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Convergence of the boundary control for the wave equation in domains with holes of critical size *

A. K. Nandakumaran

Abstract

In this paper, we consider the homogenization of the exact controllability problem for the wave equation in periodically perforated domain with holes of critical size. We show that the boundary control converges to the boundary control of the homogenized system under the assumption that the perforations are uniformly away from the boundary.

1 Introduction

In this article, we consider the following exact boundary controllability problem for the wave equation in the perforated domain $\Omega_{\varepsilon T}$:

$$\begin{gathered} y''_\varepsilon - \Delta y_\varepsilon = 0 \quad \text{in } \Omega_{\varepsilon T} \\ y_\varepsilon = v_\varepsilon \quad \text{on } \Sigma_{\varepsilon T} \\ y_\varepsilon(0) = y^0_\varepsilon,\ y'_\varepsilon = y^1_\varepsilon \quad \text{in } \Omega_\varepsilon, \end{gathered} \tag{1.1}$$

where $\Omega_{\varepsilon T} = \Omega_\varepsilon \times (0,T)$, $\Sigma_{\varepsilon T} = \Sigma_\varepsilon \times (0,T)$, $\Sigma_\varepsilon = \partial\Omega_\varepsilon$ and Ω_ε is a perforated domain obtained from Ω by removing small holes periodically distributed (with period $\varepsilon > 0$, a small parameter) of size a_ε which is of critical size. We make this precise later. The controllability problem consists in finding a control v_ε so that the corresponding solution y_ε satisfies $y_\varepsilon(T) = y'_\varepsilon(T) = 0$. The controllability problem (i.e., the existence of a control v_ε and the corresponding solution y_ε) and homogenization (limit analysis as $\varepsilon \to 0$) for wave equation have been extensively studied by various authors [1, 2, 3, 5, 6, 8].

Our aim in this article is to study the convergence of the outer boundary control $v_\varepsilon|_{\Gamma_T}$ when the holes are of critical size which seems to be open in the literature quoted above. The (strong) convergence of this control when the size is smaller than the critical one has been studied in [2]. Of course even this article does not yield the convergence of the controls on the boundary of the holes, but our expectation is that it should converge to an internal control in some sense. In the next section, we make the problem precise and state our result, while it will be proved in Section 3.

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2 Preliminaries and Main Result

Notation Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with boundary Γ. Let $Y = (-1,1)^N$ and $S \subset\subset Y$ be an open set containing the origin. Let $\varepsilon > 0$ be a small parameter and $0 < a_\varepsilon \leq \varepsilon$. Perforate the domain Ω by making holes of size a_ε from ε-periodic cells. $Y_\varepsilon = \varepsilon Y$ and $S_\varepsilon = a_\varepsilon S$. The remaining part is $Y_\varepsilon = \varepsilon Y \setminus a_\varepsilon S$. Let $Y^k = Y + k$, $k \in Z^N$ and $S^k = S + k$. Let $\alpha > 0$ be any positive real number and define

$$D_\alpha = \{x \in \Omega : d(x, \partial\Omega) \geq \alpha\}.$$

Our assumption is that, we make perforations uniformly away from the boundary, i.e., let $I_\varepsilon = \{k \in Z^N : Y_\varepsilon^k \subset \Omega \setminus D_\alpha\}$ and define

$$\Omega_\varepsilon = \Omega \setminus (\cup_{k \in I_\varepsilon} \overline{S_\varepsilon^k}).$$

The boundary of Ω_ε is $\Sigma_\varepsilon = \partial\Omega_\varepsilon = \Gamma \bigcup (\cup_{k\in I_\varepsilon} \partial S_\varepsilon^k)$. Let $T > 0$ and $\Omega_T = \Omega \times (0,T)$ and define $\Omega_{\varepsilon T}, \Sigma_{\varepsilon T}, \Gamma_T$ etc. analogously. The critical size a_ε is given by

$$a_\varepsilon = \begin{cases} C_0 \varepsilon^{N/(N-2)} & \text{if } N \geq 3 \\ \exp(-C_0/\varepsilon^2) & \text{if } = 2, \end{cases} \tag{2.2}$$

where C_0 is a constant.

We now introduce the construction of the controls v_ε given in [2, 3] using the Hilbert Uniqueness Method (HUM) introduced by J. L. Lions [5, 6].

