



A zero sum differential game in a Hilbert space [☆]

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Abstract

We study a zero sum differential game of fixed duration in a separable Hilbert space. We prove a minimax principle and establish the equivalence between the dynamic programming principle and the existence of a saddle point equilibrium. We also prove sufficient conditions for optimality.

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1. Introduction

In this paper, we study a zero sum differential game of fixed duration involving controlled semilinear evolution equations in a separable Hilbert space. The theory of zero sum differential games in Euclidean space was initiated by Isaacs [7]. He extended the notion of value, optimal strategies, saddle point equilibrium, etc. from static games to a dynamic situation. Using some formal arguments, he showed that the value function is a solution of certain nonlinear partial differential equation, now known as Hamilton–Jacobi–Isaacs (HJI for short) equation. Under the assumption that the HJI equation has a smooth solution and certain other assumptions, he proved the existence of optimal strategies and saddle point equilibrium. But the existence of a smooth solution of HJI equation is more of an exception

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than a rule as pointed out in Fleming and Soner [6]. To circumvent this difficulty, various approaches to differential games were carried out. Notable contributions were made by Fleming, Friedman, Roxin, Varaiya, Lin, Elliott, Kalton, Krasovski, Subbotin, Berkovitz, and others; see Chapter 4 in [1] and references therein. Evans and Souganidis [4] followed Elliott–Kalton approach to differential games and showed that the upper and lower values of the game were viscosity solutions to HJI equations. Many results along these lines have been carried out by many authors; see [1] and references therein. Kocan et al. [8] have studied zero sum differential games in infinite dimensional spaces and have characterized the upper and lower values in the sense of Elliott–Kalton as viscosity solution of HJI equation in infinite dimensions. Though Elliott–Kalton approach to differential games is indeed a very powerful one, certain important concepts like saddle point equilibrium, minimax principle, etc. are not well suited in this framework. In this paper, we follow the original formulation of differential games by Isaacs. We study the differential game in the framework of (open loop) relaxed strategies. In this setup, we first establish a minimax principle to characterize a saddle point equilibrium. Then we establish the equivalence between the dynamic programming principle (DPP for short) and the existence of a saddle point equilibrium via the theory of viscosity solutions introduced in [3]. Finally we establish a connection between the minimax principle and DPP via sub and super-differentials of value function and then prove sufficient conditions for optimality. We now describe the problem.

Let U_i , $i = 1, 2$, be compact metric spaces and let \mathcal{M}_i be the space of probability measures on U_i . Let H be a real and separable Hilbert space and let T be the duration of the game. Let A be a possibly unbounded linear operator generating a semigroup of contractions $S(t)$, $0 \leq t \leq T$. Note that there exist constants $M, \omega > 0$ such that

$$\|S(t)\| \leq M e^{\omega t}$$

for all $t \geq 0$. Let

$$\bar{b}: [0, T] \times H \times U_1 \times U_2 \rightarrow H.$$

We make the following assumption on \bar{b} .

(A1) \bar{b} is continuous and there exists $C_1 > 0$ such that

$$\begin{aligned} \|\bar{b}(t, x, u_1, u_2) - \bar{b}(s, y, u_1, u_2)\| &\leq C_1(|t - s| + \|x - y\|), \\ \forall u_i \in U_i, t, s \in [0, T], x, y \in H. \end{aligned}$$

Define

$$b: [0, T] \times H \times \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow H$$

by

$$b(t, x, \mu_1, \mu_2) = \int_{U_1} \int_{U_2} \bar{b}(t, x, u_1, u_2) \mu_2(du_2) \mu_1(du_1).$$

For $t \in [0, T]$, a measurable function $\mu(\cdot): [t, T] \rightarrow \mathcal{M}_i$ is called an open loop relaxed strategy for player i , $i = 1, 2$, at time t . Let \mathcal{A}_i^t denote the set of all (open loop) relaxed

strategies for player $i, i = 1, 2$, at time t . If the players choose (open loop) relaxed strategies $(\mu_1(\cdot), \mu_2(\cdot)) \in \mathcal{A}_1^t \times \mathcal{A}_2^t$, then the state of the system evolves according to

$$\begin{cases} \frac{d}{ds}X(s) + AX(s) = b(s, X(s), \mu_1(s), \mu_2(s)), & s \in (t, T], \\ X(t) = x. \end{cases} \tag{1.1}$$

Under assumptions (A1) and (A2), there is a unique mild solution for (1.1) (see [9]).

Let

$$\bar{r} : [0, T] \times H \times U_1 \times U_2 \rightarrow \mathbb{R}$$

be the running payoff function and let

$$g : H \rightarrow \mathbb{R}$$

be the terminal payoff function. We assume that

(A2) (i) The functions \bar{r} and g are continuous.

