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Exact Controllability of a Semilinear Thermoelastic System

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ABSTRACT

In this paper, we obtain the exact controllability of a semi-linear thermo-elastic system described by the partial differential equations:

$$\begin{cases} \omega_{tt} - \gamma\Delta\omega_{tt} + \Delta^2\omega + \alpha\Delta\theta + f(\theta) = 0 & \text{in } \Omega_T \\ \theta_t - \Delta\theta + \sigma\theta - \alpha\Delta\omega_t + g(\omega) = u & \text{in } \Omega_T \end{cases}$$

with Dirichlet boundary conditions when the Lipschitz constants of the non-linearities f and g are small.

Key Words: Exact controllability; Thermo-elastic system; Fixed point theorem.

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1. INTRODUCTION

In this article, we are concerned with the study of exact controllability of a semi-linear thermo-elastic system described by the partial differential equations:

$$\begin{cases} \omega_{tt} - \gamma\Delta\omega_{tt} + \Delta^2\omega + \alpha\Delta\theta + f(\theta) = 0 & \text{in } \Omega_T \\ \theta_t - \Delta\theta + \sigma\theta - \alpha\Delta\ \omega_t + g(\omega) = u & \text{in } \Omega_T \end{cases} \tag{1.1}$$

with the boundary and initial conditions:

$$\begin{cases} \omega = \dfrac{\partial\omega}{\partial\nu} = 0, \quad \theta = 0 & \text{on } \Gamma_T \\ \omega(0) = \omega_0, \quad \omega_t(0) = \omega_1, \quad \theta(0) = \theta_0 & \text{on } \Omega. \end{cases} \tag{1.2}$$

Here, u is the control function which acts only on the second equation in (1.1) satisfied by the thermal component θ and Ω is a bounded open subset of $\mathbb{R}^2$ with smooth boundary Γ and $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T = \Gamma \times (0, T)$, where $T > 0$ is the terminal time at which we would like to achieve the controllability and $\alpha > 0$, $\gamma \geq 0$, $\sigma \geq 0$ are the given parameters. The non-linearity was introduced through the non-linear functions f and g.

The exact controllability question is as follows: given initial data $\{\omega_0, \omega_1, \theta_0\}$ and terminal data $\{\omega_0^T, \omega_1^T, \theta_0^T\}$ at desired time T, in a suitable space (known as *controllability space*) does there exist a control u in an appropriate space (*control space*), so that the solution $\{\omega, \theta\}$ of the system (1.1), (1.2) satisfies $\omega(T) = \omega_0^T$, $\omega_t(T) = \omega_1^T$, $\theta(T) = \theta_0^T$.

The linear system corresponding to (1.1), (1.2), (that is, $f \equiv 0, g \equiv 0$) has been recently studied by Avalos (2000) and has the following theorem.

Theorem 1.1 (Avalos, 2000). *For all $\gamma \geq 0$, the linear system corresponding to (1.1)(1.2), that is, $f \equiv 0$, $g \equiv 0$ is exactly controllable for any arbitrary time $T > 0$ with the controllable space $H_0^2(\Omega) \times H_{0,\gamma}^1(\Omega) \times L^2(\Omega)$ and control space $L^2(0, T : H^{-1}(\Omega))$.*

Here and in the sequel we use, the standard Sobolev spaces and

$$H_{0,\gamma}^1 = \begin{cases} H_0^1(\Omega) & \text{if } \gamma > 0 \\ L^2(\Omega) & \text{if } \gamma = 0. \end{cases}$$

The inner product in $H_{0,\gamma}^1$ is given by

$$(\omega_1, \omega_2)_{H_{0,\gamma}^1} = (\omega_1, \omega_2)_{L^2} + \gamma(\nabla\omega_1, \nabla\omega_2)_{L^2}.$$

Define $H_\gamma := H_0^2 \times H_{0,\gamma}^1 \times L^2(\Omega)$ which is a Hilbert space with the standard norm.

The linear system without the control is a PDE model which describes a Kirchhoff plate subjected to a thermal damping (see Lagnese, 1989). Here ω is the displacement and θ is the temperature of the plate.

There is a vast amount of literature with various types of controllability results for the linear system with the controls acting either as distributed controls or through the boundary. We refer the reader to Avalos (2000) and the articles by Lagnese (1990), Avalos and Lasiecka (1998a,b), de Teresa and Zuazua (1996), Hansen and Zhang (1997), Lasiecka and Triggiani (1991, 1998a,b).

Regarding other semi-linear and linear systems, we refer the reader to the article by Nandakumaran and George (1995a,b) and there are more literature in these directions which we skip here.

