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# Differential Games of Fixed Duration in the Framework of Relaxed Strategies*

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**Abstract**

We study a zero sum differential game of fixed duration in the framework of relaxed strategies. We prove a mini-max principle and establish the equivalence between the dynamic programming principle and the existence of a saddle point equilibrium. Finally we establish a connection between the mini-max and dynamic programming principles.

## 1. Introduction

We study differential games of fixed duration, where the state process $x(\cdot) \in \mathbb{R}^d$ evolves according to

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$$\dot{x}(t) = b(t, x(t), u_1(t), u_2(t)),\ t \in (0, T],\ x(0) = x, \tag{1.1}$$

where $b : [0, T] \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}^d$, $U_1$ and $U_2$ are action sets of players 1 and 2 respectively, $u_i : [0, T] \to U_i$ is the (open loop) strategy of player $i, i = 1, 2$. Player 1 tries to maximize his payoff

$$R(0, x, u_1(\cdot), u_2(\cdot)) := \int_0^T r(t, x(t), u_1(t), u_2(t))\ dt + g(x(T)) \tag{1.2}$$

over his strategies $u_1(\cdot)$, whereas player 2 tries to minimize the same over his strategies $u_2(\cdot)$. Here $r$ is the running payoff function and $g$ is the terminal payoff function. The differential game is said to have a value if

$$\inf_{u_2(\cdot)} \sup_{u_1(\cdot)} R(0, x, u_1(\cdot), u_2(\cdot)) = \sup_{u_1(\cdot)} \inf_{u_2(\cdot)} R(0, x, u_1(\cdot), u_2(\cdot)).$$

A strategy $u_1^*(\cdot)$ is said to be optimal for player 1 if

$$R(0, x, u_1^*(\cdot), \tilde{u}_2(\cdot)) \geq \inf_{u_2(\cdot)} \sup_{u_1(\cdot)} R(0, x, u_1(\cdot), u_2(\cdot))$$

for any strategy $\tilde{u}_2(\cdot)$ of player 2. Similarly, a strategy $u_2^*(\cdot)$ is said to be optimal for player 2 if

$$R(0, x, \tilde{u}_1(\cdot), u_2^*(\cdot)) \leq \sup_{u_1(\cdot)} \inf_{u_2(\cdot)} R(0, x, u_1(\cdot), u_2(\cdot))$$

for any strategy $\tilde{u}_1(\cdot)$ of player 1. A pair $(u_1^*(\cdot), u_2^*(\cdot))$ of optimal strategies for both players satisfies

$$R(0, x, u_1(\cdot), u_2^*(\cdot)) \leq R(0, x, u_1^*(\cdot), u_2^*(\cdot)) \leq R(0, x, u_1^*(\cdot), u_2(\cdot))$$

for any $u_1(\cdot), u_2(\cdot)$. Thus $(u_1^*(\cdot), u_2^*(\cdot))$ constitutes a saddle point equilibrium.

For $(t, x) \in [0, T] \times \mathbb{R}^d$, define

$$V^+(t, x) := \inf_{u_2(\cdot)} \sup_{u_1(\cdot)} \left[ \int_t^T r(s, x(s), u_1(s), u_2(s))\ ds + g(x(T)) \right],$$

and

$$V^-(t,x) := \sup_{u_1(\cdot)} \inf_{u_2(\cdot)} \left[ \int_t^T r(s, x(s), u_1(s), u_2(s))ds + g(x(T)) \right]$$

where $x(\cdot)$ satisfies the differential equation (1.1) with the initial condition $x(t) = x$. The functions $V^+, V^-$ are called upper and lower value functions respectively.

The study of differential games was initiated by Isaacs [16]. Using many examples and some formal arguments, he showed that $V^+$ satisfies the equation

$$V_t^+(t,x) + \sup_{u_1} \inf_{u_2} \left[ \nabla_x V^+(t,x) \cdot b(t,x,u_1,u_2) + r(t,x,u_1,u_2) \right] = 0. \quad (1.3)$$

Similarly, $V^-$ satisfies

$$V_t^-(t,x) + \inf_{u_2} \sup_{u_1} \left[ \nabla_x V^-(t,x) \cdot b(t,x,u_1,u_2) + r(t,x,u_1,u_2) \right] = 0. \quad (1.4)$$

The equations (1.3) and (1.4) are referred to as Hamilton-Jacobi-Isaacs (HJI for short) equations. Isaacs carried out his work under the following assumption known as 'minimax' condition: for each $t \in [0,T], x, y \in \mathbb{R}^d, u_1 \in U_1, u_2 \in U_2$,

$$\begin{aligned} &\inf_{u_2} \sup_{u_1} [y \cdot b(t,x,u_1,u_2) + r(t,x,u_1,u_2)] \\ &\qquad = \sup_{u_1} \inf_{u_2} [y \cdot b(t,x,u_1,u_2) + r(t,x,u_1,u_2)] \end{aligned} \quad (1.5)$$

He showed by heuristic methods that if the 'minimax' condition (1.5) holds, then $V^-(t,x) = V^+(t,x)$ and the common value is denoted by $V(t,x)$. That is, in this case, the differential game admits a value function $V(t,x)$ and thus has a value at any $(t,x)$. Under several conditions on $b, r, g$ etc., Berkovitz [3] has given a rigorous treatment of Isaacs results. In particular, under the assumption that HJI equations have smooth solutions he has extended the usual maximum principle in optimal control. He has also given a counterexample to show the nonexistence of a value function. After this several authors have approached the differential game problem from different angles and have established the existence of a value. Notable contributions were made by Fleming [13], Friedman [15], Varaiya-Lin [22], Elliott-Kalton [8]. See [7] and the references therein. In these works different notions of strategies have been used. Assuming the 'minimax' condition Krasovskii and Subbotin [18] have established the existence of value and saddle point equilibrium by introducing the concept of stable bridges. Using Friedman's definition of strategy and the payoff in the sense of Krasovskii and Subbotin, Berkovitz [4] has established the existence of value and saddle point equilibrium under the 'minimax' condition. A significant aspect of this paper is that these results are derived without any recourse to HJI equations. Evans and Souganidis [11] have studied the differential games using Elliott-Kalton approach. In this approach a strategy for player 1 is a (nonanticipative) map from the set of all open loop strategies of player 2 to the set of all open loop strategies of player 1. A strategy for player 2 is defined analogously. Let $\Gamma$ and $\Delta$ denote the spaces of all strategies (in the sense of Elliott-Kalton) of the player 1 and player 2 respectively. Define

$$\bar{V}^+(t,x) := \sup_{\alpha_1\in\Gamma}\inf_{u_2(\cdot)}\left[\int_t^T r(s,x(s),\alpha_1[u_2](s),u_2(s))\,ds + g(x(T))\right],$$

$$\bar{V}^-(t,x) := \inf_{\alpha_2\in\Delta}\sup_{u_1(\cdot)}\left[\int_t^T r(s,x(s),u_1(s),\alpha_2[u_1](s))\,ds + g(x(T))\right].$$

