An Overview of Homogenization

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Abstract | Homogenization of partial differential equations is relatively a new area and has tremendous applications in various branches of engineering sciences like: material science, porous media, study of vibrations of thin structures, composite materials to name a few. Though the material scientists and others had reasonable idea about the homogenization process, it was lacking a good mathematical theory till early seventies. The first proper mathematical procedure was developed in the seventies and later in the last 30 years or so it has flourished in various ways both application wise and mathematically.
This is not a full survey article and on the other hand we will not be concentrating on a specialized problem. Indeed, we do indicate certain specialized problems of our interest without much details and that is not the main theme of the article. I plan to give an introductory presentation with the aim of catering to a wider audience. We go through few examples to understand homogenization procedure in a general perspective together with applications. We also present various mathematical techniques available and if possible some details about some of the techniques. *A possible definition of homogenization would be that it is a process of understanding a heterogeneous (in-homogeneous) media, where the heterogeneties are at the microscopic level, like in composite materials, by a homogeneous media. In other words, one would like to obtain a homogeneous description of a highly oscillating in-homogeneous media*. We also present other generalizations to non linear problems, porous media and so on. Finally, we will like to see a closely related issue of *optimal bounds* which itself is an independent area of research.

1. Introduction

The mathematical theory of homogenization gained its significance in the seventies when rigorous mathematics to understand the procedure was introduced. The main motivation was from the study of composite materials, more generally any medium or domain which involves microstructures. Simply speaking, homogenization is a mathematical procedure to understand *heterogeneous materials (or media)* with *highly oscillating* heterogeneties (at the microscopic level) via a *homogeneous material*. Mathematically, it is a limiting analysis. The physical problems described on such materials leads to the study of mathematical equations like: differential or integral equations, optimization problems, spectral problems, and so on, will exhibit high oscillations in the coefficients present in the equation or in the domain. This high frequency oscillations, in turn, will reflect in the solutions. Thus, even if the well - posedness of the problems were guaranteed, a numerical computation (to predict the behaviour of such heterogeneous media) of such solutions will be highly non-trivial; in fact, it is almost impossible.

The homogenization deals with the study of asymptotic analysis of such solutions and obtain the equation satisfied by the limit. This limit equation

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will characterize the **bulk/overall** behaviour of the material, which doesn't consists of microscopic heterogenities and can be solved or computed. This solved and computed solution will then, be a good approximation, in a suitable sense, to the original solution.

Composite Materials
These materials are obtained by **fine** mixing of two or more materials with different physical properties. The study of composite material is an important aspect in material science. The problems modeled on such materials leads to homogenization problems. Composite materials arise plenty in nature: e.g.; *wood, bone, lungs, soil, sand stone, granular media, any porous media* and so on. Moreover, the material scientists/engineers are engaged for many years in constructing composites (e.g. *concrete, reinforced concrete, plywood, steel* etc.) of desired properties. For example, one would like to construct good conducting materials in large quantities in an optimal way by mixing two or more materials, where the availability of given good conducting material may be limited. Or one may prefer to develop elastic materials of a combination of contradictory properties, say *stiffness and softness* in different directions or high stiffness and low weight etc. The pore structure of the bonds, trunks of wood, leaves of trees provide examples when mixtures of stiff and soft tissues can be treated as a composite. The honey-comb structures are light and possess a high bending stiffness.

Porous Media
An example of another important micro-structure is the porous media where the porosity is at a fine scale. Examples are fluid flow through porous media; like flow of ground water, oil, flow of resin in mould industries. We will see the limiting equation depends heavily on the size of the porosity at the micro scale.

Layered materials (like plywood etc.)
This also can be viewed as a composite with oscillations only in one direction.

Micro-structure of phase transition
The crystal structure of materials changes at a critical temperature while cooling. This happens at the atomic level and the structure moves from *Austenite (high temperature)* state to *Martensite (low temperature)* state.

Analysis of vibrations of thin structures
The characteristic in all these structures is that locally inhomgeneous material behaves as a homogeneous medium when the size of the inclusions is much smaller than the size of the whole sample. In such a situation the properties of the composite can be described by the *effective modulli* by special kind of averaging of the properties of the components. We will see soon by a one dimensional example that the effective property of the medium is not the arithmetic average of the same properties of the components. *The branch of mathematics that study the effective behaviour of such phenomena is known as homogenization theory.*

Fore more details, we refer the readers to the literature: Refs[1,4,7,9,13,17,18,26,28].

Shape optimization or Optimal shape design
Shape optimization is a branch of calculus of variations, where we look for optimal shapes under certain criteria. The typical problem is to minimize a class of functionals over the *characteristic functions (shapes)*. It may be cost or energy functionals. The standard variational methods do not work here due to following reasons:

(i) The minimizing sequence of characteristic functions may not converge to a characteristic function.
(ii) It may not be possible to derive optimality conditions (the sum of two characteristic functions need not be a characteristic function and hence one cannot do variations on the functionals.

To remove these difficulties, the so called *relaxation methods* are used. Enlarge the class of admissible functions and also the functional without loosing too much information. Homogenization is a systematic method for computing a relaxed formulation, where the cost functional is defined via PDEs. Furthermore homogenization gives a physical meaning to the relaxation process by associating to minimizing sequence and generalized designs, the concrete notion of infinitely fine mixtures and composite materials.