In this paper, we wish to prove exact controllability results for the semilinear system under various assumptions on f and g and using the fact that the linear system is exactly controllable due to Avalos (2000).

Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be Lipschitz continuous with Lipschitz constants β_1 and β_2 respectively, that is,

$$|f(x) - f(y)| \le \beta_1 |x - y|, \quad \forall x,\ y \in \mathbb{R} \tag{1.3}$$

$$|g(x) - g(y)| \le \beta_2 |x - y|, \quad \forall x,\ y \in \mathbb{R} \tag{1.4}$$

$$f(0) = g(0) = 0. \tag{1.5}$$

It is easy to see that, for any $v \in L^2(\Omega)$, the composition functions fov and gov are in $L^2(\Omega)$. Moreover, $\|fov\|_{L^2} \le \beta_1 \|v\|_{L^2}$ and $\|gov\|_{L^2} \le \beta_2 \|v\|_{L^2}$.

We have the following result regarding the existence and uniqueness of solution of (1.1)–(1.2).

Theorem 1.2 (Existence and Uniqueness). *Suppose that $\gamma \ge 0$ and linear functions f, g satisfy* (1.3)–(1.5) *and $u \in L^2(0, T, H^{-1}(\Omega))$, be a given function. Let the initial data $\{\omega_0, \omega_1, \theta) \in H_\gamma$. Then the non-linear system* (1.1)–(1.2) *has a unique solution (ω, θ) such that $(\omega, \omega_t, \theta) \in C([0, T], H_\gamma)$.*

We are unable to prove the exact controllability without further assumptions on the nonlinearities. We now state two controllability results.

Theorem 1.3. *Suppose that f and g satisfy* (1.3)–(1.4) *and further assume that the Lipschitz constants β_1 and β_2 are sufficiently small. Then the system* (1.1)–(1.2) *is exactly controllable with the controllable space H_γ and with a control $u \in L^2(0, T : H^{-1}(\Omega))$.*

Remark 1.4. In the context of the above theorem, one can view the semi-linear system (1.1)–(1.2) as a non-linear perturbation of the linear system. This is the motivation of the present article.

We can relax the assumption that β_1, β_2 are small, but with a boundedness condition on f and g and is given in the following theorem.

Theorem 1.5. *Let $\gamma > 0$. Suppose f and g satisfy* (1.3)–(1.4) *and assume that f and g are uniformly bounded i.e., there is a constant $M > 0$ such that*

$$|f(x)| \leq M, \quad |g(x)| \leq M, \ \forall x \in \mathbb{R}.$$

Further, assume that $f(\theta_n) \rightharpoonup f(\theta)$ weakly in $L^2(\Omega)$, whenever $\theta_n \rightharpoonup \theta$ weakly in $L^2(\Omega)$. Then the system (1.1)–(1.2) *is exactly controllable as in Theorem* 1.3.

Remark 1.6. The above theorem is also true for the case $\gamma = 0$, but then one needs the assumption that $f(\theta_n) \to f(\theta)$ strongly in $L^2(\Omega)$, whenever $\theta_n \rightharpoonup \theta$ weakly in $L^2(\Omega)$. This is not reasonable to expect for any class of non-linear functions f.

We organize the paper as follows. In Sec. 2, we present the operator formulation of the problem (1.1)–(1.2) as in Avalos and prove Theorem 1.2. In Secs. 3 and 4, we prove Theorems 1.3 and 1.5, respectively. The basic techniques of the proofs are the Fixed Point Method and Monotone Operator Theory.

2. EXISTENCE AND UNIQUENESS

We use the same notations and definitions as in Avalos (2000). Let $\mathring{A} = \triangle^2$ and $A_D = -\triangle$ be the linear operators on $L^2(\Omega) \longrightarrow L^2(\Omega)$ with the domains given by

$$D(\mathring{A}) = H^4(\Omega) \cap H_0^2(\Omega) \quad \text{and} \quad D(A_D) = H^2(\Omega) \cap H_0^1(\Omega).$$

Let $P_\gamma : H_{0,\gamma}^1 \longrightarrow H_{0,\gamma}^{-1}$ be the elliptic operator given by $P_\gamma = I + \gamma A_D$ with Dirichelet boundary condition for $\gamma > 0$ which has a bounded inverse P_γ^{-1}; i.e., $P_\gamma^{-1} \in \mathscr{L}(H_{0,\gamma}^{-1}, H_{0,\gamma}^1)$. Let $A_\gamma : D(A_\gamma) \subset H_\gamma \longrightarrow H_\gamma$ be a bounded linear operator defined by

$$A_\gamma = \begin{pmatrix} 0 & I & 0 \\ -P_\gamma^{-1}\mathring{A} & 0 & \alpha P_\gamma^{-1} A_D \\ 0 & -\alpha A_D & -A_D - \sigma I \end{pmatrix}$$

with

$$D(A_\gamma) = \{(\omega_0, \omega_1, \theta_0) \in H_0^2 \times H_0^2 \times D(A_D) : \mathring{A}\omega_0 \in H_{0,\gamma}^{-1}\}.$$

