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EXISTENCE, UNIQUENESS AND CONVERGENCE OF APPROXIMATE SOLUTIONS OF IMPULSIVE NEUTRAL DIFFERENTIAL EQUATIONS *

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Abstract. In this paper we shall study an impulsive neutral functional differential equation in a separable Hilbert space. We shall use the analytic semigroup theory of linear operators and fixed point technique to study the existence, uniqueness, and the convergence of approximate solutions to the given problem. We will also prove the existence and convergence of finite dimensional approximate solutions to the given problem. In the last an example is also illustrated.

Key Words. Impulsive neutral differential equations, Fractional power operators, Banach fixed point theorem.

AMS(MOS) subject classification. 34A60, 34K05, 34K45, 35K70.

1. Introduction.

We consider the following impulsive neutral differential equation in a separable Hilbert space $(H, \|.\|, \langle ., . \rangle)$:

$$\begin{aligned} \frac{d}{dt}[u(t) + g(t, u(t-\tau))] + Au(t) &= f(t, u_t), \ \ t \in (0, T], \ t \neq t_k, \\ \triangle u(t_k) &= I_k(u(t_k)), \ \ k = 1, 2, \dots, p, \\ u(t) &= h(t), \ t \in [-\tau, 0], \end{aligned} \tag{1}$$

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where $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$, $0 < t_1 < t_2 < \ldots < t_p \leq T < \infty$, $p \in \mathbb{N}$, $\tau > 0$, I_k's are some operators defined from H into H and $J = [0, T]$. $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ are, respectively, the right and the left limits of u at $t = t_k$. The functions $f : J \times \mathcal{C}_0 \rightarrow H$, $g : J \times H \rightarrow H$ and $h \in \mathcal{C}_0$ satisfy certain conditions to be specified later. The space $\mathcal{C}_0$ consists of functions $\psi : [-\tau, 0] \rightarrow H$ such that ψ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and $\psi(s^+)$ exist. For any continuous function u defined on the interval $[-\tau, T] \setminus \{t_1, t_2, \cdots, t_p\}$ and any $t \in J$, we denote by u_t the elements of $\mathcal{C}_0$ defined by $u_t(\theta) := u(t + \theta)$ for $\theta \in [-\tau, 0]$.

For the initial work on existence, uniqueness and regularity of solutions of impulsive differential equations under different conditions, we refer to Benchohra *et al* [1], Benchohra and Ouahabi [2], Dhage *et al* [3], Lakshmikantham *et al* [4], Ntouyas [5] and the references cited there.

Initial studies concerning existence, uniqueness and finite–time blow–up of solutions to the equation

$$\begin{aligned} u'(t) + Au(t) &= g(u(t)), \quad t \geq 0, \\ u(0) &= \phi, \end{aligned} \tag{2}$$

have been made by Heinz and von Wahl [6], Murakami [7] and Segal [8]. Bazley [9, 10] has considered the following semilinear wave equation

$$\begin{aligned} u''(t) \ + \ Au(t) \ &= \ g(u(t)), \quad t \geq 0, \\ u(0) \ = \ \phi, \quad u'(0) \ &= \ \psi, \end{aligned} \tag{3}$$

and has established the uniform convergence of approximate solutions to equation (3) by using the existence results of Heinz and von Wahl [6]. Goethel [11] has proved the convergence of approximate solutions to the problem (2), but assumed g to be defined on the whole of H. Based on the ideas of Bazley [9, 10], Miletta [12] has proved the convergence of approximate solutions of (2). Muslim [13] has proved the approximation of solutions of a history valued neutral functional differential equations in a separable Hilbert space.

In this paper, we shall use some of the ideas of Miletta, the Banach fixed point theorem, and analytic semigroup theory, to prove the existence, uniqueness, and convergence of approximate solutions to the problem (1). We shall also establish the existence and convergence of finite dimensional approximate solutions for the problem (1). Finally, we shall illustrate an example to show how easily our abstract results can be applied in practice.

2. Preliminaries and Assumptions. We note that if $-A$ is the infinitesimal generator of an analytic semigroup, then for $c > 0$ large enough,

$-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence without loss of generality we suppose that

$$\|e^{-tA}\| \leq M \quad \text{for all} \quad t \geq 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ being dense in H. We have $H_\kappa \hookrightarrow H_\alpha$ for $0 < \alpha < \kappa$ and the embedding is continuous. For more details on the fractional powers of closed linear operators we refer to Pazy [14]. It can be proved easily that $H_\alpha := D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ and it is equivalent to the graph norm of A^α.

