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An example in the homogenization of a degenerate elliptic equation

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Abstract. In this article we study the homogenization, of a particular example, of degenerate elliptic equations of second order in the setup of viscosity solutions. These results are an attempt to extend the corresponding results of Evans [8] to degenerate situations.

1. Introduction

The homogenization of nonlinear partial differential equations

$$F\left(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}\right) = 0, \tag{P$_\varepsilon$}$$

having solutions in the viscosity sense has been studied by Evans using his innovative method of perturbed test functions [7]. This method was inspired by the techniques adopted by Lions, Papanicolaou and Varadhan [13] for the homogenization of Hamilton–Jacobi equations. Evans proves homogenization results for fully nonlinear second order, uniformly elliptic, equations and for fully nonlinear first order equations satisfying a coercivity condition in his paper [8]. Typically it is shown that the solutions u_ε converge uniformly to the solution of an effective partial differential equation

$$\overline{F}(D^2u, Du, u, x) = 0, \tag{P}$$

where $\overline{F}$ is determined from an auxiliary problem which resembles an ergodic problem in the theory of differential games. We shall always use the notation $\overline{F}$ to denote the homogenized operator. Subsequently there has been more literature on the homogenization of pdes in the setup of viscosity solutions; see [1,5,10], to mention a few.

Coming back to the problem of Evans, it is to be noted that the uniform ellipticity of F is not essential to have existence of solutions to (P$_\varepsilon$). Hence, one would like to ask the question "can one have homogenization results for (P$_\varepsilon$) when F is of second order but not uniformly elliptic, i.e., admits some

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degeneracy"? This seems to be a difficult question in general as it leads to a situation where one cannot have good estimates on the sequence u_ε guaranteeing uniform convergence and possible homogenization. To the best of our knowledge there seems to be no result in the literature corresponding to this case. In this paper, we consider the family of degenerate second order equations

$$-a\left(\frac{x'}{\varepsilon}\right)\frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_1} + |Du_\varepsilon| + u_\varepsilon - f(x) = 0,$$

where x_1 denotes the first coordinate of x and x' denotes the remaining coordinates, i.e., $x = (x_1, x')$. The above family is an example of the kind of degenerate situation we are speaking of and in this case, we are able to prove a homogenization result. Roughly speaking, it seems to be possible to have homogenization if the oscillations in the coefficients and the non-degenerate directions are in complementary directions.

The paper is organised as follows. In Section 2, we introduce the family of equations and provide the assumptions under which one has existence and uniqueness of solutions to the equations. In Section 3, we define the homogenized operator and discuss its properties. In Section 4, we prove the homogenization result. Proofs of certain propositions and verification of some details are done in Appendix. We conclude with some remarks in Section 5.

2. The example

Before introducing the example, we recall the notion of viscosity solution for the Dirichlet problem corresponding to a second-order degenerate elliptic equation.

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $F: \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous where $\mathcal{S}^n$ is the set of all symmetric matrices of order n. F is said to be *degenerate elliptic* if

$$F(X,p,s,x) \leqslant F(Y,p,s,x) \quad \forall Y, X \in \mathcal{S}^n \text{ such that } Y \leqslant X,\ \forall p,s,x. \tag{2.1}$$

The inequality $Y \leqslant X$ between symmetric matrices is taken to mean that $X - Y$ is positive semi-definite.

Definition 2.1. An upper (lower) semi-continuous function $u \in \mathrm{USC}(\overline{\Omega})$ $(\mathrm{LSC}(\overline{\Omega}))$ is said to be a viscosity subsolution (supersolution) to the Dirichlet problem

$$\begin{cases} F(D^2u, Du, u, x) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{DP}$$

if for any $x \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum (minimum) at x one has

$$F(D^2\phi(x), D\phi(x), \phi(x), x) \leqslant 0 \ (\geqslant 0)$$

and $u \leqslant 0$ $(\geqslant 0)$ on $\partial\Omega$. A continuous function $u \in C(\overline{\Omega})$ is said to be a viscosity solution to the Dirichlet problem if it is both a sub- and super-solution.

We now turn to the example. Let $\varepsilon > 0$ be a parameter which eventually tends to zero. Let Ω be a bounded domain with C^2 boundary. We assume the following.

(H0) The principal curvatures (see Appendix A1) of the boundary $\partial\Omega$ (with respect to the inward unit normal) are nonnegative at every point on the boundary.

(H1) $a(\cdot)$ is a nonnegative continuous periodic function on $\mathbb{R}^{(n-1)}$ with the cell $[0,1]^{(n-1)}$ as period. We further assume that $a^{1/2}(\cdot)$ is Lipschitz continuous with Lipschitz constant L.

(H2) $f(\cdot)$ is a nonnegative Lipschitz continuous function on $\mathbb{R}^n$ with Lipschitz constant L.

Note that we have used the same letter L to denote the Lipschitz constants in (H1) and (H2). We can always do this by choosing a common constant (need not be the best possible).

