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HOMOGENIZATION OF CERTAIN SPECTRAL PROBLEMS

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ABSTRACT

This article, essentially consists of two parts, even though it is not divided in this fashion. In the beginning, we briefly introduce the notion of **Homogenization** and recall some of the results developed in the last 20 to 30 years related to spectral problems. We also introduce the various methods in homogenization without giving details as it is beyond the scope of this article. Then, we present some partial results in homogenization of spectral problems: namely, harmonic and Stokes eigenvalue problems in periodically perforated domains, that left unresolved and still remains open.

1. Introduction to Homogenization

Homogenization is a mathematical procedure to understand heterogeneous materials (or media) with highly oscillating heterogenities (at the microscopic level) via a homogeneous material. Mathematically, it is a limiting analysis. The physical problems described on such materials leads to the study of mathematical equations like: differential or integral equations, optimization problems, spectral problems, and so on, will exhibit high oscillations in the coefficients present in the equation or in the domain. This high frequency oscillations, in turn, will reflect in the solutions. Thus, even if the well posedness of the problems were guaranteed, a numerical computation (to predict the behaviour of such heterogeneous media) of such solutions will be highly non-trivial; in fact, it is almost impossible.

The homogenization deals with the study of asymptotic analysis of such solutions and obtain the equation satisfied by the limit. This limit equation will characterize the **bulk/overall** behaviour of the material, which doesn't consists of microscopic heterogenities and can be solved or computed. This solved and computed solution will then, be a good approximation, in a suitable sense, to the original solution.

There are plenty of examples of such heterogeneous media in the literature and in our daily life. To cite few examples:

1. **Composite Materials:** These material are obtained by fine mixing of two or more materials with different physical properties. The study of composite material is an important aspect in material science. The problems modeled on such materials leads to homogenization problems.

2. **Porous Media:** These can be viewed as domains obtained by removing a large number of tiny holes. Examples are fluid flow through porous media; flow of ground water or oil, flow of resins in moulds in industries.

3. **Layered materials:** like plywood etc. This also can be viewed as a composite with oscillations only in one direction.

4. **Micro-structure of phase transition:** The crystal structure of materials changes at a critical temperature while cooling. This happens at the atomic level and the structure moves from Austenite (high temperature) state to Martensite (low temperature) state.

5. **Analysis of vibrations of thin structures.**

Fore more details, we refer the readers to the literature : [4], [8], [21], [6], [12], [1], [15], [16], [23].

Now let us investigate some specific problems in the case of second order elliptic partial differential equation:

$$\begin{cases} A^\varepsilon u_\varepsilon &= f \text{ in } \Omega \\ u_\varepsilon &= 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $A^\varepsilon = -\frac{\partial}{\partial x_i}(a^\varepsilon_{ij}(x)\frac{\partial}{\partial x_j})$ is the elliptic operator with bounded coefficients $a^\varepsilon_{ij}(x)$. In fact, we assume that there exist constants $c_0, c_1 > 0$ such that

$$c_0|\xi|^2 \leq a^\varepsilon_{ij}\xi_i\xi_j \leq c_1|\xi|^2, \forall \xi \in \mathbb{R}^n. \tag{1.2}$$

Here f is a given source function and Ω is a bounded smooth domain in $\mathbb{R}^n$ occupying the material with boundary $\Gamma = \partial\Omega$. The equation (1.1) arise in plenty of physical situations like heat transfers, magnetostatics, electrostatics and so on. The coefficients $a^\varepsilon_{ij}(x)$ represent the physical property of medium; for example thermal conductivity in heat transfer, magnetic permeability in magnetostatics and dielectric constant in electrostatics. The small parameter $\varepsilon > 0$ represents the heterogenities in the small scale which creates oscillations in $a^{\varepsilon}_{ij}(x)$.

Special case (Periodic structure) : Let $Y = (0,1)^n$ be the reference unit cell and $a_{ij} : Y \longrightarrow \mathbb{R}$ be a periodic function and extend a_{ij} periodically to all of $\mathbb{R}^n$. Then, define

$$a^{\varepsilon}_{ij}(x) = a_{ij}(\frac{x}{\varepsilon}). \tag{1.3}$$

Thus $a^{\varepsilon}_{ij}(x)$ is periodic with period ε and it is oscillating rapidly as $\varepsilon \to 0$. As a particular case, let us assume, a_{ij} takes only two values $\alpha, \beta > 0$; Let $T \subset Y$ be a subdomain and

$$a(y) = \begin{cases} \alpha \text{ if } y \in T \\ \beta \text{ if } y \in Y \backslash T \end{cases} \quad \text{and } a_{ij}(y) = a(y)\delta_{ij},$$

where δ_{ij} is the Kronicker delta. In this case, we can view the material Ω as a fine mixing of two materials, when ε is small. That is, it is a composite of two materials. From the standard theory of differential equations, it is easy to see that (1.1) has a unique solution u_ε, for fixed $\varepsilon > 0$ and $u_\varepsilon \rightharpoonup u$ weakly in $H^1_0(\Omega)$, for some u. We use the standard Sobolev space notations. The problem is to characterize u as a solution of some differential equation. In the periodic case, we further know that

$$a_{ij}\left(\frac{x}{\varepsilon}\right) \rightharpoonup \bar{a}_{ij} = \int_Y a_{ij}(y)dy$$

in L^∞ weak $*$. Does u satisfy

$$-\bar{A}u = f,$$

where $\bar{A} = -\frac{\partial}{\partial x_i}(\bar{a}_{ij}\frac{\partial}{\partial x_j})$. In other words, does the **bulk** behaviour is the **averaged** behaviour ?

The result is not true even in one dimension. However, the analysis of one dimensional case can easily carried out to obtain the limiting equation. We do not do this here, except to point out that we get an additional information on the product $a(\frac{x}{\varepsilon})\frac{du_\varepsilon}{dx}$ that it is bounded in L^2. The mathematical difficulty in general, is that weak convergence do not preserve non-linearities. In particular, if u_n and v_n converges weakly to u and v, respectively, then it is not necessarily true $u_n v_n$ converges to uv, but it may still converge to some other quantity, which one would like to obtain. As an example $u_n = v_n = \sin nx \rightharpoonup 0$ weakly in L^2, but $\sin nx \cdot \sin nx \longrightarrow \frac{1}{2}$.

