Proc. Indian Acad. Sci. (Math. Sci.) Vol. 112, No. 3, August 2002, pp. 425–439.

Homogenization of a parabolic equation in perforated domain with Dirichlet boundary condition

A K NANDAKUMARAN[1,*] and M RAJESH[2]

[1]Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
[2]ANLA, U.F.R. des Sciences et Techniques, Université de Toulon et du Var, BP 132, 83957 La Garde Cedex, France
[*]Corresponding author
E-mail: nands@math.iisc.ernet.in; rajesh@univ-tln.fr

MS received 28 January 2002; revised 23 April 2002

Abstract. In this article, we study the homogenization of the family of parabolic equations over periodically perforated domains

$$\begin{aligned} \partial_t b\left(\frac{x}{d_\varepsilon}, u_\varepsilon\right) - \operatorname{div} a(u_\varepsilon, \nabla u_\varepsilon) &= f(x,t) \quad && \text{in } \Omega_\varepsilon \times (0,T)\,, \\ u_\varepsilon &= 0 && \text{on } \partial\Omega_\varepsilon \times (0,T)\,, \\ u_\varepsilon(x,0) &= u_0(x) && \text{in } \Omega_\varepsilon. \end{aligned}$$

Here, $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$ is a periodically perforated domain and d_ε is a sequence of positive numbers which goes to zero. We obtain the homogenized equation. The homogenization of the equations on a fixed domain and also the case of perforated domain with Neumann boundary condition was studied by the authors. The homogenization for a fixed domain and $b(\frac{x}{d_\varepsilon}, u_\varepsilon) \equiv b(u_\varepsilon)$ has been done by Jian. We also obtain certain corrector results to improve the weak convergence.

Keywords. Homogenization; perforated domain; correctors.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let $T > 0$ be a constant, $\Omega_T = \Omega \times (0,T)$ and let $\varepsilon > 0$ be a small parameter which eventually tends to zero. Let $Y = (-\frac{1}{2}, +\frac{1}{2})^N$ and S(closed) $\subset Y$. We define a periodically perforated domain Ω_ε as follows: First define

$$\begin{aligned} &I_\varepsilon = \{k \in \mathbb{Z}^N : \varepsilon k + a_\varepsilon S \subset \Omega\} \text{ and} \\ &S_\varepsilon = \cup_{k \in I_\varepsilon} (\varepsilon k + a_\varepsilon S), \end{aligned} \tag{1.1}$$

where a_ε is the size of an individual hole. In the case under study we have, $a_\varepsilon = \varepsilon^{N/(N-p)}$ with $2 \le p < N$. Set

$$\Omega_\varepsilon = \Omega \setminus S_\varepsilon. \tag{1.2}$$

We consider the following nonlinear parabolic equation (nonlinearity on both time and spatial components) with Dirichlet conditions on the boundary of the holes. In fact, we only consider the problem with $d_\varepsilon = \varepsilon$ (without loss of generality; see Remark 3.4 in §3):

$$\begin{aligned} \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) - \operatorname{div} a(u_\varepsilon, \nabla u_\varepsilon) &= f(x,t) \text{ in } \Omega_\varepsilon \times (0,T)\,, \\ u_\varepsilon &= 0 \qquad \text{on } \partial\Omega_\varepsilon \times (0,T)\,, \\ u_\varepsilon(x,0) &= u_0(x) \quad \text{in } \Omega_\varepsilon, \end{aligned} \tag{1.3}$$

where u_0 is a given function on Ω and f is a given function on $\Omega \times (0,T)$. For a given ε, the Cauchy problem (1.3) will also be denoted by (P_ε). It is known that under suitable assumptions on a and b (cf. assumptions (A1)–(A4) below), that the problem (P_ε) has a solution u_ε. Our aim in this paper is to study the homogenization of the equations (P_ε) as $\varepsilon \to 0$, i.e., to study the limiting behavior of u_ε as $\varepsilon \to 0$ and obtain the limiting equation satisfied by the limit.

When b is linear (i.e. $b(y,s) = s$) the asymptotic analysis of such problems has been studied quite widely [5,6,8,10,11,19]. When b is not linear, the homogenization of the equation in a fixed domain was studied by Jian [12] for $b(y,s) \equiv b(s)$ with appropriate assumptions and by the authors [17] when b is also oscillating. In the case of perforated domains, the authors [18] have obtained results on homogenization with Neumann condition on the boundary of the holes in which we also consider oscillations in the elliptic part as well. More precisely, we considered the term $a(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon)$. There is considerable difficulty in analyzing a similar problem with the Dirichlet condition on the boundary of the holes. The oscillations in the coefficients give rise to new difficulties and are hard to deal with in passing to the limit. So we only consider a without periodic oscillations. The analysis of even this case is very subtle and will use the work of Casado–Díaz [7] in the stationary case.

In this context, we would like to point out that the analysis for the stationary problem in the linear elliptic case (recently solved by Dal Maso–Murat [9]) with oscillations both in a and domain is quite involved. Now one has to develop an appropriate technique for the stationary nonlinear elliptic problem. This will then enable us to complete the homogenization for the full parabolic problem.

The layout of the paper is as follows. In §2, we give the weak formulation for the problem (P_ε). Then, we state our main results on homogenization. In §3, we obtain some crucial convergence results for one of the terms in the equation. Finally, in §4 and 5, we complete the homogenization using the results of §3 and the arguments from [7]. A corrector result is also stated in §5. An associated open problem is also discussed in §5 (see Remark 5.1).

2. Assumptions and main results

For $p > 1$, p^* will denote the conjugate exponent $p/(p-1)$. Let E_ε be $L^p(0,T; W_0^{1,p}(\Omega_\varepsilon))$. Let f belong to $L^p(0,T; W^{-1,p^*}(\Omega_\varepsilon))$. We define $u_\varepsilon \in E_\varepsilon$ to be a weak solution of (P_ε) if it satisfies:

$$b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \in L^\infty(0,T; L^1(\Omega_\varepsilon)),\ \ \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \in L^{p^*}(0,T; W_0^{1,p}(\Omega_\varepsilon))\,, \tag{2.1}$$

that is

$$\int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), \xi(x,t)\right\rangle_\varepsilon \mathrm{d}t + \int_0^T\!\!\int_{\Omega_\varepsilon} \left(b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) - b\left(\frac{x}{\varepsilon}, u_0\right)\right) \partial_t \xi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.2}$$

for all $\xi \in E_\varepsilon \cap W^{1,1}(0, T; L^\infty(\Omega_\varepsilon))$ with $\xi(T) = 0$; and,

$$\int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), \xi(x,t)\right\rangle_\varepsilon \mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} a(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \xi(x,t)\, \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} f(x,t)\xi(x,t)\, \mathrm{d}x\, \mathrm{d}t \qquad (2.3)$$

for all $\xi \in E_\varepsilon$. Here $\langle\, .\,,\, .\,\rangle_\varepsilon$ denotes the duality bracket with respect to $E_\varepsilon^*, E_\varepsilon$.