Then system (1.1), (1.2) can be written as

$$\frac{d}{dt}\begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix} = A_\gamma \begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix} + Bu + F\begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix}. \tag{2.1}$$

Here, $B : H^{-1}(\Omega) \longrightarrow H_0^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$ is the control operator (bounded linear) defined by

$$Bu = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \tag{2.2}$$

i.e., $B \in \mathscr{L}(H^{-1}, H_0^2 \times L^2 \times H^{-1})$. Further $F : H_\gamma \longrightarrow H_\gamma$ is a non-linear operator given by

$$F\begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ -P_\gamma^{-1} f(\theta) \\ -g(\omega) \end{pmatrix}. \tag{2.3}$$

Following the definition of P_γ, we get $P_\gamma^{-1} f(\theta) \in H_{0,\gamma}^1$ for any $\theta \in L^2(\Omega)$. In fact, we have $\|P_\gamma^{-1} f(\theta)\|_{H_{0,\gamma}^1} \leq \|f(\theta)\|_{L^2}$. More generally, we have for any $\theta, \tilde{\theta} \in L^2(\Omega)$,

$$\|P_\gamma^{-1} f(\theta) - P_\gamma^{-1} f(\tilde{\theta})\|_{H_{0,\gamma}^1} \leq \|f(\theta) - f(\tilde{\theta})\|_{L^2} \leq \beta_1 \|\theta - \tilde{\theta}\|_{L^2}.$$

Similarly,

$$\|g(\omega) - g(\tilde{\omega})\|_{L^2} \leq \beta_2 \|\omega - \tilde{\omega}\|_{L^2}.$$

Thus, we have the following lemma.

Lemma 2.1. *Let f and g satisfy the Lipschitz conditions as in* (1.3)–(1.4). *Then $F : H_\gamma \longrightarrow H_\gamma$ defined by* (2.3) *is Lipschitz continuous with constant* $\beta = \max\{\beta_1, \beta_2\}$.

As in Avalos (2000), Avalos and Lasiecka (1998a), for $\gamma \geq 0, A_\gamma$ generates a C_0-semi group of contractions, denoted by $\{e^{A_\gamma t}\}_{t\geq 0}$ on the Hilbert space H_γ. Hence, a mild solution of (2.1) can be represented by means of the non-linear integral equation:

$$\begin{pmatrix} \omega(t) \\ \omega_t(t) \\ \theta(t) \end{pmatrix} = e^{A_\gamma t}\begin{pmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{pmatrix} + \int_0^t e^{A_\gamma(t-s)} Bu(s)ds + \int_0^t e^{A_\gamma(t-s)} F\begin{pmatrix} \omega(s) \\ \omega_t(s) \\ \theta(s) \end{pmatrix} ds. \tag{2.4}$$

To study the solvability of (2.4), we introduce the operator $N : C([0, T]; H_\gamma) \longrightarrow C([0, T] : H_\gamma)$ defined by

$$N\begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix}(t) = e^{A_\gamma t}\begin{pmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{pmatrix} + \int_0^t e^{A_\gamma(t-s)} Bu(s) + \int_0^t e^{A_\gamma(t-s)} F\begin{pmatrix} \omega(s) \\ \eta(s) \\ \theta(s) \end{pmatrix} ds, \tag{2.5}$$

for a fixed $u \in L^2(0,T;H^{-1}(\Omega))$ and an initial condition

$$\begin{pmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{pmatrix} \in H_\gamma.$$

Here, one needs certain justification as apriori, it is not clear that the solution $(\omega(t), \omega_t(t), \theta(t)) \in H_\gamma$. This is due to the presence of the second term on the right in the above equations. Hence, initially we only have (Avalos, 2000) that the solution of the corresponding linear system $(\omega, \omega_t, \theta)$, is in $C([0,T];(D(A_\gamma^*))')$. Of course this solution is represented by the first two terms in the above equation. So, one must check the well-definedness of N. This is related to the regularity question and we briefly sketch some steps for the sake of completeness as it can be derived as in Avalos (2000) following Avalos and Lasiecka (1998a), Lagnese (1989), Zabczyk (1992). Denote