Let $m(x) = x - x^0$, $x^0 \in \mathbb{R}^N$ fixed and $T_0 = 2\|m\|_{L^\infty(\Omega)}$. Let $T > T_0$ and consider a real function $\psi \in C^1[0,T]$ such that $\psi' \leq 0$ for all $t \in [0,T]$, $\psi(t) = 1$ for all $t \in [0, \frac{T_0+T}{2}]$ and $\psi(T) = 0$. Let $\{\phi_\varepsilon^0, \phi_\varepsilon^1\} \in H_0^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ and solve the system

$$\begin{gathered} \phi_\varepsilon'' - \Delta\phi_\varepsilon = 0 \quad \text{in } \Omega_{\varepsilon T} \\ \phi_\varepsilon = 0 \quad \text{on } \Sigma_{\varepsilon T} \\ \phi_\varepsilon(0) = \phi_\varepsilon^0,\ \phi_\varepsilon'(0) = \phi_\varepsilon^1 \quad \text{in } \Omega_\varepsilon \end{gathered} \tag{2.3}$$

Then y_ε is obtained by solving

$$\begin{gathered} y_\varepsilon'' - \Delta y_\varepsilon = \quad \text{in } \Omega_{\varepsilon T} \\ y_\varepsilon = \psi(t)(m.\nu_\varepsilon)\frac{\partial\phi_\varepsilon}{\partial\nu_\varepsilon} \quad \text{on } \Sigma_{\varepsilon T} \\ y_\varepsilon(T) = y_\varepsilon'(T) = 0. \end{gathered} \tag{2.4}$$

In the above equation ν_ε denotes the exterior unit normal to the boundary. The system (2.4) has a unique solution given by the transposition method and

$$y_\varepsilon \in C^0([0,T]; L^2(\Omega_\varepsilon)) \bigcap C^1([0,T]; H^{-1}(\Omega_\varepsilon)).$$

Define $\Lambda_\varepsilon : H_0^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon) \to H^{-1}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ by

$$\Lambda_\varepsilon\{\phi_\varepsilon^0, \phi_\varepsilon^1\} = \{y_\varepsilon'(0), -y_\varepsilon(0)\}.$$

It is proved in [2, 3] that Λ_ε is an isomorphism. Thus for a given $\{y^0_\varepsilon, y^1_\varepsilon\} \in L^2(\Omega_\varepsilon) \times H^{-1}(\Omega_\varepsilon)$, define $\{\phi^0_\varepsilon, \phi^1_\varepsilon\} = \Lambda_\varepsilon^{-1}\{y^1_\varepsilon, -y^0_\varepsilon\}$. Then the solution y_ε of (2.4) associated to ϕ_ε given by (2.3) satisfies $y_\varepsilon(0) = y^0_\varepsilon$, $y'_\varepsilon(0) = y^1_\varepsilon$. Thus the controllability problem is solved. Regarding the convergence of (2.3) and (2.4), the following result can be found in [2, 3]. Throughout the paper, $\tilde{g}$ denotes the extension of g by zero in the holes.

Theorem 2.1 *Assume the notations as in Section 2.1 and Let $T > T_0$. Let $\{y^0_\varepsilon,\ y^1_\varepsilon\} = \{y^0,\ y^1\}\chi_{\Omega_\epsilon}$ with $\{y^0,\ y^1\} \in L^2(\Omega) \times L^2(\Omega)$, where χ_{Ω_ϵ} is the characteristic function of Ω_ϵ . Let y_ε be the solution of (2.4). Then as $\varepsilon \to 0$, one has*

$$\tilde{y_\varepsilon} \rightharpoonup y \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak} \star,$$

where y is the solution of

$$\begin{gathered} y'' - \Delta y + \mu y = 2\mu\psi\phi \quad \text{in } \Omega_T \\ y = \psi(t)(m.\nu)\frac{\partial \phi}{\partial \nu} \quad \text{in } \Gamma_T \\ y(0) = y^0, \quad y'(0) = y^1 \quad \text{in } \Omega, \end{gathered} \tag{2.5}$$

and ϕ is the solution of

$$\begin{gathered} \phi'' - \Delta\phi + \mu\phi = 0 \quad \text{in } \Omega_T \\ \phi = 0 \quad \text{on } \Gamma_T \\ \phi(0) = \phi^0, \quad \phi'(0) = \phi^1, \end{gathered} \tag{2.6}$$

and such that

$$\begin{gathered} \{\tilde{\phi^0_\varepsilon}, \tilde{\phi^1_\varepsilon}\} \rightharpoonup \{\phi^0, \phi^1\} \quad \text{in } H^1_0(\Omega) \times L^2(\Omega) \quad \text{weakly}, \\ \tilde{\phi_\varepsilon} \rightharpoonup \phi \quad \text{in } L^\infty(0,T;H^1_0(\Omega)) \quad \text{weak } \star\,. \end{gathered} \tag{2.7}$$

Moreover $y(T) = y'(T) = 0$.

