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Existence of Local and Global Solutions to Fractional Order Integral Equations

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Abstract

In this paper we shall study the existence of local and global solutions of the fractional order integral equations in an arbitrary Banach space by using the semigroup theory of linear operators and Schauder's fixed point theorem. We also give some examples to illustrate the applications of the abstract results.

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1 Introduction

We consider the following fractional order integral equations in a Banach space $(X, \|.\|)$:

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1}(-Au(\theta))d\theta + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} f_1(\theta, u(\theta))d\theta, \quad t \in (0, T] \tag{1.1}$$

and

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1}(-Au(\theta))d\theta + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} f_2(\theta, u(\theta), u(a(\theta)))d\theta, \quad t \in (0, T], \tag{1.2}$$

where A is a closed linear operator defined on a dense set and $0 < \beta < 1$, $0 < T < \infty$. We assume $-A$ is the infinitesimal generator of a compact analytic semigroup $\{S(t) : t \geq 0\}$ in X and $u_0 \in D(A)$. The given functions f_1, f_2 and a are satisfying certain conditions to be specified later.

Regarding initial studies on existence and uniqueness of different type of solutions to fractional order differential equations, we refer to [1, 3, 10, 11, 12, 14] and references cited in these papers.

For the earlier works on existence, uniqueness and stability of various types of solutions of differential and functional differential equations, we refer to Balachandran and Chandrasekaran [6], Byszewski [7], Byszewski and Akca [8], Byszewski and Lakshmikantham [9], Lin and Liu [17] and references cited in these papers.

In this paper, we use the Schauder's fixed point theorem and semigroup theory to prove the existence and uniqueness of mild solutions to the given problems (1.1) and (1.2).

The plan of the paper is as follows. Introduction and preliminaries are given respectively, in the first two sections. In Section 3, we prove the existence and uniqueness of local solution and in Section 4, the existence of global solution for the problems (1.1) and (1.2) is given. In the last section, we have given some examples

2 Preliminaries

We note that if $-A$ is the infinitesimal generator of a compact analytic semigroup, then for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence without loss of generality, we suppose that

$$\|S(t)\| \leq M \quad \text{for} \quad t \geq 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ being dense in X. We have $X_\kappa \hookrightarrow X_\alpha$ for $0 < \alpha < \kappa$ and the embedding is continuous. For more details on the fractional powers of closed linear operators we refer to Pazy [2]. It can be proved easily that $X_\alpha := D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ and it is equivalent to the graph norm of A^α.

We notice that $\mathcal{C}_T = C([0,T], X)$, the set of all continuous functions from $[0,T]$ into X is a Banach space under the supremum norm given by

$$\|\psi\|_T := \sup_{0 \leq \eta \leq T} \|\psi(\eta)\|, \quad \psi \in \mathcal{C}_T.$$

Throughout this paper, we shall assume the following assumptions:

(A1) $-A$ is the infinitesimal generator of a compact analytic semigroup $S(t)$.

(A2) The nonlinear map f_1 is defined from $[0,T] \times X$ into X and there exists a nondecreasing function $f_R : [0,\infty) \to [0,\infty)$ depending on $R > 0$ such that $\|f_1(t,u) - f_1(s,v)\| \leq f_R(t)[|t-s|^\nu + \|u - v\|]$ for all $t, s \in [0,T]$, a fixed ν, $0 < \nu \leq 1$ and $u, v \in B_R(X)$, where $B_R(X) = \{z \in X, \|z\| \leq R\}$.

(A3) The nonlinear map $f_2 : [0,T] \times X \times X \to X$ satisfies:
$\|f_2(t, x_1, x_2) - f_2(s, y_1, y_2)\| \leq L_r(t)[|t-s|^\nu + \sum_{i=1}^{2} \|x_i - y_i\|]$,
for all $t, s \in [0,T]$, a fixed ν, $0 < \nu \leq 1$, and $x_i,\ y_i \in B_r(X)$ for all $i = 1, 2$. Here $L_r : R_+ \to R_+$ is a nondecreasing function.