For any $t \in [0, T]$, we take

$$\begin{aligned} \mathcal{C}_t := \{u : u \text{ is a map from } [-\tau, t] \text{ into } H \text{ such that } u(s) \text{ is continuous at} \\ s \neq t_k,\ \text{left continuous at } s = t_k \text{ and the right limit } u({t_k}^+) \\ \text{exists for } k = 1, 2, \dots, p\}. \end{aligned} \tag{4}$$

It can be proved easily that $\mathcal{C}_t$ is a Banach space with the norm

$$\|u\|_t = \sup_{\theta \in [-\tau,\, t]} \|u(\theta)\|. \tag{5}$$

Also, we can see that

$$\begin{aligned} \mathcal{C}_t^\alpha := \{u : u \text{ is a map from } [-\tau, t] \text{ into } H_\alpha \text{ such that } u(s) \text{ is continuous at} \\ s \neq t_k,\ \text{left continuous at } s = t_k \text{ and the right limit } u({t_k}^+) \\ \text{exists for } k = 1, 2, \dots, p\} \end{aligned} \tag{6}$$

is a Banach space with the norm

$$\|u\|_{t,\alpha} = \sup_{\theta \in [-\tau, t]} \|u(\theta)\|_\alpha.$$

The Banach space $\mathcal{C}_o^\alpha$ is known as the history space.

We consider the following assumption on the operator A.

(H1): A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset H$ of A into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq .....$$

and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$A\phi_i = \lambda_i \phi_i \quad \text{and} \quad \langle \phi_i, \phi_j \rangle = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied then $-A$ generates an analytic semigroup $\{e^{-tA} \ : t \geq 0\}$ in H.

Further we need the following hypotheses on the maps ϕ, I_k, g and f.

(H2): $h \in \mathcal{C}_0^\alpha$ is locally hölder continuous on $[-\tau, 0]$.

Let us define

$$\bar{h}(t) = \begin{cases} h(t), & \text{if } t \in [-\tau, 0], \\ h(0), & \text{if } t \in [0, T]. \end{cases}$$

(H3): All the maps $I_k : H_\alpha \to H_\alpha, \ \ k = 1, 2, \ldots, p$ satisfy the following two conditions:

(i) $\|I_k(u)\|_\eta \leq L_k$, for all $\eta \in (0, 1)$,

(ii) $\|I_k(u_1) - I_k(u_2)\|_\alpha \leq h_k \|u_1 - u_2\|_\alpha$,

where $\alpha \geq 0$, u, u_1, $u_2 \in B_r(H_\alpha, \bar{h}(t))$, h_k and L_k's are positive constants for all $k = 1, 2, 3, \cdots, p$. For any separable Hilbert space $(Z, \ \|.\|_Z)$ and $r > 0$,

$$B_r(Z, \ \bar{h}(t)) = \{z \in Z : \|z - \bar{h}(t)\|_Z \leq r\}.$$

(H4): The nonlinear continuous map f is defined from $[0, T] \times \mathcal{C}_0^\alpha$ into H and there exists a positive constant L_f such that

$$\|f(t, u_1) - f(s, u_2)\| \leq L_f[|t - s|^{\eta_1} + \|u_1 - u_2\|_{0,\, \alpha}],$$

for all $t, s \in [0, T]$, $\eta \in (0, 1]$ and $u_1, u_2 \in B_{r_1}(\mathcal{C}_0^\alpha, \ \bar{h})$.

(H5): There exist positive constants $0 < \alpha < \beta < 1$ and $r > 0$ such that the function $A^{\beta}g$ is continuous for $(t, u) \in [0, T_0] \times H_{\alpha}$ such that

$$\|A^{\beta}g(t,x) - A^{\beta}g(s,y)\| \leq L_g\{|t-s|^{\eta_2} + \|x-y\|_{\alpha}\}$$

and

$$2\|A^{\alpha-\beta}\|L_g + 2M\sum_{k=1}^{p} h_k < 1,$$

for all $t, s \in [0, T]$, $0 < \eta_2 \leq 1$ and $x, y \in B_r(H_{\alpha}, \bar{\chi}(\theta))$ where $-\tau \leq \theta \leq T$, L_g is a positive constant and for z_0 in a Banach space $(Z, \|.\|_Z)$ and $r > 0$,