We make the following assumptions on a and b:

(A1) The function $b(y, s)$ is continuous in y and s, Y-periodic in y and nondecreasing in s and $b(y, 0) = 0$.

(A2) There exists a constant $\theta > 0$ such that for every δ and R with $0 < \delta < R$, there exists $C(\delta, R) > 0$ such that

$$|b(y, s_1) - b(y, s_2)| > C(\delta, R)|s_1 - s_2|^\theta \qquad (2.4)$$

for all $y \in Y$ and $s_1, s_2 \in [-R, R]$ with $\delta < |s_1|$.

Remark 2.1. The prototype for b is a function of the form $c(y)|s|^k \operatorname{sgn}(s)$ for some positive real number k and continuous and Y-periodic function, $c(\cdot)$, which is positive on Y. □

(A3) The mapping $(\mu, \lambda) \mapsto a(\mu, \lambda)$ defined from $\mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}^N$ is continuous in (μ, λ). Further, it is assumed that there exists positive constants α, r such that

$$a(\mu, \lambda) \cdot \lambda \geq \alpha |\lambda|^p, \qquad (2.5)$$

$$(a(\mu, \lambda_1) - a(\mu, \lambda_2)) \cdot (\lambda_1 - \lambda_2) > 0, \quad \forall\, \lambda_1 \neq \lambda_2, \qquad (2.6)$$

$$|a(\mu, \lambda)| \leq \alpha^{-1}(1 + |\mu|^{p-1} + |\lambda|^{p-1}), \qquad (2.7)$$

$$|a(\mu_1, \lambda) - a(\mu_2, \lambda)| \leq \alpha^{-1}|\mu_1 - \mu_2|^r (1 + |\mu_1|^{p-1-r} + |\mu_2|^{p-1-r} + |\lambda|^{p-1-r}) \qquad (2.8)$$

(A4) We assume that, the data, $f \in L^\infty(\Omega \times T)$.

(A5) For all $\mu, \lambda_1, \lambda_2$,

$$(a(\mu, \lambda_1) - a(\mu, \lambda_2))(\lambda_1 - \lambda_2) \geq \alpha |\lambda_1 - \lambda_2|^p. \qquad (2.9)$$

Under the assumptions (A1)–(A4), it is known that (P_ε) admits a solution u_ε (cf. [4]). The assumption (A5) will be useful in proving corrector results.

We now state our main theorem.

Theorem 2.2. *Let u_ε be a family of solutions of (P_ε). Assume that there is a constant $C > 0$, such that*

$$\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon \times (0,T))} \leq C. \qquad (2.10)$$

Then there exists a subsequence of ε, still denoted by ε, such that for all q with $0 < q < \infty$, we have

$$\widetilde{u_\varepsilon} \to u \text{ strongly in } L^q(\Omega_T) \qquad (2.11)$$

and u solves,

$$
\begin{aligned}
\partial_t \overline{b}(u) - \operatorname{div} a(u, \nabla u) + \Phi(u) &= f(x,t) \quad \textit{in } \Omega \times (0,T)\,, \\
u &= 0 \qquad\qquad \textit{on } \partial\Omega \times (0,T)\,, \\
u(x,0) &= 0 \qquad\qquad \textit{in } \Omega
\end{aligned} \tag{2.12}
$$

where

$$\overline{b}(s) \triangleq \int_Y b(y,s)\,\mathrm{d}y \tag{2.13}$$

and

$$\Phi(s) \triangleq \int_{\mathbb{R}^N} a_0(w_s, \nabla w_s) \cdot \nabla v_0 \,\mathrm{d}y, \tag{2.14}$$

where w_s is a solution of

$$
\begin{aligned}
&-\operatorname{div}(a_0(w_s, \nabla w_s)) = 0 \quad \textit{in } \mathcal{D}^{'}(\mathbb{R}^N \setminus S),\\
&w_s - s \in D^{1,p}(\mathbb{R}^N),\\
&w_s\,\phi \in W_0^{1,p}(\mathbb{R}^N \backslash S) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^N)
\end{aligned} \tag{2.15}
$$

and v_0 satisfies

$$
\begin{aligned}
&v_0 - 1 \in D^{1,p}(\mathbb{R}^N),\\
&v_0\,\phi \in W_0^{1,p}(\mathbb{R}^N \backslash S) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^N).
\end{aligned} \tag{2.16}
$$

□

In the above,

$$a_0(s,\xi) = \lim_{\varepsilon \to 0} \varepsilon^{N(p-1)/(N-p)} a(s, \xi/\varepsilon^{N/(N-p)} \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{2.17}$$

and

$$D^{1,p}(\mathbb{R}^N) = \{u \in L^{Np/(N-p)}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N)\}. \tag{2.18}$$

We shall denote the expression in (2.17) whose limit is evaluated by $a_\varepsilon(s,\xi)$.

Remark 2.3. The assumption (2.10) is true in special cases (see [13]) and it is reasonable on physical grounds (see [12]). □

Remark 2.4. The so-called 'strange term' ϕ appears even in the elliptic linear case (see Cioranescu–Murat [14]) and so it is not very surprising to see one here. □

We will state a corrector result in §5.

3. Some preliminary results

In this section we will identify the weak limit of the sequence $b(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon})$ as it will be necessary for the homogenization.

An important step in this analysis is to show that $u_\varepsilon \to u$ a.e. in $\Omega \times (0, T)$. This does not come easily as there are no *a priori* bounds on the time derivative of the sequence u_ε which will allow us to use a compactness theorem of the Aubin–Lions type. For proving this result we adapt a technique found in [4] and already used in our paper [17]. As we closely follow the treatment in [17], some of the results will only be sketched and we refer the reader to [17] for more details as and when necessary.

