

EXACT CONTROLLABILITY OF GENERALIZED HAMMERSTEIN TYPE EQUATION

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Abstract In this article, we study the exact controllability of an abstract model described by the controlled generalized Hammerstein type equation
$$x(t) = \int_0^t h(t,s)u(s)ds + \int_0^t k(t,s,x) f(s,x(s))ds; \quad 0 \leq t \leq T < \infty,$$
$$0 \leq t \leq T < \infty,$$
 where, the state $x(t)$ lies in a Hilbert space X and control $u(t)$ lies another Hilbert space V . We establish the controllability under suitable assumptions on h, k and f using the monotone operator theory.

1. Introduction

Let X and V be Hilbert spaces and $I = [0, T]$, where $0 < T < \infty$. Let $Y = L^2(0, T; X)$ be the solution space and $U = L^2(0, T; V)$ be the control

function space. We consider the following nonlinear control problem:

$$x(t) = \int_0^t h(t,s)u(s)ds + \int_0^t k(t,s,x) f(s,x(s))ds; \quad 0 \leq t \leq T < \infty. \quad (1.1)$$

Here, the state of the system $x(t) \in X$ and $u(t) \in U$ is the control at time t . The nonlinear function $f : I \times X \mapsto X$ and for each $t, s \in I, x \in Y$, the kernel $k(t,s,x) : X \mapsto X$ and $h(t,s) : V \mapsto X$ are bounded linear operators. Since the integrands in the above integrals are in the infinite dimensional Hilbert space X , the integrands are understood in the sense of Bochner integrals.

Remark 1 *Note that in equation (1.1), the kernel k depends on the unknown x , but not depends on pointwise. That is the system has to be treated separately if we consider the kernel $k(t,s,x(s))$.*

Remark 2 *The equation (1.1) satisfies the initial condition $x(0) = 0 \in X$, but one can incorporate any initial state $x(0) = x_0$ which will not alter the results.*

Definition 3 *The system (1.1) is said to be exactly controllable over the interval $[0, T]$, if for any given $x_1 \in X$, there exists a control $u \in U$ such that the corresponding solution x of (1.1) satisfies $x(T) = x_1$.*

A large amount of literature is available regarding the existence and uniqueness of the above type of equation as well as related equations. See W. Petry [10], Stuart [11], Leggett [9], Backwinkel-Schilling [8], Srikant-Joshi [12] to name a few and the references therein.

The corresponding linear control system

$$x(t) = \int_0^t h(t,s)u(s)ds; \quad 0 \leq t \leq T < \infty, \quad (1.2)$$

is quite standard and one can give various conditions to ensure the exact controllability of the linear system (1.2). So, throughout our article, we assume that the linear system is exactly controllable.

The exact controllability of related nonlinear systems are also available. See, for example, [2], [3], [4] and for approximate controllability of non-autonomous semi-linear system [7]. Joshi-George [5] have established the exact controllability for nonlinear systems in finite dimensional settings using the monotone operator theory and fixed point theorems. Our aim in this

article is to generalize these results to infinite dimensional systems. In this short article, we will only present some abstract results. The application of abstract results to specific examples both from ordinary and partial differential equations and also other sufficient conditions are the topics of a future article.

The outlay of the paper is as follows. In section 2, we give assumptions and some preliminary estimates and we transform the controllability problem to that of a solvability problem. An operator W corresponding to the linear system will be introduced and controllability depends on the compactness of this operator. We prove the compactness under various sufficient conditions in Section 3. Finally in Section 4, we establish the exact controllability result.

2. Assumptions and Estimates

Define the following operators

(2.1) for $x \in Y$, $K(x) : Y \mapsto Y$ by

$$(K(x)y)(t) = \int_0^t k(t, s, x)y(s)ds,$$

(2.2) $H : U \mapsto Y$ by $(Hu)(t) = \int_0^t h(t, s)u(s)ds$,

(2.3) $N : Y \mapsto Y$ by $(Nx)(t) = f(t, x(t))$ and

(2.4) $W : U \mapsto Y$ by $Wu = f(\cdot, x(\cdot))$, where $x(\cdot)$ is the solution of (1.1) corresponding to $u \in U$.

First, we reduce the controllability problem to a solvability problem. The results on solvability crucially depend on the compactness of W . We make the following assumptions.