The functions $\bar{V}^+$ and $\bar{V}^-$ are called upper and lower values of the game in the sense of Elliott-Kalton. Under certain conditions they have shown that $\bar{V}^+$ and $\bar{V}^-$ are unique viscosity solutions of (1.3) and (1.4) respectively. Furthermore if Isaacs 'minimax' condition holds, then by the uniqueness of viscosity solution $\bar{V}^+ \equiv \bar{V}^-$, i.e., the value (in the sense of Elliott-Kalton) exists. See [5], [10], [19], [20], [23] for related works. Note that if the 'minimax' condition (1.5) holds, then for each $(t,x,y,u_1,u_2)$ the static game with payoff given by (1.5) has a value in pure actions, a fact which is rarely met. Consider, for example, the following simple case: $U_1 = U_2 = \{1,2\}, b(\cdot,\cdot,\cdot,\cdot) = 0, r(\cdot,\cdot,i,j) = \delta_{ij}$. In this case the left hand side of (1.5) equals 1 whereas the right hand side is 0. With $g(\cdot) = 0$, it is easy to see that $V^+(t,x) = T - t$, whereas $V^-(t,x) = 0$. Any number of counterexamples along these lines can be constructed to show that the 'minimax' condition does not hold in general, and the value does not exist. In static game, one introduces mixed actions to circumvent this problem. The generalization of mixed actions in static games to differential games is relaxed strategies. A relaxed strategy for player $i, i = 1, 2$ is a $\mathcal{M}_i$-valued map on $[0, T]$, where $\mathcal{M}_i$ is the space of probability measures on $U_i$. One advantage in this setup is the convexity of the sets $\mathcal{M}_i$. Due to this the 'minimax' condition does hold, thanks to a minimax theorem by Fan [12]. Thus in the framework of relaxed strategies the differential game does have a value in the sense of Elliott-Kalton. This does not, however, imply that the differential game has a value in open loop relaxed strategies. Indeed we have the following counterexample.

Consider the following case:

$$d = 1;\ T = 1;\ b(t, x, u_1, u_2) = u_1 + u_2;$$
$$U_1 = U_2 = [-1, 1];\ r \equiv 0;\ g(x) = |x|.$$

In this case one can verify that $V^+(0, x) \equiv 1, V^-(0, x) \equiv 0$. Note that in this case even in relaxed framework, the upper and lower values remain the same since $b$ is linear in both control variables.

It may be noted that Elliott-Kalton have altered the original notion of the differential game formulated by Isaacs. They have introduced the notion of upper and lower games. In the lower game, player 1 moves first using an open loop strategy $u_1(\cdot)$, while payer 2 is allowed to use a strategy $u_2(\cdot) = \alpha(u_1(\cdot))$, where $\alpha(\cdot)$ is a certain nonanticipative strategy [1]. Therefore in the lower game, player 2 has a distinct advantage over player 1. In the upper game, the situation is reversed and player 1 gains the informational advantage over player 2. Though this notion of strategies is very useful in many applications, strict non-cooperative nature of the game is not fully maintained due to these informational disparities. Besides, the notion of saddle point equilibrium does not make sense in this approach. This motivates us to take a fresh look at open loop relaxed strategies. Under certain linearity assumptions, Elliott, Kalton and Marcus [9] have established the existence of a saddle point equilibrium in (open loop) relaxed strategies; see also Parthasarathy and Raghavan [21]. In this paper, we investigate a set of necessary and sufficient conditions for optimality of an (open loop) relaxed strategy. The rest of the paper is organized as follows. In Section 2, we introduce the basic notation and assumptions. A Mini-Max principle is derived in Section 3.

Section 4 deals with the equivalence between the dynamic programming principle and the existence of a saddle point equilibrium. The connection between the dynamic programming principle and the mini-max principle is treated in Section 5.

## 2. Preliminaries

Let $U_i$, $i = 1, 2$, be given compact metric spaces. Let $\mathcal{M}_i$, $i = 1, 2$, be the space of probability measures on $U_i$. Let $T > 0$ be fixed. Let

$$\bar{b} : [0, T] \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}^d.$$

We assume that:

**(A1)** The function $\bar{b}$ is continuous and there is a constant $C_1$ such that

$$|\bar{b}(t, x, u_1, u_2) - \bar{b}(s, y, u_1, u_2)| \leq C_1(|t-s| + |x-y|), \ \forall \ (u_1, u_2) \in U_1 \times U_2.$$

Define

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}^d$$

by

$$b(t, x, \mu_1, \mu_2) = \int_{U_1} \int_{U_2} \bar{b}(t, x, u_1, u_2) \ \mu_2(du_2) \ \mu_1(du_1).$$

For $t \in [0, T]$, a measurable function $\mu(\cdot) : [t, T] \to \mathcal{M}_i$ is called an (open loop) relaxed strategy for player $i$ at time $t$. Let $\mathcal{A}_i^t$ be the set of all relaxed strategies for player $i$ at time $t$. A strategy $\mu(\cdot) \in \mathcal{A}_i^t$ for player $i$ is said to be a pure strategy if $\mu(\cdot) = \delta_{u(\cdot)}$, where $u : [t, T] \to U_i$ is a measurable map. If the players use relaxed strategies $(\mu_1(\cdot), \mu_2(\cdot)) \in \mathcal{A}_1^t \times \mathcal{A}_2^t$, then the state of the system denoted by $x(\cdot)$ evolves according to the equation

$$\dot{x}(s) = b(s, x(s), \mu_1(s), \mu_2(s)), \ s \in (t, T], \ x(t) = x, \qquad (2.1)$$

where $x \in \mathbb{R}^d$ is the state at time $t$. Let

$$\bar{r} : [0, T] \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}$$

be the running payoff function. If $x$ is the state at time $t$ and the players choose actions $(u_1, u_2) \in U_1 \times U_2$, then player 1 receives a payoff $\bar{r}(t, x, u_1, u_2)$ from player 2. Let

$$g : \mathbb{R}^d \to \mathbb{R}$$

denote the terminal payoff function. We assume that:

**(A2)**(i) The functions $\bar{r}$ and $g$ are bounded and continuous.

(ii) There exist constants $C_2 > 0, C_3 > 0$, such that

$$|\bar{r}(t, x, u_1, u_2) - \bar{r}(s, y, u_1, u_2)| \leq C_2(|t-s| + |x-y|),$$

for all $(t, x), (s, y) \in [0, T] \times \mathbb{R}^d$, $(u_1, u_2) \in U_1 \times U_2$, and

$$|g(x) - g(y)| \leq C_3|x - y|, \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Let

$$r : [0, T] \times \mathbb{R}^d \times \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}$$

be defined by

$$r(t, x, \mu_1, \mu_2) = \int_{U_1} \int_{U_2} \bar{r}(t, x, u_1, u_2)\ \mu_2(du_2)\ \mu_1(du_1).$$