$$\mathscr{L}_T(u) = \int_0^T e^{A_\gamma(T-s)} Bu(s)ds.$$

We only need to show that $\mathscr{L}_T$ is a bounded linear operator from $L^2(0,T;H^{-1}(\Omega)) \longrightarrow H_\gamma$ as the nonlinear term causes no problem. *A priori*, we have

$$\mathscr{L}_T \in \mathscr{L}(L^2(0,T;H^{-1}(\Omega)); D(A_\gamma^*)'),$$

where $(D(A_\gamma^*))' \supset H_\gamma$. This shows that $\mathscr{L}_T^* \in \mathscr{L}(D(A_\gamma^*); L^2(0,T;H_0^1(\Omega)))$. The interesting point is that one can explicitly compute the adjoint. One has

$$\mathscr{L}_T^* \begin{pmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{pmatrix} = B^* e^{A_\gamma^*(T-.)} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{pmatrix}.$$

A straight forward computation yields

$$A_\gamma^* = \begin{pmatrix} 0 & -I & 0 \\ P_\gamma^{-1}\mathring{A} & 0 & -\alpha P_\gamma^{-1} A_D \\ 0 & \alpha A_D & -A_D - \sigma I \end{pmatrix},$$

and $D(A_\gamma^*) = D(A_\gamma)$. With a further computation of B^*, one obtains that

$$\mathscr{L}_T^* \begin{pmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{pmatrix}(t) = \psi(t),$$

where $\psi(t)$ is obtained by solving the adjoint linear system

$$\begin{cases} \phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi + \alpha\Delta\psi = 0 & \text{in } \Omega_T \\ \psi_t + \Delta\psi - \sigma\psi - \alpha\Delta\phi_t = 0 & \text{in } \Omega_T \end{cases}$$

with the boundary and terminal conditions:

$$\begin{cases} \phi = \dfrac{\partial \phi}{\partial \nu} = 0, \quad \psi = 0 & \text{on } \Gamma_T \\ \phi(T) = \phi_0, \quad \phi_t(T) = \phi_1, \quad \psi(T) = \psi_0 & \text{on } \Omega. \end{cases}$$

One has the following proposition (see Avalos, 2000; Avalos and Lasiecka, 1998a; Lagnese, 1989).

Proposition 2.2. *The component ψ of the solution to the above adjoint system satisfies $\psi \in L^2(0,\infty; H_0^1(\Omega))$. In fact, the following estimates hold for all $(\phi_0, \phi_1, \psi_0) \in H_\gamma$ and for all $0 \le t \le T$;*

$$E_\gamma(t) + \int_t^T [\| \nabla \psi(\tau)\|_2^2 + \|\psi(\tau)\|_2^2] d\tau = E_\gamma(T),$$

where

$$E_\gamma(t) = \frac{1}{2} \left\| e^{A_\gamma^*(T-t)} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{pmatrix} \right\|_{H_\gamma}^2 .$$

The above proposition shows that $\mathscr{L}_T^*$ can be extended as a bounded linear operator from H_γ into $L^2(0,T;H_0^1(\Omega))$. Thus by duality $\mathscr{L}_T$ is a bounded linear operator from $L^2(0,T;H^{-1}(\Omega))$ into H_γ. The above argument shows that we indeed have

$$N \begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix} (t) \in H_\gamma.$$

Proof of Theorem 1.2. **Step 1.** We prove that N^k is a contraction for some integer $k > 0$. Since $e^{A_\gamma t}$ is contraction and F is Lipschitz, it follows that

$$\begin{aligned} \left\| N \begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix} (t) - N \begin{pmatrix} \tilde{\omega} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix} (t) \right\|_{H_\gamma}^2 &\le \beta^2 \int_0^t \left\| \begin{pmatrix} \omega(s) \\ \eta(s) \\ \theta(s) \end{pmatrix} - \begin{pmatrix} \tilde{\omega}(s) \\ \tilde{\eta}(s) \\ \tilde{\theta}(s) \end{pmatrix} \right\|_{H_\gamma}^2 ds \\ &\le \beta^2 t \left\| \begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix} - \begin{pmatrix} \tilde{\omega} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix} \right\|_{C([0,T],H_\gamma)}^2 . \end{aligned} \tag{2.6}$$

Iterating (2.6) once again, we get

$$\left\| N^2 \begin{pmatrix} c\omega \\ \eta \\ \theta \end{pmatrix}(t) - N^2 \begin{pmatrix} \tilde{\omega} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix}(t) \right\|_{H_\gamma}^2$$

$$\leq \beta^2 \int_0^t \left\| N \begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix}(s) - N \begin{pmatrix} \tilde{\omega} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix}(s) \right\|_{H_\gamma}^2 ds$$

$$\leq \beta^2 \int_0^t \beta^2 s \left\| \begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix} - \begin{pmatrix} \tilde{\omega} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix} \right\|_{C([0,T],H_\gamma)}^2$$

$$= \frac{\beta^4 t^2}{2} \left\| \begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix} - \begin{pmatrix} \tilde{\omega} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix} \right\|_{C([0,T],H_\gamma)}^2 .$$