(ii) There are constants $C_2, C_3 > 0$ for all $t, s \in [0, T], x, y \in H, u_i \in U_i$ satisfying

$$\begin{aligned} |\bar{r}(t, x, u_1, u_2) - \bar{r}(s, y, u_1, u_2)| &\leq C_2(|t - s| + \|x - y\|), \\ |g(x) - g(y)| &\leq C_3\|x - y\|. \end{aligned}$$

Let

$$r : [0, T] \times H \times \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R}$$

be defined by

$$r(t, x, \mu_1, \mu_2) = \int_{U_1} \int_{U_2} \bar{r}(t, x, u_1, u_2) \mu_2(du_2) \mu_1(du_1).$$

When the state of the system is at x at time t and players use (open loop) relaxed strategies $(\mu_1(\cdot), \mu_2(\cdot)) \in \mathcal{A}_1^t \times \mathcal{A}_2^t$, then payoff to player 1 by player 2 is given by

$$R(t, x, \mu_1(\cdot), \mu_2(\cdot)) = \int_t^T r(s, X(s), \mu_1(s), \mu_2(s)) ds + g(X(T)),$$

where $X(\cdot)$ is the solution of (1.1). The upper and lower values V^+ and V^- are defined as follows:

$$V^+(t, x) = \inf_{\mu_2(\cdot) \in \mathcal{A}_2^t} \sup_{\mu_1(\cdot) \in \mathcal{A}_1^t} R(t, x, \mu_1(\cdot), \mu_2(\cdot)), \tag{1.2}$$

$$V^-(t, x) = \sup_{\mu_1(\cdot) \in \mathcal{A}_1^t} \inf_{\mu_2(\cdot) \in \mathcal{A}_2^t} R(t, x, \mu_1(\cdot), \mu_2(\cdot)). \tag{1.3}$$

A relaxed strategy $\mu_1^*(\cdot) \in \mathcal{A}_1^t$ is said to be optimal for player 1 at (t, x) if

$$R(t, x, \mu_1^*(\cdot), \mu_2(\cdot)) \geq V^+(t, x)$$

for any $\mu_2(\cdot) \in \mathcal{A}_2^t$. Similarly a relaxed strategy $\mu_2^*(\cdot) \in \mathcal{A}_2^t$ is said to be optimal for player 2 at (t, x) if

$$R(t, x, \mu_1(\cdot), \mu_2^*(\cdot)) \leq V^-(t, x)$$

for any $\mu_1(\cdot) \in \mathcal{A}_1^t$. Thus a pair of optimal relaxed strategies constitutes a saddle point equilibrium. The game is said to have value in relaxed strategies if $V^+(t, x) = V^-(t, x) := V(t, x)$. In such a case, V is referred to as the value function of the game.

We endow \mathcal{A}_i^t with the L^1 -weak*-topology. Using Banach–Alaoglu theorem, we can verify that \mathcal{A}_i^t is a compact metric space; see [11] for more details. Under this topology, (A1) and (A2) imply that $R(t, x, \mu_1(\cdot), \mu_2(\cdot))$ is continuous in $\mu_1(\cdot)$ for fixed t, x , and $\mu_2(\cdot)$. Similarly it is continuous in $\mu_2(\cdot)$ for fixed t, x , and $\mu_1(\cdot)$. Thus ‘inf’ and ‘sup’ in (1.2) and (1.3) may be replaced by ‘min’ and ‘max,’ respectively.

We use the following notation in the sequel: H^* denotes the dual of H , A^* denotes the adjoint of a linear operator A on H , $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_{H^*}$ stands for the inner products in H and H^* , respectively, whereas $\langle \cdot, \cdot \rangle_{H, H^*}$ stands for the duality pairing.

The rest of our paper is organized as follows. In Section 2, we establish a minimax principle to characterize a saddle point equilibrium. In Section 3, we establish the equivalence between DPP and saddle point equilibrium. We also prove the existence of saddle point equilibrium in a specific case. Finally we establish the connection between minimax principle and DPP. Section 4 contains some concluding remarks.

2. Minimax principle

In this section, we derive a minimax principle. We make the following assumption:

(A3) For $(t, u_1, u_2) \in [0, T] \times U_1 \times U_2$, $\bar{b}(t, \cdot, u_1, u_2)$, $\bar{r}(t, \cdot, u_1, u_2)$, \bar{g} are continuously Fréchet differentiable.