The non-negative constant μ is called the **strange term** in Cioranescu and Murat [4] in the study of elliptic equation in perforated domains. We recall this result in the following lemma. Similar type of test functions are also studied in [7] for other systems.

Lemma 2.2 *Let Ω_ε be as in Section 2. Then there exists a sequence $w_\varepsilon \in H^1(\Omega)$ and a non negative constant $\mu \in \mathbb{R}_+$ such that*

i. $0 \le w_\varepsilon \le 1, w_\varepsilon = 0$ *on* S_ε *for every* $\varepsilon > 0$

ii. $w_\varepsilon \rightharpoonup 1$, *weakly in* $H^1(\Omega)$ *as* $\varepsilon \to 0$

iii. $\langle -\Delta w_\varepsilon, \zeta u_\varepsilon\rangle_{H^{-1},H^1_0} \to \mu \int \zeta u,$

for every $\zeta \in \mathcal{D}(\Omega)$, every sequence u_ε such that $u_\varepsilon = 0$ on S_ε and $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$. Further if a_ε is sub critical (holes are much smaller), i.e.,

$$a_\varepsilon = \begin{cases} C_o\varepsilon^\alpha \quad \text{with } \alpha > \frac{N}{N-2} & \text{if } N \geq 3 \\ \exp(-C_0/\varepsilon^\alpha) \quad \text{with } \alpha > 2 & \text{if } N = 2, \end{cases} \tag{2.8}$$

then one can take $\mu = 0$ and the convergence in (ii) is strong in $H^1(\Omega)$.

As mentioned earlier the solution y_ε is obtained via transposition method. i.e., y_ε satisfies:

$$\int\int_{\Omega_{\varepsilon T}} y_\varepsilon f_\varepsilon = \int_0^T \int_{\partial\Omega_\varepsilon} \psi(t)(m.\nu_\varepsilon)\frac{\partial\phi_\varepsilon}{\partial\nu_\varepsilon}\frac{\partial\theta_\varepsilon}{\partial\nu_\varepsilon},$$

for all $f_\varepsilon = f.\chi_{\Omega_\varepsilon} \in L^1(0,T;L^2(\Omega_\varepsilon))$, where θ_ε is the unique solution of

$$\begin{gathered} \theta_\varepsilon'' - \Delta\theta_\varepsilon = f_\varepsilon \quad \text{in } \Omega_{\varepsilon T} \\ \theta_\varepsilon = 0 \quad \text{on } \Sigma_{\varepsilon T} \\ \theta_\varepsilon(0) = 0 = \theta_\varepsilon'(0). \end{gathered} \tag{2.9}$$

The following convergence is also true:

$$\int\int_{\Gamma_T \bigcup S_{\varepsilon T}} \psi(m.\nu_\varepsilon)\frac{\partial\phi_\varepsilon}{\partial\nu_\varepsilon}\frac{\partial\theta_\varepsilon}{\partial\nu_\varepsilon} \to \int\int_{\Gamma_T} \psi(m.\nu)\frac{\partial\phi}{\partial\nu}\frac{\partial\theta}{\partial\nu} + \int\int_{\Omega_T} 2\mu\varphi\phi\theta. \tag{2.10}$$

Here θ is the unique solution of

$$\begin{gathered} \theta'' - \Delta\theta + \mu\theta = f \\ \theta = 0 \\ \theta(0) = 0 = \theta'(0). \end{gathered} \tag{2.11}$$

The natural questions which would arise, at this stage are the convergence of the controls

$$\psi(m.\nu_\varepsilon)\frac{\partial\phi_\varepsilon}{\partial\nu}\Big|_{\Gamma_T} \quad \text{and} \quad \psi(m.\nu_\varepsilon)\frac{\partial\phi_\varepsilon}{\partial\nu_\varepsilon}\Big|_{\partial S_{\varepsilon T}}.$$

We have the following theorem which will be proved in the next section.