We first obtain *a priori* bounds under the assumption (2.10). From now on, C will denote a generic positive constant which is independent of ε.

Lemma 3.1. *Let u_ε be a family of solutions of* (P_ε) *and assume that* (2.10) *holds. Then,*

$$\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon \times (0,T))} \le C, \tag{3.1}$$

$$\sup_\varepsilon \|a(u_\varepsilon, \nabla u_\varepsilon)\|_{L^{p^*}(\Omega_\varepsilon \times (0,T))} \le C, \tag{3.2}$$

$$\sup_\varepsilon \|\partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\|_{E_\varepsilon^*} \le C. \tag{3.3}$$

Proof. Define the function $B(.,.) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$B(y,s) = b(y,s)s - \int_0^s b(y,\tau)\,\mathrm{d}\tau. \tag{3.4}$$

As in [17] we deduce that

$$\int_{\Omega_\varepsilon} B\left(\frac{x}{\varepsilon}, u_\varepsilon(x,T)\right)\,\mathrm{d}x + \int_0^T \int_{\Omega_\varepsilon} a(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_\varepsilon} B\left(\frac{x}{\varepsilon}, u_0\right)\,\mathrm{d}x + \int_0^T \int_{\Omega_\varepsilon} f\, u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$

and from this we obtain

$$\int_{\Omega} B\left(\frac{x}{\varepsilon}, u_\varepsilon(x,T)\right)\,\mathrm{d}x + \int_0^T \int_{\Omega_\varepsilon} a(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \le C \tag{3.5}$$

by (2.10) and the assumptions on b. Then, (3.1) follows from (3.5) and (2.5), as B is nonnegative, while (3.2) follows from (3.1) and (2.7). The estimate (3.3) may be obtained from (3.1), (3.2) and (2.3). Thus the lemma. □

We state the following technical lemma whose proof can be found in [17].

Lemma 3.2. *There exists a continuous, increasing function ω on $\mathbb{R}^+$ with $\omega(0) = 0$, such that, given any $C > 0, \delta > 0$, if v_1, v_2 are any two functions in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\|v_i\|_{\infty,\Omega} \le C$, $i = 1, 2$, satisfying*

$$\int_\Omega \left(b\left(\frac{x}{\varepsilon}, v_1\right) - b\left(\frac{x}{\varepsilon}, v_2\right)\right)(v_1 - v_2)\,\mathrm{d}x \le \delta \quad \forall \varepsilon > 0,$$

then

$$\int_\Omega \left| b\left(\frac{x}{\varepsilon}, v_1\right) - b\left(\frac{x}{\varepsilon}, v_2\right)\right| \mathrm{d}x \le \omega(\delta) \quad \forall \varepsilon > 0.$$

□

We now prove a crucial lemma.

Lemma 3.3. *Let u_ε be as above. Then, the sequence $\{\widetilde{u_\varepsilon}\}_{\varepsilon>0}$ is relatively compact in $L^\theta(\Omega_T)$, where θ is as in* (A2). *As a result, there is a subsequence of u_ε such that*

$$\widetilde{u_\varepsilon} \to u \text{ a.e. in } \Omega_T. \tag{3.6}$$

Proof.

Step 1: Using the arguments from [12], it can be shown that

$$h^{-1}\int_0^{T-h}\int_{\Omega_\varepsilon}\left(b\left(\frac{x}{\varepsilon}, u_\varepsilon(t+h)\right) - b\left(\frac{x}{\varepsilon}, u_\varepsilon(t)\right)\right)(u_\varepsilon(t+h) - u_\varepsilon(t))\mathrm{d}x\mathrm{d}t \le C$$

for some constant C which is independent of ε and h. Thus, as we have assumed in (A1) that $b(y, 0) = 0$, we get

$$h^{-1}\int_0^{T-h}\int_\Omega\left(b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t+h)\right) - b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t)\right)\right)(\widetilde{u_\varepsilon}(t+h) - \widetilde{u_\varepsilon}(t))\,\mathrm{d}x\,\mathrm{d}t \le C.$$

Step 2: We show that

$$\int_0^{T-h}\int_\Omega\left|b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t+h)\right) - b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t)\right)\right|\,\mathrm{d}x\,\mathrm{d}t \to 0$$

as $h \to 0$, uniformly with respect to ε. Set, for $R > 0$ and large,

$$\begin{aligned} E_{\varepsilon,R} = \Bigg\{ t \in (0, T-h) : {}& \|\widetilde{u_\varepsilon}(t+h)\|_{W^{1,p}(\Omega)} + \|\widetilde{u_\varepsilon}(t)\|_{W^{1,p}(\Omega)} \\ &+ h^{-1}\int_\Omega\left(b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t+h)\right) - b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t)\right)\right) \\ &\cdot(\widetilde{u_\varepsilon}(t+h) - \widetilde{u_\varepsilon}(t))\,\mathrm{d}x > R\Bigg\}. \end{aligned}$$

We claim that $m(E_{\varepsilon,R}) \le C/R$ independent of h. Indeed, if we set

$$E^1_{\varepsilon,R} = \{t \in (0, T) : \|\widetilde{u_\varepsilon}(t)\|_{W^{1,p}(\Omega)} > R/4\}$$

and

$$\begin{aligned} E^2_{\varepsilon,R} = \Bigg\{ t \in (0, T-h) : h^{-1}\int_\Omega & \left(b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t+h)\right)\right. \\ &\left. - b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t)\right)\right)\cdot(\widetilde{u_\varepsilon}(t+h) - \widetilde{u_\varepsilon}(t))\mathrm{d}x > R/2\Bigg\}, \end{aligned}$$

then clearly $E_{\varepsilon,R} \subset E^1_{\varepsilon,R} \cup (E^1_{\varepsilon,R} - h) \cup E^2_{\varepsilon,R}$. Now $m(E^2_{\varepsilon,R}) < C/R$ by Step 1 and $m(E^1_{\varepsilon,R}) < C/R$ by (2.10) and (3.1) for some constant C. (Indeed $m(E^1_{\varepsilon,R})(R/4)^p T \leq C$ from which this follows since $p \geq 1$.) The estimates for $m(E^1_{\varepsilon,R})$, $m(E^2_{\varepsilon,R})$ and the translation invariance of Lebesgue measure gives the estimate for $m(E_{\varepsilon,R})$.