Let the Hamiltonian

$$G : [0, T] \times H \times H^* \times \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathbb{R}$$

be defined by

$$G(t, x, p, \mu_1, \mu_2) = \langle b(t, x, \mu_1, \mu_2), p \rangle_{H, H^*} + r(t, x, \mu_1, \mu_2). \quad (2.1)$$

Let $(\mu_1^*(\cdot), \mu_2^*(\cdot))$ be a pair of optimal relaxed strategies and let $X^*(\cdot)$ be the corresponding state process with $X^*(0) = x$. Consider $p(t) = U^*(T, t)g_x(X^*(T))$, where U is the solution operator of

$$\begin{cases} \frac{\partial U(s, t)}{\partial s} + AU(s, t) = b_x(s, X^*(s), \mu_1^*(s), \mu_2^*(s))U(s, t), \\ U(t, t) = I. \end{cases} \quad (2.2)$$

We now prove the following minimax principle.

Theorem 2.1. For a.e. $t \in [0, T]$, we have the following minimax principle:

$$\begin{aligned} \min_{\mu_2 \in \mathcal{M}_2} \max_{\mu_1 \in \mathcal{M}_1} G(t, X^*(t), p(t), \mu_1, \mu_2) &= \max_{\mu_1 \in \mathcal{M}_1} G(t, X^*(t), p(t), \mu_1, \mu_2^*(t)) \\ &= \max_{\mu_1 \in \mathcal{M}_1} \min_{\mu_2 \in \mathcal{M}_2} G(t, X^*(t), p(t), \mu_1, \mu_2) = \min_{\mu_2 \in \mathcal{M}_2} G(t, X^*(t), p(t), \mu_1^*(t), \mu_2). \end{aligned} \tag{2.3}$$

Proof. We prove the theorem for the case $r \equiv 0$. The general case can be done in a standard manner by augmenting an extra space variable under our assumptions (see [2]). Now onwards, we assume $r \equiv 0$.

Fix $\mu_1 \in \mathcal{M}_1$ and let \mathcal{I} be the set of all Lebesgue points of the function $b(\cdot, X^*(\cdot), \mu_1, \mu_2^*(\cdot)) - b(\cdot, X^*(\cdot), \mu_1^*(\cdot), \mu_2^*(\cdot))$. Then \mathcal{I} is of full measure. Fix $t \in \mathcal{I}$. Let $\epsilon > 0$. Define

$$\mu_1^\epsilon(s) = \begin{cases} \mu_1^*(s) & \text{if } s \notin [t - \epsilon, t], \\ \mu_1 & \text{if } s \in [t - \epsilon, t]. \end{cases}$$

Let $X^\epsilon(\cdot)$ be the trajectory under the controls $(\mu_1^\epsilon(\cdot), \mu_2^*(\cdot))$ with the initial condition $X^\epsilon(0) = x$. Let $z(s) = U(s, t)(b(t, X^*(t), \mu_1, \mu_2^*(t)) - b(t, X^*(t), \mu_1^*(t), \mu_2^*(t)))$, i.e., $z(\cdot)$ is the unique mild solution of

$$\begin{cases} \dot{z}(s) + Az(s) = b_x(s, X^*(s), \mu_1^*(s), \mu_2^*(s))z(s), & s \geq t, \\ z(t) = b(t, X^*(t), \mu_1, \mu_2^*(t)) - b(t, X^*(t), \mu_1^*(t), \mu_2^*(t)). \end{cases}$$

We claim that $(X^\epsilon(s) - X^*(s))/\epsilon \rightarrow z(s)$ as $\epsilon \downarrow 0$ uniformly in $[t, T]$. We now prove the claim. Let $s \geq t$. Then

$$\begin{aligned} X^\epsilon(s) - X^*(s) &= \int_0^s S(s - \tau) [b(\tau, X^\epsilon(\tau), \mu_1^\epsilon(\tau), \mu_2^*(\tau)) - b(\tau, X^*(\tau), \mu_1^*(\tau), \mu_2^*(\tau))] d\tau \\ &= \int_{t-\epsilon}^s S(s - \tau) [b(\tau, X^\epsilon(\tau), \mu_1, \mu_2^*(\tau)) - b(\tau, X^*(\tau), \mu_1^*(\tau), \mu_2^*(\tau))] d\tau \\ &= \int_{t-\epsilon}^s S(s - \tau) [b(\tau, X^\epsilon(\tau), \mu_1, \mu_2^*(\tau)) - b(\tau, X^*(\tau), \mu_1, \mu_2^*(\tau))] d\tau \\ &\quad + \int_{t-\epsilon}^s S(s - \tau) [b(\tau, X^*(\tau), \mu_1, \mu_2^*(\tau)) - b(\tau, X^*(\tau), \mu_1^*(\tau), \mu_2^*(\tau))] d\tau. \end{aligned}$$