Assumptions [A]

[A₁] $\{\int_0^T \int_0^t \|k(t, s, x)\|^2 ds dt\}^{\frac{1}{2}} \equiv k(x) < k_0 < \infty$.

[A₂] $\{\int_0^T \int_0^t \|h(t, s)\|^2 ds dt\}^{\frac{1}{2}} \equiv h_0 < \infty$.

[A₃] The function f satisfies Caratheodory conditions. i.e., $t \rightarrow f(t, \cdot)$ is measurable and $x \rightarrow f(\cdot, x)$ is continuous.

[A₄] The function f satisfies the following growth condition:

$$\|f(t, x)\| \leq a_0 \|x\| + b(t),$$

where $a_0 \geq 0$ is a constant and $b(t) \geq 0$ and $b \in L^2(I)$.

Lemma 2.1 (Estimates): For each x , the operator $K(x)$ and H are bounded linear operators and N is a continuous non-linear operator and they satisfy the following estimates:

$$\|K(x)y\|_Y \leq k \|y\|_Y \quad \forall x, y \in Y. \quad (2.5)$$

$$\|Hu\| \leq h \|u\|_U \quad u \in U. \quad (2.6)$$

$$\|Nx\|_Y \leq \sqrt{2} (a_0 \|x\|_Y + b_0) \quad \forall x \in Y, \quad (2.7)$$

where $b_0 = \|b\|_{L^2(I)}$.

Proof: The estimate (2.5) follows from Cauchy-Schwartz inequality as:

$$\begin{aligned} \|K(x)y\|_Y^2 &= \int_0^T \|((Kx)(y)(t))\|_X^2 dt \\ &\leq \int_0^T \left(\int_0^t \|k(t,s,x)\| \|y(s)\|_X ds \right)^2 dt \\ &\leq \int_0^T \left(\int_0^t \|k(t,s,x)\|^2 ds \right) \left(\int_0^t \|y(s)\|^2 ds \right) dt \\ &\leq k_0^2 \|y\|_Y^2. \end{aligned}$$

The inequality (2.6) follows in a similar fashion. Now

$$\begin{aligned} \|Nx\|_Y^2 &= \int_0^T \|Nx(t)\|_X^2 dt = \int_0^T \|f(t,x(t))\|_X^2 dt \\ &\leq 2 \int_0^T [a^2 \|x(t)\|^2 + b(t)^2] dt \\ &\leq 2 [a^2 \|x\|_Y^2 + b^2] \leq 2 [a \|x\|_Y + b]^2. \end{aligned}$$

Hence (2.7). ■

Operator form of the equation: With the notations as earlier, we may write the equation (1.1) as

$$x(t) = (Hu)(t) + (K(x)(Nx))(t) \quad (2.8)$$

or, equivalently

$$x = Hu + K(x)(Nx). \quad (2.9)$$

The following theorem gives the existence of solution x of (2.9) for a given u which can be proved along the lines as in [5].

Theorem 4 (Existence and Uniqueness:) *Assume the following:*

[AK1] *There exists a constant $\mu > 0$ such that*

$$\int_0^T \left(\int_0^t k(t,s,x)x(t)ds, x(t) \right) > ds \geq \mu \int_0^T \left\| \int_0^t k(t,s,x)x(t)ds \right\|^2 dt \quad (2.10)$$

$\forall x \in Y$

[AF1] *The function f is monotone in the sense that*

$$\langle f(t,x) - f(t,y), x - y \rangle \leq 0 \quad \forall x, y \in X, t \in I. \quad (2.11)$$

Then, given $u \in U$, there exists a unique solution $x \in Y$ of (2.9) and x satisfies a growth condition

$$\|x\|_Y \leq \frac{b_0}{\mu} + \left(\frac{b_0}{\mu} + 1 \right) h_0 \|u\|_U. \quad (2.12)$$

■

Lemma 2.2 *Under the assumptions [AK1],[AF1] and the assumptions [A], the operator W is well-defined and continuous. Moreover it satisfies the following growth condition:*

$$\|Wu\|_Y \leq \sqrt{2} \left(\frac{b_0}{\mu} + 1 \right) a_0 h_0 \|u\|_U + \sqrt{2} \left(\frac{1}{\mu} + 1 \right) b_0. \quad (2.13)$$

The proof follows from the assumptions and estimate (2.12).

3. Compactness of the operator W

We make the following further assumptions in this section to guarantee the compactness of W .

