_0) + \alpha e^{\alpha t_0}\delta(x_0) \\ D_x\bar{\eta}(t_0,x_0) &= D_x\eta(t_0,x_0) + 2\varepsilon x_0 + e^{\alpha t_0}2\beta x_0 \\ D_x^2\bar{\eta}(t_0,x_0) &= D_x^2\eta(t_0,x_0) + 2\varepsilon I_d + e^{\alpha t_0}2\beta I_d, \end{aligned}$$

where I_d is the $d \times d$ identity matrix.

Since ϕ is a viscosity subsolution to (1), we have

$$\begin{aligned} \eta_t(t_0,x_0) &+ \alpha e^{\alpha t_0}\delta(x_0) + \inf_{\mu_2 \in \mathcal{M}_2} \sup_{\mu_1 \in \mathcal{M}_1} \Big[b(t_0,x_0,\mu_1,\mu_2) \cdot D_x\eta(t_0,x_0) \\ &+ r(t_0,x_0,\mu_1,\mu_2) + b(t_0,x_0,\mu_1,\mu_2) \cdot \Big(2\varepsilon x_0 + e^{\alpha t_0}2\beta x_0\Big)\Big] \\ &+ \frac{1}{2}tr(a(t_0,x_0)D_x^2\eta(t_0,x_0)) + \frac{1}{2}tr(a(t_0,x_0)(2\varepsilon I_d + e^{\alpha t_0}2\beta I_d)) \geq 0. \end{aligned} \quad (3)$$

Now by assumptions (A1) and (A2), we obtain, for some constant $C_4 > 0$,

$$\begin{aligned} |b(t_0,x_0,\mu_1,\mu_2) \cdot 2\varepsilon x_0| &\leq 2\varepsilon C_4(1+|x_0|)|x_0| \\ |b(t_0,x_0,\mu_1,\mu_2) \cdot e^{\alpha t_0}2\beta x_0| &\leq 2\beta C_4 e^{\alpha t_0}(1+|x_0|)|x_0| \\ |\frac{1}{2}tr(a(t_0,x_0)2\varepsilon I_d)| &\leq \varepsilon C_4(1+|x_0|^2) \\ |\frac{1}{2}tr(a(t_0,x_0)e^{\alpha t_0}2\beta I_d)| &\leq \beta C_4 e^{\alpha t_0}(1+|x_0|^2). \end{aligned}$$

Therefore, from (3) it follows that

$$\begin{aligned} \eta_t(t_0,x_0) &+ \inf_{\mu_2 \in \mathcal{M}_2} \sup_{\mu_1 \in \mathcal{M}_1} \Big[b(t_0,x_0,\mu_1,\mu_2) \cdot D_x\eta(t_0,x_0) + r(t_0,x_0,\mu_1,\mu_2)\Big] \\ &+ \frac{1}{2}tr(a(t_0,x_0)D_x^2\eta(t_0,x_0)) \\ &\leq 2\varepsilon C_4(1+|x_0|)|x_0| + 2\beta C_4 e^{\alpha t_0}(1+|x_0|)|x_0| + C_4\varepsilon(1+|x_0)|^2) \\ &\quad + C_4 e^{\alpha t_0}\beta(1+|x_0|^2) - \alpha e^{\alpha t_0}\beta|x_0|^2 - \alpha e^{\alpha t_0}\gamma \\ &\leq \begin{cases} 6\varepsilon C_4 + 6\beta C_4 e^{\alpha T} - \alpha\gamma & |x_0| \leq 1 \\ 6\varepsilon C_4|x_0|^2 + 6\beta C_4 e^{\alpha t_0}|x_0|^2 - \alpha e^{\alpha t_0}\beta|x_0|^2 & |x_0| \geq 1. \end{cases} \end{aligned} \quad (4)$$

Our idea is to choose suitably α, β, γ independent of the point (t_0, x_0), so as to make the right hand side of (4) nonpositive. This can be achieved as follows. First choose $\alpha > 6C_4$ and then define $\beta = \frac{6\varepsilon C_4}{\alpha - 6C_4}$, so that $\beta > 0$. Then it follows that the second term on the right of (4) is negative. Finally, choose $\gamma = \frac{6\varepsilon C_4 + 6\beta C_4 e^{\alpha T}}{\alpha}$. Also observe that β and γ are of $O(\varepsilon)$. Thus ϕ_1 is a subsolution of (1). Similarly we can prove that ψ_1 is supersolution of (1).