(A4) The function $a : [0,T] \to [0,T]$ satisfies the following two conditions:
(i) a satisfies the delay property $a(t) \leq t$ for all $t \in [0,T]$;
(ii) The function a is Lipschitz continuous; that is, there exists a positive constant L_a such that $|a(t) - a(s)| \leq L_a|t - s|$ for all $t, s \in [0,T]$.

We define the Riemann-Liouville integral of order $\beta > 0$ as follows:

$$I^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} g(\theta) d\theta.$$

3 Existence of Local Solutions

3.1 Existence of local solution to the fractional order integral equation

Definition 3.1. By a mild solution to the problem (1.1), we mean a continuous solution u of the following integral equation given below

$$\begin{aligned} u(t) = &\int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) u_0 d\theta \\ &+\beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) f(s, u(s)) d\theta ds, \end{aligned} \tag{3.1}$$

where $\zeta_\beta(\theta)$ is the probability density function [15, 16]. For the further detail on mild solution, we refer to [10, 11, 13].

Theorem 3.1. *Assume the conditions **(A1)-(A2)** are satisfied and $u_0 \in D(A)$. Then, there exists a t_0, $0 < t_0 < T$ such that the equation (1.1) has a local solution on $[0, t_0]$.*

Proof. Let $R > 0$ be such that $M\|u_0\| \le \frac{R}{2}$.
Let,

$$N_1 := \int_0^\infty \theta \zeta_\beta(\theta) d\theta, \tag{3.2}$$

$$N_2 := \int_0^\infty \theta^{1-\alpha} \zeta_\beta(\theta) d\theta \tag{3.3}$$

and

$$N_3 := \|f_1(0,0)\|. \tag{3.4}$$

Choose t_0, $0 < t_0 \le T$ such that

$$t_0 < \left[\frac{R}{2}\{N_1 M\{f_R(T)[T^\nu + R] + N_3\}\}^{-1}\right]^{\frac{1}{\beta}}. \tag{3.5}$$

We set,

$$Y = \{u \in \mathcal{C}_{t_0} : u(0) = u_0, \|u(t)\| \le R, \text{ for } 0 \le t \le t_o\}. \tag{3.6}$$

Clearly, Y is a bounded, closed and convex subset of $\mathcal{C}_{t_0}$.
For any $0 < \tilde{T} \le T$, we define a mapping F from $\mathcal{C}_{\tilde{T}}$ into $\mathcal{C}_{\tilde{T}}$ given by,

$$\begin{aligned} (F\psi)(t) &= \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) u_0 d\theta \\ &+ \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) f_1(s, \psi(s)) d\theta ds. \end{aligned} \tag{3.7}$$

Clearly, F is well defined.
To prove the theorem, first we need to show that $F : Y \to Y$. For any $\psi \in Y$, we have $(F\psi)(0) = u_0$. If $t \in [0, t_0]$, then we have,

$$\begin{aligned} \|(F\psi)(t)\| &\le \int_0^\infty \zeta_\beta(\theta) \|S(t^\beta\theta)\| \|u_0\| d\theta \\ &+ \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) \|S((t-s)^\beta \theta)\| \|f_1(s,\psi(s)) - f_1(0,0)\| d\theta ds \\ &+ \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) \|S((t-s)^\beta \theta)\| \|f_1(0,0)\| d\theta ds \\ &\le M\|u_0\| + N_1 M\{f_R(T)[T^\nu + R] + N_3\} t_0^\beta \\ &\le R. \end{aligned} \tag{3.8}$$

Hence, $F : Y \to Y$.

Now we will show that F maps Y into a precompact subset $F(Y)$ of Y. For this we will show that for fixed $t \in [0, t_0]$, $Y(t) = \{(F\psi)(t) : \psi \in Y\}$ is precompact in E and $F(Y)$ is an uniformly equicontinuous family of functions. Here, for $t = 0$, $Y(0) = \{u_0\}$ is precompact in X.