$$B_r(Z, z_0) = \{z \in Z : \|z - z_0\|_Z \leq r\}.$$

Definition 1. *A function $u \in \mathcal{C}_{\tilde{T}}$ is called a mild solution of (1) on $[-\tau, \tilde{T}]$ if it is a solution of the impulsive integral equation*

$$(7)\quad u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ \\ e^{-tA}[h(0) + g(0, \bar{h}(-\tau))] - g(t, u(t-\tau)) \\ + \int_0^t Ae^{-(t-s)A}g(s, u(s-\tau))\, ds + \int_0^t e^{-(t-s)A}f(s, u_s)ds \\ + \sum_{0<t_k<t} e^{-(t-t_k)A}I_k(u(t_k)), t \in [0, \tilde{T}], \quad 0 < \tilde{T} \leq T. \end{cases}$$

3. Existence and Convergence of Approximate Solutions. Let H_n denote the finite dimensional subspace of H spanned by $\{\phi_0, \phi_1, \cdots, \phi_n\}$ and let $P^n : H \longrightarrow H_n$ be the corresponding projection operator for $n = 0, 1, 2, \cdots$.

Let $R > 0$ be such that

$$(8)\qquad \sup_{0\leq t\leq T} \|(e^{-tA} - I)A^{\alpha}[h(0) + g(0, \bar{h}(-\tau))]\| \leq \frac{R}{2}$$

and

$$(9)\qquad L(R, \alpha, \beta, T) + C_{\alpha}L_f(R)\frac{T^{1-\alpha}}{1-\alpha} + M\sum_{k=1}^{p} C_k \leq \frac{R}{2},$$

where C_{α} is a positive constant such that $\|A^{\alpha}e^{-tA}\| \leq C_{\alpha}t^{-\alpha}$ for $t > 0$,

$$(10)\qquad \begin{aligned} L(R, \alpha, \beta, T) \quad &= \quad \|A^{\alpha-\beta}\|L_g[T^{\eta_2} + R + 2\|\bar{h}\|_{T,\,\alpha}] \\ &+ \quad C_{1+\alpha-\beta}L_g(R)\frac{T^{\beta-\alpha}}{\beta-\alpha}, \end{aligned}$$

$$L_f(R) = L_f(T^{\eta_1} + R + \|\bar{h}\|_{T,\,\alpha}) + \|f(0,0)\|, \tag{11}$$

$$L_g(R) = L_g(T^{\eta_2} + R + \|\bar{h}\|_{T,\,\alpha}) + \|A^{\beta} g(0,0)\| \tag{12}$$

and

$$C_k = h_k[R + \|\bar{h}\|_{T,\alpha}] + \|I_k(0)\|_{\alpha}. \tag{13}$$

Also, we assume that

$$M \sum_{k=1}^{p} h_k + \|A^{\alpha-\beta}\| L_g + \frac{L_g C_{1+\alpha-\beta}}{\beta-\alpha} T^{\beta-\alpha} + \frac{L_f C_{\alpha}}{1-\alpha} T^{1-\alpha} < 1. \tag{14}$$

For $n = 0, 1, 2, \cdots$, we define,

$$f_n : [0,T] \times \mathcal{C}_0^{\alpha} \longrightarrow H$$

such that

$$f_n(t,u) = f(t, P^n u)$$

and

$$g_n : [0,T] \times H_{\alpha} \longrightarrow H_{\beta}$$

such that

$$g_n(t,y) = g(t, P^n y),$$

for all $t \in [0,T],\ u \in \mathcal{C}_0^{\alpha},\ \ y \in H$ and $(P^n u)(s) = P^n(u(s))$ for all $s \in [-\tau, 0]$. Also, we define

$$I_{k,n} : H_{\alpha} \longrightarrow H_{\alpha}$$

such that

$$I_{k,n}(u) = I_k(P^n u) \text{ for all } u \in H_{\alpha},\ n = 0,1,2\cdots \text{ and } k = 1,2,\cdots,p.$$

Now we define a map $\mathcal{F}_n$ on $B_R(\mathcal{C}_T^{\alpha}, \bar{h})$ as follows

$$(\mathcal{F}_n u)(t) = \begin{cases} h(t), t \in [-\tau, 0], \\ \\ e^{-tA}[h(0) + g_n(0, h(-\tau))] - g_n(t, u(t-\tau)) \\ + \int_0^t A e^{-(t-s)A} g_n(s, u(s-\tau)) ds + \int_0^t e^{-(t-s)A} f_n(s, u_s) ds \\ + \sum_{0<t_k<t} e^{-(t-t_k)A} I_{k,n}(u(t_k)), t \in [0,T], \end{cases}$$

for $u \in B_R(\mathcal{C}_T^{\alpha}, \bar{h})$.