A notion of H-**convergence** was introduced to understand the homogenization procedure.

H-**convergence :** Define, for $c_0, c_1 > 0$, the class of matrix functions:

$$E(\Omega) = E(c_0, c_1, \Omega) = \{A = [a_{ij}] : A \text{ is symmetric and satisfies (1.2)}\}.$$

Definition 1.1. *We say a family* $\{[a_{ij}^{\varepsilon}]\}_{\varepsilon>0}$, *H-converges to* $[a_{ij}^{*}]$ *as* $\varepsilon \to 0$ *if*

$$i)\quad u_\varepsilon \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ weak}$$

$$ii)\quad a_{ij}^{\varepsilon}(x)\frac{\partial u_\varepsilon}{\partial x_j} \rightharpoonup a_{ij}^{*}(x)\frac{\partial u}{\partial x_j} \text{ in } L^2(\Omega) \text{ weak.}$$

Here u^ε *is the solution of (1.1) and* u *is the solution of*

$$\begin{aligned} A^{*}u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $A^{*} = -\frac{\partial}{\partial x_i}(a_{ij}^{*}\frac{\partial}{\partial x_j})$ *and we write* $[a_{ij}^{\varepsilon}] \xrightarrow{H} [a_{ij}^{*}]$ *or simply* $A^{\varepsilon} \xrightarrow{H} A$.

There is a very general compactness theorem (See [8], [12]) which is given below.

Theorem 1.2. (Compactness Theorem) *Let* $[a_{ij}^{\varepsilon}]_{\varepsilon>0}$, $\varepsilon \to 0$ *be any family from* $E(\Omega)$. *Then there is a subsequence* $\varepsilon_n \to 0$ *and a matrix* $[a_{ij}^{*}] \in E(\Omega)$ *such that*

$$[a_{ij}^{\varepsilon}] \xrightarrow{H} [a_{ij}^{*}].$$

At first sight, from the above theorem, it may seem that the limiting analysis has been over. But, it is far from complete in the sense that, in general, we do not know the characterization of $[a_{ij}^{*}]$. However, there are some special cases, where a_{ij}^{*} can be explicitly characterized. For example, in the periodic case as well as in composites of thin sheets (layered materials). The other direction of study is to obtain good bounds on a_{ij}^{*} which has far reaching practical applications in the construction of composite materials and hence it is more interesting for engineers and scientists. This comes in the study of optimal bounds for homogenized coefficients. We refer the readers to [9], [10],[11],[25].

We end this section by briefly introducing various methods developed in the last 30 years to handle homogenization and other non linear problems.

1. Formal asymptotic expansion : In any asymptotic problem, the first step is to look for a suitable asymptotic expansion and try to guess the correct limit from the formal analysis. Keeping the particular problem in mind one looks for :

$$u_\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \cdots,$$

where x is the slow variable and $y = \frac{x}{\varepsilon}$ is the fast variable. Then, if possible, see that u_0 is independent of y and obtain the equation satisfied by u_0 (Ref : [4]).

2. Energy method via test functions : The idea is to construct suitable test functions having same oscillations as the solutions to control the trouble creating oscillating terms to pass to the limit. In the process, the energy of the original system converges to the energy of the homogenized system (Ref : [4]).

3. Compensated Compactness : This method, actually, was introduced to pass to the limit in non-linear problems under weak convergence. We have already remarked in general, we may not be able to conclude the convergence of $u_n v_n$ to uv from the weak convergence of u_n and v_n. This may be due to the oscillations in u_n and v_n and its interactions. But if u_n and v_n oscillates in transverse directions, then the non linear functional $u_n v_n$ behaves nicely. For example if u_n and v_n are functions on complementary variables i.e., $u_n = u_n(x')$ and $v_n = v_n(x'')$, where $x = (x', x'')$, then the convergence of $u_n v_n$ to uv can be concluded easily, i.e., one needs a sort of compensation to achieve the compactness. This is the basic motivation of compensated compactness, though the theory is much more involved (Ref: [5],[16],[24]).

4. Gamma Convergence : This is a variational convergence developed to study optimization problems. Gamma convergence is a very powerful notion introduced in the seventies and have applications in several problems including homogenization problems (Ref: [8]).

5. Two Scale (Multi-scale) Convergence : This was specially introduced for studying homogenization problems. This makes the formal asymptotic analysis mathematically rigorous (Ref: [1],[19]). We see more about this later in the article.

6. Fourier (Bloch wave) method : The latest addition is the Bloch wave method. Initially, problems from fluid- solid interaction were studied using bloch wave analysis (Ref: [7]). The basic idea is to work in phase space than in the physical space represented by x variable. Essentially one diagonalize the operator A^ε and transform the equations $A^\varepsilon u^\varepsilon = f$ into a sequence of scalar equations without the derivatives. The concept of Fourier decomposition when the medium is homogeneous is that the operator can be diagonalized in the basis of plane waves. In the current periodic situation, one requires Bloch waves.

2. Spectral Problems

Now, we come to the main topic of discussion of this article, namely, the spectral problems. Consider

$$\begin{aligned} A^\varepsilon v_\varepsilon &= \frac{1}{\mu_\varepsilon} v_\varepsilon \text{ in } \Omega \\ v_\varepsilon &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where A^ε is as in (1.1). The problem (2.1) is equivalent to the study of the spectrum of the following operator:

$$S_\varepsilon : L^2(\Omega) \longrightarrow L^2(\Omega),$$

where, S_ε for $f \in L^2(\Omega)$ is defined by $S_\varepsilon f = u_\varepsilon$ as the unique solution of (1.1). The spectrum $\sigma(S_\varepsilon)$ is given by

$$\sigma(S_\varepsilon) = \{0\} \bigcup_{k\geq 1} \{\mu_\varepsilon^k\},$$

and $\mu_\varepsilon^1 \geq \mu_\varepsilon^2 \geq \cdots \mu_\varepsilon^k \geq \cdots \longrightarrow 0$ as $k \longrightarrow \infty$.