Given $\varepsilon > 0$, we have the following terminal condition:

$$\phi_1(T, x) = \phi(T, x) - \varepsilon(1 + |x|^2) - e^{\alpha T}\delta(x)$$
$$\psi_1(T, x) = \psi(T, x) + \varepsilon(1 + |x|^2) + e^{\alpha T}\delta(x).$$

It follows that

$$\phi_1(T, x) \le \psi_1(T, x) \qquad \text{for all } x \in \mathbb{R}^d.$$

Given $\varepsilon > 0$, since ϕ has only linear growth, one can choose $R(\varepsilon) > 0$, large enough such that

$$\phi_1(t, x) \le \psi_1(t, x) \qquad \text{for all } t \in [0, T], \qquad |x| \ge R(\varepsilon).$$

Now applying the comparison result for (1) on $[0, T] \times \bar{B}(0, R(\varepsilon))$ (see [12], [18]), it follows that

$$\phi_1(t, x) \le \psi_1(t, x), (t, x) \in [0, T] \times \bar{B}(0, R(\varepsilon)).$$

Simplifying (5), we get

$$\phi(t, x) \le \psi(t, x) + 2\varepsilon(1 + |x|^2) + 2e^{\alpha t}\delta(x)$$

for $(t, x) \in [0, T] \times \bar{B}(0, R(\varepsilon))$. Since β, γ are of $O(\varepsilon)$ it follows by passing to the limit as $\varepsilon \to 0$ that

$$\phi(t, x) \le \psi(t, x) \qquad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^d$$

This completes the proof of Theorem 1. □

Remark 3. *In fact, the above proof shows that the uniqueness holds in the class of functions whose growth is $|x|^{1+\delta}, 0 \le \delta < 1$, but uniqueness may not hold in the class quadratic growth functions.*

We next prove the existence of a unique viscosity solution of the HJI equation

$$\underline{F} = \overline{F} = 0$$

with the terminal condition

$$\phi(T, x) = g(x).$$

Assumption (A4): Let $g : \mathbb{R}^d \to \mathbb{R}$ be such that there is a constant $C_5 > 0$ such that

$$|g(x) - g(y)| \le C_5 |x - y|$$

Under the assumptions (A1) - (A4) we have the following theorem.

Theorem 2. *The equation (5) has a unique viscosity solution ϕ in the class $\mathcal{C}L([0,T]\times \mathbb{R}^d)$ satisfying (5).*

Proof. We use the vanishing viscosity method to establish the desired existence. First consider the equation (6). Let $\varepsilon > 0$ and consider the perturbed equations

$$\begin{aligned} \phi^\varepsilon_t + \inf_{\mu_2\in\mathcal{M}_2}\sup_{\mu_1\in\mathcal{M}_1} H(t,x,D_x\phi^\varepsilon(t,x),D_x^2\phi^\varepsilon(t,x),\mu_1,\mu_2) + \frac{\varepsilon^2}{2}\Delta\phi^\varepsilon = 0 \\ \phi^\varepsilon(T,x) = g(x). \end{aligned} \tag{5}$$

One can mimic the arguments in [2] to show that (5) has a unique classical solution ϕ^ε in $C^{1,2}([0,T]\times\mathbb{R}^d)\cap \mathrm{CL}([0,\mathrm{T}]\times\mathbb{R}^d)$. Indeed, ϕ^ε is characterized as the value function of the stochastic differential game associated with (5). More precisely,

$$\phi^\varepsilon(t,x) = \inf_{\mu_2(\cdot)}\sup_{\mu_1(\cdot)} E\left[\int_t^T r(s,X^\varepsilon(s),\mu_1(s),\mu_2(s))\;ds + g(X^\varepsilon(T))\right],$$

where $X^\varepsilon(\cdot)$ satisfies the stochastic differential equation

$$\begin{aligned} dX^\varepsilon(s) &= b(s,X^\varepsilon(s),\mu_1(s),\mu_2(s))\;ds + \begin{bmatrix}\sigma(s,X(s))\\ \varepsilon I_d\end{bmatrix} d\tilde{W}(s) \\ X^\varepsilon(t) &= x. \end{aligned} \tag{6}$$