Theorem 2.3 *Let a_ε be of critical size as in (2.2) and Ω_ε be given as in Section 2.1. Then the outer boundary controls*

$$\psi(t)(m.\nu)\frac{\partial\phi_\varepsilon}{\partial\nu} \rightharpoonup \psi(t)(m.\nu)\frac{\partial\phi}{\partial\nu} \quad \text{in } L^2(\Gamma_T) \text{ weak.}$$

where ϕ_ε, ϕ respectively are the solutions of (2.3) and (2.6).

Remark 2.4 We do not have the convergence of the internal boundary controls. Also without the assumption that the perforations are located uniformly away from the boundary, the problem still remains open.

Remark 2.5 Comparing the convergence in (2.10), it is not yet clear that whether the following convergence are true:

$$\iint_{\Gamma_T} \psi(m.\nu) \frac{\partial \phi_\varepsilon}{\partial \nu} \frac{\partial \theta_\varepsilon}{\partial \nu} \to \iint_{\Gamma_T} \psi(m.\nu) \frac{\partial \phi}{\partial \nu} \frac{\partial \theta}{\partial \nu} \tag{2.12}$$

$$\iint_{S_{\varepsilon T}} \psi(m.\nu_\varepsilon) \frac{\partial \phi_\varepsilon}{\partial \nu_\varepsilon} \frac{\partial \theta_\varepsilon}{\partial \nu_\varepsilon} \to \iint_{\Omega_T} 2\mu\varphi\phi\theta. \tag{2.13}$$

Of course the proof of Theorem 2.3 can be applied to see that $\frac{\partial \theta_\varepsilon}{\partial \nu_\varepsilon}|_{\Gamma_T}$ also converges weakly to $\frac{\partial \theta}{\partial \nu}|_{\Gamma_T}$, but this does not directly yield (2.12) (and hence (2.13)). But in the sub critical case, one gets the strong convergence of $\frac{\partial \theta_\varepsilon}{\partial \nu_\varepsilon}$ and hence (2.12) and (2.13) (see [2]) with $\mu = 0$.

3 Proof of Theorem 2.3

The proof consists of the following two steps.
Claim 1: There exists a constant $C > 0$ such that

$$\|\frac{\partial \rho_\varepsilon}{\partial \nu}\| \leq C, \tag{3.14}$$

where ρ_ε is the solution of

$$\begin{gathered} \rho_\varepsilon'' - \Delta \rho_\varepsilon = f_\varepsilon \quad \text{in } \Omega_{\varepsilon T} \\ \rho_\varepsilon = 0 \quad \text{on } \Gamma_T \bigcup S_{\varepsilon T} \\ \rho_\varepsilon(0) = \rho_\varepsilon^0, \quad \rho_\varepsilon'(0) = \rho_\varepsilon^1, \end{gathered} \tag{3.15}$$

where

$$\begin{gathered} f_\varepsilon \to f \quad \text{in } L^1(0,T;L^2(\Omega)) \quad \text{strong} \\ \rho_\varepsilon^0 \rightharpoonup \rho^0 \quad \text{in } H_0^1(\Omega) \quad \text{weak} \\ \rho_\varepsilon^1 \rightharpoonup \rho^1 \quad \text{in } L^2(\Omega) \quad \text{weak.} \end{gathered} \tag{3.16}$$

From the homogenization results of [1], one has

$$\begin{gathered} \rho_\varepsilon \rightharpoonup \rho \quad \text{in } L^\infty(0,T,H_0^1) \quad \text{weak} \\ \rho_\varepsilon' \rightharpoonup \rho' \quad \text{in } L^\infty(0,T;L^2) \quad \text{weak,} \end{gathered} \tag{3.17}$$

where ρ satisfies

$$\begin{gathered} \rho'' - \Delta \rho + \mu\rho = f \\ \rho = 0 \\ \rho(0) = \rho^0, \quad \rho'(0) = \rho^1. \end{gathered} \tag{3.18}$$

Claim 2: $\frac{\partial \rho_\varepsilon}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu}$ in $L^2(\Gamma_T)$ weak.

Proof Claim 1: Recall $D_\alpha = \{x \in \Omega : d(x, \partial\Omega) \leq \alpha\}$ and choose $q \in C^1(\bar{\Omega})^N$ such that

$$q = \begin{cases} \nu & \text{on } \partial\Omega \\ 0 & \text{in } \Omega \backslash D_\alpha, \end{cases} \tag{3.19}$$

where ν is the unit normal to Γ. Since D_α does not contains any holes, we have $q = 0$ on S_ε, that is on the boundary of the holes.