Let $t > 0$ be fixed. For an arbitrary $\epsilon \in (0, t)$, define a mapping F_ϵ on Y by the formula,

$$\begin{aligned}(F_\epsilon\psi)(t) &= \int_0^\infty \zeta_\beta(\theta) S(t^\beta\theta) u_0 d\theta \\ &+ \beta \int_0^{t-\epsilon}\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta) S((t-s)^\beta\theta) f_1(s,\psi(s)) d\theta ds \\ &= S(\epsilon)\int_0^\infty \zeta_\beta(\theta) S(t^\beta\theta - \epsilon) u_0 d\theta \\ &+ S(\epsilon)\beta \int_0^{t-\epsilon}\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta) S((t-s)^\beta\theta-\epsilon) f_1(s,\psi(s)) d\theta ds. \quad (3.9)\end{aligned}$$

Since $S(\epsilon)$ is compact for every $\epsilon > 0$, hence the set $Y_\epsilon(t) = \{(F_\epsilon\psi)(t) : \psi \in Y\}$ is precompact in X for every $\epsilon \in (0, t)$, where $t \in (0, t_0]$.

Also, we have

$$\begin{aligned}\|(F\psi)(t) - (F_\epsilon\psi)(t)\| &= \|\beta \int_{t-\epsilon}^t \theta(t-s)^{\beta-1}\zeta_\beta(\theta) S((t-s)^\beta\theta) f_1(s,\psi(s)) ds\| \\ &\le \epsilon^\beta R_1, \quad (3.10)\end{aligned}$$

for all $t \in (0, t_0]$, $\psi \in Y$ and $R_1 = N_1 M\{f_R(T)[T^\nu + R] + N_3\}$. Consequently, the set $Y(t)$, where $t \geq 0$ is precompact in X.

For any $t_1, t_2 \in (0, t_0]$ with $t_1 < t_2$ and $\psi \in Y$, we have

$$\begin{aligned}&(F\psi)(t_2) - (F\psi)(t_1) = \int_0^\infty \zeta_\beta(\theta)[S(t_2^\beta\theta) - S(t_1^\beta\theta)] u_0 d\theta \\ &+\beta \int_{t_1}^{t_2}\int_0^\infty \theta(t_2-s)^{\beta-1}\zeta_\beta(\theta) S((t_2-s)^\beta\theta) f_1(s,\psi(s)) d\theta ds \\ &+(-\beta)\int_0^{t_1}\int_0^\infty \theta[(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}]\zeta_\beta(\theta) S((t_2-s)^\beta\theta) f_1(s,\psi(s)) d\theta ds \\ &+\beta \int_0^{t_1}\int_0^\infty \theta(t_1-s)^{\beta-1}\zeta_\beta(\theta)[S((t_2-s)^\beta\theta) - S((t_1-s)^\beta\theta)] f_1(s,\psi(s)) d\theta ds \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Hence,

$$\|(F\psi)(t_2) - (F\psi)(t_1)\| \le \|I_1\| + \|I_2\| + \|I_3\| + \|I_4\|. \quad (3.11)$$

We have

$$\begin{aligned}I_1 &= \int_0^\infty \zeta_\beta(\theta)[S(t_2^\beta\theta) - S(t_1^\beta\theta)] u_0 d\theta \\ &= \int_0^\infty \zeta_\beta(\theta)[\int_{t_1}^{t_2} \beta\theta t^{\beta-1} A S(t^\beta\theta) u_0 dt] d\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}\|I_1\| &\leq \int_0^\infty \zeta_\beta(\theta)\int_{t_1}^{t_2} \beta\theta t^{\beta-1}\|A^{-\alpha}\|\|A^\alpha S(t^\beta\theta)\|\|Au_0\|dtd\theta \\ &\leq C^* C_\alpha \beta \int_0^\infty \theta^{1-\alpha}\zeta_\beta(\theta)\int_{t_1}^{t_2} t^{\beta(1-\alpha)-1}\|Au_0\|dtd\theta.\end{aligned}$$