Here $\tilde{W}(\cdot)$ is $2d$-dimensional standard Brownian motion and $\mu_i(.)$ is a $\mathcal{M}_i$- valued process, progressively measurable with respect to the natural filterations of $X^\varepsilon(.)$. Using the stochastic representation (6) and the assumptions (A1) - (A4), it can be shown that ϕ^ε is pointwise bounded and there exists a constant $C_6 > 0$, independent of ε such that

$$|\phi^\varepsilon(t_1,x_1) - \phi^\varepsilon(t_2,x_2)| \le C_6(|t_1-t_2|^{1/2} + |x_1-x_2|).$$

Therefore, ϕ^ε is equicontinuous and uniformly pointwise bounded. Thus, by Arzela-Ascoli theorem, there exists a sequence $\{\varepsilon_n\}$ converging to 0 and $\phi \in C([0,T]\times\mathbb{R}^d)$ such that $\phi^{\varepsilon_n}\to\phi$ uniformly on compact subsets of $[0,T]\times\mathbb{R}^d$ and ϕ satisfies

$$|\phi(t_1,x_1) - \phi(t_2,x_2)| \le C_5(|t_1-t_2|^{1/2} + |x_1-x_2|).$$

By the stability results of viscosity solutions ([12], [18]), ϕ is a viscosity solution of (6) satisfying (5). Clearly $\phi\in CL([0,T]\times\mathbb{R}^d)$.

Now, we show that ϕ is a viscosity solution to (5). Let $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$ and $\eta\in C^{1,2}([0,T)\times\mathbb{R}^d)$ be such that $\phi-\eta$ has a local maximum at (t_0,x_0). Then

$$\begin{aligned} &\eta_t(t_0,x_0) + \sup_{\mu_1\in\mathcal{M}_1}\inf_{\mu_2\in\mathcal{M}_2} H(t_0,x_0,D_x\eta(t_0,x_0),D_x^2\eta(t_0,x_0),\mu_1,\mu_2) \\ &\quad = \eta_t(t_0,x_0) + \inf_{\mu_2\in\mathcal{M}_2}\sup_{\mu_1\in\mathcal{M}_1} H(t_0,x_0,D_x\eta(t_0,x_0),D_x^2\eta(t_0,x_0),\mu_1,\mu_2) \le 0. \end{aligned}$$

The equality follows by Fan's minimax theorem [11] and inequality holds, since ϕ is a viscosity subsolution to (6). Thus ϕ is also a viscosity subsolution to (6). Similarly, we can show that ϕ is a viscosity supersolution to (5). Hence ϕ is a viscosity solution to (5) satisfying (5). The uniqueness follows from Theorem 1. □

3. Application to Stochastic Differential Game

We consider a zero-sum stochastic differential game on the finite horizon where the state $X(\cdot)$ is an $\mathbb{R}^d$-valued controlled degenerate diffusion given by

$$\begin{aligned} dX(t) &= \bar{b}(t, X(t), u_1(t), u_2(t))\, dt + \sigma(X(t))\, dW(t), \qquad t \in (0,T] \\ X(0) &= x, \end{aligned} \tag{1}$$

where $W(\cdot)$ is a standard d-dimensional Brownian motion; $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$; $u_i : [0,T] \to U_i$ is the strategy of player i, $i = 1, 2$, U_1 and U_2 are the action sets of players 1 and 2 respectively.

Player 1 tries to maximize his expected payoff

$$R(0, x, u_1(\cdot), u_2(\cdot)) := E\left[\int_0^T \bar{r}(t, X(t), u_1(t), u_2(t))\, dt + g(X(T))\right]$$

over his strategies $u_1(\cdot)$, whereas player 2 tries to minimize the same over his strategies $u_2(\cdot)$. Here $\bar{r}$ is the running payoff function and g is the terminal payoff function. The stochastic differential game (SDG for short) has a value if

$$\inf_{u_2(\cdot)} \sup_{u_1(\cdot)} R(0, x, u_1(\cdot), u_2(\cdot)) = \sup_{u_1(\cdot)} \inf_{u_2(\cdot)} R(0, x, u_1(\cdot), u_2(\cdot)).$$