We consider the Dirichlet problem

$$\begin{cases} -a\left(\dfrac{x'}{\varepsilon}\right)\dfrac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_1} + |Du_\varepsilon| + u_\varepsilon - f(x) = 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{P$_\varepsilon$}$$

The above family constitutes our example and we shall prove a homogenization result for this family as $\varepsilon \to 0$. But first, for each $\varepsilon > 0$, we examine the question of existence and uniqueness of solutions to the equation (P$_\varepsilon$) using Perron's method. We set

$$F(X,p,s,x,y) = -a(y')X_{11} + |p| + s - f(x), \tag{2.2}$$

where $|\cdot|$ shall stand for the l^1 norm in $\mathbb{R}^n$ to make matters simple in the sequel. We observe that F has the following properties.

(P1) F is degenerate elliptic.

(P2) For all X, p, x and for $s, r \in \mathbb{R}$

$$F\left(X,p,s,x,\frac{x}{\varepsilon}\right) - F\left(X,p,r,x,\frac{x}{\varepsilon}\right) \geqslant s - r.$$

In fact, the above inequality is an equality.

(P3) For all $x, z \in \mathbb{R}^n$, for $\beta > 0$ and $X, Z \in S^n$ satisfying

$$-\beta\begin{pmatrix} I & O \\ O & -I \end{pmatrix} \leqslant \begin{pmatrix} X & O \\ O & -Z \end{pmatrix} \leqslant \beta\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

one has

$$F\left(Z,\beta(x-z),s,z,\frac{z}{\varepsilon}\right) - F\left(X,\beta(x-z),s,x,\frac{x}{\varepsilon}\right) \leqslant \beta L^2\frac{|x-z|^2}{\varepsilon^2} + L|x-z|.$$

(P4) $F(0,0,0,x,x/\varepsilon) \leqslant 0$ for all $x \in \Omega$.

(P5) There exists a constant $C > 0$ such that

$$F\left(0,0,C,x,\frac{x}{\varepsilon}\right) \geqslant 0 \quad \text{for all } x \in \Omega.$$

It is not difficult to verify the above properties; for the sake of completeness the verifications are done in Appendix (see A3).

The properties (P1)–(P3) imply, by [6, Theorem 3.3], a *comparison principle* for sub- and supersolutions of (P_ε). To be precise, if $u \in \mathrm{USC}(\overline{\Omega})$ be a subsolution and $v \in \mathrm{LSC}(\overline{\Omega})$ be a supersolution of the Dirichlet problem (P_ε) (note that $u \leqslant 0 \leqslant v$ on $\partial\Omega$ by definition) then $u \leqslant v$ in $\overline{\Omega}$. In particular, if a viscosity solution to (P_ε) exists then it is *unique*.

The existence will follow from Theorem 4.1 [6] provided we construct a subsolution and a supersolution vanishing on the boundary. It is obvious from (P4) that 0 is a subsolution to (P_ε). If C is as in (P5) then it is clear that C is a viscosity supersolution of (P_ε) in Ω. We need to construct a supersolution which vanishes identically on $\partial\Omega$. For $\lambda > 0$, let $\Omega_\lambda = \{x \in \Omega\colon d(x, \partial\Omega) < 1/\lambda\}$. Choose M such that $M(1 - \mathrm{e}^{-1}) > C$. Define

$$v(x) = \begin{cases} M\left(1 - \mathrm{e}^{-\lambda d(x,\partial\Omega)}\right) \wedge C & \text{in } \overline{\Omega}_\lambda, \\ C & \text{in } \Omega \setminus \overline{\Omega}_\lambda. \end{cases}$$

Clearly $v = 0$ on $\partial\Omega$. We verify that v is a supersolution in Ω for λ sufficiently large.

Set $d(x) = d(x, \partial\Omega)$ for short. Choose $0 < \lambda_0$, initially, such that d is twice differentiable in $\overline{\Omega}_{\lambda_0}$. This is possible since $\partial\Omega$ is C^2. By the condition (H0) we can choose a $\lambda > \lambda_0$ such that $D^2 d(x)$ is negative semi-definite in $\overline{\Omega}_\lambda$ (see Appendix A2). Also $|Dd(x)| = 1$, in fact, $Dd(x) = n(y)$ where y is the point nearest to x on the boundary and $n(\cdot)$ is the inward unit normal at a point on the boundary. Let $v_0 = M(1 - \mathrm{e}^{-\lambda d(x)})$. Then for $x \in \overline{\Omega}_\lambda$

$$\begin{aligned} F\left(D^2 v_0(x), Dv_0(x), v_0(x), x, \frac{x}{\varepsilon}\right) &\geqslant -a\left(\frac{x'}{\varepsilon}\right) M\,\mathrm{e}^{-\lambda d(x)}\left(-\lambda^2 Dd(x) \otimes Dd(x) + \lambda D^2 d(x)\right)_{11} \\ &\quad + M\,\mathrm{e}^{-\lambda d(x)} \lambda |Dd(x)| + M\left(1 - \mathrm{e}^{-\lambda d(x)}\right) - f(x) \\ &\geqslant M\,\mathrm{e}^{-1}\lambda - \|f\|_\infty \end{aligned}$$

since $-a(x'/\varepsilon)M\,\mathrm{e}^{-\lambda d(x)}(-\lambda^2 Dd(x) \otimes Dd(x) + \lambda D^2 d(x))_{11} \geqslant 0$ and $M(1 - \mathrm{e}^{-\lambda d(x)}) \geqslant 0$. Thus, finally, if we further choose λ such that $\lambda \geqslant \|f\|_\infty / (M\,\mathrm{e}^{-1})$ we have

$$F\left(D^2 v_0(x), Dv_0(x), v_0(x), x, \frac{x}{\varepsilon}\right) \geqslant 0 \quad \text{for } x \in \overline{\Omega}_\lambda.$$