Now set $E^{'}_{\varepsilon,R}$ to be the complement of $E_{\varepsilon,R}$ in $(0, T-h)$. Hence, for $t \in E^{'}_{\varepsilon,R}$, by Lemma 3.2, we have

$$\int_\Omega \left| b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t+h)\right) - b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t)\right) \right| \mathrm{d}x < \omega(hR), \tag{3.7}$$

where, obviously, the modulus of continuity function does not depend on ε. Therefore,

$$\begin{aligned}
&\int_0^{T-h} \int_\Omega \left| b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t+h)\right) - b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t)\right) \right| \\
&\quad = \int_{E_{\varepsilon,R}} \int_\Omega \left| b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t+h)\right) - b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t)\right) \right| \\
&\qquad + \int_{E^{'}_{\varepsilon,R}} \int_\Omega \left| b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t+h)\right) - b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}(t)\right) \right| \\
&\quad \leq C/R + T\omega(hR)
\end{aligned}$$

for all ε, R and h. Now, choose $R = h^{-1/2}$ and let $h \to 0$ to complete the proof of Step 2.

Step 3: By assumption (A2), it follows from Step 2 that

$$\int_0^{T-h} \int_\Omega |\widetilde{u_\varepsilon}(t+h) - \widetilde{u_\varepsilon}(t)|^\theta \,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ as } h \to 0 \tag{3.8}$$

uniformly with respect to ε.

Step 4: In this crucial step, we demonstrate the relative compactness of the sequence $\{\widetilde{u_\varepsilon}\}_{\varepsilon>0}$ in $L^\theta(\Omega_T)$. This is an argument to reduce it to the time independent case. Set,

$$v_\varepsilon(x,t) = \begin{cases} \widetilde{u_\varepsilon}(x,t) & \text{if } t \in (0, T-h) \backslash E_{\varepsilon,R} \\ 0 & \text{otherwise} \end{cases}. \tag{3.9}$$

Choose h so that T is an integral multiple of h. We have

$$\begin{aligned}
&\frac{1}{h}\int_0^h \mathrm{d}s \int_0^T \mathrm{d}t \int_\Omega |\widetilde{u_\varepsilon}(t) - \sum_{i=1}^{T/h} \chi_{((i-1)h,ih)}(t) v_\varepsilon((i-1)h+s)|^\theta \,\mathrm{d}x \\
&\quad = \frac{1}{h}\sum_{i=1}^{T/h} \int_0^h \mathrm{d}s \int_{(i-1)h}^{ih} \mathrm{d}t \int_\Omega |\widetilde{u_\varepsilon}(t) - v_\varepsilon((i-1)h+s)|^\theta \,\mathrm{d}x \\
&\quad = \frac{1}{h}\sum_{i=1}^{T/h} \int_{(i-1)h}^{ih} \mathrm{d}s \int_{(i-1)h}^{ih} \mathrm{d}t \int_\Omega |\widetilde{u_\varepsilon}(t) - v_\varepsilon(s)|^\theta \,\mathrm{d}x \\
&\quad = \frac{1}{h}\sum_{i=1}^{T/h} \int_{(i-1)h}^{ih} \mathrm{d}t \int_{(i-1)h-t}^{ih-t} \mathrm{d}s \int_\Omega |\widetilde{u_\varepsilon}(t) - v_\varepsilon(s+t)|^\theta \,\mathrm{d}x
\end{aligned}$$

$$\leq \frac{1}{h}\sum_{i=1}^{T/h}\int_{(i-1)h}^{ih}\mathrm{d}t\int_{-h}^{h}\mathrm{d}s\int_{\Omega}|\widetilde{u_\varepsilon}(t)-v_\varepsilon(s+t)|^\theta\,\mathrm{d}x$$

$$=\frac{1}{h}\int_0^T\mathrm{d}t\int_{-h}^{h}\mathrm{d}s\int_{\Omega}|\widetilde{u_\varepsilon}(t)-v_\varepsilon(s+t)|^\theta\,\mathrm{d}x$$

$$=\frac{1}{h}\int_{-h}^{h}\mathrm{d}s\int_0^T\mathrm{d}t\int_{\Omega}|\widetilde{u_\varepsilon}(t)-v_\varepsilon(s+t)|^\theta\,\mathrm{d}x$$

$$=\frac{1}{h}\int_{-h}^{h}\mathrm{d}s\int_S\mathrm{d}t\int_{\Omega}|\widetilde{u_\varepsilon}(t)-u_\varepsilon(s+t)|^\theta\,\mathrm{d}x$$

$$+\frac{1}{h}\int_{-h}^{h}\mathrm{d}s\int_{S'}\mathrm{d}t\int_{\Omega}|\widetilde{u_\varepsilon}(t)|^\theta\,\mathrm{d}x,$$

where for each $s\in[-h,h]$, $S=\{t\in(0,T): s+t\in(\max(0,-s),\min(T,T-s))\setminus E_{\varepsilon,R}\}$ and $S'(\subset[0,h]\cup[T-h,T]\cup E_{\varepsilon,R}))$ is its complement. The inequality from equality is obtained by replacing a bigger interval for the s variable. Indeed, if $t\in[(i-1)h,ih]$ and $s\in[(i-1)h-t,ih-t]$, then $s\in[-h,h]$. Thus

$$\frac{1}{h}\int_0^h\int_0^T\int_{\Omega}|\widetilde{u_\varepsilon}(t)-\sum_{i=1}^{T/h}\chi_{((i-1)h,ih)}(t)v_\varepsilon((i-1)h+s)|^\theta\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$\leq\frac{1}{h}\int_{-h}^{h}\int_{\max(0,-s)}^{\min(T,T-s)}\int_{\Omega}|\widetilde{u_\varepsilon}(t)-v_\varepsilon(s+t)|^\theta\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$+\frac{1}{h}\int_{-h}^{h}\int_{S'}\int_{\Omega}|\widetilde{u_\varepsilon}(t)|^\theta\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$\leq\sup_{|s|\leq h}\int_{\max(0,-s)}^{\min(T,T-s)}\int_{\Omega}|\widetilde{u_\varepsilon}(t)-\widetilde{u_\varepsilon}(s+t)|^\theta\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{h}\int_{-h}^{h}\int_{S'}\int_{\Omega}|\widetilde{u_\varepsilon}(t)|^\theta\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$