Theorem 2.3. **(Homogenization)** *For each k, the eigenvalues μ_ε^k converges to some μ^k as $\varepsilon \to 0$ and along a subsequence (normalized v_ε^k), the eigenvectors v_ε^k converges to v^k in $L^2(\Omega)$. Further (μ^k, v^k) satisfies $A^* v = \frac{1}{\mu} v$ in Ω, $v = 0$ on $\partial\Omega$, where A^* is the homogenized operator.*

Thus we can write

$$\lim_{\varepsilon\to 0} \sigma(S_\varepsilon) = \sigma(S),$$

where S is the operator associated with A^*.

Initially, the above result was proved by S.Kesavan [13]. It is, now, a consequence of the more general result. Of course, one required to show the convergence of S_ε to S.

Theorem 2.4. *Let S_ε, S be compact self adjoint operators in $L^2(\Omega)$ and $S_\varepsilon \longrightarrow S$ uniformly, i.e, $\|S_\varepsilon - S\| \longrightarrow 0$. Then $\lim_{\varepsilon\to 0} \sigma(S_\varepsilon) = \sigma(S)$.*

Note that the set limit is understood in the sense that $\sigma(S)$ is the set of accumulation points λ of the sequences $\lambda_\varepsilon \in \sigma(S_\varepsilon)$ as $\varepsilon \longrightarrow 0$. If S_ε do not converge uniformly, we cannot conclude the theorem. However, if S_ε converges strongly to a limit operator S (i.e., $S_\varepsilon f \longrightarrow Sf$ in $L^2(\Omega)$, $\forall f \in L^2(\Omega)$), not necessarily compact, with spectrum $\sigma(S)$, (Ref: [3]), then

$$\lim_{\varepsilon\to 0} \sigma(S_\varepsilon) \supset \sigma(S).$$

If S_ε converges to S weakly; i.e., $S_\varepsilon f \rightharpoonup Sf$ weakly in $L^2(\Omega), \forall f \in L^2(\Omega)$, in general, we cannot conclude anything about the limit spectrum.

However, regarding the analysis of $\sigma(S_\varepsilon)$ of our problem, there are more recent results, using Bloch wave analysis. The spectral analysis which we have seen is known as **low frequency** limit, where we fix k and take $\varepsilon \to 0$. Of course,

$$\lim_{\varepsilon\to 0, k\to\infty} \mu_\varepsilon^k = 0.$$

However, it does not say much about the rate at which $\mu_\varepsilon^k \to 0$ as $\varepsilon \to 0, k \to \infty$. This **high frequency** analysis was studied in [3] using Bloch wave method. In fact, a complete analysis of $\lim_{\varepsilon\to 0} a_\varepsilon^{-2} \sigma(S_\varepsilon)$ was carried out. If $a_\varepsilon \longrightarrow 0$ be such that $\lim_{\varepsilon\to 0}\frac{a_\varepsilon}{\varepsilon} = 0$ or $+\infty$, then, $\lim_{\varepsilon\to 0} a_\varepsilon^{-2}\sigma(S_\varepsilon) = \mathbb{R}_+$. When $a_\varepsilon \sim 0(\varepsilon)$, then

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\sigma(S_\varepsilon) = \sigma_{bloch}\bigcup\sigma_{boundary}.$$

Here σ_{bloch} consists of spectral values coming via the bloch decomposition of certain associated operators and $\sigma_{boundary}$consists spectral values coming from a boundary layer phenomena. Both multi-scale methods and bloch wave analysis were used to carry out this work. There are many open questions like : we do not know in general, the values in $\sigma_{boundary}$ are the spectral values of some boundary operators, whether there are gaps between σ_{bloch} and $\sigma_{boundary}$.

Now, we move on to other spectral problem complicated due to the geometry of the domain.

Perforated domain: Let $\Omega_\varepsilon = \Omega\backslash\bigcup T_{\varepsilon k}$ be a periodically perforated domain obtained from Ω by removing a large number of holes of size ε in a periodic fashion. Let $Y = (-\frac{1}{2},\frac{1}{2})^n$ and T be an open set such that $\bar{T}\subset Y$ with boundary $S = \partial T$. Let $Y^* = Y\backslash T$ and $Y_k = Y + k, Y_k^* = Y^* + k, k\in\mathbb{Z}^n$ be all the translated cells. Consider the index sets

$$I_\varepsilon = \{k\in\mathbb{Z}^n : \varepsilon Y_k\subset\Omega\},\quad J_\varepsilon = \{k\in\mathbb{Z}^n, \varepsilon Y_k\cap\Gamma\neq\phi\}.$$

So $\{\varepsilon T_k : k\in I_\varepsilon\}$ are the interior holes of size ε in Ω and $\{\varepsilon T_k : k\in J_\varepsilon\}$ are the boundary holes. Thus Ω_ε is given by $\Omega_\varepsilon = \Omega\backslash\bigcup_{k\in I_\varepsilon\cup J_\varepsilon}\varepsilon T_k$.

We consider the eigenvalue problem

$$\begin{aligned} -\Delta u_\varepsilon &= \lambda_\varepsilon u_\varepsilon \quad \text{in } \Omega_\varepsilon \\ u_\varepsilon &= 0 \qquad\ \ \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

The operator $-\Delta$ is elliptic in Ω_ε with $a_{ij} = \delta_{ij}$ in Ω_ε. By defining $a_{ij} = 0$ in the holes, the problem can be viewed as a similar problem considered earlier, but it is no longer uniformly elliptic. In fact, for each k, the set $\{\lambda_\varepsilon^k\}_{\varepsilon>0}$ is not bounded (it was bounded in the earlier case). It is proved in [28], that $\lambda_\varepsilon^k \sim O(\varepsilon^{-2})$. More precisely,

$$\lambda_\varepsilon^k = \varepsilon^{-2}\lambda^1 + \mu_\varepsilon^k,$$

where λ^1 is the first eigenvalue of the Laplacian in the reference cell Y^*. The set $\{\mu_\varepsilon^k\}_k$ is the spectrum associated with an elliptic operator in Ω_ε, but $\{\mu_\varepsilon^k\}_{\varepsilon>0}$ is bounded for each k and homogenization can be studied. As a special case, we have

$$\varepsilon^2\lambda_\varepsilon^k \longrightarrow \lambda^1 \text{ as } \varepsilon\to 0,\ \text{for any } k.$$

This problem exhibits a strong interplay between the size of the perforations (ε, here) and period of the structure, which is also ε. One way to get bounded spectrum is to consider smaller perforations, namely the holes $a_\varepsilon T_{\varepsilon k}$ instead of εT_k. In fact, there exists a critical size of the holes a_ε for which the spectrum is bounded. A similar result, with small holes for the elasticity system was carried out in [17].