Now, multiplying (3.15) by $q_k \frac{\partial \rho_\varepsilon}{\partial x_k}$ (where we use the repeated indices convention) and integrating by parts, we get

$$\begin{aligned} &\int_{\Omega_\varepsilon} [\rho_\varepsilon' q_k \frac{\partial \rho_\varepsilon}{\partial x_k}]_0^T - \int_{\Omega_{\varepsilon T}} \rho_\varepsilon' q_k \frac{\partial \rho_\varepsilon'}{\partial x_k} + \int_{\Omega_{\varepsilon T}} \nabla \rho_\varepsilon \nabla q_k \frac{\partial \rho_\varepsilon}{\partial x_k} \\ &\quad + \int_{\Omega_{\varepsilon T}} \nabla \rho_\varepsilon . q_k \nabla \frac{\partial \rho_\varepsilon}{\partial x_k} - \int_{\Gamma_{\varepsilon T} \bigcup S_{\varepsilon T}} \frac{\partial \rho_\varepsilon}{\partial \nu} q_k \frac{\partial \rho_\varepsilon}{\partial x_k} = \int_{\Omega_{\varepsilon T}} f_\varepsilon q_k \frac{\partial \rho_\varepsilon}{\partial x_k}, \end{aligned} \tag{3.20}$$

which can be written under the form

$$I_1 + I_2 + I_3 + I_4 - I = I_5.$$

Since $\rho_\varepsilon = 0$ on Γ_T, we have $\frac{\partial \rho_\varepsilon}{\partial x_k} = \nu_k \frac{\partial \rho_\varepsilon}{\partial \nu}$. By the choice of q, it follows that

$$I = \int_{\Gamma_T} |\frac{\partial \rho_\varepsilon}{\partial \nu}|^2 d\sigma dt.$$

We estimate the other terms as follows:

$$\begin{aligned} I_4 &= \frac{1}{2} \int_{\Omega_{\varepsilon T}} q_k \frac{\partial}{\partial x_k} (\nabla \rho_\varepsilon . \nabla \rho_\varepsilon) \\ &= -\frac{1}{2} \int_{\Omega_{\varepsilon T}} \operatorname{div} q . |\nabla \rho_\varepsilon|^2 + \frac{1}{2} \int_{\Gamma_T} \nabla \rho_\varepsilon \nabla \rho_\varepsilon \\ &= -\frac{1}{2} \int_{\Omega_{\varepsilon T}} \operatorname{div} q . |\nabla \rho_\varepsilon|^2 + \frac{1}{2} \int_{\Gamma_T} |\frac{\partial \rho_\varepsilon}{\partial \nu}|^2. \end{aligned}$$

$$\begin{aligned} I_2 &= -\frac{1}{2} \int_{\Omega_{\varepsilon T}} q_k \frac{\partial}{\partial x_k} (\rho_\varepsilon'^2) \\ &= \frac{1}{2} \int_{\Omega_{\varepsilon T}} \operatorname{div} q . |\rho_\varepsilon'|^2 - \frac{1}{2} \int_{\Gamma_T \bigcup S_{\varepsilon T}} q_k \rho_\varepsilon'^2 \nu_k, \end{aligned}$$

The second term vanishes as $\rho'_\varepsilon = 0$ on $\Gamma_T \bigcup S_{\varepsilon T}$, so we get from (3.20):

$$\begin{aligned}
\frac{1}{2}\int_{\Gamma_T} |\frac{\partial \rho_\varepsilon}{\partial \nu}|^2 &= \int_{\Omega_\varepsilon} [\rho'_\varepsilon(T) q_k \frac{\partial \rho_\varepsilon}{\partial x_k}(T) - \rho'_\varepsilon(0) q_k \frac{\partial \rho_\varepsilon}{\partial x_k}(0)] \\
&\quad + \frac{1}{2}\int_{\Omega_{\varepsilon T}} \operatorname{div} q.(\rho'^2_\varepsilon - |\nabla \rho_\varepsilon|^2) + \int_{\Omega_{\varepsilon T}} \nabla \rho_\varepsilon \nabla q_k \frac{\partial \rho_\varepsilon}{\partial x_k} \\
&\quad - \int_{\Omega_{\varepsilon T}} f_\varepsilon q_k \frac{\partial \rho_\varepsilon}{\partial x_k} \\
&\le C[(\|\rho'_\varepsilon(T)\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla \rho_\varepsilon(T)\|^2_{L^2(\Omega_\varepsilon)}) \\
&\quad + (\|\rho'_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla \rho_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)})] \\
&\quad + C[\int_{\Omega_{\varepsilon T}} (|\rho'_\varepsilon|^2 + |\nabla \rho_\varepsilon|^2)] + C[\int_{\Omega_{\varepsilon T}} |f_\varepsilon||\nabla \rho_\varepsilon|].
\end{aligned} \tag{3.21}$$