A strategy $u_1^*(\cdot)$ is said to be optimal for player 1 if

$$R(0, x, u_1^*(\cdot), \tilde{u}_2(\cdot)) \geq \inf_{u_2(\cdot)} \sup_{u_1(\cdot)} R(0, x, u_1(\cdot), u_2(\cdot))$$

for any strategy $\tilde{u}_2(\cdot)$ of player 2. Similarly, a strategy $u_2^*(\cdot)$ is said to be optimal for player 2 if

$$R(0, x, \tilde{u}_1(\cdot), u_2^*(\cdot)) \leq \sup_{u_1(\cdot)} \inf_{u_2(\cdot)} R(0, x, u_1(\cdot), u_2(\cdot))$$

for any strategy $\tilde{u}_1(\cdot)$ of player 1. A pair of optimal strategies for both players satisfies

$$R(0, x, u_1(\cdot), u_2^*(\cdot)) \leq R(0, x, u_1^*(\cdot), u_2^*(\cdot)) \leq R(0, x, u_1^*(\cdot), u_2(\cdot))$$

for any $u_i(\cdot) \in U_i$, $i = 1, 2$. Thus $(u_1^*(\cdot), u_2^*(\cdot))$ constitutes a saddle point equilibrium. Conversely, a pair of saddle point strategies $(u_1^*(\cdot), u_2^*(\cdot))$ is clearly a pair of optimal strategies for both players. For $(t, x) \in [0,T] \times \mathbb{R}^d$, define

$$V^+(t, x) = \inf_{u_2(\cdot)} \sup_{u_1(\cdot)} E\left[\int_t^T \bar{r}(s, X(s), u_1(s), u_2(s))\, ds + g(X(T))\right],$$

and

$$V^{-}(t,x)=\sup_{u_1(\cdot)}\inf_{u_2(\cdot)} E\left[\int_t^T \bar{r}(s,X(s),u_1(s),u_2(s))\,ds+g(X(T))\right],$$

where $X(\cdot)$ satisfies (1) with the initial condition $X(t)=x$. The functions V^+, V^- are called the upper and lower value functions respectively of the SDG.

The study of differential games (i.e., SDG with $\sigma\equiv 0$) was initiated by Isaacs [17]. Though, there is a vast literature in differential games (see [21] and the references therein), the corresponding literature for SDG seems to be rather limited. The non-degenerate case (i.e., when $\sigma\sigma^*$ is uniformly elliptic) is treated in [2], [7], [14], [15], [16]. Generalizing from differential games, one would expect that V^+ satisfies the equation

$$\begin{aligned} V_t^{+}(t,x) &+ \frac{1}{2}\mathrm{tr}(a(t,x)D_x^2V^+(t,x)) \\ &+ \sup_{u_1}\inf_{u_2}\left[\bar{b}(t,x,u_1,u_2)\cdot D_xV^+(t,x)+\bar{r}(t,x,u_1,u_2)\right]=0, \quad (2)\end{aligned}$$

where $a(t,x)=\sigma(t,x)\sigma^*(t,x)$. Similarly, V^- should satisfy the equation

$$\begin{aligned} V_t^{-}(t,x) &+ \frac{1}{2}\mathrm{tr}(a(t,x)D_x^2V^-(t,x)) \\ &+ \inf_{u_2}\sup_{u_1}\left[\bar{b}(t,x,u_1,u_2)\cdot D_xV^-(t,x)+\bar{r}(t,x,u_1,u_2)\right]=0. \quad (3)\end{aligned}$$

The equation (2) and (3) are referred to as (second order) Hamilton-Jacobi-Isaacs (HJI for short) equation. When the matrix a is uniformly elliptic, using the framework of relaxed strategies, Borkar and Ghosh [2] have shown that $V^+\equiv V^- := V$, i.e., the value function exists and it is the unique 'classical' (i.e., $C^{1,2}$) solution of (2) and (3) satisfying the terminal condition $V(T,x)=g(x)$. They have also established the existence of optimal strategies for both players. The degenerate case (i.e., when the matrix a is not uniformly elliptic), however, is far more involved. In this situation, the existence of classical solutions of (2), (3) is more of an exception than rule even in optimal control problem (i.e., one person game) as pointed out in [12]. Some significant contributions in this case are due to Fleming and Souganidis [13] and Lions and Souganidis [22]. In these works, it has been shown that V^+ and V^- are unique viscosity solutions of (2) and (3) respectively, satisfying the terminal condition $V^+(T,x)=g(x)=V^-(T,x)$. In fact, these works are generalization of the corresponding work on differential games by Evans and Souganidis [10]. In these works ([10], [13], [22]), the authors have used strategies and (upper and lower) values in the sense of Elliott-Kalton [8]. When the state dynamics and the cost functional are linear in x, Elliott, Kalton and Markus [9] and Parthasarathy and Raghavan [23] showed the existence of saddle point in relaxed strategies framework, for the deterministic case, i.e., when $\sigma\equiv 0$.