Coming back to the first case; $a_\varepsilon = O(\varepsilon)$, where period and perforations are of the same order, we would like to remark that the proof in the Laplacian case depends heavily on the simplicity of the first eigenvalue λ^1 and the positivity of the corresponding eigenvector.

These are no longer true for other systems like: bilaplacian, Stokes, elasticity eigenvalue problems and so on. All these problems remains open till today. Of course, when the holes are small, in all these cases, one can show that the spectrum $\{\lambda_\varepsilon^k\}_{\varepsilon>0}$, for each k, is bounded and homogenization process can be carried out.

We, now, present some partial results when $a_\varepsilon = O(\varepsilon)$ in the case of bilaplacian and Stokes systems.

3. Bilaplacian eigenvalue problem

We consider the biharmonic problem in the periodically perforated domain Ω_ε namely,

$$\begin{cases} \Delta^2 u_\varepsilon = \lambda_\varepsilon u_\varepsilon \text{ in } \Omega_\varepsilon \\ u_\varepsilon = \frac{\partial u_\varepsilon}{\partial v} = 0 \text{ on } \partial\Omega_\varepsilon \\ \int u_\varepsilon^2 = 1. \end{cases} \tag{3.1}$$

Here $\frac{\partial}{\partial v}$ is the normal derivative at the boundary $\partial\Omega_\varepsilon$. For fixed $\varepsilon > 0$, (3.1) is a standard elliptic eigenvalue problem, which has a variational formulation in $H_0^2(\Omega_\varepsilon)$. Thus there is a sequence of solutions $\{\lambda_\varepsilon^l, u_\varepsilon^l\} \in \mathbf{R} \times H_0^2(\Omega_\varepsilon)$ such that

$$0 \leq \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \cdots \lambda_\varepsilon^l \cdots \longrightarrow \infty \text{ as } l \to \infty \tag{3.2}$$

and $\{u_\varepsilon^l\}$ is an orthonormal sequence in $L^2(\Omega_\varepsilon)$. The eigenvalues λ_ε^l can be characterized using Rayleigh quotient formulation as

$$\begin{aligned} \lambda_\varepsilon^l &= \min\{\max_{v\in S_l} R_\varepsilon(v) : S_l \subset H_0^2(\Omega_\varepsilon), dim\ S_l = l\} \\ &= \min\{R_\varepsilon(v) : (v, u_\varepsilon^i) = 0, \forall i = 1, \cdots, l-1\}, \end{aligned} \tag{3.3}$$

where

$$R_\varepsilon(v) = \frac{\int_{\Omega_\varepsilon} |\Delta v|^2}{\int_{\Omega_\varepsilon} |v|^2}.$$

We have the following results.

Theorem 3.5. *Let $\{\lambda_\varepsilon^l\}$ be the spectrum of (3.1) satisfying (3.2). Then there exist constants $C_1 > 0, C_2 = C_2(l) > 0$, independent of ε, such that*

$$C_1 < \varepsilon^4 \lambda_\varepsilon^l < C_2, \quad l = 1, 2, \cdots \tag{3.4}$$

Theorem 3.6. *Let $\{u_\varepsilon^l, \lambda_\varepsilon^l\}$ be the entire spectrum of (3.1) and $\tilde{u}_\varepsilon^l$ be the extension of u_ε^l by zero to all of Ω. Then there exists a subsequence of ε, again denoted by ε, such that there exist a functions $u_0^l \in L^2(\Omega, L_p^2(Y))$ and scalars $\lambda^l, 0 < \lambda^l < \infty$ such that*

$$\varepsilon^4 \lambda_\varepsilon^l \to \lambda^l \tag{3.5}$$

and

$$\tilde{u}_\varepsilon^l \to u^l = \int_{Y^*} u_0^l(x,y)dy \text{ in } L^2(\Omega) \quad \text{weak.} \tag{3.6}$$

Moreover λ^l, u_0^l satisfy

$$\begin{cases} \Delta_y^2 u_0 = \lambda u_0 \text{ in } \Omega \times Y^* \\ u_0 = \dfrac{\partial u_0}{\partial v} = 0 \text{ on } S \\ u_0, \nabla u_0 \text{ are } Y-\text{ periodic.} \end{cases} \tag{3.7}$$

Remark 3.7. *The Theorem 3.5. does not provide a complete answer, because it is not proved that $u_0 \not\equiv 0$. So the first result to be proved in this direction is $u_0 \not\equiv 0$ and we believe that $\int_{\Omega\times Y^*} u_0^2 dxdy = 1$. If this is proved, then from (3.7), it follows that u_0 can be represented as $u_0(x,y) = \tilde{u}(x)\phi(y)$, where ϕ is an eigenvector of the following problem (3.8). In this case λ is an eigenvalue.*

$$\begin{cases} \Delta^2 \phi = \lambda \phi \text{ in } Y^* \\ \phi = \dfrac{\partial \phi}{\partial v} = 0 \text{ on } S \\ \phi, \nabla \phi \text{ are } Y-\text{periodic} \\ \displaystyle\int \phi^2 = 1. \end{cases} \tag{3.8}$$

Thus, we have the following theorem.

Theorem 3.8. *Let $\{u_\varepsilon^l, \lambda_\varepsilon^l\}, u_0$ be as in Theorem 3.6. and assume that $u_0 \not\equiv 0$. Then along a subsequence of ε, we have*

$$\varepsilon^4 \lambda_\varepsilon^l \longrightarrow \lambda^l \text{ as } \varepsilon \longrightarrow 0,$$

where λ^l is some eigenvalue of (3.8). If ϕ is the corresponding eigenvector, then

$$u_0(x,y) = \tilde{u}(x)\phi(y),$$

for some function $\tilde{u}(x)$.