2. Elliptic Problems

We now introduce homogenization problem related to a class of elliptic operators. We begin with the mathematical description and later we will see how such problems arise in literature. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Define, for $\alpha, \beta > 0$, the class of matrix functions:

$$E(\Omega) = E(\alpha, \beta, \Omega) = \{A = [a_{ij}(x)] : A \text{ is symmetric and satisfies (2.1)}\}.$$

We assume the matrix A satisfies

$$\alpha|\xi|^2 \leq \langle A(x)\xi, \xi\rangle = a_{ij}\xi_i\xi_j \leq \beta|\xi|^2, \forall \xi \in \mathbb{R}^n. \tag{2.1}$$

The first inequality is nothing but the uniform ellipticity.

The aim is to introduce certain convergence in the above class relevant to the homogenization theory. This will become more clear as we go further. Given an element $A \in E(\Omega)$, introduce the elliptic boundary value problem

$$\begin{aligned} \mathcal{A}u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.2}$$

Here $\mathcal{A} = -\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$ is the PDE operator associated to A.

Definition 2.1 (*G*-convergence or *H*-convergence). *We say a family $\{[a_{ij}^{\varepsilon}]\}_{\varepsilon>0}$, H-converges to $[a_{ij}^{*}]$ as $\varepsilon \to 0$ if*

i) $u_\varepsilon \rightharpoonup u$ *in* $H_0^1(\Omega)$ *weak*

ii) $a_{ij}^{\varepsilon}(x)\frac{\partial u_\varepsilon}{\partial x_j} \rightharpoonup a_{ij}^{*}(x)\frac{\partial u}{\partial x_j}$ *in* $L^2(\Omega)$ *weak.*

Here u^ε, u are, respectively, the solution of (2.2) corresponding to the operators $\mathcal{A}^\varepsilon$, $\mathcal{A}^*$ and we write $[a_{ij}^{\varepsilon}] \overset{H}{\rightharpoonup} [a_{ij}^{*}]$ or simply $A^\varepsilon \overset{H}{\rightharpoonup} A^*$. ∎

There is a very general compactness theorem (See[9,13]) which is given below.

Theorem 2.2 (Compactness Theorem). *Let $[a_{ij}^{\varepsilon}]_{\varepsilon>0}$, $\varepsilon \to 0$ be any family from $E(\Omega)$. Then there is a subsequence $\varepsilon_n \to 0$ and a matrix $[a_{ij}^{*}] \in E(\Omega)$ such that*

$$[a_{ij}^{\varepsilon}] \overset{H}{\rightharpoonup} [a_{ij}^{*}].$$

Remark 2.3. *If we do not include the symmetry assumption in the class $E(\Omega)$, one can still have a compactness theorem. In this case we can only conclude that $A^* \in E(\alpha, \frac{\beta^2}{\alpha}, \Omega)$. Observe that $\frac{\beta^2}{\alpha} \geq \alpha$.*

At first sight, from the above theorem, it may seem that the limiting analysis has been over. But, it is far from complete in the sense that, in general, we do not know the characterization of $[a_{ij}^{*}]$. The general theorem is not very useful as far as applications are concerned as the problems arising from physical and engineering models and needs to calculate the limiting coefficients which represents the physical modeling. However, there are some special cases, where a_{ij}^{*} can be explicitly characterized. For example, in the periodic case as well as in composites of thin sheets (layered materials). The following two questions are of utmost importance; *(i) Either explicitly give methods to evaluate A^* or (ii) Give appropriate lower and upper estimates (known as optimal bounds) on the limiting matrix.* Few examples where (i) can be answered will be given soon. We do present certain optimal estimates in the last section of this article. Before the examples, let me say few words on a related issue.

G-Closure Problem

In general the homogenized cannot be calculated, but by the general theorem, various sub-sequential *G-limits* or *H-limits* exist. The major open problem for the material scientists is to describe all the limits. More precisely, *describe the set of all effective property tensors of the composite which can be obtained from the given amount of the component materials.* This is the famous *G-closure* problem. This is largely depend on the following three information; namely the properties of the component materials, their volume fractions and more importantly the micro-structure. In general the first two are known to us, where as the third one is delicate and not available. Indeed, given a micro-structure, finding the effective modulli is the process of homogenization. But in practice the micro-structure is not known and describing G-closure set is open.

The next step is to obtain bounds on G-closure set from above and below which are independent of the micro-structure. More precisely, bounds are valid for a composite material of any structure with fixed volume fractions of the components. Also look for the effective property tensors of particular structures from component material (e.g. laminate composite, laminate composite of laminate composite, Hashin-Strikman type structure etc.). We refer the readers to Refs [10–12,30].