In this section, we study SDG with degenerate diffusions using relaxed controls. A relaxed control for player i, $i=1,2$, is an $\mathcal{M}_i$-valued map on $[0,T]$, where $\mathcal{M}_i$ is the space of probability measures on U_i. Let $b, r, \bar{b}, \bar{r}$ be as in Section 1.

The state of the system $X(\cdot)$ evolves according to the controlled stochastic differential equation of Ito type

$$\begin{aligned} dX(s) &= b(s, X(s), \mu_1(s), \mu_2(s))\, ds + \sigma(s, X(s))\, dW(s), \quad s \in (t, T], \\ X(t) &= x \in \mathbb{R}^d. \end{aligned} \tag{4}$$

Here $W(\cdot)$ is a standard d-dimensional Wiener process. $\mu_i(\cdot) \in \mathcal{A}_i$, where $\mathcal{A}_i$ is the set of all $\mathcal{M}_i-$ valued standard brownian motions such that $\mu_i(t) = f_i(t, W(.))$, where f_i is progressively measurable with respect to the filtrations generated by $W(.)$.

The payoff function is given by

$$R(t, x, \mu_1(\cdot), \mu_2(\cdot)) = E\left[\int_t^T r(s, X(s), \mu_1(s), \mu_2(s))\, ds + g(X(T))\right],$$

where $X(\cdot)$ is given by (4). Let

$$V^+(t,x) = \inf_{\mu_2(\cdot)\in\mathcal{A}_2} \sup_{\mu_1(\cdot)\in\mathcal{A}_1} R(t, x, \mu_1(\cdot), \mu_2(\cdot)),$$

$$V^-(t,x) = \sup_{\mu_1(\cdot)\in\mathcal{A}_1} \inf_{\mu_2(\cdot)\in\mathcal{A}_2} R(t, x, \mu_1(\cdot), \mu_2(\cdot)).$$

The functions V^+ and V^- are called upper and lower value functions of the game respectively.

A strategy $\mu_1^*(\cdot) \in \mathcal{A}_1$ is said to be optimal for player 1 at (t, x), if

$$R(t, x, \mu_1^*(\cdot), \mu_2(\cdot)) \geq V^+(t, x),$$

for any $\mu_2(\cdot) \in \mathcal{A}_2$. Similarly, a strategy $\mu_2^*(\cdot) \in \mathcal{A}_2$ is said to be optimal for player 2 at (t, x), if

$$R(t, x, \mu_1(\cdot), \mu_2^*(\cdot)) \leq V^-(t, x),$$

for any $\mu_1(\cdot) \in \mathcal{A}_1$. The game is said to have a value if $V^+(t,x) = V^-(t,x) := V(t,x)$, for all (t, x). In such a case, V is called the value function of the game.

Remark 4. *One of the biggest advantages in the setup of relaxed strategies is the convexity of the sets $\mathcal{M}_i$. Due to this, we have already seen in section 2 that a viscosity solution of (5) is also a viscosity solution of (6) due to the Fan's minimax principle.*

In general, even under the relaxed strategy set up, V^+ and V^- are not equal. Consider the following simple example. Let the state dynamics be given by $\dot{x} = u + v$ where $u, v \in [-1, 1]$. Let the payoff functional be $R(t, x, u(\cdot), v(\cdot)) = |x(1)|$. Then it is easy to see that $V^+(0,0) \neq V^-(0,0)$. But we can show that they are equal when $\bar{b}$ and $\bar{r}$ are linear in the variable x. See Elliott, Kalton and Markus **[9]** and Parthasarathy and Raghavan **[23]** for the case $\sigma \equiv 0$. In such cases, we can have the following Dynamic Programing Principle, whose proof is omitted.