When the state of the system is $x$ at time $t$, and the players use relaxed strategies $(\mu_1(\cdot), \mu_2(\cdot)) \in \mathcal{A}_1^t \times \mathcal{A}_2^t$, then the payoff of player 1 is given by

$$R(t, x, \mu_1(\cdot), \mu_2(\cdot)) := \int_t^T r(s, x(s), \mu_1(s), \mu_2(s))\, ds + g(x(T)), \quad (2.2)$$

where $x(\cdot)$ is given by (2.1). Let

$$V^+(t, x) := \inf_{\mu_2(\cdot) \in \mathcal{A}_2^t} \sup_{\mu_1(\cdot) \in \mathcal{A}_1^t} R(t, x, \mu_1(\cdot), \mu_2(\cdot)), \quad (2.3)$$

$$V^-(t, x) := \sup_{\mu_1(\cdot) \in \mathcal{A}_1^t} \inf_{\mu_2(\cdot) \in \mathcal{A}_2^t} R(t, x, \mu_1(\cdot), \mu_2(\cdot)). \quad (2.4)$$

The functions $V^+$ and $V^-$ are called upper value function and lower value function respectively. A relaxed strategy $\mu_1^*(\cdot) \in \mathcal{A}_1^t$ is said to be optimal for player 1 at $(t, x)$ if

$$R(t, x, \mu_1^*(\cdot), \mu_2(\cdot)) \geq V^+(t, x) \quad (2.5)$$

for any $\mu_2(\cdot) \in \mathcal{A}_2^t$. Similarly a relaxed strategy $\mu_2^*(\cdot) \in \mathcal{A}_2^t$ is said to be optimal for player 2 at $(t, x)$ if

$$R(t, x, \mu_1(\cdot), \mu_2^*(\cdot)) \leq V^-(t, x) \tag{2.6}$$

for any $\mu_1(\cdot) \in \mathcal{A}_1^t$. The (differential) game is said to have a value in relaxed strategies if for all $(t, x)$

$$V^+(t, x) = V^-(t, x) := V(t, x).$$

In such a case, the function $V$ is referred to as the value function of the game.

We endow $\mathcal{A}_i^t$ with the $L^1$-weak*-topology. Using Banach-Alaoglu theorem, we can verify that $\mathcal{A}_i^t$ is a compact metric space. For more details, see [24]. Under this topology, $R(0, x, \mu_1(\cdot), \mu_2(\cdot))$ is continuous in $\mu_1(\cdot)$ for fixed $x$ and $\mu_2(\cdot)$. Similarly it is continuous in $\mu_2(\cdot)$ for fixed $x$ and $\mu_1(\cdot)$.

# 3. Mini-Max Principle

In this section we derive a Mini-Max principle (MMP for short). To this end we make the following assumption.

**(A3)** For each $(t, x, u_1, u_2) \in [0, T] \times \mathbb{R}^d \times U_1 \times U_2$, the functions $\bar{b}(t, \cdot, u_1, u_2)$ and $\bar{r}(t, \cdot, u_1, u_2)$ are continuously differentiable. Also, the function $g$ is continuously differentiable.

Let $(\mu_1^*(\cdot), \mu_2^*(\cdot)) \in \mathcal{A}_1^0 \times \mathcal{A}_2^0$ be a pair of optimal relaxed strategies i.e., a saddle point equilibrium for $(0, x)$ and $x^*(\cdot)$, the corresponding process with $x^*(0) = x$. Let $p(\cdot)$ denote the co-state process satisfying the following equation

$$\left.\begin{aligned} \dot{p}(t) &= -b_x'(t, x^*(t), \mu_1^*(t), \mu_2^*(t))p(t) - \nabla_x r(t, x^*(t), \mu_1^*(t), \mu_2^*(t)), \\ p(T) &= \nabla_x g(x^*(T)), \end{aligned}\right\} \tag{3.1}$$

where $b_x$ is the usual Jacobian matrix and $b'_x$ denotes the transpose of $b_x$. Let the Hamiltonian $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}$ be defined as

$$H(t,x,p,\mu_1,\mu_2) = b(t,x,\mu_1,\mu_2)\cdot p + r(t,x,\mu_1,\mu_2).$$

We now prove the following MMP.

**Theorem 3.1.** Assume (A1) - (A3). Let $(\mu_1^*(\cdot),\mu_2^*(\cdot))$ be a pair of optimal relaxed strategies and $x^*(\cdot)$ the corresponding state process with $x^*(0)=x$. Let the co-state process $p(\cdot)$ be defined as in (3.1). Then for a.e., $t\in[0,T]$,

$$\begin{aligned}\min_{\mu_2\in\mathcal{M}_2}\max_{\mu_1\in\mathcal{M}_1} H(t,x^*(t),p(t),\mu_1,\mu_2) &= \max_{\mu_1\in\mathcal{M}_1} H(t,x^*(t),p(t),\mu_1,\mu_2^*(t))\\ &= \max_{\mu_1\in\mathcal{M}_1}\min_{\mu_2\in\mathcal{M}_2} H(t,x^*(t),p(t),\mu_1,\mu_2)\\ &= \min_{\mu_2\in\mathcal{M}_2} H(t,x^*(t),p(t),\mu_1^*(t),\mu_2).\end{aligned} \tag{3.2}$$

**Proof.** Let $\varepsilon>0$. Let $\mu_1(\cdot)\in\mathcal{A}_1^0$ be arbitrary. Let $x^\varepsilon(\cdot)$ be the solution of (2.1) under $(\mu_1^*(\cdot)+\varepsilon(\mu_1(\cdot)-\mu_1^*(\cdot)),\mu_2^*(\cdot))$ with $x^\varepsilon(0)=x$. Then

$$\begin{aligned}&\frac{d}{d\varepsilon}\left[R(0,x,\mu_1^*(\cdot)+\varepsilon(\mu_1(\cdot)-\mu_1^*(\cdot)),\mu_2^*(\cdot))\right]\Big|_{\varepsilon=0}\\ &=\frac{d}{d\varepsilon}\left[\int_0^T r(t,x^\varepsilon(t),\mu_1^*(t)+\varepsilon(\mu_1(t)-\mu_1^*(t)),\mu_2^*(t))dt+g(x^\varepsilon(T))\right]\Big|_{\varepsilon=0}\\ &=\int_0^T[\nabla_x r(t,x^*(t),\mu_1^*(t),\mu_2^*(t))z(t)+r(t,x^*(t),\mu_1(t),\mu_2^*(t))\\ &-r(t,x^*(t),\mu_1^*(t),\mu_2^*(t))]dt+\nabla_x g(x^*(T))z(T),\end{aligned} \tag{3.3}$$

where $z(t)$ is given by

$$\left.\begin{aligned}\dot z(t) &= b'_x(t,x^*(t),\mu_1^*(t),\mu_2^*(t))z(t)+b(t,x^*(t),\mu_1(t),\mu_2^*(t))\\ &\quad -b(t,x^*(t),\mu_1^*(t),\mu_2^*(t))\\ z(0) &= 0.\end{aligned}\right\} \tag{3.4}$$