Now

$$\begin{aligned} &\left| X^\epsilon(s) - X^*(s) \right. \\ &\quad \left. - \int_{t-\epsilon}^s S(s - \tau) [b(\tau, X^*(\tau), \mu_1, \mu_2^*(\tau)) - b(\tau, X^*(\tau), \mu_1^*(\tau), \mu_2^*(\tau))] d\tau \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \int_{t-\epsilon}^s S(s-\tau) [b(\tau, X^\epsilon(\tau), \mu_1, \mu_2^*(\tau)) - b(\tau, X^*(\tau), \mu_1, \mu_2^*(\tau))] d\tau \right| \\
&\leq CM e^{\omega T} \int_{t-\epsilon}^s |X^\epsilon(\tau) - X^*(\tau)| d\tau \\
&\leq CM e^{\omega T} \int_{t-\epsilon}^s |X^\epsilon(\tau) - X^*(\tau) \\
&\quad - \int_{t-\epsilon}^{\tau} S(\tau-\sigma) [b(\sigma, X^*(\sigma), \mu_1, \mu_2^*(\sigma)) - b(\sigma, X^*(\sigma), \mu_1^*(\sigma), \mu_2^*(\sigma))] d\sigma \Big| d\tau \\
&\quad + CM^2 e^{2\omega T} \int_{t-\epsilon}^s \int_{t-\epsilon}^{\tau} |b(\sigma, X^*(\sigma), \mu_1, \mu_2^*(\sigma)) \\
&\quad - b(\sigma, X^*(\sigma), \mu_1^*(\sigma), \mu_2^*(\sigma))| d\sigma d\tau.
\end{aligned}$$

Since $t \in \mathcal{I}$,

$$\int_{t-\epsilon}^s \int_{t-\epsilon}^{\tau} |b(\sigma, X^*(\sigma), \mu_1, \mu_2^*(\sigma)) - b(\sigma, X^*(\sigma), \mu_1^*(\sigma), \mu_2^*(\sigma))| d\sigma d\tau \leq o(\epsilon)$$

for all $s \in [t - \epsilon, t]$. Thus

$$\begin{aligned}
&\left| X^\epsilon(s) - X^*(s) \right. \\
&\quad \left. - \int_{t-\epsilon}^s S(s-\tau) [b(\tau, X^*(\tau), \mu_1, \mu_2^*(\tau)) - b(\tau, X^*(\tau), \mu_1^*(\tau), \mu_2^*(\tau))] d\tau \right| \leq o(\epsilon).
\end{aligned}$$

Note that under assumptions (A1) and (A3), both $X^\epsilon(\cdot)$ and $X^*(\cdot)$ are differentiable with respect to the initial condition $X^\epsilon(t) = x$, $X^*(t) = x$. Now divide the above expression by ϵ and let $\epsilon \downarrow 0$. Then it follows that $(X^\epsilon(s) - X^*(s))/\epsilon \rightarrow z(s)$ as $\epsilon \downarrow 0$ uniformly in $[t, T]$.

Since $(\mu_1^*(\cdot), \mu_2^*(\cdot))$ is a saddle point equilibrium, we have

$$\frac{1}{\epsilon} [g(X^\epsilon(T)) - g(X^*(T))] \leq 0. \tag{2.4}$$

Using the claim, we now obtain

$$\langle p, z(T) \rangle \leq 0.$$

Thus

$$\langle p, U(T, t)b(t, X^*(t), \mu_1, \mu_2^*(t)) - b(t, X^*(t), \mu_1^*(t), \mu_2^*(t)) \rangle \leq 0,$$

and hence

$$\langle p(t), b(t, X^*(t), \mu_1, \mu_2^*(t)) - b(t, X^*(t), \mu_1^*(t), \mu_2^*(t)) \rangle \leq 0.$$

Thus for a.e. t ,

$$G(t, X^*(t), p(t), \mu_1, \mu_2^*(t)) \leq G(t, X^*(t), p(t), \mu_1^*(t), \mu_2^*(t)).$$

Note that here μ_1 is arbitrary. Similarly we can show that for a.e. t and for all μ_2 ,

$$G(t, X^*(t), p(t), \mu_1^*(t), \mu_2) \geq G(t, X^*(t), p(t), \mu_1^*(t), \mu_2^*(t)).$$

Using these two inequalities we obtain (2.3). \square

3. Dynamic programming and saddle point equilibrium

In this section, we prove the equivalence between DPP and the existence of saddle point equilibrium. We first state a lemma whose proof is omitted (see [2] for a proof of this result in the case of control problem).