Lemma 1. *Let $\bar{b}, \bar{r}$ and σ be linear in their first argument, Then for $0 \leq t \leq t + \Delta < T$ and $x \in \mathbb{R}^d$,*

$$V^+(t,x) = \inf_{\mu_2(\cdot)\in\mathcal{A}_2} \sup_{\mu_1(\cdot)\in\mathcal{A}_1} E\Big[\int_t^{t+\Delta} r(s, X(s), \mu_1(s), \mu_2(s))\, ds + V^+(t+\Delta, X(t+\Delta))\Big], \tag{5}$$

where $X(\cdot)$ is the process given by (4) corresponding to $(\mu_1(\cdot), \mu_2(\cdot))$ with the initial condition $X(t) = x$. Similarly,

$$V^-(t,x) = \sup_{\mu_1(\cdot)\in\mathcal{A}_1} \inf_{\mu_2(\cdot)\in\mathcal{A}_2} E\Big[\int_t^{t+\Delta} r(s, X(s), \mu_1(s), \mu_2(s))\, ds + V^-(t+\Delta, X(t+\Delta))\Big]. \tag{6}$$

By defining the strategies and values in the sense of Elliott- Kalton, Fleming and Souganidis [13] are able to prove the Dynamic Programing Principle for general $\bar{b}$ and $\bar{r}$. We now give the details of these without proofs.

A strategy for the first player is a map $\alpha : \mathcal{A}_2 \to \mathcal{A}_1$ which is nonanticipative in the sense that $\mu_2(s) = \tilde{\mu}_2(s)$ for all $s \leq t$, then $\alpha[\mu_2](s) = \alpha[\tilde{\mu}]_2(s)$ for all $s \leq t$. The set of strategies for the first player is denoted by γ_1. Similarly, the set γ_2 of strategies for the player 2 is defined. The upper and lower values are defined as follows:

$$V^+(t,x) = \sup_{\alpha_1\in\gamma_1} \inf_{\mu_2(\cdot)\in\mathcal{A}_2} R(t, x, \alpha_1[\mu_2](\cdot), \mu_2(\cdot)),$$
$$V^-(t,x) = \inf_{\alpha_2\in\gamma_2} \sup_{\mu_1(\cdot)\in\mathcal{A}_1} R(t, x, \mu_1(\cdot), \alpha_2[\mu_1](\cdot)).$$

Now the dynamic programming principle takes the following form.

Lemma 2. *For $0 \leq t \leq t + \Delta < T$ and $x \in \mathbb{R}^d$,*

$$V^+(t,x) = \sup_{\alpha_1\in\gamma_1} \inf_{\mu_2(\cdot)\in\mathcal{A}_2} E\Big[\int_t^{t+\Delta} r(s, X(s), \alpha_1[\mu_2](s), \mu_2(s))\, ds + V^+(t+\Delta, X(t+\Delta))\Big], \tag{7}$$

where $X(\cdot)$ is the process given by (4) corresponding to $(\mu_1(\cdot), \mu_2(\cdot))$ with the initial condition $X(t) = x$. Similarly,

$$V^-(t,x) = \inf_{\alpha_2\in\gamma_2} \sup_{\mu_1(\cdot)\in\mathcal{A}_1} E\Big[\int_t^{t+\Delta} r(s, X(s), \mu_1(s), \alpha_2[\mu_1](s))\, ds + V^-(t+\Delta, X(t+\Delta))\Big]. \tag{8}$$

Using Lemma (1) or Lemma (2), the following result can easily be proved.

Theorem 3. *The lower value function V^- is a viscosity solution of HJI upper equation (6). Similarly the upper value function V^+ is a viscosity solution of HJI lower equation (5).*

Theorem 4. *The value function V exists and is the unique viscosity solution of (5) in $CL([0,T] \times \mathbb{R}^d)$ satisfying (5).*

Proof. By Theorem 3, V^- is viscosity solution of HJI equation (6). We now show that V^- is also viscosity solution of HJI equation (5).
Let $\phi \in C^{1,1}([0,T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$ be such that $V^- - \phi$ has a local maximum at (t_0, x_0). Then

$$\phi_t(t_0, x_0) + \inf_{\mu_2 \in \mathcal{M}_2} \sup_{\mu_1 \in \mathcal{M}_1} H(t_0, x_0, \nabla_x \phi(t_0, x_0), \mu_1, \mu_2) \leq 0.$$