Thus v_0 is a viscosity supersolution of (P_ε) in $\overline{\Omega}_\lambda$ and hence by inf stability $v = v_0 \wedge C$ is a viscosity supersolution in $\overline{\Omega}_\lambda$. To have that v extended by C outside $\overline{\Omega}_\lambda$ is a viscosity supersolution in Ω it remains to check that it is so for $x \in \Omega$ such that $d(x) = 1/\lambda$. But for such an x, by the choice of M above, there exists a neighbourhood of x such that $v \equiv C$ in that neighbourhood. Thus we are able to conclude that v is a viscosity supersolution of (P_ε) satisfying $v \equiv 0$ on $\partial\Omega$.

We, therefore, obtain the following result directly from the above by an application of [6, Theorem 4.1].

Proposition 2.2. *Assume* (H1), (H2) *and that the boundary satisfies the curvature condition* (H0). *For each $\varepsilon > 0$ there exists a unique viscosity solution u_ε of* (P_ε)

$$0 \leqslant u_\varepsilon \leqslant v. \tag{2.3}$$

As a consequence of (2.3) *we also have that the solutions* u_ε *satisfy the uniform bound*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_\infty < \infty. \tag{2.4}$$

3. $\overline{F}$ and its properties

We now introduce the homogenized operator $\overline{F}$ corresponding to our example by means of the cell problem (3.1) below. The choice of $\overline{F}$ will be completely justified by the homogenization result in Section 4.

Proposition 3.1. *For each* (X,p,s,x) *there exists a pair* (ρ,w) *where* ρ *is a constant and* w *is bounded, uniformly continuous on* $\mathbb{R}^{(n-1)}$ *such that* w *is a viscosity solution of*

$$-a(y')X_{11} + |p_1| + |p' + D_{y'}w| + s - f(x) = \rho \quad \textit{in}\ \mathbb{R}^{(n-1)}. \tag{3.1}$$

Moreover, ρ *is unique.*

Proof. The proof follows the lines of [8, Lemma 2.1] but in $\mathbb{R}^{(n-1)}$ and is given in Appendix (see A4) for the sake of completeness. □

We are now able to define $\overline{F}$ as follows.

Definition 3.2. For (X,p,s,x) given, let (ρ,w) be as above. Noting that ρ is unique for given (X,p,s,x), we define the homogenized operator by

$$\overline{F}(X,p,s,x) \doteq \rho = \rho(X,p,s,x).$$

We now study some properties of $\overline{F}$.

Proposition 3.3. $\overline{F}$ *defined as above has the following properties.*

(i) $\overline{F}(X,p,s,x)$ *depends only on* X_{11} *in the variable* X.
(ii) $\overline{F}$ *is Lipschitz continuous in all its variables.*
(iii) *For all* X, p, x *and all* $r,s \in \mathbb{R}$ *one has*

$$\overline{F}(X,p,s,x) - \overline{F}(X,p,r,x) \geqslant (s-r).$$

In fact, the above holds with an equality.

(iv) *For all* p, r, x *and all* $X,Z \in S^n$ *such that* $X \geqslant Z$ *one has*

$$\overline{F}(X,p,r,x) \leqslant \overline{F}(Z,p,r,x).$$

Further, if $a(\cdot) \geqslant \alpha > 0$, *one has*

$$\overline{F}(X,p,r,x) + \alpha(X-Z)_{11} \leqslant \overline{F}(Z,p,r,x).$$

(v) *For all* $x, z \in \mathbb{R}^n$, $r \in \mathbb{R}$, $\beta > 0$ *and* $X, Z \in S^n$ *satisfying*

$$-\beta\begin{pmatrix} I & O \\ O & -I \end{pmatrix} \leqslant \begin{pmatrix} X & O \\ O & -Z \end{pmatrix} \leqslant \beta\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

one has

$$\overline{F}(Z, \beta(x-z), r, z) - \overline{F}(X, \beta(x-z), r, x) \leqslant L|x-z|,$$

where L is the Lipschitz constant of f in the definition of F.

(vi) $\overline{F}(0,0,0,x) \leqslant 0$ *for all* $x \in \Omega$.

(vii) $\overline{F}(0,0,C,x) \geqslant 0$ *for all* $x \in \Omega$ *where C is as in property* (P5) *in Section* 2.

(viii) *For* M, λ *sufficiently large* $v_0 = M(1 - \mathrm{e}^{-\lambda d(x,\partial\Omega)})$ *satisfies*

$$\overline{F}(D^2 v_0(x), Dv_0(x), v_0(x), x) \geqslant 0 \quad \text{for all } x \in \overline{\Omega}_\lambda.$$

The proof of the proposition is along the same lines as that of [8, Lemma 2.2] and it shall be given in Appendix (see A5).