$$\leq T\,w(hR)+C(2h+1/R)$$

which can be taken small, say less than δ (for all ε), by fixing h small and $R=h^{-1/2}$. Therefore, there exists $s_\varepsilon\in(0,h)$ such that

$$\int_{\Omega_T}\left|\widetilde{u_\varepsilon}(t)-\sum_{i=1}^{T/h}\chi_{((i-1)h,ih)}(t)v_\varepsilon((i-1)h+s_\varepsilon)\right|^\theta\,\mathrm{d}x\,\mathrm{d}t \tag{3.10}$$

is small uniformly in ε.

Note that for h fixed as above, we have a finite combination of the sequences $\{v_\varepsilon((i-1)h+s_\varepsilon)\}_{\varepsilon>0}$ which are independent of time. Therefore, in order to prove the relative compactness of the sequence $\sum_{i=1}^{T/h}\chi_{((i-1)h,ih)}v_\varepsilon((i-1)h+s_\varepsilon)$ in $L^p(\Omega_T)$, which we denote by $w_{\varepsilon,h}$ for fixed h, it is enough to prove the relative compactness of the sequences

$v_\varepsilon((i-1)h+s_\varepsilon)$ in $L^p(\Omega)$ for $i=1,2,...,T/h$. But, this follows from the compact inclusion of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ as these sequences are bounded in $W^{1,p}(\Omega)$ (by the definition of $E_{\varepsilon,R}$) for each i. Then, (3.10) and the relative compactness of $w_{\varepsilon,h}$ in $L^p(\Omega_T)$ for each fixed h, imply that the sequence $\widetilde{u_\varepsilon}$ is totally bounded in $L^\theta(\Omega_T)$ and hence relatively compact there. □

Remark 3.4. The first inequality in Step 1 is one of the crucial inequalities. Once this inequality is true for a general $b_\varepsilon(x,u_\varepsilon)$ instead of $b(\frac{x}{\varepsilon},u_\varepsilon)$, then the rest of our methods and techniques can be carried out for more general parabolic equation with the parabolic term $\partial_t b_\varepsilon(x,u_\varepsilon)$. For example, the results are true with the parabolic term $\partial_t b(\frac{x}{d_\varepsilon},u_\varepsilon)$, where $d_\varepsilon>0$ and $d_\varepsilon\to 0$ as $\varepsilon\to 0$. □

From Lemma 3.2 above, the continuity of b and the assumption (2.10), we derive the following corollaries.

COROLLARY 3.5

We have, $b(\frac{x}{\varepsilon},\widetilde{u_\varepsilon})-b(\frac{x}{\varepsilon},u)\to 0$ *strongly in* $L^q(\Omega_T)$ $\forall\, q,\ 0<q<\infty$.

Proof. By the *a priori* bound (2.10), it is enough to consider the function b on $Y\times[-M,M]$ for a large $M>0$. As b is continuous, it is uniformly continuous on $Y\times[-M,M]$. Therefore, given $h_0>0$, there exists a $\delta>0$ such that

$$|b(y,s)-b(y',s')|<h_0,$$

whenever $|y-y'|+|s-s'|<\delta$.

Now, since $\widetilde{u_\varepsilon}\to u$ a.e in Ω_T, by Egoroff's theorem, given $h_1>0$, there exists $E\subset\Omega_T$ such that its Lebesgue measure $m(E)<h_1$ and $\widetilde{u_\varepsilon}$ converges uniformly to u on $(\Omega_T\setminus E)$, which we denote by E'. Therefore, we can find $\varepsilon_1>0$ such that

$$\|\widetilde{u_\varepsilon}-u\|_{\infty,E'}<\delta\quad\forall\,\varepsilon<\varepsilon_1. \tag{3.11}$$

Therefore, for $\varepsilon<\varepsilon_1$ we have

$$\begin{aligned}
&\int_{\Omega_T}\left|b\left(\frac{x}{\varepsilon},\widetilde{u_\varepsilon}\right)-b\left(\frac{x}{\varepsilon},u\right)\right|^q\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{E'}\left|b\left(\frac{x}{\varepsilon},\widetilde{u_\varepsilon}\right)-b\left(\frac{x}{\varepsilon},u\right)\right|^q\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_{E}\left|\left(b\left(\frac{x}{\varepsilon},\widetilde{u_\varepsilon}\right)-b\left(\frac{x}{\varepsilon},u\right)\right)\right|^q\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le h_0^q m(\Omega_T)+2^q\sup(|b|^q)\,m(E)\\
&\quad\le h_0^q\,m(\Omega_T)+2^q\sup(|b|^q)\,h_1.
\end{aligned}$$

This completes the proof as h_0 and h_1 can be chosen arbitrarily small. □

The following result follows easily.

COROLLARY 3.6

We have the following convergences:

$$b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}\right) \rightharpoonup \overline{b}(u) \text{ weakly in } L^q(\Omega_T),$$

$$\chi\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}\right) \rightharpoonup b^*(u) \text{ weakly in } L^q(\Omega_T).$$

for $q > 1$. *Further*, $\overline{b}(u) = b^*(u)$.

Proof. We note that

$$b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}\right) = \left(b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}\right) - b\left(\frac{x}{\varepsilon}, u\right)\right) + b\left(\frac{x}{\varepsilon}, u\right)$$
$$\to 0 + \overline{b}(u)$$

by Corollary 3.5 and the averaging principle for periodic functions.

Similarly,

$$\chi\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}\right) = \chi\left(\frac{x}{\varepsilon}\right) \left(b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}\right) - b\left(\frac{x}{\varepsilon}, u\right)\right) + \chi\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}, u\right)$$
$$\to 0 + b^*(u).$$

From $\chi(\frac{x}{\varepsilon})\, b(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}) = b(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon})$, we readily obtain the last of the conclusions in the corollary. □

4. Homogenization

This section is devoted to the proof of Theorem 2.2.