Repeating similarly, we can see that

$$\left\| N^k \begin{pmatrix} c\omega \\ \eta \\ \theta \end{pmatrix} - N^k \begin{pmatrix} \tilde{\omega} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix} \right\|_{C([0,T],H_\gamma)} \leq \frac{\beta^{2k} T^k}{k!} \left\| \begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix} - \begin{pmatrix} \tilde{\omega} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix} \right\|_{C([0,T],H_\gamma)} .$$

For sufficiently large k, we can make the constant $(\beta^2 T)^k/k!$ less than 1, which proves that N^k is a contraction.

Step 2. By the generalized Banach Contraction Principle (Joshi and Bose, 1984), it follows that N has a unique fixed point $(\omega, \eta, \theta) \in C([0,T] : H_\gamma)$. Then from the system of equations (2.1) (first equation), we get $\eta = \omega_t$. Hence the theorem. ∎

Remark 2.3. Observe that the existence and uniqueness do not require the smallness of the Lipschitz constants β_1, β_2.

Following Theorem (1.2), we define a solution operator $S : L^2(0, T : H^{-1}(\Omega)) \longrightarrow L^2(0, T : H_\gamma)$ by

$$Su = \begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix}, \tag{2.7}$$

where $(\omega, \omega_t, \theta)$ is the unique solution of (2.1) corresponding to the control function u.

Lemma 2.4. *Under the assumptions* (1.3)–(1.4), *the solution operator* S *is well-defined and Lipschitz continuous.*

Proof. Well-definedness of S follows from the existence and uniqueness theorem. Now, let $u, \tilde{u} \in L^2(0, T; H^{-1}(\Omega))$ and $(\omega, \omega_t, \theta), (\tilde{\omega}, \tilde{\omega}_t, \tilde{\theta})$ be the corresponding solutions. Since $e^{A_\gamma t}$ is a contraction, it follows from the integral representation (2.4), that

$$\begin{aligned}
\|Su(t) - S\tilde{u}(t)\|_{H_\gamma} &\leq \int_0^t e^{A_\gamma(t-s)} B(u(s) - \tilde{u}(s))ds \\
&\quad + \int_0^t \left\| F\begin{pmatrix} \omega(s) \\ \omega_t(s) \\ \theta(s) \end{pmatrix} - F\begin{pmatrix} \tilde{\omega}(s) \\ \tilde{\omega}(s) \\ \theta(s) \end{pmatrix} \right\|_{H_\gamma} ds \\
&\leq \|B\| \int_0^t \|u(s) - \tilde{u}(s)\|_{H^{-1}(\Omega)} \\
&\quad + \beta \int_0^t \left\| \begin{pmatrix} \omega(s) \\ \omega_t(s) \\ \theta(s) \end{pmatrix} - \begin{pmatrix} \tilde{\omega}(s) \\ \tilde{\omega}(s) \\ \theta(s) \end{pmatrix} \right\|_{H_\gamma} ds \\
&\leq \|B\| \int_0^t \|u(s) - \tilde{u}(s)\|_{H^{-1}(\Omega)} + \beta \int_0^t \|Su(s) - S\tilde{u}(s)\|_{H_\gamma} ds.
\end{aligned}$$

Hence, from the Grownwall's inequality, we get

$$\begin{aligned}
\|Su(t) - S\tilde{u}(t)\|_{H_\gamma} &\leq \|B\| e^{\beta t} \int_0^t \|u(s) - \tilde{u}(s)\|_{H^{-1}(\Omega)}\, ds \\
&\leq \|B\| e^{\beta T} \|u - \tilde{u}\|_{L^2(0,T;H^{-1}(\Omega))} \cdot t^{1/2}
\end{aligned}$$

Thus, by squaring and integrating, we get

$$\|Su - S\tilde{u}\|_{L^2(0,T;H_\gamma)} \leq \frac{T}{\sqrt{2}} \|B\| e^{\beta T} \|u - \tilde{u}\|_{L^2(0,T;H^{-1}(\Omega))}.$$

Hence, S is Lipschitz continuous. ■

3. PROOF OF THEOREM 1.3

From Eq. (2.4), it follows that the system (1.1), (1.2) is exactly controllable in H_γ if and only if, for given any initial data $\{\omega_0, \omega_1, \theta_0\}$, and final state $\{\omega_0^T, \omega_1^T, \theta_0^T\}$ in H_γ, there exists a control $u \in L^2(0, T; H^{-1}(\Omega))$ such that

$$\begin{pmatrix} \omega_0^T \\ \omega_1^T \\ \theta_0^T \end{pmatrix} = N \begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix} (T), \tag{3.1}$$

where, N is defined as in (2.5). Here, $(\omega, \omega_t, \theta)$ is the unique solution of the system (1.1), (1.2) corresponding to a control u, i.e., $(\omega, \omega_t, \theta) = Su$.