We like to draw some conclusions from the proposition before concluding this section. The properties (i)–(v) in the proposition imply, by [6, Theorem 4.1], a comparison principle for sub- and supersolutions of the equation

$$\begin{cases} \overline{F}(D^2u, Du, u, x) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathrm{P_{hom}}$}$$

In particular, we also have *uniqueness* of solutions to the above equation. We can have existence using Perron's method since 0 is a subsolution following (vi) and v defined before will be a supersolution following (vii) and (viii).

4. Homogenization

We now state and prove our main theorem.

Theorem 4.1. *The solutions* u_ε *of* (P_ε) *converge uniformly to the solution of the homogenized equation* ($\mathrm{P_{hom}}$) *where* $\overline{F}$ *is as in Definition* 3.2.

Proof. We note that the bound (2.4) does not indicate in any way the uniform convergence of the sequence u_ε. But in fact, we shall be able to prove this following the technique of Barles and Perthame [4]. One uses the bounds (2.4) first to define

$$u^*(x) = \limsup_{z \to x,\, \varepsilon \to 0} u_\varepsilon(z) \quad \text{and} \quad u_*(x) = \liminf_{z \to x,\, \varepsilon \to 0} u_\varepsilon(z). \tag{4.1}$$

Then by the very definition it follows that $u^*(x)$ is upper semicontinuous and $u_*(x)$ is lower semicontinuous and

$$u^* \geqslant u_*. \tag{4.2}$$

The proof will be completed if we show that $u^*(x)$ is a viscosity subsolution of (P_{hom}) and that $u_*(x)$ is a viscosity supersolution of (P_{hom}) verifying $u^* = u_* = 0$ on $\partial\Omega$. Indeed it will follow, by the comparison principle, that

$$u^* \leqslant u_*. \tag{4.3}$$

By these inequalities, we will have a common value $u^* = u_* = u$ which is both a sub- and super-solution to (P_{hom}) and hence, a viscosity solution. The equality of the two limits in (4.1) also implies that the sequence u_ε converges uniformly to u.

We shall only show that $u^*(x)$ is a viscosity subsolution of (P_{hom}) and the other part can be proved similarly.

Let $x_0 \in \Omega$ and let $\phi \in C_0^\infty(\Omega)$ be such that $u^* - \phi$ has a strict local maximum at x_0 and $u^*(x_0) = \phi(x_0)$. We need to show that $\overline{F}(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0) \leqslant 0$. However, taking into account (i) Proposition 3.3, we can write this simply as

$$\overline{F}\left(\frac{\partial^2\phi}{\partial x_1\partial x_1}(x_0), D\phi(x_0), \phi(x_0), x_0\right) \leqslant 0. \tag{4.4}$$

Assume the contrary, that is, let

$$\overline{F}\left(\frac{\partial^2\phi}{\partial x_1\partial x_1}(x_0), D\phi(x_0), \phi(x_0), x_0\right) \geqslant \delta > 0, \tag{4.5}$$

for some $\delta > 0$. Let w be the viscosity supersolution of (3.1) with

$$X_{11} = \frac{\partial^2\phi}{\partial x_1\partial x_1}(x_0), \quad p = D\phi(x_0), \quad s = \phi(x_0) \quad \text{and}$$

$$\rho = \overline{F}\left(\frac{\partial^2\phi}{\partial x_1\partial x_1}(x_0), D\phi(x_0), \phi(x_0), x_0\right).$$

Let

$$\phi_\varepsilon(x) \doteq \phi(x) + \varepsilon w\left(\frac{x'}{\varepsilon}\right). \tag{4.6}$$

Claim. *ϕ_ε satisfies the following equation in the viscosity sense*

$$-a\left(\frac{x'}{\varepsilon}\right)\frac{\partial^2\phi_\varepsilon}{\partial x_1\partial x_1} + \left|D\phi_\varepsilon(x)\right| + \phi_\varepsilon(x) - f(x) \geqslant \frac{\delta}{2} \tag{4.7}$$

in the ball $B(x_0, r)$ for r and ε sufficiently small.

We complete the proof of the theorem assuming that the claim is true. From (P_ε) and (4.7), it follows by the comparison principle that

$$u_\varepsilon(z) - \phi_\varepsilon(z) \leqslant \max_{\partial B(x_0,r)}(u_\varepsilon - \phi_\varepsilon)$$

for all $z \in B(x_0, r)$ and all $\varepsilon > 0$. Passing to the limit as $\varepsilon \to 0$ in the above we get

$$u^*(z) - \phi(z) \leqslant \max_{\partial B(x_0,r)} (u^* - \phi) \quad \text{for all } z \in B(x_0, r)$$

which, by taking r sufficiently small, contradicts our assumption that $u^* - \phi$ has a strict local maximum at x_0. Thus, (4.4) must be valid or in other words we have shown that u^* is a viscosity subsolution of $(\mathrm{P_{hom}})$. □

Proof of the claim. Set $H(p, s, x) = |p| + s - f(x)$ for convenience of writing. Let M be a constant such that $\|w\|_\infty \leqslant M$. Note that there exists $\alpha_0 > 0$ such that

$$\left|H(p, s, x) - H(q, t, z)\right| \leqslant \frac{\delta}{4} \quad \text{whenever } \max\left(|p - q|, |s - t|, |x - z|\right) \leqslant \alpha_0. \tag{4.8}$$

Fix any ε such that $\varepsilon M \leqslant \alpha_0$. Let r be a positive number to be fixed later.