Composite of two materials; A mathematical modeling

Let two homogeneous materials with physical property (say conductivity) γ_1 and γ_2 occupies a fixed domain Ω which is a bounded set in $\mathbb{R}^n$. At a fine mixing of these two materials, it will occupy some parts of Ω which we denote by Ω_1^ε and Ω_2^ε, respectively. Here ε represents the micro geometry of the mixing which in general is unavailable or unknown, but fine-ness is indicated by the smallness of ε. The volume ratio of the materials fixed and constant, that is $\theta_1 = \frac{|\Omega_1^\varepsilon|}{|\Omega|}$ and $\theta_2 = \frac{|\Omega_2^\varepsilon|}{|\Omega|}$ are independent of ε. Of course one can think of more complicated problems where these quantities itself may change. The property of the medium at the micro level is then given by

$$\gamma^\varepsilon(x) := \gamma_1 \chi_{\Omega_1^\varepsilon} + \gamma_2 \chi_{\Omega_2^\varepsilon}.$$

Here χ_A is the characteristic function of the subset A of Ω. As is well known to the material scientists that for a fine mixing, that is when ε is small, the composite may behave like a homogeneous material with certain physical property (conductivity) γ. *The aim is to describe this γ as limit of γ^ε in some suitable sense which is precisely the homogenized limit.* In this case the elliptic equation, which is the modeling of various physical phenomena in such a composite, takes the form as in (2.2) with $\mathcal{A} = \mathcal{A}^\varepsilon(x) = \gamma^\varepsilon(x) I$.

Remark 2.4. *The two material described has an isotropic microstructure. But I would like to remind that the composite limit need not be isotropic in the sense that $\mathcal{A}^*$ need not be of the form γ I. This will be clear from the examples to be studied soon.*

More generally, an elliptic problem in a composite of two or more materials with isotropic or non isotropic can be written in the form as in (2.2), with appropriate boundary conditions.

We end this section with a notion of correctors.

Correctors

The compactness theorem tells us that the solution $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ weak, which means that $u_\varepsilon \rightharpoonup u$ strongly in $L^2(\Omega)$, but the ∇u_ε converges weakly in $L^2(\Omega)$. But the weak convergence is not good for numerical computations and the gradient is equally an important physical quantity. In general, the weak convergence cannot be improved and can be seen from the one dimensional example to be presented below. The aim of correctors is to add suitable lower order terms to obtain 'some sort of' strong approximation to ∇u_ε which will be extremely useful in practical applications.

3. Examples

Example 3.1 (One Dimensional case). *Let $a^\varepsilon \in L^\infty(c,d)$. Consider one-dimensional problem:*

$$\begin{cases} -\frac{d}{dx}(a^\varepsilon \frac{du^\varepsilon}{dx}) = f \text{ in } (c,d) \\ u_\varepsilon(0) = u_\varepsilon(1) = 0 \end{cases} \tag{3.1}$$

where $f \in L^2(c,d)$ is given.

Assume that

$$0 < \alpha \le a^\varepsilon(x) \le \beta \quad \text{a.e. x.}$$

We set

$$\xi_\varepsilon = a^\varepsilon(x)\frac{du_\varepsilon}{dx},$$

so that

$$-\frac{d\xi_\varepsilon}{dx} = f\,. \tag{3.2}$$

Since a^ε is bounded in $L^\infty(c,d)$ and u_ε is bounded in $H_0^1(c,d)$, we have $\frac{du_\varepsilon}{dx}$ is bounded in $L^2(c,d)$. Thus, ξ_ε is bounded in $L^2(c,d)$. We can derive upto a subsequence, $\frac{du_\varepsilon}{dx} \rightharpoonup \frac{du}{dx}$ and $\xi_\varepsilon \rightharpoonup \xi$ weakly in $L^2(c,d)$ and $a^\varepsilon \rightharpoonup a^*$ in $L^\infty(c,d)$ weak*. Hence, we get the equation $-\frac{d\xi}{dx} = f$. It remains to find the relation between ξ and u to complete the analysis.

From the relation between ξ_ε, u_ε and a^ε we may tend to conclude that $\xi = a^* \frac{du_\varepsilon}{dx}$. Unfortunately this is not true in general as the weak convergence do not preserve nonlinearities. This is the major issue not only homogenization problems, it is an issue in other non linear problems as well.

The present one dimensional situation can be handled with a simple trick. By (3.2), ξ_ε is bounded in $H^1(c,d)$. Thus, upto a subsequence, $\xi_\varepsilon \rightharpoonup \xi$ weakly in $H^1(c,d)$ and so strongly in $L^2(c,d)$. We may write

$$\frac{du_\varepsilon}{dx} = \frac{1}{a^\varepsilon}\xi_\varepsilon.$$

Note that $\{\frac{1}{a^\varepsilon}\}$ is bounded in $L^\infty(c,d)$ and so converges upto a subsequence to some b in $L^\infty(c,d)$ weak*. Now, we can pass to the limit on the RHS as ξ_ε having strong convergence to get $\xi = \frac{1}{b}\frac{du_\varepsilon}{dx}$. Finally we get the equation

$$\begin{cases} -\frac{d}{dx}\left(\frac{1}{b}\frac{du}{dx}\right) = f \quad \text{in } (c,d) \\ u(0) = u(1) = 0\,. \end{cases}$$

Remark 3.2. *Contrary to intuition, we got the limiting coefficient as $\frac{1}{b}$ and not the intuitive limit a^*. We remark that in general these two quantities are not equal.*

Periodic case

In this situation, one can explicitly compute the limiting coefficient. Periodic case is settled in general case as well, but the proof is different. Let $Y = [0,1]$ and let $a \in L^\infty(Y)$ be a Y-periodic function, i.e. $a(0) = a(1)$ and satisfies the ellipticity and boundedness. Using the scaling map $x \mapsto x/\varepsilon$ from $[0,\varepsilon] \to [0,1]$ we define $a_\varepsilon(x) = a(x/\varepsilon)$ on $[0,\varepsilon]$ and then extend it periodically to all of $\mathbb{R}$. We continue to denote this function by $a(x/\varepsilon)$. Then $a^\varepsilon(x) = a(x/\varepsilon)$ and $\frac{1}{a^\varepsilon(x)} = \frac{1}{a}^\varepsilon(x)$ are bounded in $L^\infty(c,d))$.