We may use the following notations and definitions. Let

$$h = \begin{pmatrix} \omega_0^T \\ \omega_1^T \\ \theta_0^T \end{pmatrix} - e^{A_\gamma T} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{pmatrix}. \tag{3.2}$$

Let $\mathscr{L}_T : L^2(0, T; H^{-1}(\Omega)) \longrightarrow H_\gamma$ be the linear operator defined by

$$\mathscr{L}_T u = \int_0^T e^{A_\gamma (T-s)} Bu(s) ds. \tag{3.3}$$

Let $K : L^2(0, T; H^{-1}(\Omega)) \longrightarrow H_\gamma$ be the non-linear operator given by

$$Ku = \int_0^T e^{A_\gamma (T-s)} F((Su)(s)) ds. \tag{3.4}$$

With these notations, (3.1) can be written as

$$h = \mathscr{L}_T u + Ku. \tag{3.5}$$

It was shown in Avalos (2000) that $\mathscr{L}_T$ is a bounded linear operator. The controllability of the linear system is equivalent to the onto-ness of $\mathscr{L}_T$. This was shown by proving an observability inequality for the adjoint operator $\mathscr{L}_T^* \in \mathscr{L}(H_\gamma; L^2(0, T; H_0^1(\Omega)))$,

$$\|\mathscr{L}_T^* v\| \geq C_T \|v\|, \quad \forall v \in H_\gamma, \tag{3.6}$$

for some $C_T > 0$. Thus, $\mathscr{L}_T^*$ is $1 - 1$. We, now, look for a control u in (3.5) of the form

$$u = \mathscr{L}_T^* v$$

for some $v \in H_\gamma$. Then (3.5) becomes

$$h = \mathscr{L}_T \mathscr{L}_T^* v + K \mathscr{L}_T^* v$$

That is,

$$h = \mathscr{R} v,$$

where, $\mathscr{R} = \mathscr{L}_T \mathscr{L}_T^* + K \mathscr{L}_T^*$ is an operator from $H_\gamma \longrightarrow H_\gamma$. Now

$$\begin{aligned} \langle \mathscr{R} v, v \rangle &= \langle \mathscr{L}_T \mathscr{L}_T^* v, v \rangle + \langle K \mathscr{L}_T^* v, v \rangle \\ &= \langle \mathscr{L}_T^* v, \mathscr{L}_T^* v \rangle + \langle K \mathscr{L}_T^* v, v \rangle. \end{aligned} \tag{3.7}$$

Claim. The operator $\mathscr{R}$ is strongly monotone if β_1, β_2 are sufficiently small.

Assuming the claim is true, the proof of Theorem 1.3 is complete because, then $\mathscr{R}$ is onto (see Joshi and Bose, 1984).

Proof of the Claim. First we see that K is Lipschitz continuous. For,

$$\begin{aligned}\|Ku - K\tilde{u}\|_{H_\gamma}^2 &\leq \beta\|Su - S\tilde{u}\|_{L^2(0,T;H_\gamma)} \\ &\leq \beta e^{\beta T}\|B\|\frac{T}{\sqrt{2}}\|u - \tilde{u}\|, \end{aligned} \tag{3.8}$$

thanks to the Lipschitz continuity of S.

Now,

$$\langle Rv - R\tilde{v}, v - \tilde{v}\rangle = \langle \mathscr{L}_T^*(v - \tilde{v}), \mathscr{L}_T^*(v - \tilde{v})\rangle + \langle K\mathscr{L}_T^* v - K\mathscr{L}_T^*\tilde{v}, v - \tilde{v}\rangle.$$

We estimate the second term as:

$$|\langle K\mathscr{L}_T^* v - K\mathscr{L}_T^*\tilde{v}, v - \tilde{v}\rangle| \leq \delta\|\mathscr{L}_T^* v - \mathscr{L}_T^*\tilde{v}\|\|v - \tilde{v}\| \leq \delta\|\mathscr{L}_T^*\|\|v - \tilde{v}\|^2,$$

where,

$$\delta = \beta e^{\beta T}\|B\|\frac{T}{\sqrt{2}}.$$

Therefore, from the observability inequality (3.6), it follows that

$$\langle \mathscr{R}v - \mathscr{R}\tilde{v}, v - \tilde{v}\rangle \geq (C_T - \delta\|\mathscr{L}_T^*\|)\|v - \tilde{v}\|^2.$$