Hence,

$$\begin{aligned}\|I_1\| &\leq C^*\frac{N_2}{(1-\alpha)}C_\alpha\|Au_0\|(t_2^{\beta(1-\alpha)} - t_1^{\beta(1-\alpha)}) \\ &\leq C^*C_\alpha\|Au_0\|N_2\beta(t_1+\delta(t_2-t_1))^{\beta(1-\alpha)-1}(t_2-t_1) \\ &\leq C^*C_\alpha\|Au_0\|N_2\beta\delta^{\beta(1-\alpha)-1}(t_2-t_1)^{\beta(1-\alpha)}, \end{aligned} \tag{3.12}$$

where C_α is some positive constant satisfying $\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}$ for all $t \geq 0$, $C^* = \|A^{-\alpha}\|$ and $0 < \delta < 1$.

Also,

$$\|I_2\| \leq \frac{L_f(R)}{(1-\alpha)}C^*C_\alpha N_2(t_2-t_1)^{\beta(1-\alpha)} \tag{3.13}$$

and

$$\|I_3\| \leq \beta N_2 L_f(R)C^*C_\alpha\int_0^{t_1}(t_1-s)^{\lambda-1}[(t_1-s)^{-\lambda\mu}-(t_2-s)^{-\lambda\mu}]ds, \tag{3.14}$$

where $L_f(R) = \{f_R(T)[T^\nu + R] + N_3\},\ \lambda = 1-\beta\alpha$ and $\mu = \frac{1-\beta}{1-\beta\alpha}$.

After some calculation, we get

$$\|I_3\| \leq \beta N_2 L_f(R)C^*C_\alpha\mu{\delta_1}^{\mu-1}(1-c)^{-\lambda(1-\mu)-1}(t_2-t_1)^{\lambda(1-\mu)}, \tag{3.15}$$

where $c = (1-(\frac{\mu}{\lambda})^{\frac{1}{\lambda\mu}})$ and $0 < \delta_1 \leq 1$.

Similarly, we get

$$\begin{aligned}\|I_4\| &\leq \beta N_2 C^* L_f(R)\frac{C_{1+\alpha}}{\alpha}\int_0^{t_1}(t_1-s)^{\beta-1}[(t_1-s)^{-\beta\alpha}-(t_2-s)^{-\beta\alpha}]ds \\ &\leq \beta N_2 L_f(R)C^*\frac{C_{1+\alpha}}{\alpha}{\delta_2}^{\alpha-1}(1-c_1)^{-\beta(1-\alpha)-1}(t_2-t_1)^{\beta(1-\alpha)},\end{aligned}$$

where $c_1 = (1-(\frac{\alpha}{\beta})^{\frac{1}{\alpha\beta}}),\ 0 < \delta_2 \leq 1$ and $C_{1+\alpha}$ is some positive constant satisfying $\|A^{\alpha+1}S(t)\| \leq C_{1+\alpha}t^{-1-\alpha}$ for all $t \geq 0$.

Thus from the above calculations we observe that the right hand side of the inequality (3.11) tends to zero when $t_2 - t_1 \to 0$. Hence, $F(Y)$ is a family of equicontinuous functions. Also, $F(Y)$ is bounded. Thus from the Arzela-Ascoli theorem (cf. see Dieudonne [5]), $F(Y)$ is precompact. The existence of a fixed point of F in Y is the consequence of *Schauder's* fixed point theorem.

Hence, there exists $u \in Y$, such that for all $t \in [0, t_0]$, we have

$$u(t) = \int_0^\infty \zeta_\beta(\theta)S(t^\beta\theta)u_0 d\theta + \beta\int_0^t \int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)S((t-s)^\beta\theta)f_1(s,u(s))d\theta ds, \tag{3.16}$$

where $u(0) = u_0$.