We have the following general lemma and for a proof see[5].

Lemma 3.3. *Let $a^\varepsilon(x) = a(\frac{x}{\varepsilon})$ be as above, then $a^\varepsilon(x) \rightharpoonup \int_0^1 a(y)dy$ in $L^\infty(c,d)$ weak*.*

From the above lemma for the periodic case, we get $b = \int_0^1 \frac{1}{a(y)}dy$. It is clear that in general $\int_0^1 \frac{1}{a(y)}dy \neq \frac{1}{\int_0^1 a(y)dy}$.

Example 3.4 (General Periodic case). *The higher dimensional case is very delicate and as explained earlier, there is no explicit representation for the homogenized coefficients in the general case. However, when there is a periodic structure for the geometry, one can obtain the formula and prove it mathematically.*

Let $a_{ij}^{\varepsilon}(x) : \mathbb{R}^n \to \mathbb{R}$ be ε- periodic given by $a_{ij}^{\varepsilon}(x) = a_{ij}(\frac{x}{\varepsilon})$ where a_{ij} is defined on the unit cell $Y = [0,1]^n$ as a periodic function, that is it takes the same values on the opposite sides. Then extend a_{ij} periodically to all of $\mathbb{R}^n$ and we denote the extension by a_{ij} itself as it doesn't cause any confusion. Thus $a_{ij}^{\varepsilon}(x)$ gives the material property tensor of the microstructure.

Remark 3.5. *The above Lemma 3.3 holds good in the higher dimensional case as well.*

In this example, as expected the limiting coefficients a_{ij}^* are constants indicating the homogeniety of the limit structure and is given by

$$a_{ij}^* = \frac{1}{|Y|}\int_Y \left[a_{ij}(y) - a_{ik}\frac{\partial \chi^j}{\partial y_k}(y)\right], \qquad (3.3)$$

The unknown functions χ^j for $1 \le j \le n$ are obtained by solving n elliptic problems in periodic cell Y as: for fixed k, define χ^k by

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial \chi^k}{\partial y_j}\right) = \frac{-\partial a_{ik}}{\partial y_i}, \ \chi^k \ Y-\text{periodic}.$$

Equivalently

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial(\chi^k - y_k)}{\partial y_j}\right) = 0, \ \chi^k \ Y-\text{periodic}.$$

We remark that the functions χ^k actually captures the oscillations in the solutions. This particular point may not be very transparent at this stage and requires further analysis. We do not go through the proof, however we do indicate some of the methods developed to study such problems.

Example 3.6 (Layered Composite; Rank 1 materials). *Indeed these composites can be treated as a special case, but it has its own practical significance. These are created by stacking together, say alternatively thin homogeneous materials of different physical properties. That is the material property changes rapidly only in one direction.*

By a suitable rigid motion, mathematically it amounts to the condition that the coefficients a_{ij} and a_{ij}^{ε} depends only on one variable; that is $a_{ij}(y) = a_{ij}(y_1)$. The homogenized coefficient has a very simple representation as follows:

$$\frac{1}{a_{11}^*} = \mathcal{M}\left(\frac{1}{a_{11}}\right)$$
$$a_{1j}^* = a_{11}^*\mathcal{M}\left(\frac{a_{1j}}{a_{11}}\right),\ a_{j1}^* = a_{11}^*\mathcal{M}\left(\frac{a_{j1}}{a_{11}}\right),\ 2 \le j \le n$$
$$a_{ij}^* = \frac{a_{1i}^* a_{j1}^*}{a_{11}^*} + \mathcal{M}\left(a_{ij} - \frac{a_{1i}a_{j1}}{a_{11}}\right),\ 2 \le i,j \le n.$$

Here $\mathcal{M}(f)$ represents the average of the function f over the unit cell Y.

We now briefly describe various methods developed in the last 3 decades.

1. Formal asymptotic expansion

In any asymptotic problem, the first step is to look for a suitable asymptotic expansion and try to guess the correct limit from the formal analysis. The normal expansion like in any other asymptotic problems is as follows:

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \cdots.$$

Indeed this expansion leads to the anticipated, but wrong answer

$$\begin{cases} -\frac{d}{dx}\left(a^*\frac{du}{dx}\right) = f & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Keeping the particular problem in mind one looks for:

$$u_\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \cdots,$$

where x is the **slow** variable and $y = \frac{x}{\varepsilon}$ is the **fast** variable. Then, if possible, see that u_0 is independent of y and obtain the equation satisfied by u_0 (Refs [4,5]).

2. Energy method via test functions

The idea is to construct suitable test functions having same oscillations as the solutions to control the trouble creating oscillating terms to pass to the limit. In the process, the energy of the original system converges to the energy of the homogenized system[4,5]. This was essentially carried out by J. L. Lions for the periodic case.