Theorem 1. *Assume that the conditions (H1)–(H5) are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then, there exists a unique $u_n \in B_R(\mathcal{C}_T^\alpha, \bar{h})$ such that $\mathcal{F}_n u_n = u_n$ for each $n = 0, 1, 2, 3, \cdots$, i.e., u_n satisfies the approximate integral equation*

$$u_n(t) = \begin{cases} h(t), \quad t \in [-\tau, 0], \\ \\ e^{-tA}[h(0) + g_n(0, h(-\tau))] - g_n(t, u_n(t-\tau)) \\ + \int_0^t Ae^{-(t-s)A} g_n(s, u_n(s-\tau))ds \\ + \int_0^t e^{-(t-s)A} f_n(s, u_{ns})ds \\ + \sum_{0<t_k<t} e^{-(t-t_k)A} I_{k,n}(u_n(t_k)), \quad t \in [0, \tilde{T}]. \end{cases} \tag{15}$$

Proof. To prove this theorem, first we need to show that $\mathcal{F}_n : B_R(\mathcal{C}_T^\alpha, \bar{h}) \to B_R(\mathcal{C}_T^\alpha, \bar{h})$. Clearly, $\mathcal{F}_n : \mathcal{C}_T^\alpha \to \mathcal{C}_T^\alpha$ and $\mathcal{F}_n : B_R(\mathcal{C}_T^\alpha, \bar{h}) \to \mathcal{C}_T^\alpha$.

Now for $t \in [-\tau, 0]$, we have

$$(\mathcal{F}_n u)(t) - \bar{h}(t) = 0.$$

For $t \in (0, T]$, we find

$$\begin{aligned} &\|(\mathcal{F}_n u)(t) - \bar{h}(t)\|_\alpha \\ &\le \|(e^{-tA} - I)A^\alpha(\bar{h}(0) + g_n(0, \bar{h}(-\tau)))\| \\ &\quad + \|A^{\alpha-\beta}\|\|A^\beta g_n(0, \bar{h}(-\tau)) - A^\beta g_n(t, u(t-\tau))\| \\ &\quad + \int_0^t \|A^{1+\alpha-\beta} e^{-(t-s)A}\|\|A^\beta g_n(s, u(s-\tau))\|ds \\ &\quad + \int_0^t \|e^{-(t-s)A}A^\alpha\|\|f_n(s, u_s)\|ds + \sum_{0<t_k<t} \|e^{-(t-t_k)A}\|\|A^\alpha I_{k,n}(u(t_k))\| \\ &\le \frac{R}{2} + L(R, \alpha, \beta, T) + C_\alpha L_f(R)\frac{T^{1-\alpha}}{1-\alpha} + M\sum_{k=1}^p C_k, \end{aligned} \tag{16}$$

where $L(R, \alpha, \beta, T)$, $L_f(R)$, $L_g(R)$ and C_k are respectively given by the equations (10), (11), (12) and (13). Hence,

$$\|\mathcal{F}_n u - \bar{h}\|_{T,\alpha} \le R.$$

Thus, $\mathcal{F}_n : B_R(\mathcal{C}_T^\alpha, \bar{h}) \to B_R(\mathcal{C}_T^\alpha, \bar{h})$.

For any $u, v \in B_R(\mathcal{C}_T^\alpha, \bar{h})$ and $t \in [-\tau, 0]$, we have, $\mathcal{F}_n u(t) - \mathcal{F}_n v(t) = 0$.