Applying the similar arguments as above, we see that the function u given by equation (3.16) is uniformly Hölder continuous on $[0, t_0]$. With the help of the condition (**A2**), we can show that the map $t \longmapsto f_1(t, u(t))$ is Hölder continuous on $[0, t_0]$. This completes the proof of the theorem. □

3.2 Existence of local solution to the fractional order functional integral equation

Definition 3.2. By a mild solution of the problem (1.2), we mean a continuous solution u of the following integral equation given below

$$u(t) = \int_0^\infty \zeta_\beta(\theta)S(t^\beta\theta)u_0 d\theta + \beta\int_0^t \int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)S((t-s)^\beta\theta)f(s,u(s),u(a(\theta)))d\theta ds, \tag{3.17}$$

where $\zeta_\beta(\theta)$ is the probability density function [15, 16]. For the further detail on mild solution, we refer to [10, 11, 13].

Applying the similar arguments to those in the proof of Theorem 3.1 from this paper, we obtain the following.

Theorem 3.2. *Assume the conditions **(A1), (A3)-(A4)** are satisfied and $u_0 \in D(A)$. Then, there exists a t_0, $0 < t_0 < T$ such that the equation (1.2) has a local solution on $[0, t_0]$.*

4 Existence of Global Solutions

4.1 Existence of global solution to the fractional order integral equation

Theorem 4.1. *Suppose that $0 \in \rho(-A)$ and $-A$ generates a compact analytic semigroup $S(t)$ with $\|S(t)\| \le M$, for $t \ge 0$, $u_0 \in D(A)$ and the function $f_1 : [0,\infty) \times X \to X$ satisfies the condition (**A2**). If there is a continuous nondecreasing real valued function $k(t)$ such that*

$$\|f_1(t,x)\| \le k(t)(1 + \|x\|) \ \text{for} \ t \ge 0, \quad x \in X, \tag{4.1}$$

then the equation (1.1) has a unique solution u which exists for all $t \ge 0$.

Proof. By Theorem 3.1, we can continue the solution of equation (1.1) as long as $\|u(t)\|$ stays bounded. It is therefore sufficient to show that if u exists on $[0, T)$, then $\|u(t)\|$ is bounded as $t \uparrow T$.

For $t \in [0, T)$, we have

$$\begin{aligned} u(t) &= \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) u_0 d\theta \\ &+ \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) f_1(s, u(s)) d\theta ds. \end{aligned} \tag{4.2}$$

From the above equation, we get

$$\|u(t)\| \leq M\|u_0\| + \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) \|S((t-s)^\beta \theta)\| \|f_1(s, u(s))\| d\theta ds.$$

Hence

$$\|u(t)\| \leq C_1 + C_2 \int_0^t (t-s)^{(\beta-1)} \|u(s)\| ds, \tag{4.3}$$

where $C_1 = M\|u_0\| + N_1 M k(T) T^\beta$ and $C_2 = \beta N_1 M k(T)$. Hence from Lemma 6.7 [Chapter 5 in Pazy [2]], u is a global solution.

To complete the proof of the theorem we only need to show that u is unique on the whole interval.

Let u_1 and u_2 be two solutions of the given fractional integral equation (1.1). Then, by a similar argument as above, we see that

$$\|u_1(t) - u_2(t)\| \leq f_R(T) \beta N_1 M \int_0^t (t-s)^{(\beta-1)} \|u_1(s) - u_2(s)\| ds.$$

Hence from Lemma 6.7 [Chapter 5, Pazy [2]], the solution u is unique. This completes the proof of the theorem. □

4.2 Existence of global solution to the fractional order functional integral equation

Applying the similar arguments to those in the proof of Theorem 4.1 from this paper, we obtain the following