Thus, $\mathscr{R}$ is strongly monotone if $C_T - \delta\|\mathscr{L}_T^*\| > 0$ which can be achieved if we choose δ small enough, i.e., β is small enough. Hence the claim and the theorem. ■

An Alternate Approach to Theorem 1.3. In the earlier approach, we have transformed the problem to the solvability of an operator equation for the control. As a complement to the above approach, one can introduce an explicit control and then the problem will become a fixed point problem for the solution.

It follows from the controllability of the linear system that the operator $\mathscr{L}_T$ has a generalized inverse. In fact, due to the onto-ness of $\mathscr{L}_T$, we see that $(\mathscr{L}_T\mathscr{L}_T^*)^{-1}$ is a bounded linear operator by open mapping theorem. Thus, the generalized inverse is given by the bounded linear operator

$$W = \mathscr{L}_T^*(\mathscr{L}_T\mathscr{L}_T^*)^{-1}. \tag{3.9}$$

Define a control u implicitly as:

$$u = W\left\{ h - \int_0^T e^{A_\gamma(T-s)} F \begin{pmatrix} \omega(s) \\ \omega_t(s) \\ \theta(s) \end{pmatrix} ds \right\}, \tag{3.10}$$

where, $(\omega, \omega_t, \theta)$ is the solution of (2.4) corresponding to the control u and h is as in (3.2). Substituting this in (2.4), we get

$$\begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix}(t) = e^{A_\gamma t} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{pmatrix} + \int_0^t e^{A_\gamma(t-s)} BW \left\{ h - \int_0^T e^{A_\gamma(T-\tau)} F \begin{pmatrix} \omega(\tau) \\ \omega_t(\tau) \\ \theta(\tau) \end{pmatrix} d\tau \right\} ds$$
$$+ \int_0^t e^{A_\gamma(t-s)} F \begin{pmatrix} \omega(s) \\ \omega_t(s) \\ \theta(s) \end{pmatrix} ds. \tag{3.11}$$

Obviously, $(\omega(0), \omega_t(0), \theta(0)) = (\omega_0, \omega_1, \theta_0)$ and $(\omega(T), \omega_t(T), \theta(T)) = (\omega_0^T, \omega_1^T, \theta_0^T)$. Hence the non-linear system is exactly controllable with the control defined by (3.10) provided Eq. (3.11) has a solution.

We employ the method of fixed point theory. Define a nonlinear operator $K : C([0,T]; H_\gamma) \longrightarrow C([0,T]; H_\gamma)$, where

$$K \begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix}$$

represents the right hand side of (3.11) with

$$\begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix}$$

replaced by

$$\begin{pmatrix} \omega \\ \eta \\ \theta \end{pmatrix}.$$

It can be seen that K is a contraction provided β_1, β_2 are small enough. Then K has a fixed point and we can recover $\eta = \omega_t$. We skip the details.

4. PROOF OF THEOREM 1.5

As in the proof of Theorem 1.3 (second or alternate approach), we show that (3.11) has a solution without the smallness of β_1, β_2, but with uniform boundedness of f and g.

Let K be as above. It is easy to see, with the boundedness of f and g, that there exists a constant $M_1 > 0$ such that

$$\left\| K \begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix} \right\|_{C([0,T];H_\gamma)} \leq M_1.$$

Let $\mathscr{B}$ be the closed convex set defined by

$$\mathscr{B} = \left\{ \begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix} \in C([0,T];H_\gamma) : \left\| \begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix} \right\| \leq M_1 \right\}.$$

Then K maps $\mathscr{B}$ into itself. We have the following claim.

Claim. The family $\mathscr{F} = K\mathscr{B}$ is equi-continuous and, for each $t, \mathscr{F}_t = \{K(\mathbf{w})(t) : \mathbf{w} \in \mathscr{B}\}$ is pre-compact in H_γ