Assumptions [B]

[B₁] *There exists $\tilde{k} > 0$ such that*

$$\left\| \int_s^t k(t,\tau,x)x(\tau)d\tau \right\|_X \leq \tilde{k}(t-s) \|x\|_Y, \quad 0 \leq s < t \leq T.$$

[B₂] There exists $\bar{h} > 0$ such that

$$\left\| \int_s^t h(t, \tau) u(\tau) d\tau \right\|_X \leq \bar{h}(t-s) \|u\|_U, \quad 0 \leq s < t \leq T.$$

[B₃] The operators k and h satisfy the uniform continuity in the following sense: Given $\varepsilon > 0$ there exists $h > 0$ small such that

$$\|k(r+h, s, x) - k(r, s, x)\|_{BL(X)} \leq \varepsilon$$

and

$$\|h(r+h, s) - h(r, s)\|_{BL(X)} \leq \varepsilon, \quad 0 \leq r \leq r+h \leq T.$$

[B₄] There exists a space \hat{X} such that $X \mapsto \hat{X}$ is a compact imbedding.

[B₅] Assume that f can be extended to $I \times \hat{X} \mapsto X$ such that f is Caratheodory and $x \mapsto f(\cdot, x(\cdot))$ is continuous from $L^2(I; \hat{X}) \mapsto L^2(I; X)$

Theorem 5 *Under the assumptions (B), the operator W is compact.*

Proof: Let $\{u_n\}$ be a bounded sequence in U . We have to show that $\{Wu_n\} = \{f(\cdot, x_n(\cdot))\}$ has a convergent subsequence. First of all $\{f(\cdot, x_n(\cdot))\}$ is bounded in Y by Lemma 2.2. Therefore there exists a constant $M > 0$ such that

$$\int_0^T \|f(t, x_n(t))\|_X^2 dt \leq M^2.$$

We show that the family $\{x_n(\cdot)\}$ is equi-continuous in $C(I; X)$. Let $t = r + h_0$. We have

$$\begin{aligned} \|x_n(t) - x_n(r)\| &\leq \left\| \int_0^r \{k(t, \tau, x_n) - k(r, \tau, x_n)\} f(\tau, x_n(\tau)) d\tau \right\| \\ &\quad + \left\| \int_r^t k(t, \tau, x_n) f(\tau, x_n(\tau)) d\tau \right\| \\ &\quad + \left\| \int_0^r \{h(t, \tau) - h(r, \tau)\} u_n(\tau) d\tau \right\| + \left\| \int_r^t h(t, \tau) u_n(\tau) d\tau \right\| \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now by (B₃) and (B₁) respectively, we get

$$I_1 \leq \varepsilon \int_0^T \|f(\tau, x_n(\tau))\|_X d\tau \leq \varepsilon \tau^{\frac{1}{2}} M \leq \varepsilon M T^{\frac{1}{2}}$$

and

$$I_2 \leq \tilde{h} h_0 \|f(\cdot, x_n(\cdot))\|_Y.$$

Similarly I_3 and I_4 can be estimated as

$$I_3 \leq \varepsilon T^{\frac{1}{2}} \|u_n\|_U \text{ and } I_4 \leq \tilde{h} h_0 \|u_n\|_U.$$

The above estimates shows that $\{x_n(\cdot)\}$ is equi-continuous in $C(I; X)$. Further, $\{x_n(\cdot)\}$ is also uniformly bounded in $C(I; X)$. Now, using the compact inclusion $X \hookrightarrow \hat{X}$ and applying general form of Arzela-Ascoli theorem [1], we deduce that $\{x_n(\cdot)\}$ is relatively compact in $C(I; \hat{X})$. Thus along a subsequence $\{x_{n_k}\}$ converges in $C(I; \hat{X})$ and so converges in $L^2(0, T; \hat{X})$. Then from the assumption (B_5) , it follows that $f(\cdot, x_{n_k}(\cdot))$ converges in Y . Thus the operator is compact and the proof is complete. ■

Remark 6 *If $h(t, s)$ is a compact operator, then it is easy to show that W is compact. In such situations, the exact controllability in the whole space may be impossible (see [13]).*

Remark 7 *It is possible to give various more specific conditions under which the operator W is compact, but we do not go into these details in this short article.*

We now move on to the exact controllability under the assumption that the operator W is compact.