Theorem 4.2. *Suppose that $0 \in \rho(-A)$ and $-A$ generates a compact analytic semigroup $S(t)$ with $\|S(t)\| \leq M$, for $t \geq 0$, $u_0 \in D(A)$ and the function $f_2 : [0, \infty) \times X \times X \to X$ satisfies the conditions (**A3**)-(**A4**). If there is a continuous nondecreasing real valued function $k(t)$ such that*

$$\|f_2(t, x, y)\| \leq k(t)(1 + \|x\| + \|y\|) \ \text{ for } \ t \geq 0, \quad x, y \in X, \tag{4.4}$$

then the fractional order functional integral equation (1.2) has a unique solution u which exists for all $t \geq 0$.

5 Examples

Let $X = L^2((0,1);\mathbb{R})$. We consider the following fractional order integro-differential equation,

$$\begin{aligned} w(t,x) &= w(0,x) + \frac{1}{\Gamma(\beta)}\int_0^t (t-\theta)^{\beta-1}(\partial_x^2 w(\theta,x))d\theta \\ &+\frac{1}{\Gamma(\beta)}\int_0^t (t-\theta)^{\beta-1}F(\theta,w(\theta,x))d\theta, \\ w(t,0) &= w(t,1) = 0,\ t\in[0,T],\ 0<T<\infty, \\ w(0,x) &= u_0, \end{aligned} \tag{5.1}$$

where F is a given sufficiently smooth function satisfies the Hölder condition.

We define an operator $A : D(A) \to X$ as follows,

$$Au = -u'',$$

where the domain of A is given by

$$D(A) = H^2(0,1)\cap H_0^1(0,1). \tag{5.2}$$

Then, $-A$ is the infinitesimal generators of an strongly continuous semigroup $S(t)$ which is compact and analytic.

The equation (5.1) can be reformulated as the following abstract equation in $X = L^2((0,1);\mathbb{R})$:

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-\theta)^{\beta-1}(-Au(\theta))d\theta \\ &+\frac{1}{\Gamma(\beta)}\int_0^t (t-\theta)^{\beta-1}f_1(\theta,u(\theta))d\theta,\ \ t\in(0,T], \end{aligned} \tag{5.3}$$

where $u(t) = w(t,.)$ that is $u(t)(x) = w(t,x)$, $t\in[0,T]$, $x\in(0,1)$ and the function $f_1 : [0,T]\times X \to X$ is given by

$$f_1(t,u(t))(x) = F(t,w(t,x)). \tag{5.4}$$

We can take $f_1(t,u) = h(t)g(u)$, where h is Hölder continuous and $g : X \to X$ is Lipschitz continuous on X. We can take $g(u) = sinu,\ g(u) = \xi u,\ g(u) = arctanu$, where ξ is constant.

In the case of functional integral equation, we can take

$$f_2(t,u(t),u(a(t)))(x) = F_2(t,w(t,x),w(a(t),x)), \tag{5.5}$$

where F_2 is a sufficiently smooth given function satisfies the Hölder condition in the first variable.

In particular we can take

$$f_2(t,u_1,u_2) = f_0(t) + b(t)\sum_{i=1}^{2}\|u_i\|u_i,$$

where the functions $f_0 : [0, T] \to X$ and $b : [0, T] \to R$ are Hölder continuous.

For the function a we can take

(i) $a(t) = kt$, where $t \in [0, T]$ and $0 < k \leq 1$.

(ii) $a(t) = kt^n$ for $t \in I = [0, 1]$ $k \in (0, 1]$ and $n \in \mathbb{N}$;

(iii) $a(t) = k \sin t$ for $t \in I = [0, \frac{\pi}{2}]$, and $k \in (0, 1]$.

It may be verified that all the assumptions of Theorem 3.1 are satisfied which ensures the existence of solutions of (5.3) as well as that of (5.1). Thus, all the results of the sections 3 and 4 can be apply to the problems (5.3) and (5.1).

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