Let $z \in B(x_0, r)$ and let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\phi_\varepsilon - \psi$ has a strict minimum at z with $\phi_\varepsilon(z) = \psi(z)$. We easily conclude, using the fact that $\phi_\varepsilon - \psi$ is smooth in the x_1 direction,

$$D_{x_1}\phi_\varepsilon(z) = D_{x_1}\psi(z) \quad \text{or} \quad D_{x_1}\phi(z) = D_{x_1}\psi(z) \tag{4.9}$$

and,

$$\frac{\partial^2}{\partial x_1 \partial x_1}(\phi_\varepsilon - \psi)(z) \geqslant 0 \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x_1 \partial x_1}(z) \geqslant \frac{\partial^2 \psi}{\partial x_1 \partial x_1}. \tag{4.10}$$

We also have

$$|\phi(z) - \psi(z)| \leqslant \varepsilon M. \tag{4.11}$$

We are now ready to establish the claim. As

$$\phi_\varepsilon(x) - \psi(x) = \varepsilon\left[w\left(\frac{x'}{\varepsilon}\right) - \frac{1}{\varepsilon}(\psi(x) - \phi(x))\right]$$

we have that $w(y') - \frac{1}{\varepsilon}(\psi(\varepsilon y) - \phi(\varepsilon y))$ has a local minimum at $y_\varepsilon = z/\varepsilon$ and hence also its restriction to $\mathbb{R}^{n-1}$ has a local minimum at y'_ε. So by the definition of w we have

$$\begin{aligned}&-a(y'_\varepsilon)\frac{\partial^2 \phi}{\partial x_1 \partial x_1}(x_0) + H\left(\left(D_{x_1}\phi(x_0), D_{x'}\phi(x_0) + \left((D_{x'}\psi)(\varepsilon y_\varepsilon) - (D_{x'}\phi)(\varepsilon y_\varepsilon)\right)\right), \phi(x_0), x_0\right) \\ &\quad \geqslant \overline{F}\left(\frac{\partial^2 \phi}{\partial x_1 \partial x_1}(x_0), D\phi(x_0), \phi(x_0), x_0\right) \geqslant \delta\end{aligned}$$

that is

$$-a\left(\frac{z'}{\varepsilon}\right)\frac{\partial^2 \phi}{\partial x_1 \partial x_1}(x_0) + H\left(D_{x_1}\phi(x_0), D_{x'}\phi(x_0) + \left(D_{x'}\psi(z) - D_{x'}\phi(z)\right), \phi(x_0), x_0\right) \geqslant \delta.$$

As ϕ is smooth, a is bounded and H is Lipschitz continuous, we can choose r sufficiently small such that

$$-a\left(\frac{z'}{\varepsilon}\right)\frac{\partial^2\phi}{\partial x_1\partial x_1}(z)+H\big(D_{x_1}\phi(z),D_{x'}\phi(z)+\big(D_{x'}\psi(z)-D_{x'}\phi(z)\big),\phi(z),z\big)\geqslant\frac{3\delta}{4}.$$

Now using the relations (4.9), (4.10) and the fact that a is nonnegative we obtain

$$-a\left(\frac{z'}{\varepsilon}\right)\frac{\partial^2\psi}{\partial x_1\partial x_1}(z)+H\big(D\psi(z),\phi(z),z\big)\geqslant\frac{3\delta}{4}.$$

It now follows from (4.11), the choice of ε and (4.8) that

$$-a\left(\frac{z'}{\varepsilon}\right)\frac{\partial^2\psi}{\partial x_1\partial x_1}(z)+H\big(D\psi(z),\psi(z),z\big)\geqslant\frac{\delta}{2}.$$

This proves the claim. □

Boundary condition. It remains to show that $u^*=0=u_*$ on $\partial\Omega$. Since $0\leqslant u_\varepsilon\leqslant v$ we have

$$0\leqslant\liminf_{z\to x,\ \varepsilon\to0}u_\varepsilon(z)\leqslant\limsup_{z\to x,\ \varepsilon\to0}u_\varepsilon(z)\leqslant\limsup_{z\to x,\ \varepsilon\to0}v(z)$$

that is

$$0\leqslant u_*(x)\leqslant u^*(x)\leqslant v(x)$$

as v is continuous from its definition. As $v\equiv0$ on $\partial\Omega$ we obtain the desired conclusion.

Remark 4.2. The curvature condition (H0) played a role only in the existence of a supersolution to (P_ε) and to (P_{hom}). So the homogenization result holds true without this condition if, by some other means, one is able to show the existence of a bounded sequence of solutions to (P_ε). We also remark that we get, for free, the existence of a solution to the homogenized problem by the limiting process.

5. Remarks

The above example is particularly easy to deal with as even though the problem is partially degenerate to begin with, the cell (ergodic) problem (3.1) is purely degenerate (a first order problem) and could be solved. If instead the ergodic problem defining $\overline{F}$ were partially degenerate in the strict sense it may not even have a solution(cf. Arisawa and Lions [2]). So one cannot expect a homogenization result for partially degenerate equations in general but only for good classes a problem which needs further investigation.