Lemma 3.1. *Assume (A1) and (A2). Then the value functions V^+ and V^- are continuous and Lipschitz continuous in the space variable. Furthermore, if the operator A is analytic, then they are jointly Lipschitz continuous.*

We now prove the DPP under the assumption that a saddle point equilibrium exists.

Theorem 3.2. *Assume (A1) and (A2) and that a saddle point equilibrium exists for (t, x) . Then for $0 \leq t < t + \Delta < T$,*

$$V^+(t, x) = \min_{\mu_2(\cdot) \in \mathcal{A}_2^t} \max_{\mu_1(\cdot) \in \mathcal{A}_1^t} \left[\int_t^{t+\Delta} r(s, X(s), \mu_1(s), \mu_2(s)) ds + V^+(t + \Delta, X(t + \Delta)) \right], \tag{3.1}$$

where $X(\cdot)$ is solution of (1.1) under $(\mu_1(\cdot), \mu_2(\cdot))$ with $X(t) = x$. Similarly,

$$V^-(t, x) = \max_{\mu_1(\cdot) \in \mathcal{A}_1^t} \min_{\mu_2(\cdot) \in \mathcal{A}_2^t} \left[\int_t^{t+\Delta} r(s, X(s), \mu_1(s), \mu_2(s)) ds + V^-(t + \Delta, X(t + \Delta)) \right]. \tag{3.2}$$

Proof. Let $(\mu_1^*(\cdot), \mu_2^*(\cdot)) \in \mathcal{A}_1^t \times \mathcal{A}_2^t$ be a saddle point equilibrium for (t, x) . Denote the right-hand side of (3.1) by $W(t, x)$. For any $\mu_2(\cdot) \in \mathcal{A}_2^t$, we have

$$W(t, x) \leq \max_{\mu_1(\cdot) \in \mathcal{A}_1^t} \left[\int_t^{t+\Delta} r(s, X(s), \mu_1(s), \mu_2(s)) ds + V^+(t + \Delta, X(t + \Delta)) \right], \quad (3.3)$$

where $X(\cdot)$ is the solution of (1.1) under $(\mu_1(\cdot), \mu_2(\cdot))$ with $X(t) = x$. Let $(\tau, \bar{x}) \in [0, T] \times H$ and $\bar{\mu}_2 \in \mathcal{A}_2^\tau$. Define $V_{\bar{\mu}_2}^+$ by

$$V_{\bar{\mu}_2}^+(\tau, \bar{x}) = \max_{\mu_1(\cdot) \in \mathcal{A}_1^\tau} \left[\int_\tau^T r(s, \bar{X}(s), \mu_1(s), \bar{\mu}_2(s)) ds + g(\bar{X}(T)) \right],$$

where $\bar{X}(\cdot)$ is the solution of (1.1) under $(\mu_1(\cdot), \bar{\mu}_2(\cdot))$ with $\bar{X}(\tau) = \bar{x}$. Now using DPP for optimal control (see [10]), we have

$$V_{\bar{\mu}_2}^+(\tau, \bar{x}) = \max_{\mu_1(\cdot) \in \mathcal{A}_1^\tau} \left[\int_\tau^{\tau+\Delta} r(s, \bar{X}(s), \mu_1(s), \bar{\mu}_2(s)) ds + V_{\bar{\mu}_2}^+(\tau + \Delta, \bar{X}(\tau + \Delta)) \right] \quad (3.4)$$

for any $\tau < \tau + \Delta < T$. Also, we have

$$V^+(\tau, \bar{x}) \leq V_{\bar{\mu}_2}^+(\tau, \bar{x}). \quad (3.5)$$

From (3.3)–(3.5), we obtain

$$\begin{aligned} W(t, x) &\leq \max_{\mu_1(\cdot) \in \mathcal{A}_1^t} \left[\int_t^{t+\Delta} r(s, X(s), \mu_1(s), \mu_2(s)) ds + V_{\mu_2}^+(t + \Delta, X(t + \Delta)) \right] \\ &= V_{\mu_2}^+(t, x). \end{aligned}$$