4. Exact controllability

We, first reduce the problem to a solvability problem. Define the control operator $C : U \mapsto X$ by

$$Cu = \int_0^T h(t, s) u(s) ds. \quad (4.1)$$

The operator C is bounded linear and in fact, is a control operator for the linear system

$$x(t) = \int_0^t h(t, s) u(s) ds, \quad x(0) = 0. \quad (4.2)$$

Let $N(C) = \{u \in U : Cu = 0\}$ be the null space and $Z = [N(C)]^\perp = \{u \in U : (u, v) = 0 \ \forall v \in N(C)\}$.

Definition 8 We call a bounded linear operator $S : X \mapsto Z$, a **Steering Operator** if S steers the linear system (4.2) from 0 to x_1 . In other words, if $u = Sx_1$, ($x_1 \in X$), then

$$x(T) = \int_0^T h(T, s)(Sx_1)(s)ds = x_1$$

Clearly $CS = I$, identity operator on X . Thus, if there exists a steering operator S , then $u = Sx_1$ acts as a control and the linear system (4.2) is controllable. Conversely, if the linear system is controllable, then for any $x_1 \in X$ there exists $u \in U$ such that $Cu = x_1$; i.e., C is onto. Thus, we can define a generalized inverse $C^\# = (C|_Z)^{-1} : X \mapsto Z$ and $S = C^\#$ will be a steering operator. Thus, one gets the following result.

Theorem 9 The linear system (4.2) is exactly controllable if and only if there exists a steering operator.

We now assume the controllability of the linear system and proceed to prove the exact controllability of the nonlinear system. Define an operator $F : Z \mapsto X$ by

$$Fu = \int_0^T k(T, s, x)(Wu)(s)ds, \quad (4.3)$$

where given $u \in U$, let $x \in Y$ be the corresponding solution of (1.1). Let S be the steering operator of the linear system. Let $x_1 \in X$ and $u_0 = Sx_1$ be the control which steers the linear system from 0 to x_1 . The exact controllability of (1.1) is equivalent to the existence of $u \in Z$ (let x be the corresponding solution (1.1)) such that

$$x_1 = x(T) = \int_0^T k(T, s, x)(Wu)(s)ds + \int_0^T h(t, s)u(s)ds.$$

That is

$$x_1 = Fu + Cu.$$

Applying S on both sides, we get

$$u_0 = SFu + u.$$

Thus, the problem of controllability reduces to solvability problem of the operator equation :

$$\begin{cases} \text{Solve } u \in Z \\ (I + SF)u = u_0. \end{cases} \quad (4.4)$$

We now state our controllability result. For the sake of generality, we state the theorem by imposing indirect conditions on W and F . The explicit conditions on k, h, f can be given to verify the conditions on W and F . This will be doing in a future paper along with applications to explicit problems.

Theorem 10 *Assume that the linear system (4.2) is exactly controllable with a steering operator S and the non-linear operator W defined in Section 2 is well defined and compact. Further, assume that the composition operator SF , where F is defined by (4.3), satisfies:*

$$\|SFu\| \leq a_0 \|u\| + b_0, \text{ with } a_0 < 1, b_0 \geq 0.$$

Then the system (4.4) is solvable in Z .

Proof: We look for the solvability of the operator $R : Z \mapsto U$, where

$$Ru = [I + SF]u.$$

Then

$$\begin{aligned} \langle Ru, u \rangle &= \langle u, u \rangle + \langle SFu, u \rangle \\ &\geq \|u\|^2 - a_0 \|u\| - b_0 \|u\|, \end{aligned}$$

which implies

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle Ru, u \rangle}{\|u\|} = \infty.$$

Thus, R is coercive operator. Since W is compact and from the definition (4.3), it follows that F is compact. Thus the operator SF is compact as S is continuous.

Now, R is compact perturbation of the identity operator and hence R is of type (M). See [6] for a definition of type(M). Since any coercive operator of type(M) is onto [6], the proof of the theorem is complete. ■

Corollary 11 *Assume the linear system is controllable with a steering operator S . Assume the conditions [AK1] and [AF1] and the assumptions (B). Then the nonlinear system is controllable if*

$$\|S\| k_0(b_0 + \mu)a_0 h_0 < \mu.$$

Remark 12 *As mentioned earlier, we have our main theorem in a general form and it is possible to give various conditions in terms of the system components to verify the same. One can put nonlinear evolution systems with internal controls in the above frame to study the controllability. It is also possible to use the above results to study exact controllability problems associated with partial differential equations with boundary controls.*

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