Appendix

A1. Principal curvatures of $\partial\Omega$

For each point $p \in \partial\Omega$ choose a rectangular coordinate system with the inward normal at p as the positive x_n direction. As $\partial\Omega$ is a C^2 surface it is possible to represent it locally as the graph of a C^2 function. More precisely, let (p', p_n) represent p in the coordinate system. Then there exists a neighbourhood U of p' and a neighbourhood V of p_n and a C^2 function $\phi : U \to V$ such that

$$(x', x_n) \in \partial\Omega \cap U \times V \quad \text{if and only if} \quad x_n = \phi(x').$$

Definition. The eigenvalues of $D^2\phi(p')$, denoted by $k_1(p), \ldots, k_{n-1}(p)$, are called the *principal curvatures* of $\partial\Omega$ at p (see [9]).

The curvature condition roughly tells us that the boundary is locally the graph of a convex function (with the positive x_n direction same as the inward normal). Examples of such domains are balls. A non-example is an annular domain. In this case, the curvature condition is satisfied on the outer boundary but on the inner boundary all the curvatures are negative.

A2. Ramification of the curvature condition

We show, under the curvature condition (H0), that the Hessian of the distance function to the boundary is negative semi-definite in $\overline{\Omega}_\lambda$ for λ sufficiently large. Given $x \in \Omega_\mu$, let $y \in \partial\Omega$ be such that $|x - y| = d(x)$. Then the Hessian of d at x is given by [9]

$$[D^2 d(x)] = \operatorname{diag}\left[\frac{-k_1(y)}{1 - k_1(y)d}, \ldots, \frac{-k_{n-1}(y)}{1 - k_{n-1}(y)d}, 0\right]. \tag{A.1}$$

Thus, by the curvature condition (H0), the Hessian of the distance function is negative semi-definite in $\overline{\Omega}_\lambda$ if we fix λ such that $\lambda > \max_{y\in\partial\Omega} \max_i(k_i(y))$. This is possible since the k_i's are bounded from above on $\partial\Omega$. To see this one notices that the boundary can be covered by a finite number of charts(closed) on each of which the principal curvatures are upper bounded by continuity.

A3. Verification of properties (P1)–(P5)

(P1) follows from the non-negativity of a. (P2) is obvious. Next note that

$$-\beta\begin{pmatrix} I & O \\ O & -I \end{pmatrix} \leqslant \begin{pmatrix} X & O \\ O & -Z \end{pmatrix} \leqslant \beta\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

implies

$$-\beta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \leqslant \begin{pmatrix} X_{11} & 0 \\ 0 & -Z_{11} \end{pmatrix} \leqslant \beta\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Thus for the vector $\binom{a^{1/2}(x'/\varepsilon)}{a^{1/2}(z'/\varepsilon)}$ the right most inequality, in the sense of quadratic forms, yields

$$-a\left(\frac{z'}{\varepsilon}\right)Z_{11}+a\left(\frac{x'}{\varepsilon}\right)X_{11}\leqslant\beta\left(a^{1/2}\left(\frac{z'}{\varepsilon}\right)-a^{1/2}\left(\frac{x'}{\varepsilon}\right)\right)^2\leqslant\beta L^2\frac{|x'-z'|^2}{\varepsilon^2}\leqslant\beta L^2\frac{|x-z|^2}{\varepsilon^2}.$$

Now, (P3) follows from this and the Lipschitz continuity of f. Next we see that $F(0,0,0,x,x/\varepsilon)=-f(x)\leqslant 0$ by the nonnegativity of f. Finally, $F(0,0,C,x,x/\varepsilon)=C-f(x)\geqslant 0$ by fixing C such that $C\geqslant\|f\|_\infty$.

A4. Proof of Proposition 3.1

For $\delta>0$ and (X,p,s,x) given we consider the equation

$$-a(y')X_{11}+|p_1|+|p'+D_{y'}w^\delta|+s-f(x)+\delta w^\delta=0\quad\text{in }\mathbb{R}^{(n-1)}. \tag{A.2}$$

Define

$$G(q,r,y';X,p,s,x,\delta)\doteq -a(y')X_{11}+|p_1|+|p'+q|+s-f(x)+\delta r,$$

where (X,p,s,x,δ) play the role of parameters. (X,p,s,x) are arbitrary but fixed and $0<\delta<1$ which we will allow to tend to zero eventually. We will suppress these parameters for convenience of presentation. It is clear that

(i) for any q, y' and r_1, r_2

$$G(q,r_1,y')-G(q,r_2,y')\geqslant\delta(r_1-r_2),\quad\text{in fact, equal,}$$

(ii) $\lim_{|q|\to\infty}G(q,r,y')=\infty$ locally uniformly in the other variables,
(iii) $G(q,r,y')$ is uniformly continuous on $\mathbb{R}^{(n-1)}\times\mathbb{R}\times\mathbb{R}^{(n-1)}$ and
(iv) there exists $K_\delta>0$ such that

$$G(0,-K_\delta,y')\leqslant 0\leqslant G(0,K_\delta,y')\quad\text{in }\mathbb{R}^{(n-1)}.$$

Indeed this is true for $K_\delta=(\|a\|_\infty|X_{11}|+|p|+|s-f(x)|)/\delta$ for given (X,p,s,x).