3. Compensated Compactness

This method, actually, was introduced to pass to the limit in non-linear problems under weak convergence. We have already remarked in general, we may not be able to conclude the convergence

of $u_n v_n$ to uv from the weak convergence of u_n and v_n. This may be due to the oscillations in u_n and v_n and its interactions. But if u_n and v_n oscillates in transverse directions, then the non linear functional $u_n v_n$ behaves nicely. For example if u_n and v_n are functions on complementary variables i.e., $u_n = u_n(x')$ and $v_n = v_n(x'')$, where $x = (x', x'')$, then the convergence of $u_n v_n$ to uv can be concluded easily. i.e., one needs a sort of compensation to achieve the compactness. This is the basic motivation of compensated compactness, though the theory is much more involved[6,18,29]. We state a fundamental lemma towards this direction. If we look at our problem, we see that our interest is in the convergence of $\sigma_i^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}$ and $\sigma_i^\varepsilon = a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}$.

Lemma 3.7 (Div-Curl Lemma). *Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $L^2(\Omega)^N$ weakly. Further assume $\{div\, u_n\}$ and $\{curl\, v_n\}$ remains in a compact subset of $H^{-1}(\Omega)$. Then*

$$u_n v_n \to uv \text{ in distribution.}$$

We remark that this can be used to prove H-compactness and the proof involves lot of specific details, in particular periodic case can be derived.

4. Gamma Convergence

This is a variational convergence developed to study optimization problems. Gamma convergence is a very powerful notion introduced in the seventies and have applications in several problems including homogenization problems[9].

5. Two Scale (Multi-scale) Convergence

This was specially introduced for studying homogenization problems. It makes the formal two scale asymptotic analysis mathematically rigorous. The two-scale limit captures the oscillations involved in a weakly convergent sequence. Here I would like to bring to the attention of the readers that rapid oscillations and concentrations are main cause which prevents the weakly convergent sequence to become strongly convergent. In this direction, we state the following theorem due to Nguetseng[24] (see also Allaire[1] and Nandakumaran[20]).

Lemma 3.8 (Two-scale Convergence). *Let $\{u_\varepsilon\}$ be a uniformly bounded sequence in $L^2(\Omega)$. Then there is a subsequence of ε, denoted again by ε, and*

$$u_0 = u_0(x, y) \in L^2(\Omega, L_p^2(Y))$$

such that

$$\int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \to \int_{\Omega\times Y} u_0(x,y)\psi(x,y)\,dxdy \tag{3.4}$$

as $\varepsilon \to 0$, for all $\psi \in C_c(\bar{\Omega}, C_p(Y))$. Moreover

$$\int_\Omega u_\varepsilon(x)v(x)\omega\left(\frac{x}{\varepsilon}\right) \to \int_{\Omega\times Y} u_0(x,y)v(x)\omega(y)\,dxdy \tag{3.5}$$

as $\varepsilon \to 0, \forall v \in C_c(\bar{\Omega})$ and $\forall \omega \in L_p^2(Y)$. Further, if u is the L^2 weak limit of u_ε, then by taking $\omega \equiv 1$ in ((3.5)) we get

$$u(x) = \int_Y u_0(x,y)\,dy. \tag{3.6}$$

Here $L_p^2(Y)$ denotes the space of L^2-periodic functions and $C_p(Y)$ denotes the space of continuous periodic functions on Y.

6. Fourier (Bloch wave) method

The latest addition is the Bloch wave method. Initially, problems from fluid–solid interaction were studied using bloch wave analysis[8]. The basic idea is to work in phase space than in the physical space represented by x variable. Essentially one diagonalize the operator A^ε and transform the equations $A^\varepsilon u^\varepsilon = f$ into a sequence of scalar equations without the derivatives. The concept of Fourier decomposition when the medium is homogeneous is that the operator can be diagonalized in the basis of plane waves. In the current periodic situation, one requires Bloch waves.

4. Nonlinear Problems

One can also study homogenization of various non-linear problems. We present a special class of non-linear problem which we have studied[21–23].

Now we consider the following initial-boundary value problem described in the beginning of the talk.

$$\begin{aligned} \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) - \operatorname{div} a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) &= f(x,t) \quad \text{in } \Omega_T, \\ a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right).\nu &= 0 \quad \text{on } \Gamma_{2,T}, \\ u_\varepsilon &= g \quad \text{on } \Gamma_{1,T}, \\ u_\varepsilon(x,0) &= u_0 \quad \text{in } \Omega. \end{aligned} \tag{4.1}$$

Weak formulation gives the solution $u_\varepsilon \in E := L^p(0,T;V)$, where $V = \{v \in W^{1,p}(\Omega) : v = 0 \text{ on } \Gamma_1\}$ and let V^* be the dual of V. Moreover

$$b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \in L^\infty(0,T;L^1(\Omega)), \text{ and}$$
$$\partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \in L^{p^*}(0,T;V^*).$$

Homogenized Equation and the Main Theorem

Let u_ε be a family of solutions of (4.1). Assume that there is a constant $C > 0$, such that

$$\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega_T)} \le C \tag{4.2}$$