Proof of Theorem 3.1: The first inequality in (3.2) follows from the following lemma.

Lemma 3.9. *There exists a constant C, independent of ε, such that*

$$\int_{\Omega_\varepsilon} u^2 \leq C\varepsilon^4 \int_{\Omega_\varepsilon} |\Delta u|^2, \ \forall\, u \in H_0^2(\Omega_\varepsilon).$$

PROOF: Take any $u \in H_0^2(\Omega_\varepsilon)$ and consider the restriction $u_{\varepsilon k} = u|_{\varepsilon Y_k}$ for $k \in I_\varepsilon \cup J_\varepsilon$. Using the homothetic mapping $x \in \varepsilon Y_k \to y = \frac{x-x_k}{\varepsilon} \in Y$, define $v_{\varepsilon k}(y) = u_{\varepsilon k}(x) = u_{\varepsilon k}(\varepsilon y + x_k)$, where x_k is the center of εY_k. Now using the standard Poincarè inequality, since $v_{\varepsilon k} \in H^2(Y^*)$ and $v_{\varepsilon k} = \frac{\partial v_{\varepsilon k}}{\partial \nu} = 0$ on S, we get

$$\int_{Y^*} v_{\varepsilon k}^2 dy \leq C \int_{Y^*} |\Delta v_{\varepsilon k}|^2 dy,$$

where C is independent of ε. Hence

$$\int_{\varepsilon Y_k^*} u_{\varepsilon k}^2 dx \leq C\varepsilon^4 \int_{\varepsilon Y_k^*} |\Delta u_{\varepsilon k}|^2 dx.$$

Summing over all $k \in I_\varepsilon \bigcup J_\varepsilon$ and observing that Ω can be covered by $m(\Omega)\varepsilon^{-N}$, $\varepsilon-$ periodic cells, it follows that

$$\int_{\Omega_\varepsilon} u^2 \leq \sum_{k \in I_\varepsilon \bigcup J_\varepsilon} \int_{\varepsilon Y_k^*} u_{\varepsilon k}^2 dx \leq C\varepsilon^4 \sum_{k \in I_\varepsilon \bigcup J_\varepsilon} \int_{\varepsilon Y_k^*} |\Delta u_{\varepsilon k}|^2 dx \leq C\varepsilon^4 \int_{\Omega_\varepsilon} |\Delta u|^2.$$

This complete the proof of Lemma 3.9..

From the lemma and (3.3) it follows that

$$\varepsilon^4 \lambda_\varepsilon^1 \geq \varepsilon^4 \lambda_\varepsilon^1 \geq C_1 > 0.$$

To prove the second inequality in (3.4), introduce the following eigenvalue problem in $T_1 \backslash \bar{T}$, where T_1 is an open set with smooth boundary S_1, such that $\bar{T} \subset T_1 \subset \bar{T_1} \subset Y$:

$$\begin{cases} \Delta^2 \psi = \mu \psi \text{ in } T_1 \backslash \bar{T} \\ \psi = \dfrac{\partial \psi}{\partial \nu} = 0 \text{ on } S_1 \bigcup S \\ \int \psi^2 = 1. \end{cases} \tag{3.9}$$

Define $\psi = 0$ in $Y \backslash T_1$ and extend periodically to all of $\mathbb{R}^N$. Corresponding to the eigenvectors $\psi_1, \psi_2, \cdots$, define

$$\psi_i^\varepsilon(x) = \begin{cases} \psi_i(\frac{x-x_k}{\varepsilon}) \ \text{ if } x \in \varepsilon Y_k \text{ for } k \in I_\varepsilon \\ 0 \text{ if } x \in \varepsilon Y_k \bigcap \Omega, \text{ for } k \in J_\varepsilon. \end{cases}$$

Then, obviously, $\psi_i^\varepsilon \in H_0^2(\Omega_\varepsilon)$. We would like to remark that if we introduce the problem (3.9) in the domain Y^* with a periodicity condition, then we will not get $\psi_k^\varepsilon = 0$ on $\partial\Omega$, which is one of the crucial point to be noted.

Claim: Fix $l \geq 1$, let $S_l^\varepsilon = span\{\psi_1^\varepsilon, \cdots, \psi_l^\varepsilon\}$. Then S_l^ε is an l-dimensional subspace of $H_0^2(\Omega_\varepsilon)$ and it is L^2-orthogonal.

Proof of the claim : If $0 = \sum_{i=1}^l c_i\psi_i^\varepsilon$ in Ω_ε, then $\sum_{i=1}^l c_i\psi_i^\varepsilon = 0$ in εY_k for any $k \in I_\varepsilon$, which shows that $\sum_{i=1}^l c_i\psi_i = 0$ in $T_1\backslash\bar{T}$. Hence $c_i = 0$ $\forall i = 1, \cdots, l$ because $\{\psi_1, \cdots, \psi_l\}$ is independent. The L^2-orthogonality follows from the orthogonality of $\{\psi_1, \cdots, \psi_l\}$ and since

$$\int_{\Omega_\varepsilon} \psi_i^\varepsilon\psi_j^\varepsilon = \sum_{k\in I_\varepsilon}\int_{\varepsilon Y_k}\psi_i^\varepsilon\psi_j^\varepsilon = \sum_{k\in I_\varepsilon}\varepsilon^n\int_{T_1\backslash\bar{T}}\psi_i\psi_j = 0.$$

Thus the claim.
Now it is easy to see that

$$\|\psi_i^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \sim m(\Omega)\|\psi_i\|^2_{L^2(Y^*)} = m(\Omega)$$

and

$$\|\Delta\psi_i^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \sim m(\Omega)\varepsilon^{-4}\|\Delta\psi_i\|^2_{L^2(Y^*)} = m(\Omega)\varepsilon^{-4}\mu_i.$$

Now from (3.3), we have

$$\lambda_\varepsilon^l \leq \max_{v\in S_l^\varepsilon} R_\varepsilon(v), \tag{3.10}$$

where S_l^ε is defined as above. For any $v \in S_l^\varepsilon, v = \sum_{i=1}^l c_i\psi_i^\varepsilon$, and using L^2 orthogonality, we get

$$\|v\|^2_{L^2(\Omega_\varepsilon)} = \sum_{i=1}^l c_i^2\|\psi_i^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \geq C\sum_{i=1}^l c_i^2$$

and

$$\|\Delta v\|^2_{L^2(\Omega_\varepsilon)} \leq C\sum_{i=1}^l c_i^2\|\Delta\psi_i^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C\mu_l\varepsilon^{-4}\sum_{i=1}^l c_i^2.$$

Substituting these estimates in (3.10), we get the second inequality in (3.4). This completes the proof of the Theorem 3.5..