Since $\mu_2(\cdot)$ is arbitrary, we get

$$W(t, x) \leq V^+(t, x).$$

We now prove the reverse inequality. Since $(\mu_1^*(\cdot), \mu_2^*(\cdot))$ is a saddle point at (t, x) , we have

$$V^+(t, x) \leq \min_{\mu_2(\cdot) \in \mathcal{A}_2^t} \left[\int_t^T r(s, X^*(s), \mu_1^*(s), \mu_2(s)) ds + g(X^*(T)) \right], \quad (3.6)$$

where $X^*(\cdot)$ is the solution of (1.1) under $(\mu_1^*(\cdot), \mu_2(\cdot))$ with $X^*(t) = x$. Let $(\tau, \bar{x}) \in [0, T] \times H$ and $\bar{\mu}_1 \in \mathcal{A}_1^\tau$. Define $\bar{V}_{\bar{\mu}_1}^+$ by

$$\bar{V}_{\bar{\mu}_1}^+(\tau, \bar{x}) = \min_{\mu_2(\cdot) \in \mathcal{A}_2^\tau} \left[\int_\tau^T r(s, \bar{X}(s), \bar{\mu}_1(s), \mu_2(s)) ds + g(\bar{X}(T)) \right],$$

where $\bar{X}(\cdot)$ is the solution of (1.1) under $(\bar{\mu}_1(\cdot), \mu_2(\cdot))$ with $\bar{X}(\tau) = \bar{x}$. We have,

$$\bar{V}_{\bar{\mu}_1}^+(\tau, \bar{x}) \leq V^+(\tau, \bar{x}).$$

Again by DPP for optimal control, we have for any $\tau < \tau + \Delta < T$,

$$\begin{aligned} \bar{V}_{\bar{\mu}_1}^+(\tau, \bar{x}) &= \min_{\mu_2(\cdot) \in \mathcal{A}_2^\tau} \left[\int_{\tau}^{\tau+\Delta} r(s, \bar{X}(s), \bar{\mu}_1(s), \mu_2(s)) ds + \bar{V}_{\bar{\mu}_1}^+(\tau + \Delta, \bar{X}(\tau + \Delta)) \right] \\ &\leq \min_{\mu_2(\cdot) \in \mathcal{A}_2^\tau} \left[\int_{\tau}^{\tau+\Delta} r(s, \bar{X}(s), \bar{\mu}_1(s), \mu_2(s)) ds + V^+(\tau + \Delta, \bar{X}(\tau + \Delta)) \right] \\ &\leq \min_{\mu_2(\cdot) \in \mathcal{A}_2^\tau} \max_{\mu_1(\cdot) \in \mathcal{A}_1^\tau} \left[\int_{\tau}^{\tau+\Delta} r(s, X(s), \mu_1(s), \mu_2(s)) ds \right. \\ &\quad \left. + V^+(\tau + \Delta, x(\tau + \Delta)) \right] \\ &= W(\tau, x), \end{aligned}$$

where $X(\cdot)$ is the solution of (1.1) under $(\mu_1(\cdot), \mu_2(\cdot))$ with $X(\tau) = \bar{x}$. Plugging these into (3.6) with $\bar{\mu}_1(\cdot) = \mu_1^*(\cdot)$, we obtain

$$V^+(t, x) \leq \bar{V}_{\mu_1^*}^+(t, x) \leq W(t, x).$$

Hence (3.1) holds. Similarly (3.2) can be proved. \square

Assuming that DPP holds, it is easy to see that the lower and upper value functions are the viscosity solutions of the HJI equations given by

$$v_t(t, x) - \langle A(t)x, v_x(t, x) \rangle_{H, H^*} + \sup_{\mu_1 \in \mathcal{M}_1} \inf_{\mu_2 \in \mathcal{M}_2} G(t, x, Dv(t, x), \mu_1, \mu_2) = 0, \tag{3.7}$$

$$v_t(t, x) - \langle A(t)x, Dv(t, x) \rangle_{H, H^*} + \inf_{\mu_2 \in \mathcal{M}_2} \sup_{\mu_1 \in \mathcal{M}_1} G(t, x, Dv(t, x), \mu_1, \mu_2) = 0. \tag{3.8}$$

Note that in infinite-dimensional spaces there are several definitions of viscosity solutions. Here we use the definition of viscosity solutions in the sense of [3,8]. We refer to [8] for more details about this. Note that Eqs. (3.7) and (3.8) are the same in view of Fan’s mini-max theorem [5]. Thus if we have the uniqueness of viscosity solutions of these equations, the lower and upper value functions are the same and thus the game has value. We now show the equivalence between the existence of saddle point equilibrium and DPP assuming the uniqueness of viscosity solutions of (3.7) and (3.8). Note that in order to have the uniqueness, we need more conditions and we refer to [3] for these details.