Under, the assumptions (A1)–(A4), there exists a subsequence of ε, still denoted by ε, such that for all q with $0 < q < \infty$, we have,

$$u_\varepsilon \to u \text{ strongly in } L^q(\Omega_T)$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega_T)$$

$$b\Big(\frac{x}{\varepsilon}, u_\varepsilon\Big) - b\Big(\frac{x}{\varepsilon}, u\Big) \to 0 \text{ strongly in } L^q(\Omega_T)$$

$$b\Big(\frac{x}{\varepsilon}, u_\varepsilon\Big) \to \overline{b}(u) \text{ weakly in } L^q(\Omega_T) \text{ for } q > 1,$$

and u solves,

$$\begin{gathered}\partial_t \overline{b}(u) - \operatorname{div} A(u, \nabla u) = f \quad \text{in } \Omega_T,\\ A(u, \nabla u).\nu = 0 \quad \text{on } \Gamma_{2,T},\\ u = g \quad \text{on } \Gamma_{1,T},\\ u(x,0) = u_0 \quad \text{in } \Omega,\end{gathered}$$

The assumption is true in special cases and it is reasonable on physical grounds. The functions $\overline{b}$ and A are defined by

$$\overline{b}(s) = \int_Y b(y,s)\, dy$$

and for $\mu \in \mathbb{R}, \lambda \in \mathbb{R}^n$,

$$A(\mu,\lambda) = \int_Y a(y,\mu,\lambda + \nabla \Phi_{\mu,\lambda}(y))\, dy,$$

where $\Phi_{\mu,\lambda} \in W^{1,p}_{per}(Y)$ solves the periodic boundary value problem

$$\int_Y a(y,\mu,\lambda + \nabla \Phi_{\mu,\lambda}(y)).\nabla \phi(y)\, dy = 0$$

for all $\phi \in W^{1,p}_{per}(Y)$. Here $Y = (0,1)^n$.

Correctors: let $U_1 \in L^p(\Omega_T; W^{1,p}_{per}(Y))$ be the solution of the variational problem,

$$\int_{\Omega_T}\int_Y a(y, u, \nabla_x u + \nabla_y U_1(x,t,y)).\nabla_y \Psi(x,t,y) = 0,$$

for all $\Psi \in L^p(\Omega_T; W^{1,p}_{per}(Y))$. If u, U_1 are sufficiently smooth, i.e. belong to $C^1(\Omega_T)$ and $C(\Omega_T; C^1_{per}(Y))$, then

$$u_\varepsilon - u - \varepsilon U_1\Big(x,t,\frac{x}{\varepsilon}\Big) \to 0$$

and

$$\nabla u_\varepsilon - \nabla u - \nabla_y U_1\Big(x,t,\frac{x}{\varepsilon}\Big) \to 0,$$

strongly in $L^p(\Omega_T)$.

We have improved the convergence of ∇u_ε by adding the corrector term. If U_1 were to be differentiable in x, then $\nabla(u - \varepsilon \nabla U_1(x,t,\frac{x}{\varepsilon}))$ would approximate ∇u_ε.

5. Optimal Bounds

So far, we were discussing about the limiting analysis of micro-structures. The explicit calculation of homogenized coefficients is available only for certain situations. Though the general theorem assures us the existence of limiting equation, it is no loner possible, in general, to get them explicitly which has utmost importance for engineers and experimental scientists. They need more information about the homogenized coefficients. This is the subject of optimal bounds.

Having failed to capture exact coefficients, one proceeds to get estimates known as *optimal bounds* on the coefficients or effective tensors. These bounds will be very useful in most of the experimental situations, where a complete knowledge of the composite geometry may not be available. Even when an accurate determination of complex micro-geometry is possible, obtaining this information and parameterizing it can be a very time consuming process. Cross-sectional photographs may give only a limited information and it is difficult to predict the 3-dimensional micro-geometry. Many situations the optimal bounds may be a very good approximation. More importantly, the validation of numerical schemes can be done via such bounds. For example, if the numerical solution is outside the bounds, it can be rejected outright.

In this section, we present certain bounds available in the literature without any proofs. The main aim, of course, to construct bounds independent of the geometry of the domain.

First consider two material composite.

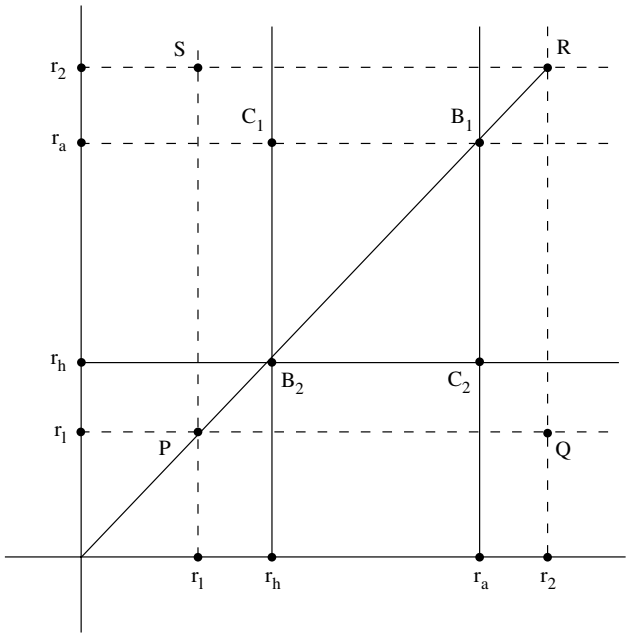
1. Bounds from General Theory (G-convergence): The Theorem 2.2 tells that if the homogenized coefficient matrix $\mathcal{A}$ is in $E(\alpha,\beta,\Omega)$ (we skip ε in this discussion), then the homogenized coefficient matrix $\mathcal{A}^*$ is also in $E(\alpha,\beta,\Omega)$. Thus we have the crude estimate