Now we state a two-scale convergence theorem due to Nguetseng [19] (see also Allaire [1] and Nandakumaran [18]).

Lemma 3.10. *(Nguetseng [19]) Let $\{u_\varepsilon\}$ be a uniformly bounded sequence in $L^2(\Omega)$. Then there is a subsequence of ε, denoted again by ε, and*

$$u_0 = u_0(x, y) \in L^2(\Omega, L_p^2(Y))$$

such that

$$\int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon}) \to \int_{\Omega\times Y} u_0(x,y)\psi(x,y)dxdy \tag{3.11}$$

as $\varepsilon \to 0$, *for all* $\psi \in C_c(\bar\Omega, C_p(Y))$. *Moreover*

$$\int_\Omega u_\varepsilon(x)v(x)\omega(\frac{x}{\varepsilon}) \to \int_{\Omega\times Y} u_0(x,y)v(x)\omega(y)dxdy \tag{3.12}$$

as $\varepsilon \to 0, \forall v \in C_c(\bar\Omega)$ *and* $\forall\omega \in L^2_p(Y)$. *Further, if* u *is the* L^2 *weak limit of* u_ε, *then by taking* $\omega \equiv 1$ *in (3.12) we get*

$$u(x) = \int_Y u_0(x,y)dy. \tag{3.13}$$

Here $L^2_p(Y)$ denotes the space of L^2- periodic functions and $C_p(Y)$ denotes the space of continuous periodic functions on Y.

Proof of Theorem 3.6.: From Theorem 3.5., it follows that there is a subsequence of ε, such that

$$\varepsilon^4\lambda^l_\varepsilon \to \lambda^l$$

as $\varepsilon \to 0$ and $C_1 \le \lambda^l \le C_2(l) \le \infty$. Also, we have

$$\|\tilde u^l_\varepsilon\|_{L^2(\Omega)} = 1$$

Multiply (3.1) by u_ε, integrate by parts and putting $\xi^l_\varepsilon = \varepsilon^2\Delta u^l_\varepsilon$, we get

$$\|\tilde\xi^l_\varepsilon\|^2_{L^2(\Omega)} = \varepsilon^4\lambda^l_\varepsilon \le C_2(l) < \infty,$$

where $\tilde\xi^l_\varepsilon$ is the extension of ξ^l_ε by zero outside Ω_ε. Using Lemma 3.9., one can associate $u^l_0(x,y)$, $\xi^l_0(x,y) \in L^2(\Omega, L^2_p(Y))$ to $\tilde u^l_\varepsilon$, $\tilde\xi^l_\varepsilon$, respectively, satisfying properties (3.11)-(3.13). Thus, we have

$$u^l(x) = \int_Y u^l_0(x,y)dy,$$

where $\tilde u^l_\varepsilon \rightharpoonup u^l$ in $L^2(\Omega)$ weak. We, now, prove that λ^l, u^l_0 will satisfy (3.7). For simplicity, we skip the superscript l. Take $\phi \in \mathcal{D}(\Omega), w \in \mathcal{D}(Y^*)$. Use the test functions $v = \phi w^\varepsilon = \phi(x)w(\frac{x}{\varepsilon})$ in the equations (3.1) to get:

$$\begin{aligned}\varepsilon^4\lambda_\varepsilon\int_{\Omega_\varepsilon} u_\varepsilon\phi w^\varepsilon &= \varepsilon^4\lambda_\varepsilon\int_\Omega \tilde u_\varepsilon\phi w^\varepsilon \\ &\to \lambda\int_{\Omega\times Y} u_0(x,y)\phi(x)w(y)dxdy\end{aligned}$$

and

$$\varepsilon^4 \int_{\Omega_\varepsilon} \Delta u_\varepsilon \Delta(\phi w^\varepsilon) = \varepsilon^4 \int_{\Omega_\varepsilon} \Delta u_\varepsilon (\Delta\phi w^\varepsilon + 2\nabla\phi.\nabla w^\varepsilon + \phi\Delta w^\varepsilon).$$

The first two terms on the right hand side go to zero and the third term converges to $\int_{\Omega\times Y} \xi_0(x,y)\phi(x)\Delta_y w(y)dxdy$. Thus, we have

$$\int_\Omega \left[\int_Y \xi_0(x,y)\Delta_y w(y)dy\right]\phi(x)dx = \lambda\int_\Omega \left[\int_Y u_0(x,y)w(y)dy\right]\phi(x)dx,$$

which holds for all $\phi \in \mathcal{D}(\Omega), w \in \mathcal{D}(Y^*)$. Therefore

$$\Delta_y \xi_0(x,y) = \lambda u_0(x,y) \text{ in } \Omega \times Y^*. \tag{3.14}$$

Claim: $\xi_0(x,y) = \Delta_y u_0(x,y)$ in $\Omega \times Y$.
Once the claim is proved, we get

$$\Delta_y^2 u_0(x,y) = \lambda u_0(x,y) \text{ in } \Omega \times Y^*. \tag{3.15}$$

Proof of the Claim: Observe that the claim is in $\Omega \times Y$, not just in $\Omega \times Y^*$, but (3.14) and hence (3.15) holds only in $\Omega \times Y^*$. Take $v \in \mathcal{D}(\Omega), w \in \mathcal{D}(Y)$, then

$$\int_\Omega \tilde{\xi}_\varepsilon v w^\varepsilon \to \int_{\Omega\times Y} \xi_0(x,y)v(x)w(y)dxdy.$$