Note that while differentiating with respect to $\varepsilon$ under the integral sign in (3.3), the derivative of $r$ with respect to the third variable will not appear as it is a measure and due to the definition of $r$ in terms of $\bar{r}$ as in Section 2. Using integration by parts, we have

$$
\begin{aligned}
&\int_0^T \nabla_x r(t, x^*(t), \mu_1^*(t), \mu_2^*(t)) \cdot z(t)\, dt + \nabla_x g(x^*(T)) \cdot z(T) \\
&= \int_0^T [-\dot{p}(t) - b_x'(t, x^*(t), \mu_1^*(t), \mu_2^*(t))p(t)]\, z(t)\, dt + p(T) \cdot z(T) \\
&= \int_0^T p(t) \cdot [b(t, x^*(t), \mu_1(t), \mu_2^*(t)) - b(t, x^*(t), \mu_1^*(t), \mu_2^*(t))]\, dt. \qquad (3.5)
\end{aligned}
$$

From (3.3) and (3.5), it follows that

$$
\begin{aligned}
&\frac{d}{d\varepsilon}[R(0, x, \mu_1^*(\cdot) + \varepsilon(\mu_1(\cdot) - \mu_1^*(\cdot)), \mu_2^*(\cdot))]\big|_{\varepsilon=0} \\
&= \int_0^T [H(t, x^*(t), p(t), \mu_1(t), \mu_2^*(t)) - H(t, x^*(t), p(t), \mu_1^*(t), \mu_2^*(t))]\, dt \\
&\leq 0. \qquad (3.6)
\end{aligned}
$$

The last inequality in (3.6) follows from the fact that $(\mu_1^*(\cdot), \mu_2^*(\cdot))$ is a pair of saddle point strategies. Let $\tilde{\mu}_1(\cdot) \in \mathcal{A}_1^0$ be given by

$$
\tilde{\mu}_1(s) = \begin{cases} \mu_1^*(s), & s \in [0, t] \cup [t + \varepsilon, T] \\ \mu_1, & s \in (t, t + \varepsilon) \end{cases}
$$

for a fixed $\mu_1 \in \mathcal{M}_1$. Then using $\tilde{\mu}_1(\cdot)$ in place of $\mu_1(\cdot)$ in (3.6), we get for a.e., $t \in [0, T]$ and any $\mu_1 \in \mathcal{M}_1$,

$$
H(t, x^*(t), p(t), \mu_1, \mu_2^*(t)) \leq H(t, x^*(t), p(t), \mu_1^*(t), \mu_2^*(t)). \qquad (3.7)
$$

Similarly, we can show that for any $\mu_2 \in \mathcal{M}_2$, for a.e., $t \in [0, T]$,

$$
H(t, x^*(t), p(t), \mu_1^*(t), \mu_2) \geq H(t, x^*(t), p(t), \mu_1^*(t), \mu_2^*(t)). \qquad (3.8)
$$

Combining (3.7) and (3.8), we get the desired result (3.2).

# 4. Dynamic Programming

In this section, we show that the value function $V$ is the viscosity solution of HJI equations under the assumption that a saddle point equilibrium in relaxed strategies exists. Then we derive sufficient conditions for optimality. Using (A1) and (A2), the following result can be easily proved.

**Lemma 4.1.** Assume (A1), (A2). The functions $V^+$ and $V^-$, as in (2.3) and (2.4) respectively are Lipschitz continuous on $[0,T] \times \mathbb{R}^d$.

Now we prove the equivalence between the Dynamic Programming Principle (DPP for short) and the existence of saddle point equilibrium. First we prove the DPP under the assumption that a saddle point equilibrium exists.

**Theorem 4.1. (DPP)** Assume (A1), (A2) and that a saddle point equilibrium exists in relaxed strategies for $(t,x) \in [0,T] \times \mathbb{R}^d$. Then for $0 \le t < t+\Delta < T$,

$$V^+(t,x)$$
$$= \min_{\mu_2(\cdot)\in\mathcal{A}_2^t} \max_{\mu_1(\cdot)\in\mathcal{A}_1^t} \left[\int_t^{t+\Delta} r(s,x(s),\mu_1(s)\mu_2(s))\,ds + V^+(t+\Delta,x(t+\Delta))\right] \tag{4.1}$$

where $x(\cdot)$ is the solution of (2.1) under $\mu_1(\cdot),\mu_2(\cdot)$ with $x(t)=x$. Similarly,

$$V^-(t,x) = \max_{\mu_1(\cdot)\in\mathcal{A}_1^t} \min_{\mu_2(\cdot)\in\mathcal{A}_2^t} \left[\int_t^{t+\Delta} r(s,x(s),\mu_1(s),\mu_2(s))\,ds + V^-(t+\Delta,x(t+\Delta))\right]. \tag{4.2}$$

**Proof.** Let $(\mu_1^*(\cdot),\mu_2^*(\cdot)) \in \mathcal{A}_1^t \times \mathcal{A}_2^t$ be a saddle point equilibrium for $(t,x)$. Denote the RHS of (4.1) by $W(t,x)$. For any $\mu_2(\cdot) \in \mathcal{A}_2^t$, we have

$$W(t,x) \le \max_{\mu_1(\cdot)\in\mathcal{A}_1^t} \left[\int_t^{t+\Delta} r(s,x(s),\mu_1(s),\mu_2(s))\,ds + V^+(t+\Delta,x(t+\Delta))\right] \tag{4.3}$$

where $x(\cdot)$ is the solution of (2.1) under $(\mu_1(\cdot), \mu_2(\cdot))$ with $x(t) = x$. Let $(\tau, \bar{x}) \in [0, T] \times \mathbb{R}^d$ and $\bar{\mu}_2 \in \mathcal{A}_2^\tau$. Define $V_{\bar{\mu}_2}^+$ by

$$V_{\bar{\mu}_2}^+(\tau, \bar{x}) = \max_{\mu_1(\cdot) \in \mathcal{A}_1^\tau} \left[ \int_\tau^T r(s, \bar{x}(s), \mu_1(s), \bar{\mu}_2(s))\, ds + g(\bar{x}(T)) \right],$$

where $\bar{x}(\cdot)$ is the solution of (2.1) under $(\mu_1(\cdot), \bar{\mu}_2(\cdot))$ with $\bar{x}(\tau) = \bar{x}$. Now using DPP for optimal control, we have

$$\begin{aligned} V_{\bar{\mu}_2}^+(\tau, \bar{x}) = \max_{\mu_1(\cdot) \in \mathcal{A}_1^\tau} \Bigg[ \int_\tau^{\tau+\Delta} & r(s, \bar{x}(s), \mu_1(s), \bar{\mu}_2(s))\, ds \\ & + V_{\bar{\mu}_2}^+(\tau + \Delta, \bar{x}(\tau + \Delta)) \Bigg] \end{aligned}$$

for any $\tau < \tau + \Delta < T$. Also, we have

$$V^+(\tau, \bar{x}) \le V_{\bar{\mu}_2}^+(\tau, \bar{x}).$$

Plugging these into (4.3), we obtain

$$\begin{aligned} W(t, x) &\le \max_{\mu_1(\cdot) \in \mathcal{A}_1^t} \left[ \int_t^{t+\Delta} r(s, x(s), \mu_1(s), \mu_2(s))\, ds + V_{\mu_2}^+(t + \Delta, x(t + \Delta)) \right] \\ &= V_{\mu_2}^+(t, x). \end{aligned}$$