Proof. **Step 1** (Equi-continuity). Let $G = K(\mathbf{w})$ be any element in $\mathscr{F}$, where

$$\mathbf{w} = \begin{pmatrix} \omega \\ \omega_t \\ \theta \end{pmatrix} \in \mathscr{B}$$

be arbitrary. We have to estimate $\|K(\mathbf{w})(t) - K(\mathbf{w})(t^*)\|_{H_\gamma}$. Write

$$K(\mathbf{w})(t) - K(\mathbf{w})(t^*) = (e^{A_\gamma t} - e^{A_\gamma t^*})\mathbf{w}_0 + I_1 + I_2$$

where

$$I_1 = \int_0^T \left[e^{A_\gamma(t-s)}BB^* e^{A_\gamma^*(T-t)} - e^{A_\gamma(t^*-s)}BB^* e^{A_\gamma^*(T-t^*)}\right] \times (\mathscr{L}_T\mathscr{L}_T^*)^{-1}\left[\mathbf{w}_0^T - e^{A_\gamma T}\mathbf{w}_0 - \int_0^T e^{A_\gamma(T-\tau)}F(\mathbf{w}(\tau))d\tau\right]ds$$

and

$$I_2 = \int_0^t e^{A_\gamma(t-s)}F(\mathbf{w}(s))ds - \int_0^{t^*} e^{A_\gamma(t^*-s)}F(\mathbf{w}(s))ds.$$

The term I_1 can be estimated and can make arbitrarily small, independent of $\mathbf{w} \in \mathscr{B}$, when t^* in close to t. In fact, one can make precise estimates which we do not work out here. The second term I_2 can be written as, for $t > t^*$,

$$I_2 = \int_0^{t^*} \left(e^{A_\gamma(t-s)} - e^{A_\gamma(t^*-s)}\right)F(\mathbf{w}(s))ds + \int_{t^*}^t e^{A_\gamma(t-s)}F(\mathbf{w}(s))ds,$$

which can be made small as t^* goes to t, independent of $\mathbf{w}$ under the boundedness of f and g. Thus $\mathscr{F}$ is equi-continuous.

Step 2 (Pre-compactness). Let $t > 0$ be fixed and $\{K(\mathbf{w}^n)(t) : \mathbf{w}^n \in \mathscr{B}\}$ be a bounded sequence in H_γ. Since $\mathbf{w}^n \in \mathscr{B}, \{\mathbf{w}^n\}$ is a bounded sequence in $C([0, T]; H_\gamma)$. Thus, for any $\tau \in [0, T]$, the sequence $\{\mathbf{w}^n(\tau)\}$ is bounded in H_γ and hence it has a weakly convergent subsequence (represent by the same sequence) such that

$$\mathbf{w}^n(\tau) \rightharpoonup \mathbf{w}(\tau) \text{ in } H_\gamma \text{ weak.}$$

Now

$$\begin{aligned} \|F(\mathbf{w}^n)(\tau) - F(\mathbf{w}(\tau))\|^2_{H_\gamma} &= \left\| \begin{pmatrix} 0 \\ -P_\gamma^{-1}(f(\theta_n) - f(\theta))(\tau) \\ -(g(\omega_n) - g(\omega)(\tau)) \end{pmatrix} \right\|^2_{H_\gamma} \\ &\leq \|P_\gamma^{-1}(f(\theta_n) - f(\theta))\|^2_{H^1_{0,\gamma}} + \|g(\omega_n) - g(\omega)\|^2_{L^2} \end{aligned} \tag{4.1}$$

By Lipschitz continuity of g, we get

$$\|g(\omega_n(\tau)) - g(\omega(\tau))\|^2_{L^2} \leq C\|\omega_n(\tau) - \omega(\tau)\|^2_{L^2} \to 0,$$

as $n \to \infty$. Now for $\gamma > 0$,

$$\|P_\gamma^{-1}(f(\theta_n) - f(\theta))\|^2_{H^1_{0,\gamma}} \leq \|P_\gamma^{-1}(f(\theta_n) - f(\theta))\|^2_{H^1_0},$$

which goes to zero by assumption in the theorem as P_γ is then a uniformly elliptic operator for $\gamma > 0$.

Using bounded convergence theorem, it can be seen that

$$K(\mathbf{w}^n)(t) \longrightarrow K(\mathbf{w})(t) \text{ strongly in } H_\gamma.$$

This proves the pre-compactness and hence the claim.

From the claim, it follows that K is a compact operator and is a self map from $\mathscr{B} \longrightarrow \mathscr{B}$ (see Royden, 1995). Thus, from the Schauder's fixed point theorem (Joshi and Bose, 1984), K has a fixed point in $\mathscr{B}$ which is the solution of (3.11). Hence the theorem. ■

Remark 4.1. When $\gamma = 0$, we have P_γ is the identity operator on L^2. Hence, we have to show that $\|f(\theta_n(\tau)) - f(\theta(\tau))\|_{L^2}$ goes to zero, which is not reasonable for weakly convergent sequences θ_n. Hence, we are unable to substantiate the Theorem 1.5 when $\gamma = 0$.

Remark 4.2. Without the boundedness of f, g and smallness of β_1, β_2, the problem remains open.

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