Theorem 3.3. *Assume (A1) and (A2) and that Eqs. (3.7) and (3.8) have unique viscosity solutions. Then the following are equivalent:*

- (i) *There exists a saddle point equilibrium in relaxed strategies;*
- (ii) *DPP holds, i.e., (3.1) and (3.2) are true.*

Proof. In view of Theorem 3.2, it suffices to prove that (ii) implies (i). We prove this for $t = 0$. The proof is analogous for any t . We have by continuity,

$$\inf_{\mu_2(\cdot) \in \mathcal{A}_2^0} \sup_{\mu_1(\cdot) \in \mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \min_{\mu_2(\cdot) \in \mathcal{A}_2^0} \max_{\mu_1(\cdot) \in \mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot))$$

and

$$\sup_{\mu_1(\cdot) \in \mathcal{A}_1^0} \inf_{\mu_2(\cdot) \in \mathcal{A}_2^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \max_{\mu_1(\cdot) \in \mathcal{A}_1^0} \min_{\mu_2(\cdot) \in \mathcal{A}_2^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)).$$

By the uniqueness assumption and Fan's minimax theorem, $V^+(t, x) = V^-(t, x) = V(t, x)$. Hence

$$\min_{\mu_2(\cdot) \in \mathcal{A}_2^0} \max_{\mu_1(\cdot) \in \mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \max_{\mu_1(\cdot) \in \mathcal{A}_1^0} \min_{\mu_2(\cdot) \in \mathcal{A}_2^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)).$$

Choose $(\mu_1^*(\cdot), \mu_2^*(\cdot)) \in \mathcal{A}_1^0 \times \mathcal{A}_2^0$ such that

$$\min_{\mu_2(\cdot) \in \mathcal{A}_2^0} \max_{\mu_1(\cdot) \in \mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \max_{\mu_1(\cdot) \in \mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2^*(\cdot))$$

and

$$\max_{\mu_1(\cdot) \in \mathcal{A}_1^0} \min_{\mu_2(\cdot) \in \mathcal{A}_2^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \min_{\mu_2(\cdot) \in \mathcal{A}_2^0} R(0, x, \mu_1^*(\cdot), \mu_2(\cdot)).$$

Clearly $(\mu_1^*(\cdot), \mu_2^*(\cdot))$ is a pair of saddle point strategies for $(0, x)$. \square

In the next theorem, we prove the existence of a saddle point equilibrium in a special case. We make the following assumption:

(A4) Let \bar{b}, \bar{r} be independent of x and let g be a bounded linear functional on H .

Theorem 3.4. Assume (A1), (A2), and (A4). Then there exists a saddle point equilibrium in (open loop) relaxed strategies.

Proof. For a fixed $x \in H$, $\mu_1(\cdot) \in \mathcal{A}_1^0$, the map

$$\mu_2(\cdot) \mapsto R(0, x, \mu_1(\cdot), \mu_2(\cdot))$$

is continuous in weak* topology. Similarly the map

$$\mu_1(\cdot) \mapsto R(0, x, \mu_1(\cdot), \mu_2(\cdot))$$

is continuous in weak* topology. Now under (A4), it is easy to see that the sets

$$\begin{aligned} &\{\mu_2(\cdot) \in \mathcal{A}_2^0: R(0, x, \bar{\mu}_1(\cdot), \mu_2(\cdot)) \geq l\}, \\ &\{\mu_1(\cdot) \in \mathcal{A}_1^0: R(0, x, \mu_1(\cdot), \bar{\mu}_2(\cdot)) \geq l\} \end{aligned}$$

are convex for all $l \in \mathbb{R}$, $\bar{\mu}_1(\cdot) \in \mathcal{A}_1^0$, $\bar{\mu}_2(\cdot) \in \mathcal{A}_2^0$. Hence by Fan's minimax theorem [5], the desiring result follows. \square

We now prove sufficient condition under some smoothness assumptions.

Theorem 3.5. Assume (A1)–(A3) and let A be generator of analytic semigroup. Let DPP hold. Suppose the equation

$$\begin{aligned} & v_t(t, x) - \langle A(t)x, v_x(t, x) \rangle_{H, H^*} + \inf_{\mu_2 \in \mathcal{M}_2} \sup_{\mu_1 \in \mathcal{M}_1} G(t, x, v_x(t, x), \mu_1, \mu_2) \\ &= v_t(t, x) - \langle A(t)x, v_x(t, x) \rangle_{H, H^*} + \sup_{\mu_1 \in \mathcal{M}_1} \inf_{\mu_2 \in \mathcal{M}_2} G(t, x, v_x(t, x), \mu_1, \mu_2) = 0 \end{aligned} \tag{3.9}$$

has a bounded smooth solution satisfying

$$v(T, x) = g(x). \tag{3.10}$$

(i) Suppose $\mu_1^*(\cdot) \in \mathcal{A}_1^0$ is such that for any $\mu_2(\cdot) \in \mathcal{A}_2^0$, if $X^*(\cdot)$ denotes the corresponding state process with $X^*(0) = x$ and if

$$\begin{aligned} & \max_{\mu_1 \in \mathcal{M}_1} \min_{\mu_2 \in \mathcal{M}_2} G(t, X^*(t), W_x(t, X^*(t)), \mu_1, \mu_2) \\ &= \min_{\mu_2 \in \mathcal{M}_2} G(t, X^*(t), W_x(t, X^*(t)), \mu_1^*(t), \mu_2) \end{aligned} \tag{3.11}$$

for a.e. t , then $\mu_1^*(\cdot)$ is optimal for player 1 for $(0, x)$.