Since $\mu_2(\cdot)$ is arbitrary, we get

$$W(t, x) \le V^+(t, x).$$

We now prove the reverse inequality. Since $(\mu_1^*(\cdot), \mu_2^*(\cdot))$ is a saddle point at $(t, x)$, we have

$$V^+(t, x) \le \min_{\mu_2(\cdot) \in \mathcal{A}_2^t} \left[ \int_t^T r(s, x^*(s), \mu_1^*(s), \mu_2(s))\, ds + g(x^*(T)) \right], \quad (4.4)$$

where $x^*(\cdot)$ is the solution of (2.1) under $(\mu_1^*(\cdot), \mu_2(\cdot))$ with $x(t) = x$. Let $(\tau, \bar{x}) \in [0, T] \times \mathbb{R}^d$ and $\bar{\mu}_1 \in \mathcal{A}_2^\tau$. Define $\overline{V}_{\bar{\mu}_1}^+$ by

$$\overline{V}_{\bar{\mu}_1}^+(\tau, \bar{x}) = \min_{\mu_2(\cdot) \in \mathcal{A}_2^\tau} \left[ \int_\tau^T r(s, \bar{x}(s), \bar{\mu}_1(s), \mu_2(s))\, ds + g(\bar{x}(T)) \right],$$

where $\bar{x}(\cdot)$ is the solution of (2.1) under $(\bar{\mu}_1(\cdot), \mu_2(\cdot))$ with $\bar{x}(\tau) = \bar{x}$. We have,

$$\overline{V}^+_{\bar{\mu}_1}(\tau, \bar{x}) \leq V^+(\tau, \bar{x}).$$

Again by DPP for optimal control, we have for any $\tau < \tau + \Delta < T$

$$\begin{aligned}
\overline{V}^+_{\bar{\mu}_1}(\tau, \bar{x}) &= \min_{\mu_2(\cdot)\in\mathcal{A}_2^\tau} \left[\int_\tau^{t+\Delta} r(s, \bar{x}(s), \bar{\mu}_1(s), \mu_2(s))\, ds + \overline{V}^+_{\bar{\mu}_1}(\tau + \Delta, \bar{x}(\tau + \Delta))\right] \\
&\leq \min_{\mu_2(\cdot)\in\mathcal{A}_2^\tau} \left[\int_\tau^{t+\Delta} r(s, \bar{x}(s), \bar{\mu}_1(s), \mu_2(s))\, ds + V^+(\tau + \Delta, \bar{x}(\tau + \Delta))\right] \\
&\leq \min_{\mu_2(\cdot)\in\mathcal{A}_2^\tau} \max_{\mu_1(\cdot)\in\mathcal{A}_1^\tau} \left[\int_\tau^{t+\Delta} r(s, x(s), \mu_1(s), \mu_2(s))\, ds + V^+(\tau + \Delta, x(\tau + \Delta))\right] \\
&= W(t, x),
\end{aligned}$$

where $x(\cdot)$ is the solution of (2.1) under $(\mu_1(\cdot), \mu_2(\cdot))$ with $x(\tau) = \bar{x}$. Plugging these into the above inequality with $\bar{\mu}_1(\cdot) = \mu_1^*(\cdot)$, we obtain

$$V^+(t, x) \leq \bar{V}^+_{\mu_1^*}(t, x) \leq W(t, x).$$

Hence (4.1) holds. Similarly (4.2) can be proved.

**Remark 4.1.** (i) In view of Theorem 4.2., if a pair of relaxed strategies $(\mu_1(\cdot), \mu_2(\cdot)) \in \mathcal{A}_1^t \times \mathcal{A}_2^t$ is optimal for both players for $(t, x)$, then it is optimal for $(s, x(s))$ for any $s \in (t, T)$, where $x(\cdot)$ is the solution of (2.1).

(ii) Suppose $\bar{b}(t, x, u_1, u_2) = A(t)x + f(t, u_1, u_2)$, where $A(\cdot)$ is continuous $d \times d$ matrix and $f$ is a continuous function on $[0, T] \times U_1 \times U_2$. Also suppose that $\bar{r}(t, x, u_1, u_2)$ is independent of the second argument and $g$ is linear. Then a saddle point equilibrium in relaxed strategies does exist [9], [21] and hence DPP is true in this case.

Given DPP holds, we can mimic the arguments of Evans and Souganidis [11] to obtain the following result.

**Theorem 4.2.** Assume (A1), (A2) and the DPP. Then the lower value function $V^-$ is a viscosity solution of HJI equation

$$\phi_t + \inf_{\mu_2} \sup_{\mu_1} H(t, x, \nabla_x \phi, \mu_1, \mu_2) = 0. \tag{4.5}$$

Similarly the upper value function $V^+$ is a viscosity solution of HJI equation

$$\psi_t + \sup_{\mu_1} \inf_{\mu_2} H(t, x, \nabla_x \psi, \mu_1, \mu_2) = 0. \tag{4.6}$$

Using Theorem 4.2, Fan's minimax theorem [12] and the uniqueness of viscosity solutions [1], we get the following result.

**Theorem 4.3.** Assume (A1), (A2) and the DPP. Then the value function $V$ exists and is the unique bounded continuous viscosity solution of

$$\phi_t + \inf_{\mu_2} \sup_{\mu_1} H(t, x, \nabla_x \phi, \mu_1, \mu_2) = \phi_t + \sup_{\mu_1} \inf_{\mu_2} H(t, x, \nabla_x \phi, \mu_1, \mu_2) = 0 \tag{4.7}$$

satisfying

$$\phi(T, x) = g(x). \tag{4.8}$$

Combining Theorems 4.1, 4.2 and 4.3, we obtain the following theorem.

**Theorem 4.4.** Assume (A1), (A2). Then the following are equivalent.
(i) There exists a saddle point equilibrium in relaxed strategies.
(ii) DPP holds i.e., (4.1) and (4.2) are true.