To estimate the right-hand side of the above inequality, we multiply (3.15) by ρ'_ε and integrate from 0 to t to get

$$\frac{1}{2}\int_0^t \int_{\Omega_\varepsilon} \frac{\partial}{\partial t}(\rho'^2_\varepsilon + \nabla \rho_\varepsilon \nabla \rho_\varepsilon) = \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon \rho'_\varepsilon,$$

i.e.,

$$E_\varepsilon(t) - E_\varepsilon(0) = \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon \rho'_\varepsilon,$$

where

$$E_\varepsilon(t) = \frac{1}{2}\int_{\Omega_\varepsilon} (|\rho'_\varepsilon(t)|^2 + |\nabla \rho_\varepsilon(t)|^2)\, dx.$$

Now

$$\begin{aligned}
|\int_0^t \int_{\Omega_\varepsilon} f_\varepsilon \rho'_\varepsilon| &\le \int_0^t \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\rho'_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\le \|\rho'_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \|f_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))}.
\end{aligned} \tag{3.22}$$

Therefore,

$$\begin{aligned}
E_\varepsilon(t) &\le E_\varepsilon(0) + \|\rho'_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \|f_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))} \\
&\le C[|\nabla \rho^0_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \|\rho^1_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \|f_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))}].
\end{aligned}$$

Obviously, we also have

$$\begin{aligned}
\int_{\Omega_{\varepsilon T}} |f_\varepsilon||\nabla \rho_\varepsilon| &\le \|\nabla \rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \|f_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))} \\
&\le C\|f_\varepsilon\|_{L^1(0,T;L^2(\Omega_\varepsilon))}.
\end{aligned}$$

So, using these inequalities in (3.21), it follows that

$$\begin{aligned}
\int_{\Gamma_T} |\frac{\partial \rho_\varepsilon}{\partial \nu}|^2 &\le C[\|\nabla \rho^0_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\rho^1_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|f_\varepsilon\|_{L^1(0,T:L^2(\Omega_\varepsilon))}] \\
&\le \text{constant},
\end{aligned} \tag{3.23}$$

for all the choices of ρ^0_ε, ρ'_ε and f_ε as in (3.16). This completes the proof of Claim 1.

Proof of Claim 2: From claim 1, it follows that

$$\frac{\partial \rho_\varepsilon}{\partial \nu} \rightharpoonup \eta \quad \text{in } L^2(\Gamma_T) \quad \text{weak.}$$

We have now to identify η. Let $g \in \mathcal{D}(0,T), v \in C^\infty(\bar{\Omega})$. Multiplying (3.15) by $gv\omega_\varepsilon$, where ω_ε are the test functions given by Lemma 2.2, we get

$$\begin{aligned} &-\int_{\Omega_{\varepsilon T}} \rho_\varepsilon' g' v \omega_\varepsilon + \int_{\Omega_{\varepsilon T}} \nabla \rho_\varepsilon g \nabla v \omega_\varepsilon \\ &+ \int_{\Omega_{\varepsilon T}} \nabla \rho_\varepsilon g v \nabla \omega_\varepsilon - \int_{\Gamma_T \bigcup S_{\varepsilon T}} \frac{\partial \rho_\varepsilon}{\partial \nu} g v \omega_\varepsilon = \int_{\Omega_{\varepsilon T}} f_\varepsilon g v \omega_\varepsilon \end{aligned} \tag{3.24}$$

which can be written in the form

$$I_1 + I_2 + I_3 - I = I_4.$$

Note that since the holes are away from the boundary, we have $\omega_\varepsilon = 1$ in a neighborhood of Γ (this is from the construction of ω_ε). Since $\omega_\varepsilon = 0$ on S_ε, from (3.17) and Lemma 2.2 we can easily get the convergence of I, I_1, I_2 and I_4 as follows:

$$\begin{gathered} I \to \int_{\Gamma_T} \eta g v, \quad I_1 \to \int_{\Omega_T} \rho' g' v, \\ I_2 \to \int_{\Omega_T} \nabla \rho g \nabla v, \\ I_4 \to \int_{\Omega_T} f g v. \end{gathered}$$