$$\alpha I \le \mathcal{A}^* \le \beta I. \tag{5.1}$$

Interpretation

Consider 2-material isotropic composite with properties γ_1 and γ_2 with $0 < \gamma_1 < \gamma_2$. Let λ_1, λ_2 be the eigenvalues of $\mathcal{A}^*$, then (5.1) states that the pair (λ_1, λ_2) is in the square $[\gamma_1, \gamma_2] \times [\gamma_1, \gamma_2]$ (see Fig. 1).

Figure 1:



2. Reuss–Voigt Bounds

Voigt proposed that the *arithmetic mean* would be a good approximation to the effective modulli, where as Reuss suggested the *harmonic mean*. In fact in one dimension, we have seen that the harmonic mean is attained and in laminates, the first component is the harmonic value and all other components are arithmetic means if the laminates is in the x_1-direction.

Define the harmonic and arithmetic averages of two materials with volume ratios m_1 and m_2, respectively as $\gamma_h = \frac{\gamma_1\gamma_2}{m_1\gamma_2+m_2\gamma_1}$ and $\gamma_a = m_1\gamma_1 + m_2\gamma_2$. Then the eigenvalues of the effective modulli lie in the square $[\gamma_h,\ \gamma_a]\times[\gamma_h,\ \gamma_a]$. Note that this square is inside the square given by the general theory (see Fig. 1).

More generally, if we have composite of k-materials with material property tensors $\mathcal{A}_1,\cdots,\mathcal{A}_k$ with volume ratios $m_1,\cdots,m_k$, then we have the *Reuss-Voigt* inequalities

$$\langle\mathcal{A}^{-1}\rangle^{-1} \le \mathcal{A}^* \le \langle\mathcal{A}\rangle. \tag{5.2}$$

Here $\langle\mathcal{A}\rangle = \sum m_i\mathcal{A}_i$ is the arithmetic mean and $\langle\mathcal{A}^{-1}\rangle = \sum m_i\mathcal{A}_i^{-1}$ is the harmonic average. We remark that the above inequality can be proved using variational principle.

3. Hashin–Shtrikman Bounds

This is derived in connection with problems mechanics and it is also known from physics and chemistry etc as *Clausius–Mossoti*, *Lorentz–Lorenz*, *Maxwell–Garnett* bounds.

Consider 2-composite material as earlier and note that $m_2 = 1-m_1$ and put $\theta = m_1$.

Assumption

The homogenized matrix is isotropic. That is $\mathcal{A}^* = \gamma^* I$.

The assumption is crucial and we remark that even if the component materials are isotropic, the effective modulli need not be isotropic. Indeed under the assumption and by the Reuss-voigt estimate, the eigenvalues coincide with γ^* and $(\gamma^*,\ \gamma^*)$ lies on the diagonal B_1B_2. The following Hashin-Shtrikman bounds gives better estimate:

$$\gamma_- \le \gamma^* \le \gamma_+ \tag{5.3}$$

where

$$\gamma_- = \frac{\theta\gamma_1+(1-\theta)\gamma_2+\gamma_2}{\theta\gamma_2+(1-\theta)\gamma_1+\gamma_1}\,\gamma_1,$$
$$\gamma_+ = \frac{\theta\gamma_1+(1-\theta)\gamma_2+\gamma_1}{\theta\gamma_2+(1-\theta)\gamma_1+\gamma_2}\,\gamma_2.$$

The above estimate shows that $(\gamma^*,\ \gamma^*)$ lies on the diagonal EF (see Fig. 2) which is contained in B_1B_2. This will be explained in the next general estimate without the previous assumption.

4. General Case

Let the materials with properties γ_1 and γ_2 occupying the regions Ω_1^ε and Ω_2^ε respectively so that $\Omega = \Omega_1^\varepsilon \biguplus \Omega_2^\varepsilon$, then $\gamma^\varepsilon = \gamma_1\chi_1^\varepsilon + \gamma_2\chi_2^\varepsilon$ is the material property of composite at the microscopic level and $\mathcal{A}^\varepsilon = \gamma^\varepsilon I$ is the coefficient matrix. Here $\chi_i^\varepsilon = \chi_{\Omega_i^\varepsilon}$ is the characteristic function of Ω_i^ε. Since χ_i^ε is $L^\infty(\Omega)$ bounded uniformly in ε, we assume that $\chi_1^\varepsilon \to \theta(x)$ in $L^\infty(\Omega)$ $weak*$, and hence $\chi_2^\varepsilon \to 1-\theta(x)$ in $L^\infty(\Omega)$ $weak*$.

We remark that $\theta(x)$ is a function which need not be a characteristic function.