For passing to the limit in eq. (2.3) we need to take test functions which vanish on the holes. In fact, we take the test functions to be $v_\varepsilon\, \phi\, \psi$, where $\phi \in \mathcal{D}(\Omega)$, $\psi \in C_0^1(0, T)$ and $\{v_\varepsilon\}$ is a bounded family of functions which satisfies

$$v_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad v_\varepsilon = 0 \text{ in the holes and}$$
$$v_\varepsilon \rightharpoonup 1 \text{ weakly in } W^{1,p}(\Omega).$$

The construction of such functions has been established in [14] for the linear problems (i.e., with $p = 2$) and in [7] for nonlinear problems. We, in fact, choose v_0 as in (2.16) and define v_ε by

$$v_\varepsilon(x) = v_0(y_\varepsilon(x)),$$

where $y_\varepsilon(x) = x - \varepsilon k\left(\frac{x}{\varepsilon}\right)/\varepsilon^{N/(N-p)}$ is a change of variable and k is defined as in §2. Then it can be easily seen, by the choice of the size of the perforations a_ε, that v_ε satisfies the required properties.

Employing these test functions in (2.3) we have

$$\int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), v_\varepsilon\, \phi\, \psi\right\rangle_\varepsilon \mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} a(u_\varepsilon, \nabla u_\varepsilon)\cdot \nabla(v_\varepsilon\, \phi\, \psi)\mathrm{d}x\, \mathrm{d}t$$
$$= \int_0^T \int_{\Omega_\varepsilon} f(x,t) v_\varepsilon\, \phi\, \psi\, \mathrm{d}x\, \mathrm{d}t. \tag{4.1}$$

Rewriting

$$\int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), v_\varepsilon\, \phi\, \psi\right\rangle_\varepsilon \mathrm{d}t = -\int_0^T \int_{\Omega_\varepsilon} b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) v_\varepsilon\, \phi\, \partial_t \psi\, \mathrm{d}x\, \mathrm{d}t$$
$$= -\int_0^T \int_{\Omega} b\left(\frac{x}{\varepsilon}, \widetilde{u_\varepsilon}\right) v_\varepsilon\, \phi\, \partial_t \psi\, \mathrm{d}x\, \mathrm{d}t\,, \tag{4.2}$$

we compute its limit knowing the weak limit of $b(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon)$. This has been done in the previous Corollary 3.5 using which we get

$$\lim_{\varepsilon\to 0}\int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), v_\varepsilon\, \phi\, \psi\right\rangle_\varepsilon \mathrm{d}t = -\int_0^T \int_{\Omega} \overline{b}(u)\phi\, \partial_t \psi\, \mathrm{d}t$$
$$= \int_0^T \langle \partial_t \overline{b}(u), \phi\, \psi\rangle\, \mathrm{d}t. \tag{4.3}$$

Also,

$$\int_0^T \int_{\Omega_\varepsilon} f\, v_\varepsilon\, \phi\psi\, \mathrm{d}x\, \mathrm{d}t \overset{\varepsilon\to 0}{\to} \int_0^T \int_{\Omega} f\, \phi\, \psi\, \mathrm{d}x\, \mathrm{d}t. \tag{4.4}$$

It remains to compute the limit

$$\lim_{\varepsilon\to 0}\int_0^T \int_{\Omega_\varepsilon} a(u_\varepsilon, \nabla u_\varepsilon)\cdot \nabla(v_\varepsilon\, \phi)\, \psi\, \mathrm{d}x\, \mathrm{d}t. \tag{4.5}$$

This is a difficult computation and has been done by Casado–Díaz in his paper [7] where he considers the homogenization of the nonlinear Dirichlet problem

$$\begin{aligned} &-\operatorname{div} a(u_\varepsilon, \nabla u_\varepsilon) = f \text{ in } \Omega_\varepsilon,\\ &u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon) \end{aligned} \tag{4.6}$$

in perforated domain Ω_ε. The two-scale convergence method (cf. [1,15,16]) used by us in [17,18] is not helpful here. The two-scale convergence method has been seen in a new light by Arbogast *et al* [2]. The key idea is that, given a sequence of functions u_ε they introduce a sequence of two-variable functions $\widehat{u}_\varepsilon(x, y) = u_\varepsilon(\varepsilon k(\frac{x}{\varepsilon} + \varepsilon^\alpha y))$, where $\alpha = 1$ and $y \in Y$, $k = k(x) \in \mathbb{Z}^N$ such that $x \in (k - 1/2, k + 1/2)^N$. For the situation under consideration, Casado–Díaz chooses $\alpha = N/(N-p)$ and proves a compactness lemma for the modified sequence $\widehat{u}_\varepsilon$. This allows him to compute the limit of a similar quantity as in (4.5). By a similar computation we can show that

$$\lim_{\varepsilon\to 0}\int_0^T \int_{\Omega_\varepsilon} a(u_\varepsilon, \nabla u_\varepsilon)\cdot \nabla(v_\varepsilon\, \phi)\, \psi\, \mathrm{d}x\, \mathrm{d}t$$
$$= \int_0^T \int_{\Omega} a(u, \nabla u)\cdot \nabla\phi\, \psi\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_{\Omega} \Phi(u)\phi\, \psi\, \mathrm{d}x\, \mathrm{d}t. \tag{4.7}$$

It follows from (4.3), (4.4), and (4.7) that the homogenized equation is that given by (2.12). We briefly sketch some steps in the proof of (4.7).

5. Computation of $\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon} \boldsymbol{a}(\boldsymbol{u}_\varepsilon, \nabla \boldsymbol{u}_\varepsilon)\cdot\nabla(\boldsymbol{v}_\varepsilon\,\boldsymbol{\phi})\,\boldsymbol{\psi}$

The entire computation will not be done here, but only outlined as it differs little from that of Casado–Díaz [7]. Except for the time dependence of the sequences involved little else is different.

Step 1: As the sequence u_ε is difficult in the calculation, a sequence z_ε is defined which has the same behavior as the original sequence near the holes helping to capture the 'μ_{a_0}-capacity' but is otherwise the same as u. Let a_0 be as in (2.17).