Using the boundary conditions of u_ε, we get

$$\begin{aligned}\int_\Omega \tilde{\xi}_\varepsilon v w^\varepsilon = \int_{\Omega_\varepsilon} \xi_\varepsilon v w^\varepsilon &= \varepsilon^2 \int_{\Omega_\varepsilon} \Delta u_\varepsilon v w^\varepsilon = \varepsilon^2 \int_{\Omega_\varepsilon} u_\varepsilon \Delta(v w^\varepsilon) \\ &= \varepsilon^2 \int_{\Omega_\varepsilon} u_\varepsilon (\Delta v w^\varepsilon + 2\nabla v.\nabla w^\varepsilon + v\Delta w^\varepsilon).\end{aligned}$$

Again the first two terms go to zero and the third term converges to $\int_{\Omega\times Y} u_0(x,y)v(x)\Delta_y w(y)dxdy$. Combining, we get (3.14) and this completes the proof of the claim and hence (3.15).

Now, since $u_0(x,.) \in H^2(Y)$, to prove the boundary conditions on S, it is enough to prove that $u_0 \equiv 0$ in T. This is quite easy; for consider $w \in \mathcal{D}(Y)$ such that $supp(w) \subset T$ and $v \in \mathcal{D}(\Omega)$. Then, we have

$$0 = \int_\Omega \tilde{u}_\varepsilon v w^\varepsilon \to \int_{\Omega\times Y} u_0(x,y)v(x)w(y)dxdy.$$

Since $w = 0$ in $Y \backslash T$, we get

$$\int_{\Omega} \left[\int_{T} u_0(x,y) w(y) dy \right] v(x) dx = 0,$$

for all $v \in \mathcal{D}(\Omega)$ and for all $w \in \mathcal{D}(T)$. This shows that $u_0 \equiv 0$ in T. The proof of the Theorem 3.6. is complete.

We complete this section by stating the following questions.

1. Is $u_0 \not\equiv 0$?
2. Is λ^l, the first eigenvalue of the problem (3.8)?
3. What can we say about the further coefficients in the expansion of λ_ε. In particular, study the behaviour of $\lambda_\varepsilon^l - \varepsilon^4 \lambda^l$. We conjecture that $\lambda_\varepsilon^l - \varepsilon^4 \lambda^l = O(\varepsilon^{-2})$.

In the next section, we study the Stokes eigenvalue problem and obtain, somewhat similar results as in this section. We only sketch the results.

4. Stokes Eigenvalue Problem

Here, we assume due to technical reasons that the holes do not intersect the boundary Γ of Ω. That is $\Omega_\varepsilon = \Omega \backslash \bigcup_{k \in I_\varepsilon} \varepsilon T_k$. Consider the following system:

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = \lambda_\varepsilon u_\varepsilon \text{ in } \Omega_\varepsilon \\ div\ u_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ u_\varepsilon = 0 \text{ in } \partial\Omega_\varepsilon. \end{cases} \tag{4.1}$$

The above system is a mathematical model that describes the vibrations of an elastic body occupying a region Ω_ε. To study (4.1), we introduce the following spaces:

$$V = V(\Omega) = \{v \in H_0^1(\Omega)^N : div\ v = 0 \text{ in } \Omega\}$$

and $V_\varepsilon = V(\Omega_\varepsilon)$. Then the variational formulation of (4.1) is given by: find $u_\varepsilon = V_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} \nabla u_{\varepsilon k}.\nabla v_k = \lambda_\varepsilon \int_{\Omega_\varepsilon} u_{\varepsilon k} v_k, \ \forall\, v \in V_\varepsilon, \ \int_{\Omega_\varepsilon} u_\varepsilon^2 := \int_{\Omega_\varepsilon} |u_\varepsilon|^2 = 1. \tag{4.2}$$

This, again is a standard elliptic eigenvalue problem in the variational formulation and it has a sequence of solutions $u_\varepsilon^l, \lambda_\varepsilon^l$. The pressure term is absent in (4.2), but the existence of p_ε^l satisfying (4.1) is well known in the literature. One can refer to Payne [20], Temam [26], Serrin [22] etc. For the homogenization of Stokes problem in perforated domains, one can see Tartar [26], Allaire [2], Nandakumaran [18].

We have the following estimate on the eigenvalues.

Theorem 4.11. *There exist constants $C_1 > 0, C_2(l) > 0$, independent of ε, such that*

$$C_1 < \varepsilon^2 \lambda_\varepsilon^l \leq C_2(l),\ l = 1, 2, \cdots \tag{4.3}$$

PROOF: The first inequality follows from the following lemma which is similar to Lemma 3.9.. See Tartar [26].

Lemma 4.12. *There exists a constant C, independent of ε, such that*

$$||u||_{L^2(\Omega_\varepsilon)} \leq C\varepsilon||\nabla u||_{L^2(\Omega_\varepsilon)},\ \forall\ u \in H_0^1(\Omega_\varepsilon).$$

To prove the second inequality, we use the same technique as in the proof of Theorem 3.5.. Introduce the following eigenvalue problem:

$$\begin{cases} -\Delta\psi + \nabla q = \mu\psi \text{ in } T_1 \setminus \bar{T} \\ div\ \psi = 0 \text{ in } T_1 \setminus \bar{T} \\ \psi = 0 \text{ on } S_1 \bigcup S,\ \displaystyle\int \psi^2 = 1. \end{cases} \tag{4.4}$$

Now consider the eigenvalues $\{\psi_1, \cdots, \psi_l\}$ and follow similar steps in the proof of Theorem 3.5. to obtain the result.