So by [3, Theorem 2.12] we have the existence of a unique bounded Lipschitz continuous viscosity solution to (A.2) in $\mathbb{R}^{(n-1)}$ satisfying

$$\|w^\delta\|_\infty\leqslant K_\delta\quad\text{for each }\delta.$$

We remark that w^δ is periodic with period $[0,1]^{(n-1)}$ by the periodicity of G and the uniqueness of solutions to (A.2). We have by the choice of K_δ and the above inequality that

$$\sup_\delta\|\delta w^\delta\|_\infty<\infty. \tag{A.3}$$

Using (A.3) and (ii) above one obtains, from Eq. (A.2), that

$$\sup_{\delta} \|Dw^{\delta}\|_{\infty} < \infty.$$

Set $v^{\delta} \doteq w^{\delta} - \min_{\mathbb{R}^{(n-1)}} w^{\delta}$. Then v^{δ} is Lipschitz continuous, periodic and satisfies

$$\sup_{\delta} \|v^{\delta}\|_{C^{0,1}(\mathbb{R}^{(n-1)})} < \infty.$$

Thus, we may extract a subsequence indexed by δ' such that

$$v^{\delta'} \to w \quad \text{uniformly in } \mathbb{R}^{(n-1)}.$$

Hence,

$$\delta' w^{\delta'} \to -\rho \quad \text{a constant, uniformly in } \mathbb{R}^{(n-1)}.$$

Passing to the limit in (A.2), using the above, we obtain that (ρ, w) satisfies

$$-a(y')X_{11} + |p_1| + |p' + D_{y'}w| + s - f(x) = \rho \quad \text{in } \mathbb{R}^{(n-1)}$$

in the viscosity sense.

A5. Proof of Proposition 3.3

(i) is obvious.

The remaining properties are all obtained by an application of the comparison principle.

Let w^{δ} and η^{δ} be bounded Lipschitz continuous functions satisfying the following equations in the viscosity sense

$$-a(y')X_{11} + |p_1| + |p' + D_{y'}w^{\delta}| + s - f(x) + \delta w^{\delta} = 0 \quad \text{in } \mathbb{R}^{(n-1)},$$
$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}\eta^{\delta}| + r - f(z) + \delta \eta^{\delta} = 0 \quad \text{in } \mathbb{R}^{(n-1)}.$$

We rewrite the first of the equations as follows

$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}w^{\delta}| + r - f(z) + \delta w^{\delta} + \big(a(y')Z_{11} - a(y')X_{11}\big) + |p_1| - |q_1| \\ + |p' + D_{y'}w^{\delta}| - |q' + D_{y'}w^{\delta}| + s - r + f(z) - f(x) = 0 \quad \text{in } \mathbb{R}^{(n-1)} \tag{A.4}$$

which gives

$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}w^{\delta}| + r - f(z) + \delta w^{\delta} + \big(M_0|Z_{11} - X_{11}|\big) + |p - q| \\ + s - r + f(z) - f(x) \geqslant 0 \quad \text{in } \mathbb{R}^{(n-1)}.$$

Setting $\zeta^\delta = w^\delta + (M_0|Z_{11} - X_{11}| + |p-q| + s - r + f(z) - f(x))/\delta$ we conclude that

$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}\zeta^\delta| + r - f(z) + \zeta^\delta \geqslant 0$$

in the viscosity sense. By the comparison principle for (A.2) one obtains $\zeta^\delta \geqslant \eta^\delta$ for all $\delta > 0$. Or in other words

$$\delta w^\delta + M_0|Z_{11} - X_{11}| + |p-q| + s - r + f(z) - f(x) \geqslant \delta\eta^\delta.$$

Passing to the limit as $\delta \to 0$ in the above we get

$$\overline{F}(X,p,s,x) - \overline{F}(Z,q,r,z) \leqslant M_0|Z_{11} - X_{11}| + |p-q| + s - r + f(z) - f(x).$$

In particular,

$$\overline{F}(X,p,s,x) - \overline{F}(Z,q,r,z) \leqslant M_0|Z_{11} - X_{11}| + |p-q| + |s-r| + L|z-x|$$

and similarly,

$$\overline{F}(Z,q,r,z) - \overline{F}(X,p,s,x) \leqslant M_0|Z_{11} - X_{11}| + |p-q| + |s-r| + L|z-x|.$$

Thus we have (ii).

If in (A.4) we take $Z = X$, $q = p$ and $z = x$ we will have

$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}w^\delta| + r - f(z) + \delta w^\delta + s - r = 0 \quad \text{in } \mathbb{R}^{(n-1)}$$

and therefore by uniqueness for (A.2) we have

$$\delta w^\delta + s - r = \delta\eta^\delta$$

leading to the conclusion (iii).