First define $z_0 : \mathbb{R}^N \times (0,T)\times\mathbb{R}^N \to \mathbb{R}$ to be a solution of

$$\begin{aligned}
&-\operatorname{div}_y a_0(z_0(x,t,y), \nabla_y z_0(x,t,y)) = 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N\setminus S) \quad \text{a.e. } x,t,\\
&z_0(x,t,\,.\,) - u(x,t) \in L^p(\mathbb{R}^N\times(0,T); D^{1,p}(\mathbb{R}^N)),\\
&z_0(x,t,\,.\,)\,\phi(\cdot) \in W_0^{1,p}(\mathbb{R}^N\setminus S) \quad \forall\phi\in\mathcal{D}(\mathbb{R}^N) \text{ a.e. } x,t.
\end{aligned} \tag{5.1}$$

Let $\hat{h}_\varepsilon$ be a bounded sequence in $L^\infty(\mathbb{R}^N)$ satisfying

$$\begin{aligned}
&\hat{h}_\varepsilon \in W^{1,1}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\\
&\hat{h}_\varepsilon = 0 \text{ a.e. in } \mathbb{R}^N\setminus B_{r_\varepsilon},\\
&(\hat{h}_\varepsilon - 1)\,\phi \in W_0^{1,N}(\mathbb{R}^N\setminus S) \quad \forall\phi\in\mathcal{D}(\mathbb{R}^N),\\
&\hat{h}_\varepsilon \to 1 \text{ a.e. in } \mathbb{R}^N,\\
&\int_{\mathbb{R}^N} |\nabla \hat{h}_\varepsilon|^N\,\mathrm{d}y \to 0,
\end{aligned} \tag{5.2}$$

where r_ε is a sequence of real numbers tending to ∞ as $\varepsilon \to 0$ in such a way that $r_\varepsilon \varepsilon^{p/(N-p)} \to 0$. Such a sequence $\hat{h}_\varepsilon$ can be obtained by solving a suitable p-Laplacian in $B_{r_\varepsilon}\setminus S$. We then set

$$h_\varepsilon = \hat{h}_\varepsilon(y_\varepsilon(x)), \tag{5.3}$$

where $y_\varepsilon(x) = (x - k(\frac{x}{\varepsilon}))/\varepsilon^{N/(N-p)}$ and $k(\frac{x}{\varepsilon})$ denotes the multi-integer k such that $x \in \varepsilon(k+Y)$. The sequence h_ε helps to join the behavior near the holes and the behavior away from the holes.

Define

$$\widetilde{z_\varepsilon}(x,t) = \frac{1}{\varepsilon^N}\int_{C_\varepsilon(x)} z_0(\rho, t, y_\varepsilon(x))\,\mathrm{d}\rho, \tag{5.4}$$

where $C_\varepsilon(x)$ denotes the ε-cell to which x belongs,

$$\nabla\widetilde{z_\varepsilon}(x,t) = \frac{1}{\varepsilon^{N+N/(N-p)}}\int_{C_\varepsilon(x)} \nabla_y z_0(\rho,t,y_\varepsilon(x))\,\mathrm{d}\rho \tag{5.5}$$

and

$$z_\varepsilon(x,t) = h_\varepsilon(x)\widetilde{z_\varepsilon}(x,t) + (1-h_\varepsilon(x))\,u(x,t). \tag{5.6}$$

It can be shown that z_ε has the following properties

$$
\begin{aligned}
&z_\varepsilon \in L^p(0,T;W^{1,p}(\mathbb{R}^N)),\\
&z_\varepsilon\,\phi \in L^p(0,T;W_0^{1,p}(\Omega_\varepsilon)) \quad \forall\phi\in\mathcal{D}(\Omega),\\
&z_\varepsilon \rightharpoonup u \ \text{ in } \ L^p(0,T;W^{1,p}(\Omega)),\\
&\nabla z_\varepsilon - \nabla u - \nabla\widetilde{z_\varepsilon} \to 0 \text{ in } L^p((0,T)\times\mathbb{R}^N).
\end{aligned}
\tag{5.7}
$$

Step 2: *Main property of* z_ε. For any sequence $w_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon) \rightharpoonup w$ in $W_0^{1,p}(\Omega)$ and $\psi \in C_0^1(0,T)$, the sequence z_ε satisfies

$$
\int_0^T\int_\Omega a(z_\varepsilon,\nabla z_\varepsilon)\cdot\nabla w_\varepsilon\psi\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_0^T\int_\Omega a(u,\nabla u)\cdot\nabla w\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \Phi(u)\,w\,\psi\,\mathrm{d}x\,\mathrm{d}t. \tag{5.8}
$$

The limit is computed by separating the contribution from near the holes and that away from the holes; i.e. we write

$$
\int_0^T\int_\Omega a(z_\varepsilon,\nabla z_\varepsilon)\cdot\nabla w_\varepsilon\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{A_\varepsilon} a(z_\varepsilon,\nabla z_\varepsilon)\cdot\nabla w_\varepsilon\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega\setminus A_\varepsilon} a(z_\varepsilon,\nabla z_\varepsilon)\cdot\nabla w_\varepsilon\psi\,\mathrm{d}x\,\mathrm{d}t, \tag{5.9}
$$

where $A_\varepsilon \triangleq \cup_{k\in\mathbb{Z}^N} B(\varepsilon k, r_\varepsilon\varepsilon^{N/(N-p)})$. It is easy to see that

$$
\int_0^T\int_{\mathbb{R}^N\setminus A_\varepsilon} a(z_\varepsilon,\nabla z_\varepsilon)\cdot\nabla w_\varepsilon\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega a(u,\nabla u)\cdot\nabla w\psi\,\mathrm{d}x\,\mathrm{d}t + O(\varepsilon) \tag{5.10}
$$

from the definition of z_ε, the weak convergence of w_ε and the fact that the measure of the sets A_ε tend to zero.