Then, it remains to pass to the limit in I_3. Of course, we can write formally,

$$I_3 = \int_0^T g \langle -\Delta \omega_\varepsilon, \rho_\varepsilon v \rangle$$

At this point, we cannot apply Lemma 2.2, because $v \notin \mathcal{D}(\Omega)$, but we can overcome this using the fact that $\rho_\varepsilon = 0$ on the boundary Γ_T. We proceed as follows: Let $\delta > 0$, put

$$A_\delta = \{x \in \mathbb{R}^N \backslash \Omega : d(x, \partial\Omega) < \delta\}$$

and let $\Omega_1 = \bar{\Omega} \bigcup A_\delta$ be the δ neighborhood of Ω. Extend ρ_ε to $\tilde{\rho_\varepsilon}$ by zero in A_δ and since $\rho_\varepsilon = 0$ on Γ_T, we have $\tilde{\rho_\varepsilon} \in L^\infty(0,T; H_0^1(\Omega_1))$ and

$$\tilde{\rho_\varepsilon} \rightharpoonup \tilde{\rho} \quad \text{in } L^\infty(0,T; H_0^1(\Omega_1) \quad \text{weak} * .$$

Moreover,

$$\tilde{\rho} = \begin{cases} \rho & \text{in } \Omega \\ 0 & \text{in } A_\delta. \end{cases}$$

Let $\tilde{v}$ be any extension of v such that $\tilde{v} \in \mathcal{D}(\Omega_1)$ and extend ω_ε by 1 outside Ω and again denote the extension by ω_ε. We can apply Lemma 2.3 and since $\tilde{\rho}\tilde{v}\big|_\Omega = \rho v \in H_0^1(\Omega)$, we get

$$\langle -\Delta\omega_\varepsilon, \tilde{\rho}_\varepsilon \tilde{v}\rangle_{H_0^{-1}(\Omega_1), H_0^1(\Omega_1)} \to \mu \int_{\Omega_1} \tilde{\rho}\tilde{v} = \mu \int_\Omega \rho v.$$

Now,

$$\begin{aligned} I_3 &= \int_{\Omega_{\varepsilon T}} \nabla\rho_\varepsilon g v \nabla\omega_\varepsilon = \int_{\Omega_{1T}} \nabla\tilde{\rho}_\varepsilon g\tilde{v}\nabla\omega_\varepsilon \\ &= \int_0^T g\langle -\Delta\omega_\varepsilon, \tilde{\rho}_\varepsilon\tilde{v}\rangle_{H^{-1}(\Omega_1), H_0^1(\Omega_1)} \\ &\to \int_{\Omega_{1T}} \mu g\tilde{v}\tilde{\rho} = \int_{\Omega_T} \mu g v \rho. \end{aligned}$$

So, passing to the limit in (3.24), we get

$$\int_{\Gamma_T} \eta g v = -\int_{\Omega_T} \rho' g' v + \int_{\Omega_T} \nabla\rho g \nabla v + \int \mu\rho g v - \int f g v. \tag{3.25}$$

On the other hand, multiplying (3.18) by gv, we get

$$-\int_{\Omega_T} \rho' g' v + \int_{\Omega_T} \nabla\rho g\nabla v - \int_{\Gamma_T} \frac{\partial\rho}{\partial\nu} g v + \mu\int_{\Omega_T} \rho g v = \int_{\Omega T} f g v. \tag{3.26}$$

From (3.25) and (3.26), it follows that

$$\int_{\Gamma_T} \eta g v = \int_{\Gamma_T} \frac{\partial\rho}{\partial\nu} g v, \ \ \forall \ \ g \in \mathcal{D}(0,T), v \in C^\infty(\bar{\Omega}),$$

which implies that

$$\eta = \frac{\partial\rho}{\partial\nu}.$$

Hence, Claim 2 is proved, which ends the proof of the Theorem 2.3.

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A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science
Bangalore - 560 012, India.
e-mail: nands@math.iisc.ernet.in