Discussion: Let $\tilde{u}_\varepsilon$ be the extension by zero in the holes. We have the following estimates

$$||\nabla\tilde{u}_\varepsilon||_{L^2(\Omega)} \leq C\varepsilon^{-1}.$$

Using the technique of Tartar [26], one can obtain an extension $\tilde{p}_\varepsilon$ of p_ε such that

$$||\nabla\tilde{p}_\varepsilon||_{H^{-1}(\Omega)} \leq C\varepsilon^{-2} \text{ and } ||\tilde{p}_\varepsilon||_{L^2(\Omega)/\mathbb{R}} \leq C\varepsilon^{-2}.$$

Moreover, their exists $p \in L^2(\Omega)/\mathbb{R}$ such that

$$\varepsilon^2\nabla\tilde{p}_\varepsilon \to \nabla p \text{ in } H^{-1}(\Omega) \text{ strong}$$

$$\varepsilon^2\tilde{p}_\varepsilon \to p \text{ in } L^2(\Omega) \text{ strong}.$$

Now using convergence result of Nguetseng [19], we can deduce the following result (we omit the proof).

Theorem 4.13. *There exist u_0, p, p_1 such that*

$u_0 \in L^2(\Omega, L_p^2(Y)), \nabla_y u_0 \in L^2(\Omega, L_p^2(Y)), p \in L^2(\Omega)/\mathbb{R}, p_1 : \Omega \to H_p^1(Y^*) |_{\mathbb{R}}$

satisfying the following system :

$$\begin{cases} -\Delta_y u_0 + \nabla_y p_1 + \nabla_x p = \lambda u_0 \textit{ in } \Omega \times Y^* \\ div_y u_0 = 0 \textit{ in } Y^*, u_0 = 0 \textit{ on } S \\ u_0, p_1 \textit{ are } Y-\textit{periodic} \\ div_x \displaystyle\int_{Y^*} u_0 = 0 \textit{ in } \Omega, \nu \cdot \int_{Y^*} u_0(x,y)dy = 0 \textit{ on } \Gamma. \end{cases} \tag{4.5}$$

Moreover,

$$\begin{cases} \varepsilon^2 \lambda_\varepsilon^l \to \lambda \\ \tilde{u}_\varepsilon^l \rightharpoonup u(x) = \int_{Y^*} u_0(x,y)dy \text{ in } L^2(\Omega) \text{ weak} \\ \varepsilon^2 \tilde{p}_\varepsilon \to p \text{ in } L^2(\Omega)/\mathbb{R} \text{ strong}. \end{cases} \tag{4.6}$$

The system (4.5) can, also, be obtained using a formal asymptotic expansion. It is an open problem to study the system (4.5). The steady system (4.5) with a function $f \in L^2(\Omega, L^2(Y^*)^N)$, instead of λu_0 in (4.5), has been studied in Lions [14] in the following way. Let

$$V(Y^*) = \{w \in (H^1(Y^*))^N \ : \ div\, w = 0 \text{ in } Y^*, w = 0 \text{ on } S, \ w \text{ is periodic}\}$$

and define the space

$$W = \{v \in L^2(\Omega, V(Y^*)) : div_x \int_{Y^*} v(x,y)dy = 0 \text{ in } \Omega, v \cdot \int_{Y^*} u_0(x,y)dy = 0 \text{ on } \Gamma\}$$

with the norm

$$\|v\|_W^2 = \int_\Omega \|v(x,\cdot)\|_{H^1}^2 dx.$$

For any $u, v \in W$, define

$$a_{Y^*}(u,v) = \int_{Y^*} \nabla_y u \cdot \nabla_y v \text{ and } a(u,v) = \int_\Omega a_{Y^*}(u,v)dx.$$

Then (4.5) with $f \in L^2(\Omega, L^2(Y^*)^N)$, instead of λu_0, has the variational formulation:

$$a(u_0, v) = \int_{\Omega \times Y^*} f_i v_i, \ \ \forall v \in W. \tag{4.7}$$

The problem (4.7) has a unique solution u_0 and existence of p and p_1, follow as in Lions [14].

On the other hand, (4.5) is equivalent to : find $(\lambda, u_0) \in \mathbb{R} \times W$ such that

$$a(u_0, v) = \lambda(u_0, v), \forall v \in W. \tag{4.8}$$

Here observe that there is no ellipticity in the x variable and hence we cannot apply the compact operator theory to get the spectrum. We, also would like to remark that unlike in the case of biLaplacian, in the present limit system, the variable x is also present explicitly. It is an open problem to study the spectrum of the system.

Is it possible to obtain a system which is only defined in the unit cell Y^* and is λ the first eigenvalue? Under the following assumption, we answer the former question.

Assumption(A): $p =$ constant and $u(x) = \int_{Y^*} u_0(x,y)dy = 0.$

The assumption (A) is equivalent to :

Assumption(B) : $\int_S (-\frac{\partial u_0}{\partial v} + p_1 \cdot v)ds = 0.$

To see this, integrate (4.5) with respect to y to obtain

$$-\int_S \frac{\partial u_0}{\partial v} + \int_S p_1 \cdot v + |Y^*|\nabla_x p(x) = \lambda u.$$

So obviously (A)⇒(B). On the other hand, if we assume (B), then

$$|Y^*|\nabla_x p = \lambda u \text{ in } \Omega,$$

so that by taking divergence on both sides and using $div\ u = 0$ in Ω, we get $\Delta p = 0$ in Ω. Moreover,

$$\frac{\partial p}{\partial v} = v \cdot \nabla_x p = \frac{\lambda}{|Y^*|} v \cdot u = 0 \text{ on } \Gamma.$$

Thus, we have a Neumann elliptic problem for the pressure p and hence $p =$ constant by uniqueness which in turn implies $u \equiv 0$.

Now to derive a system in Y^*, under the above assumption, first put $\tilde{u}_0(y) = \int_\Omega u_0(x,y)dx$, $\tilde{p}_1(y) = \int_\Omega p_1(x,y)dx$. Then $\tilde{u}_0\, \tilde{p}_1$ satisfy

$$\begin{aligned}
&-\Delta_y \tilde{u}_0 + \nabla_y \tilde{p}_1 = \lambda \tilde{u}_0 \text{ in } Y^* \\
&div_y\ \tilde{u}_0 = 0 \text{ in } Y^*, \\
&\tilde{u}_0 = 0 \text{ on } S,\ \tilde{u}_0 \text{ is } Y-\text{ periodic.}
\end{aligned}$$

This is a standard stokes eigenvalue problem in the unit cell Y^*.

So under the assumption (A) or (B), we have a situation similar to Laplacian and biLaplacian and one can ask similar question as earlier.

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