The limit of the term defined over A_ε is calculated by tailoring the two-scale convergence technique in such a way that it 'sees' the holes though they are not of size ε. This is the content of Lemma 3.1 in the paper by Casado–Díaz [7]. It becomes necessary to introduce the two-scale sequence

$$
\widehat{z_\varepsilon}(x,y) \triangleq z_\varepsilon\left(\varepsilon k\left(\frac{x}{\varepsilon}\right) + \varepsilon^{N/(N-p)}y\right) \tag{5.11}
$$

using which the first term in (5.9) can be written as

$$
\int_0^T\int_{\mathbb{R}^N\times B_{r_\varepsilon}} a_\varepsilon(\widehat{z_\varepsilon}, M_\varepsilon(\nabla_y z_0))\cdot\nabla_y\widehat{w_\varepsilon}\,\psi\,\mathrm{d}x\,\mathrm{d}y, \tag{5.12}
$$

where

$$
M_\varepsilon(\nabla_y z_0) = \frac{1}{\varepsilon^N}\int_{C_\varepsilon(x)} \nabla_y z_0(\rho,y)\,\mathrm{d}\rho. \tag{5.13}
$$

Written this way, (5.12) can be shown to converge to $\int_0^T \int_\Omega \Phi(u)\, w\, \psi \,\mathrm{d}x \,\mathrm{d}t$ by Lemma 3.1 in [7].

Step 3: It is then shown that the sequence z_ε has the desired approximation properties (Steps 4–6 in [7]) so that $\int_0^T \int_\Omega a(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla w_\varepsilon \psi \,\mathrm{d}x \,\mathrm{d}t$ and $\int_0^T \int_\Omega a(z_\varepsilon, \nabla z_\varepsilon)\cdot\nabla w_\varepsilon \psi \,\mathrm{d}x \,\mathrm{d}t$ have the same limit, thus completing the proof.

Remark 5.1. In the above, the diffusion term $a(\,.\,,\,.\,)$ did not itself vary with ε. It is desirable to consider the case when it is of the form $a(\frac{x}{\varepsilon},\,.\,,\,.\,)$. One may even study just the stationary Dirichlet problem (4.6) with such a coefficient term $a(\frac{x}{\varepsilon},\,.\,,\,.\,)$. This seems to be an open problem and as we have remarked earlier Dal Maso–Murat [9] have obtained results in the linear case with more general coefficients.

The case when (4.6) is the Euler–Lagrange equation of a variational problem, has been solved by Ansini and Braides [3] by the method of Γ-convergence. The capacitary term obtained by them is simply the capacity function which corresponds to the homogenized operator a_{hom}, when it is positively homogeneous of degree $p-1$ in the gradient. □

We end by stating a corrector result whose proof may be established following [7].

Theorem 5.2. *Assume that $a(s,\xi)$ does not depend on s. Then, for a subsequence of ε, still denoted by ε, we have the following corrector result: for every $f \in L^{p^*}(0,T;W^{-1,p}(\Omega_\varepsilon))$, the solution u_ε of* (1.3) *satisfies*

$$\widetilde{u_\varepsilon} - z_\varepsilon \to 0 \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega))$$

where z_ε is defined in (5.6). □

Acknowledgement

The authors would like to thank the referee for the suggestions and comments as it has improved the exposition of this article.

References

[1] Allaire G, Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992) 1482–1518

[2] Arbogast T, Douglas J and Hornung U, Derivation of double porosity model of single phase flow via homogenization theory, *SIAM J. Math. Anal.* **21** (1990) 823–836

[3] Ansini N and Braides A, Separation of scales and almost periodic effects in the asymptotic behaviour of perforated media, *Acta Appl.* (to appear)

[4] Alt H W and Luckhaus S, Quasilinear elliptic-parabolic differential equations, *Math. Z.* **183** (1983) 311–341

[5] Avellaneda M and Lin F H, Compactness methods in the theory of homogenization, *Comm. Pure Appl. Math.* **40** (1987) 803–847

[6] Brahim-Omsmane S, Francfort G A and Murat F, Correctors for the homogenization of wave and heat equations, *J. Math. Pure Appl.* **71** (1992) 197–231

[7] Casado-Díaz J, Two-scale convergence for nonlinear Dirichlet problems in perforated domains, *Proc. R. Soc. Edinburgh* **A130** (2000) 249–276

[8] Clark G W and Showalter R E, Two-scale convergence of a model for flow in a partially fissured medium, *Electr. J. Diff. Equ.* **1999(2)** (1999) 1–20

[9] Dal Maso G and Murat F, Asymptotic behaviour and correctors for linear Dirichlet problems with simultaneously varying operators and domains (in preparation)

[10] Douglas Jr J, Peszyńska M and Showalter R E, Single phase flow in partially fissured media, *Transport in Porous Media*, **28** (1995) 285–306

[11] Hornung U, Applications of the homogenization method to flow and transport in porous media, Notes of Tshingua Summer School on Math. Modelling of Flow and Transport in Porous Media (ed.) Xiao Shutie (1992) (Singapore: World Scientific) pp. 167–222

[12] Jian H, On the homogenization of degenerate parabolic equations, *Acta Math. Appl. Sinica* **16(1)** (2000) 100–110

[13] Ladyzenskaya O A, Solonnikov V and Ural'tzeva N N, Linear and Quasilinear Equations of Parabolic Type *Am. Math. Soc. Transl. Monogr.* **23** (1968) Providence R.I.

[14] Cioranescu D and Murat F, Un terme ètrange venu d'ailleurs, in: Non linear Partial *Differential Equations and their Applications, Collège de France Seminar*, Vols II and III (eds) H Brézis and J L Lions (1982) Research Notes in Mathematics (Pitman) Vols 60 and 70, pp. 98–138 and pp. 78–154

[15] Nguetseng G, A general convergence result of a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20** (1989) 608–623

[16] Nandakumar A K, Steady and evolution Stokes equations in a porous media with non-homogeneous boundary data: a homogenization process, *Diff. Int. Equ.* **5(1)** (1992) 73–93

[17] Nandakumaran A K and Rajesh M, Homogenization of a nonlinear degenerate parabolic differential equation, *Electr. J. Diff. Equ.* **2001(17)** (2001) 1–19

[18] Nandakumaran A K and Rajesh M, Homogenization of a parabolic equation in perforated domain with Neumann boundary condition, *Proc. Indian Acad. Sci. (Math. Sci.)* **112(1)** (2002) 195–207

[19] Oleinik O A, Kozlov S M and Zhikov V V, On G-convergence of parabolic operators, *Ouspekhi Math. Naut.* **36(1)** (1981) 11–58

[20] Rajesh M, Correctors for flow in a partially fissured medium, *Electr. J. Diff. Equ.* **1999(27)** (1999) 1–15