**Proof.** In view of Theorem 4.1, it suffices to prove (ii) implies (i). We prove this for $t = 0$. The proof is analogous for any $t$. We have by continuity

$$\inf_{\mu_2(\cdot)\in\mathcal{A}_2^0} \sup_{\mu_1(\cdot)\in\mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \min_{\mu_2(\cdot)\in\mathcal{A}_2^0} \max_{\mu_1(\cdot)\in\mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot))$$

and

$$\sup_{\mu_1(\cdot)\in\mathcal{A}_1^0} \inf_{\mu_2(\cdot)\in\mathcal{A}_2^0} R(0, x, \mu_1(\cdot), \mu_1(\cdot)) = \max_{\mu_1(\cdot)\in\mathcal{A}_1^0} \min_{\mu_2(\cdot)\in\mathcal{A}_2^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)).$$

Hence by Theorem 4.3

$$\min_{\mu_2(\cdot)\in\mathcal{A}_2^0} \max_{\mu_1(\cdot)\in\mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \max_{\mu_1(\cdot)\in\mathcal{A}_1^0} \min_{\mu_2(\cdot)\in\mathcal{A}_2^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)).$$

Let $(\mu_1^*(\cdot), \mu_2^*(\cdot)) \in \mathcal{A}_1^0 \times \mathcal{A}_2^0$ be such that

$$\min_{\mu_2(\cdot)\in\mathcal{A}_2^0} \max_{\mu_1(\cdot)\in\mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \max_{\mu_1(\cdot)\in\mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2^*(\cdot))$$

and

$$\max_{\mu_1(\cdot)\in\mathcal{A}_2^0} \min_{\mu_2(\cdot)\in\mathcal{A}_1^0} R(0, x, \mu_1(\cdot), \mu_2(\cdot)) = \min_{\mu_2(\cdot)\in\mathcal{A}_2^0} R(0, x, \mu_1^*(\cdot), \mu_2(\cdot)).$$

Clearly $(\mu_1^*(\cdot), \mu_2^*(\cdot))$ is optimal for both players for $(0, x)$. Thus it is a pair of saddle point strategies.

Next we investigate sufficient conditions for optimality. We first prove the 'classical' version of the sufficient conditions for optimality.

**Theorem 4.5.** Assume (A1), (A2). Let DPP hold. Suppose the equation (4.7) has a bounded solution $W \in C^{1,1}([0,T] \times \mathbb{R}^d)$ satisfying (4.8).
(i) Suppose $\mu_1^*(\cdot) \in \mathcal{A}_1^0$ is such that for any $\mu_2(\cdot) \in \mathcal{A}_2^0$, if $x^*(\cdot)$ denotes the corresponding state with $x^*(0) = x$, and if

$$\begin{aligned} &\max_{\mu_1\in\mathcal{M}_1} \min_{\mu_2\in\mathcal{M}_2} H(t, x^*(t), \nabla_x W(t, x^*(t)), \mu_1, \mu_2) \\ &= \min_{\mu_2\in\mathcal{M}_2} H(t, x^*(t), \nabla_x W(t, x^*(t)), \mu_1^*(t), \mu_2) \end{aligned} \tag{4.9}$$

for $t$ a.e., then $\mu_1^*(\cdot)$ is optimal for player 1 for $(0, x)$.
(ii) Suppose $\mu_2^*(\cdot) \in \mathcal{A}_2^0$ is such that for any $\mu_1(\cdot) \in \mathcal{A}_1^0$, if $x^*(\cdot)$ denotes the corresponding state with $x^*(0) = x$, and if

$$\begin{aligned} &\min_{\mu_2\in\mathcal{M}_1} \max_{\mu_1\in\mathcal{M}_2} H(t, x^*(t), \nabla_x W(t, x^*(t)), \mu_1, \mu_2) \\ &= \max_{\mu_1\in\mathcal{M}_2} H(t, x^*(t), \nabla_x W(t, x^*(t)), \mu_1, \mu_2^*(t)) \end{aligned} \tag{4.10}$$

for $t$ a.e., then $\mu_2^*(\cdot)$ is optimal for player 2 for $(0, x)$.

**Proof.** We prove only (i). The rest can be done in a similar manner. First note that by the uniqueness of the viscosity solution of (4.7)-(4.8), we have $W = V$, the value function. Now for $t$ a.e.,

$$\begin{aligned}\frac{d}{dt}V(t,x^*(t)) &= V_t(t,x^*(t)) + b(t,x^*(t),\mu_1^*(t),\mu_2(t))\cdot\nabla_x V(t,x^*(t))\\ &= V_t(t,x^*(t)) + H(t,x^*(t),\nabla_x V(t,x^*(t)),\mu_1^*(t),\mu_2(t))\\ &\quad - r(t,x^*(t),\mu_1^*(t),\mu_2(t))\\ &\geq V_t(t,x^*(t)) + \min_{\mu_2\in\mathcal{M}_2} H(t,x^*(t),\nabla_x V(t,x^*(t)),\mu_1^*(t),\mu_2)\\ &\quad - r(t,x^*(t),\mu_1^*(t),\mu_2(t))\\ &= V_t(t,x^*(t)) + \max_{\mu_1\in\mathcal{M}_1}\min_{\mu_2\in\mathcal{M}_2} H(t,x^*(t),\nabla_x V(t,x^*(t)),\mu_1,\mu_2)\\ &\quad - r(t,x^*(t),\mu_1^*(t),\mu_2(t)),\ \text{by (4.9)}\\ &= -r(t,x^*(t),\mu_1^*(t),\mu_2(t)),\ \text{by (4.7)}. \end{aligned} \tag{4.11}$$

Integrating (4.11) from 0 to $T$ and rearranging, we have

$$R(0,x,\mu_1^*(\cdot),\mu_2(\cdot)) \geq V(0,x).$$

Thus $\mu_1^*(\cdot)$ is optimal for player 1 for $(0,x)$.

**Remark 4.2.** Suppose that the value function $V \in C^{1,1}([0,T]\times\mathbb{R}^d)$. Let $\overline{\mu}_1 : [0,T]\times\mathbb{R}^d \to \mathcal{M}_1$, $\overline{\mu}_2 : [0,T]\times\mathbb{R}^d \to \mathcal{M}_2$ be measurable maps such that for all $x\in\mathbb{R}^d$ and for a.e. $t$

$$\begin{aligned}\max_{\mu_1\in\mathcal{M}_1}\min_{\mu_2\in\mathcal{M}_2} & H(t,x,\nabla_x V(t,x),\mu_1,\mu_2)\\ &= \min_{\mu_2\in\mathcal{M}_2} H(t,x,\nabla_x V(t,x),\overline{\mu}_1(t,x),\mu_2)\\ &= \min_{\mu_2\in\mathcal{M}_2}\max_{\mu_1\in\mathcal{M}_1} H(t,x,\nabla_x V(t,x),\mu_1,\mu_2)\\ &= \max_{\mu_1\in\mathcal{M}_1} H(t,x,\nabla_x V(t,x),\mu_1,\overline{\mu}_2(t,x)). \end{aligned} \tag{4.12}$$

The existence of such maps is guaranteed by a standard measurable selection theorem [2]. If (2.1) under feedback strategies defined by $\tilde{\mu}_1(t) = \overline{\mu}_1(t,x(t))$, $\tilde{\mu}_2(t) = \overline{\mu}_2(t,x(t))$ has a unique solution, then it can be shown as in the proof of Theorem 4.5, that $(\tilde{\mu}_1(\cdot),\tilde{\mu}_2(\cdot))$ is a pair of saddle point strategies. To establish the existence of a unique solution of (2.1) under feedback strategies may, however, be a very difficult problem.