Assume that $a(\cdot) \geqslant \alpha > 0$. We again go back to (A.4) with $X \geqslant Z$, $q = p$, $s = r$ and $z = x$ we get

$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}w^\delta| + r - f(z) + \delta w^\delta + \big(a(y')Z_{11} - a(y')X_{11}\big) = 0 \quad \text{in } \mathbb{R}^{(n-1)}$$

and in particular since $a(y')Z_{11} - a(y')X_{11} \leqslant \alpha(Z_{11} - X_{11})$ we get

$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}w^\delta| + r - f(z) + \delta w^\delta + \alpha(Z_{11} - X_{11}) \geqslant 0 \quad \text{in } \mathbb{R}^{(n-1)}.$$

So, by comparison,

$$\delta w^\delta + \alpha(Z_{11} - X_{11}) \geqslant \delta\eta^\delta$$

and letting $\delta \to 0$ we get the second part of (iv). We obtain the first part as a special case.

Let $x, z \in \mathbb{R}^n$, $\beta > 0$ and let $X, Z \in \mathcal{S}^n$ be such that

$$-\beta\begin{pmatrix} I & O \\ O & -I \end{pmatrix} \leqslant \begin{pmatrix} X & O \\ O & -Z \end{pmatrix} \leqslant \beta\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Note that $X \leqslant Z$ by the rightmost inequality. Take $p = \beta(x - z) = q,\ s = r$ in (A.4) to get

$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}w^\delta| + r - f(z) + \delta w^\delta + \big(a(y')Z_{11} - a(y')X_{11}\big) + f(z) - f(x) = 0 \quad \text{in } \mathbb{R}^{(n-1)}$$

which gives

$$-a(y')Z_{11} + |q_1| + |q' + D_{y'}w^\delta| + r - f(z) + \delta w^\delta \leqslant L|z - x| \quad \text{in } \mathbb{R}^{(n-1)}.$$

Again by comparison

$$\delta w^\delta \leqslant L|x - z| + \delta\eta^\delta$$

and letting $\delta \to 0$ we get (v).

Let now w^δ be the solution of (A.2) with $X = 0$, $p = 0$ and $s = 0$ that is

$$|D_{y'}w^\delta| - f(x) + \delta w^\delta = 0 \quad \text{in } \mathbb{R}^{(n-1)}$$

which implies

$$|D_{y'}w^\delta| + \delta w^\delta \geqslant 0 \quad \text{in } \mathbb{R}^{(n-1)}.$$

Obviously 0 is a solution and in particular a subsolution to

$$|D_{y'}\zeta^\delta| + \delta\zeta^\delta = 0 \quad \text{in } \mathbb{R}^{(n-1)}.$$

Hence, by comparison, $\delta w^\delta \geqslant 0$. Letting $\delta \to 0$ we get (vi).

Similarly if w^δ is a solution to (A.2) with $X = 0$, $p = 0$ and $s = C$ we get

$$|D_{y'}w^\delta| + C - f(x) + \delta w^\delta = 0 \quad \text{in } \mathbb{R}^{(n-1)}$$

which, since $C \geqslant \|f\|_\infty$, implies

$$|D_{y'}w^\delta| + \delta w^\delta \leqslant 0 \quad \text{in } \mathbb{R}^{(n-1)}.$$

Again by comparison with 0, now considered as a supersolution, we get $\delta w^\delta \leqslant 0$ and letting $\delta \to 0$ we get (vii).

Let now $x \in \overline{\Omega}_\lambda$ and let w^δ be the solution of

$$-a(y')\big(D^2 v_0(x)\big)_{11} + |D_{x_1}v_0(x)| + |D_{x'}v_0(x) + D_{y'}w^\delta| + v_0(x) - f(x) + \delta w^\delta = 0 \quad \text{in } \mathbb{R}^{(n-1)}.$$

As before $D^2v_0(x)$ is negative semi-definite and we obtain from this

$$M\lambda\, \mathrm{e}^{-\lambda d(x)}\left|(Dd(x))_1\right| + \left|M\lambda\, \mathrm{e}^{-\lambda d(x)}(Dd(x))' + D_{y'}w^\delta\right| - f(x) + \delta w^\delta \leqslant 0 \quad \text{in } \mathbb{R}^{(n-1)},$$

where $(Dd(x))_1$ and $(Dd(x))'$ denote respectively the first and remaining components of $Dd(x)$. On the other hand, for λ large, 0 is a supersolution of

$$M\lambda\, \mathrm{e}^{-\lambda d(x)}\left|(Dd(x))_1\right| + \left|M\lambda\, \mathrm{e}^{-\lambda d(x)}(Dd(x))' + D_{y'}w^\delta\right| - f(x) + \delta w^\delta = 0 \quad \text{in } \mathbb{R}^{(n-1)}.$$

Indeed,

$$M\lambda\, \mathrm{e}^{-\lambda d(x)}\left|(Dd(x))_1\right| + \left|M\lambda\, \mathrm{e}^{-\lambda d(x)}(Dd(x))'\right| - f(x) \geqslant 0 \quad \text{in } \mathbb{R}^{(n-1)}$$

for λ such that $\lambda \geqslant \|f\|_\infty/(M\,\mathrm{e}^{-1})$. Thus by the comparison principle

$$0 \geqslant \delta w^\delta \quad \text{or} \quad 0 \leqslant -\delta w^\delta$$

from which (viii) follows.

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