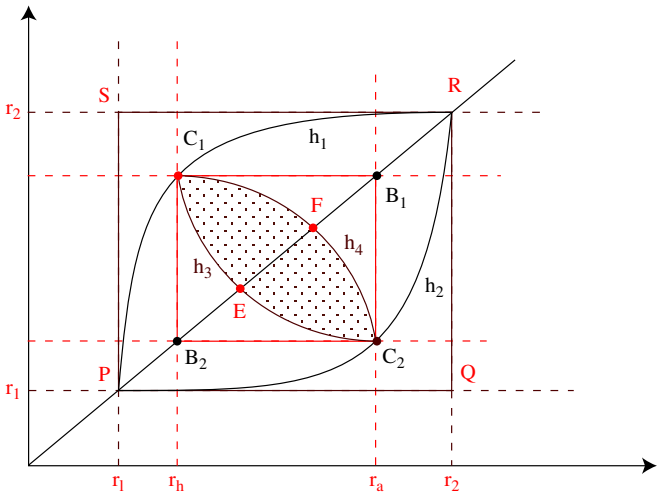
Theorem 5.1 (Murat–Tartar). *Let γ^ε be given as earlier so that $\mathcal{A}^\varepsilon \in E(\gamma_1,\gamma_2,\Omega)$. Let $\mathcal{A}^*$ be the homogenized limit with eigenvalues $\lambda_1(x),\cdots,\lambda_n(x)$. Then they satisfy the following inequalities:*

$$\begin{cases} \gamma_h(x) \le \lambda_i(x) \le \gamma_a(x),\ 1 \le i \le n \\ \sum_{i=1}^n \frac{1}{\lambda_i(x)-\gamma_1} \le \frac{1}{\gamma_h-\gamma_1} + \frac{n-1}{\gamma_a-\gamma_1} \\ \sum_{i=1}^n \frac{1}{\gamma_2-\lambda_i(x)} \le \frac{1}{\gamma_2-\gamma_h} + \frac{n-1}{\gamma_2-\gamma_a}, \end{cases} \tag{5.4}$$

where $\gamma_a(x) = \theta(x)\gamma_1 + (1-\theta(x))\gamma_2$ and $\gamma_h(x) = (\frac{\theta(x)}{\gamma_1} + \frac{1-\theta(x)}{\gamma_2})^{-1}$.

Conversely, if the eigenvalues of some matrix of functions $\mathcal{A}$ satisfy (5.4) for some function $\theta(x)$, $0 \le \theta(x) \le 1$ a.e. in Ω, then there exist a sequence of matrices $\mathcal{A}^\varepsilon$ of the form $\gamma^\varepsilon I$ which H-converges to $\mathcal{A}$. In conclusion the estimates given are optimal.

Figure 2:



Geometric Interpretation when $n = 2$

As θ varies from 0 to 1, the point $C_1(\theta) := (\gamma_h, \gamma_a) = (\frac{\gamma_1\gamma_2}{(\gamma_2-\gamma_1)\theta+\gamma_1}, \gamma_2-(\gamma_2-\gamma_1)\theta)$ traces a part of the hyperbola $x = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-y}$ denoted by h_1, from the point $R = C_(0) = (\gamma_2, \gamma_2)$ to the point $P = C_(1) = (\gamma_1, \gamma_1)$. The point $C_2(\theta) := (\gamma_a, \gamma_h)$ is the reflection of C_1 with respect to the diagonal line which again traces a part of the hyperbola as in Fig. 2.

So for a fixed volume fraction θ, we get two points $C_1 = C_1(theta)$ and $C_2 = C_2(theta)$ on the hyperbolae as in the fig. 2. Let $B_1 = B_1(theta) = (\gamma_a, \gamma_a)$ and $B_2 = B_2(theta) = (\gamma_h, \gamma_h)$.

In conclusion, we have:

1. As remarked earlier, the general theorem says that the eigen values are contained in the square $PORS$, where as the Reuss-Voigt (the first inequality in the above theorem) gives it is in the smaller square $B_1C_1B_2C_2$.

2. The other two inequalities in the theorem, the eigenvalues are actually contained in shaded region FC_1EC_2 which is bounded by the two hyperbolae (obtained by equating the inequalities):

$$\frac{1}{x-\gamma_1}+\frac{1}{y-\gamma_1}=\frac{1}{\gamma_h-\gamma_1}+\frac{1}{\gamma_a-\gamma_1}$$

and

$$\frac{1}{\gamma_2-x}+\frac{1}{\gamma_2-y}=\frac{1}{\gamma_2-\gamma_h}+\frac{1}{\gamma_2-\gamma_a}.$$

Indeed, under the isotropic assumption as in Hashin-Shtrikman of the homogenized coefficients, the eigenvalues are equal and hence it lies on the line EF.

The converse, merely indicates that every point in the shaded region is achievable by a micro-structure and hence optimality. In fact, Hashin-Shtrikman constructed certain micro-structures, known as *Hashin-Shtrikman assemblages of coated circles* to obtain the limiting values E and F. Further, a minor modification of the assemblages lead to any value from E to F and more generally, using assemblages of coated ellipsoids instead of circles one can attain any point in the shaded region. When one moves towards the points C_1 or C_2, the ellipses become more and more stretched in one of the directions and the elliptic assemblages will turn into a stratified (laminate) composite of Rank 1.

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