**Remark 4.3.** The sufficient conditions for optimality in the Theorem 4.5 is based on a classical solution of the HJI equation. But note that even in optimal control problem, which is a one player game, the existence of a $C^1$ solution is more of an exception than a rule as pointed out in [11]. Following [26], We proceed to establish analogous sufficient conditions for optimality via viscosity solution of HJI equation. In this framework, the gradient of $V$ in (4.9), (4.10) will be replaced by the sub-differential or super-differential.

Let $D^+_{t,x}V(t,x)$ and $D^-_{t,x}V(t,x)$ denote respectively the super-differential and sub-differential of $V$ at $(t,x)$. Since $V$ is Lipschitz continuous, for a.e. $(t,x)$

$$D^+_{t,x}V(t,x) = D^-_{t,x}V(t,x) = \{V_t(t,x), \nabla_x V(t,x)\}.$$

**Theorem 4.6.** Assume (A1), (A2). Let DPP hold. (i) Let $\mu_1^*(\cdot) \in \mathcal{A}_1^0$ be such that for any $\mu_2(\cdot) \in \mathcal{A}_2^0$, if $x^*(\cdot)$ denotes the solution of (2.1) with $x^*(0) = x$, and if for a.e., $t \in [0,T]$, there exists $(\eta_t^*, \zeta_t^*) \in D^+_{t,x}V(t,x^*(t))$ such that

$$\eta_t^* + \min_{\mu_2 \in \mathcal{M}_2} H(t, x^*(t), \zeta_t^*, \mu_1^*(t), \mu_2) \geq 0, \tag{4.13}$$

then $\mu_1^*(\cdot)$ is optimal for player 1 for $(0,x)$.
(ii) Let $\mu_2^*(\cdot) \in \mathcal{A}_2^0$ be such that for any $\mu_1(\cdot) \in \mathcal{A}_1^0$, if $x^*(\cdot)$ denotes the solution of (2.1) with $x^*(0) = x$ and if for a.e., $t \in [0,T]$, there exists $(\eta_t^*, \zeta_t^*) \in D^-_{t,x}V(t,x^*(t))$ such that

$$\eta_t^* + \max_{\mu_1 \in \mathcal{M}_1} H(t, x^*(t), \zeta_t^*, \mu_1, \mu_2^*(t)) \leq 0, \tag{4.14}$$

then $\mu_2^*(\cdot)$ is optimal for player 2 for $(0,x)$.

**Proof.** We prove (ii); (i) can be proved using analogous arguments. We have, for almost every $s \in [0,T]$,

$$\frac{d}{dt}V(t,x^*(t)) \mid_{t=s} = V'(s,x^*(s))(1, b(s,x^*(s),\mu_1(s),\mu_2^*(s))), \tag{4.15}$$

where $V'(t,x)(t_1,x_1)$ denotes the directional derivative of $V$ at $(t,x)$ along $(t_1,x_1)$. Since $(\eta_t^*, \zeta_t^*) \in D^-_{t,x}(t,x^*(t))$ for almost all $t \in [0,T]$, it follows from (4.15) that

$$\begin{aligned}\frac{d}{dt}V(t,x^*(t))\mid_{t=s} &\le \eta_s^* + \zeta_s^* \cdot b(s,x^*(s),\mu_1(s),\mu_2^*(s))\\ &= \eta_s^* + H(s,x^*(s),\zeta_s^*,\mu_1(s),\mu_2^*(s)) - r(s,x^*(s),\mu_1(s),\mu_2^*(s))\\ &\le \eta_s^* + \max_{\mu_1\in\mathcal{M}_1} H(s,x^*(s),\zeta_s^*,\mu_1,\mu_2^*(s)) - r(s,x^*(s),\mu_1(s),\mu_2^*(s))\\ &\le -r(s,x^*(s),\mu_1(s),\mu_2^*(s)), \text{ by (4.14)}. \end{aligned} \tag{4.16}$$

Integrating (4.16) from 0 to $T$ and rearranging the terms, it follows that

$$R(0,x,\mu_1(\cdot),\mu_2^*(\cdot)) \le V(0,x).$$

Thus $\mu_2^*(\cdot)$ is optimal for player 2 for $(0,x)$.

**Remark 4.4.** (i) If (4.13) is satisfied, it follows that

$$\eta_t^* + \max_{\mu_1\in\mathcal{M}_1}\min_{\mu_2\in\mathcal{M}_2} H(t,x^*(t),\zeta_t^*,\mu_1,\mu_2) \ge 0. \tag{4.17}$$

On the other hand, since $V$ is a viscosity solution of (4.7)-(4.8), it follows that

$$\eta_t^* + \max_{\mu_1\in\mathcal{M}_1}\min_{\mu_2\in\mathcal{M}_2} H(t,x^*(t),\zeta_t^*,\mu_1,\mu_2) \le 0. \tag{4.18}$$

Thus from (4.13), (4.17) and (4.18), it follows that for a.e. $t$

$$\max_{\mu_1\in\mathcal{M}_1}\min_{\mu_2\in\mathcal{M}_2} H(t,x^*(t),\zeta_t^*,\mu_1,\mu_2) = \min_{\mu_2\in\mathcal{M}_2} H(t,x^*(t),\zeta_t^*,\mu_1^*(t),\mu_2). \tag{4.19}$$

(ii) In case $D^+_{t,x}V(t,x^*(t))$ is empty and if (4.13) holds for $a.e.t$, for $(\eta_t^*,\zeta_t^*) \in D^-_{t,x}V(t,x^*(t))$, then $\mu_1^*(\cdot)$ is optimal for player 1.
(iii) Again if $D^-_{t,x}(t,x^*(t))$ is empty and if (4.14) holds for $a.e.t$, for $(\eta_t^*,\zeta_t^*) \in D^+_{t,x}V(t,x^*(t))$, then $\mu_2^*(\cdot)$ is optimal for player 2. In such a situation it can be shown as in (4.19) that for $a.e.t$

$$\min_{\mu_2\in\mathcal{M}_2}\max_{\mu_1\in\mathcal{M}_1} H(t,x^*(t),\zeta_t^*,\mu_1,\mu_2) = \max_{\mu_1\in\mathcal{M}_1} H(t,x^*(t),\zeta_t^*,\mu_1,\mu_2^*(t)). \tag{4.20}$$

(iv) If $D^+_{t,x}V(t,x^*(t)) \cap D^-_{t,x}V(t,x^*(t))$ is empty, then analogous results can be derived using closed super- and sub- differentials which are always nonempty for Lipschitz continuous functions.