(ii) Suppose $\mu_2^*(\cdot) \in \mathcal{A}_2^0$ is such that for any $\mu_1(\cdot) \in \mathcal{A}_1^0$, if $X^*(\cdot)$ denotes the corresponding state process with $X^*(0) = x$ and if

$$\begin{aligned} & \min_{\mu_2 \in \mathcal{M}_2} \max_{\mu_1 \in \mathcal{M}_1} G(t, X^*(t), W_x(t, X^*(t)), \mu_1, \mu_2) \\ &= \max_{\mu_1 \in \mathcal{M}_1} G(t, X^*(t), W_x(t, X^*(t)), \mu_1, \mu_2^*(t)) \end{aligned} \tag{3.12}$$

for a.e. t , then $\mu_2^*(\cdot)$ is optimal for player 2 for $(0, x)$.

Proof. We prove only part (i). Part (ii) can be proved in an analogous way. First note that by Proposition 6.3 in [8], a classical solution of (3.8) is a viscosity solution of (3.8). From the uniqueness of viscosity solution, we have $W = V$. Now $X^*(t)$ is Lipschitz on $[\varepsilon, T]$ for any $\varepsilon > 0$ [9]. Thus for any $\varepsilon > 0$, we have for t a.e. in $[\varepsilon, T]$,

$$\begin{aligned} \frac{d}{dt} V(t, X^*(t)) &= V_t(t, X^*(t)) - \langle A(t)X^*(t), V_x(t, X^*(t)) \rangle_{H, H^*} \\ &\quad + b(t, X^*(t), \mu_1^*(t), \mu_2(t)) V_x(t, X^*(t)) \\ &= V_t(t, X^*(t)) + G(t, X^*(t), V_x(t, X^*(t)), \mu_1^*(t), \mu_2(t)) \\ &\quad - r(t, X^*(t), \mu_1^*(t), \mu_2(t)) \\ &\geq V_t(t, X^*(t)) + \min_{\mu_2 \in \mathcal{M}_2} G(t, X^*(t), V_x(t, X^*(t)), \mu_1^*(t), \mu_2) \\ &\quad - r(t, X^*(t), \mu_1^*(t), \mu_2(t)) \\ &= V_t(t, X^*(t)) + \max_{\mu_1 \in \mathcal{M}_1} \min_{\mu_2 \in \mathcal{M}_2} G(t, X^*(t), V_x(t, X^*(t)), \mu_1, \mu_2) \\ &\quad - r(t, X^*(t), \mu_1^*(t), \mu_2(t)) \\ &= -r(t, X^*(t), \mu_1^*(t), \mu_2(t)). \end{aligned} \tag{3.13}$$

Integrating (3.13) from ε to T and rearranging, we have

$$R(\varepsilon, X(\varepsilon), \mu_1^*(\cdot), \mu_2(\cdot)) \geq V(\varepsilon, X(\varepsilon)).$$

Now letting $\varepsilon \rightarrow 0$, we get

$$R(0, x, \mu_1^*(\cdot), \mu_2(\cdot)) \geq V(0, x).$$

Thus $\mu_1^*(\cdot)$ is optimal for player 1 for $(0, x)$. \square

4. Conclusions

We have studied a differential game of fixed duration where the state equation is governed by a semilinear controlled evolution equation in a separable Hilbert space. We have established necessary conditions for optimality by proving a minimax theorem. We have established the equivalence between dynamic programming principle and existence of a saddle point equilibrium. Finally we have derived some sufficient conditions for optimality. Throughout our paper, we have assumed that the operator A occurring in the state equation is time independent. We would like to point out that the minimax principle proved in Section 2 can be extended in a routine manner even if the operator A has time dependence. Viscosity solutions, however, run into difficulties if A is dependent on time. Thus a result of Section 3, viz., the dynamic programming principle implies the existence of saddle point equilibrium, established via viscosity solutions, cannot easily be extended to the case of time dependent A . This needs further investigation.

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