Now by Fan's minimax theorem [11],

$$\inf_{\mu_2 \in \mathcal{M}_2} \sup_{\mu_1 \in \mathcal{M}_1} H(t_0, x_0, D_x\phi(t_0, x_0), \mu_1, \mu_2) = \sup_{\mu_1 \in \mathcal{M}_1} \inf_{\mu_2 \in \mathcal{M}_2} H(t_0, x_0, D_x\phi(t_0, x_0), \mu_1, \mu_2)$$

Therefore, V^- is subsolution of (5). Similarly V^- is supersolution of (5). Thus V^- is solution of (5). Analogously, we can also show that V^+ is a viscosity solution of (6).

Thus V^-, V^+ are solutions of (5) satisfying (5) and hence uniqueness given by Theorem 2, we have $V^- = V^+ = V$ exists as the unique solution of (5) satisfying (5). □

We now establish the existence of saddle point equilibrium when the Lemma 1 is true. in particular, the result is true when $\bar{b}, \bar{r}$ and σ are linear in x.

Theorem 5. *For any $x \in \mathbb{R}^d$, there exists $(\mu_1^*(\cdot), \mu_2^*(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that $\mu_i^*(\cdot)$ is optimal for player $i, i = 1, 2$, for $(0, x)$.*

Proof. For each $i, i = 1, 2$, let $\mathcal{A}_i$ be endowed with the (weak*)topology described in Chapter 2 in [1]. Under this topology, $\mathcal{A}_i$ is a compact metric space. Using Lemma II.1.3, p.20 in [1], it follows that for $\mu_1(\cdot) \in \mathcal{A}_1$, $x \in \mathbb{R}^d$ fixed, the map $\mu_2(\cdot) \mapsto R(0, x, \mu_1(\cdot), \mu_2(\cdot))$ from $\mathcal{A}_2$ to $\mathbb{R}$ is continuous. Similarly, the map $\mu_1(\cdot) \mapsto R(0, x, \mu_1(\cdot), \mu_2(\cdot))$ is continuous for fixed $(x, \mu_2(\cdot))$.
Therefore

$$\inf_{\mu_2(\cdot) \in \mathcal{A}_2} \sup_{\mu_1(\cdot) \in \mathcal{A}_1} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \min_{\mu_2(\cdot) \in \mathcal{A}_2} \max_{\mu_1(\cdot) \in \mathcal{A}_1} R(0, x, \mu_1(\cdot), \mu_2(\cdot))$$

and

$$\sup_{\mu_1(\cdot) \in \mathcal{A}_1} \inf_{\mu_2(\cdot) \in \mathcal{A}_2} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \max_{\mu_1(\cdot) \in \mathcal{A}_1} \min_{\mu_2(\cdot) \in \mathcal{A}_2} R(0, x, \mu_1(\cdot), \mu_2(\cdot)).$$

Hence using Theorem 4, we have

$$\min_{\mu_2(\cdot) \in \mathcal{A}_2} \max_{\mu_1(\cdot) \in \mathcal{A}_1} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \max_{\mu_1(\cdot) \in \mathcal{A}_1} \min_{\mu_2(\cdot) \in \mathcal{A}_2} R(0, x, \mu_1(\cdot), \mu_2(\cdot)).$$

Choose $(\mu_1^*(\cdot), \mu_2^*(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\min_{\mu_2(\cdot)\in\mathcal{A}_2} \max_{\mu_1(\cdot)\in\mathcal{A}_1} R(0,x,\mu_1(\cdot),\mu_2(\cdot)) = \max_{\mu_1(\cdot)\in\mathcal{A}_1} R(0,x,\mu_1(\cdot),\mu_2^*(\cdot)),$$

$$\max_{\mu_1(\cdot)\in\mathcal{A}_1} \min_{\mu_2(\cdot)\in\mathcal{A}_2} R(0,x,\mu_1(\cdot),\mu_2(\cdot)) = \min_{\mu_2(\cdot)\in\mathcal{A}_2} R(0,x,\mu_1^*(\cdot),\mu_2(\cdot)).$$

Clearly $(\mu_1^*(\cdot), \mu_2^*(\cdot))$ is a pair of saddle point strategies for $(0, x)$. □

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