## 5. Connection between DPP and MMP

We first make a formal connection between the costate process $p(t)$ in the MMP and the value function $V$ via dynamic programming. Suppose the value function $V$ is $C^{2,2}([0,T]\times \mathbb{R}^d)$. Consider the situation described in Remark 4.2. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be the outer maximizing and outer minimizing selectors respectively of (4.12). Let (2.1) have a unique solution $\overline{x}(\cdot)$ under the feedback strategies $\overline{\mu}_1, \overline{\mu}_2$. Then

$$\begin{aligned} V_t(t,\overline{x}(t)) + \nabla_x V(t,\overline{x}(t))\cdot b(t,\overline{x}(t),\overline{\mu}_1(t,\overline{x}(t)),\overline{\mu}_2(t,\overline{x}(t))) \\ +r(t,\overline{x}(t),\overline{\mu}_1(t,\overline{x}(t)),\overline{\mu}_2(t,\overline{x}(t))) = 0. \end{aligned}$$

Differentiating the above, we get

$$\begin{aligned} \nabla_x V_t(t,\overline{x}(t)) &+ V_{xx}(t,\overline{x}(t)) b(t,\overline{x}(t),\overline{\mu}_1(t,\overline{x}(t)),\overline{\mu}_2(t,\overline{x}(t))) \\ &+ b'_x(t,\overline{x}(t),\overline{\mu}_1(t,\overline{x}(t)),\overline{\mu}_2(t,\overline{x}(t)))\nabla_x V(t,\overline{x}(t)) \qquad (5.1) \\ &+ \nabla_x r(t,\overline{x}(t),\overline{\mu}_1(t,\overline{x}(t)),\overline{\mu}_2(t,\overline{x}(t))) = 0. \end{aligned}$$

Set

$$\overline{p}(t) = \nabla_x V(t,\overline{x}(t)), \qquad (5.2)$$

then

$$\begin{aligned} \dot{\overline{p}}(t) &= \nabla_x V_t(t,\overline{x}(t)) + V_{xx}(t,\overline{x}(t)) b(t,\overline{x}(t),\overline{\mu}_1(t,\overline{x}(t)),\overline{\mu}_2(t,\overline{x}(t))) \\ &= -b'_x(t,\overline{x}(t),\overline{\mu}_1(t,\overline{x}(t)),\overline{\mu}_2(t,\overline{x}(t)))\overline{p}(t) \\ &\quad -\nabla_x r(t,\overline{x}(t),\overline{\mu}_1(t,\overline{x}(t)),\overline{\mu}_2(t,\overline{x}(t))), \\ \overline{p}(1) &= g_x(\overline{x}(1)), \end{aligned}$$

where $V_{xx}$ is the Hessian matrix. Thus we get the costate process $\overline{p}(\cdot)$ and the costate equation (3.1). Note that the above arguments are purely formal as in most cases $V$ is not even $C^1$. Besides (2.1) will not, in general, admit a unique solution under feedback strategies. Following Zhou [25], we now derive the above type of connection between MMP and DPP via the super-differential and sub-differential of $V$.

From Theorem 3.2 in [25], the following result follows. We omit the details.

**Theorem 5.1.** Let $(x^*(\cdot), \mu_1^*(\cdot), \mu_2^*(\cdot))$ be as in Theorem 3.1.. Then for a.e., $t \in [0,T]$,

$$D_x^- V(t, x^*(t)) \subset \{p(t)\} \subset D_x^+ V(t, x^*(t)).$$

Theorem 5.1 describes the non-smooth and rigorous connection between DPP and MMP. Note that Theorem 5.1 only connects the costate process with the super- and sub-differential of the value function $V$. We next proceed to make a connection with the Hamiltonian as well. Towards this end we first prove the following result.

**Lemma 5.1.** Let $(x^*(\cdot), \mu_1^*(\cdot), \mu_2^*(\cdot))$ be as before. Then for a.e., $t \in [0,T]$, if $(\eta_t, \zeta_t) \in D_{t,x}^+ V(t, x^*(t)) \cup D_{t,x}^- V(t, x^*(t))$, then

$$\begin{aligned} \eta_t + H(t, x^*(t), \zeta_t, \mu_1^*(t), \mu_2^*(t)) &= \eta_t + \min_{\mu_2 \in \mathcal{M}_2} \max_{\mu_1 \in \mathcal{M}_1} H(t, x^*(t), \zeta_t, \mu_1, \mu_2) \\ &= \eta_t + \max_{\mu_1 \in \mathcal{M}_1} \min_{\mu_2 \in \mathcal{M}_2} H(t, x^*(t), \zeta_t, \mu_1, \mu_2) \\ &= 0. \end{aligned} \tag{5.3}$$

**Proof.** Let $(\eta_t, \zeta_t) \in D_{t,x}^- V(t, x^*(t))$. Then by a straightforward computation (see the proof of Proposition 3.1 in [25] for more details), it follows that

$$\eta_t + H(t, x^*(t), \zeta_t, \mu_1^*(t), \mu_2^*(t)) = 0.$$

By Theorem 5.1, it follows that $\zeta_t = p(t)$. Thus by Theorem 3.1, the equation (5.3) holds in this case. On the other hand, if $(\eta_t, \zeta_t) \in D_{t,x}^+ V(t, x^*(t))$, then it can again be shown that

$$\eta_t + H(t, x^*(t), \zeta_t, \mu_1^*(t), \mu_2^*(t)) = 0. \tag{5.4}$$

Since $V$ is viscosity solution of (4.7)-(4.8), it follows that

$$\eta_t + \inf_{\mu_2 \in \mathcal{M}_2} \sup_{\mu_1 \in \mathcal{M}_1} H(t, x^*(t), \zeta_t, \mu_1, \mu_2) \leq 0. \tag{5.5}$$

On the other hand, if player 1 announces that he is using strategy $\mu_1^*(\cdot)$, then for player 2 it is an optimal control problem. In this situation $V$

is his value function and $\mu_2^*(\cdot)$ an optimal control. Thus it follows as in Proposition 3.1 in [25] that

$$\eta_t + \inf_{\mu_2 \in \mathcal{M}_2} H(t, x^*(t), \zeta_t, \mu_1^*, \mu_2) = 0.$$

Therefore

$$\eta_t + \sup_{\mu_1 \in \mathcal{M}_1} \inf_{\mu_2 \in \mathcal{M}_2} H(t, x^*(t), \zeta_t, \mu_1, \mu_2) \geq 0 . \tag{5.6}$$

Thus (5.3) again follows by combining (5.4)-(5.6).

In view of Theorem 5.1, Lemma 5.1, we can mimic the the arguments in the proof of Theorem 3.2 in [25] to get the following result. We again omit the details.

**Theorem 5.2.** Let $(x^*(\cdot), \mu_1^*(\cdot), \mu_2^*(\cdot))$ be as before. Then for a.e., $t \in [0, T]$,

$$D^-_{t,x}V(t, x^*(t)) \subset \{-H(t, x^*(t), p(t), \mu_1^*(t), \mu_2^*(t)), p(t)\} \subset D^+_{t,x}V(t, x^*(t)).$$

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