Thus, if $t \in (0, T]$ and $u, v \in B_R(\mathcal{C}_T^\alpha, \bar{h})$ then

$$\begin{aligned}
&\|\mathcal{F}_n u(t) - \mathcal{F}_n v(t)\|_\alpha \\
&\leq \|A^{\alpha-\beta}\| \|A^\beta g_n(t, u(t-\tau)) - A^\beta g_n(t, v(t-\tau))\| \\
&\quad + \int_0^t \|e^{-(t-s)A} A^{1+\alpha-\beta}\| \|A^\beta g_n(s, u(s-\tau)) - A^\beta g_n(s, v(s-\tau))\| ds \\
&\quad + \int_0^t \|e^{-(t-s)A} A^\alpha\| \|f_n(s, u_s) - f_n(s, v_s)\| ds \\
&\quad + \sum_{0<t_k<t} \|e^{-(t-t_k)A}\| \|A^\alpha (I_{k,n}(u(t_k)) - I_{k,n}(v(t_k)))\| \\
&\leq \|A^{\alpha-\beta}\| L_g \|u - v\|_{T,\,\alpha} + \int_0^t C_{1+\alpha-\beta}(t-s)^{\beta-\alpha-1} L_g \|u - v\|_{T,\,\alpha} ds \\
&\quad + \int_0^t C_\alpha (t-s)^{-\alpha} L_f \|u_s - v_s\|_{0,\,\alpha} ds + M \sum_{k=1}^p h_k \|u - v\|_{T,\,\alpha} \\
&\leq [M \sum_{k=1}^p h_k + \|A^{\alpha-\beta}\| L_g + \frac{L_g C_{1+\alpha-\beta}}{\beta - \alpha} T^{\beta-\alpha} + \frac{L_f C_\alpha}{1-\alpha} T^{1-\alpha}] \|u - v\|_{T,\,\alpha}.
\end{aligned}$$

Therefore, taking the supremum with respect to t over $[-\tau,\ T]$, we get

$$\|\mathcal{F}_n u - \mathcal{F}_n v\|_{T,\,\alpha} < \|u - v\|_{T,\,\alpha}.$$

Hence, there exists a unique $u_n \in B_R(\mathcal{C}_T^\alpha, \bar{h})$ such that $\mathcal{F}_n u_n = u_n$, which satisfies the approximate integral equation (15). ∎

Corollary 1. *Let the assumptions (H1)-(H5) are hold and $h(t) \in D(A)$, for all $t \in [-\tau,\ 0]$. Then, $u_n(t) \in D(A^\nu)$ for all $t \in [-\tau,\ T]$, where $0 \leq \nu \leq \beta < 1$.*

Proof. From Theorem 1, we have the existence of a unique $u_n \in B_R(\mathcal{C}_T^\alpha, \bar{h})$ satisfying the integral equation (15). By Theorem 2.4 [Chapter 1, Pazy [14]], $\int_0^t e^{-tA} x ds \in D(A)$ for any $x \in H$. Also for any $x \in H$, $e^{-tA} x \in D(A^\nu)$ for $\nu \geq 0$. The Corollary now follows from these facts and the fact that $D(A) \subseteq D(A^\nu)$ for $0 \leq \nu \leq 1$. ∎

Corollary 2. *Let the assumptions (H1)-(H5) are hold and $h(t) \in D(A)$, for all $t \in [-\tau, 0]$. Then, there exists a positive constant M_0, independent of n, such that*

$$\|A^\nu u_n(t)\| \leq M_0$$

for all $-\tau \leq t \leq T_0$ and $0 < \alpha < \nu < \beta < 1$.

Proof. Applying A^ν on the both sides of the approximate integral equation (15), we get

$$\|u_n(t)\|_\nu \le \|h(t)\|_\nu \le \|h\|_{0,\,\nu}, \quad \text{for all } \ t \in [-\tau, 0].$$

Also for $t \in (0, T]$, we find

$$\begin{aligned}
\|u_n(t)\|_\nu \quad &\le \quad \|e^{-tA}\|[\|h(0)\|_\nu + \|A^{\nu-\beta}\|\|g_n(0, h(-\tau))\|_\beta] \\
&+ \quad \|A^{\nu-\beta}\|\|g_n(t, u_n(t-\tau))\|_\beta \\
&+ \quad \int_0^t \|A^{1+\nu-\beta}e^{-(t-s)A}\|\|g_n(s, u_n(s-\tau))\|_\beta ds \\
&+ \quad \int_0^t \|A^\nu e^{-(t-s)A}\|\|f_n(s, u_{ns})\|ds \\
&+ \quad \sum_{0<t_k<t} \|e^{-(t-t_k)A}\|\|I_{k,n}(u_n(t_k))\|_\nu \\
&\le \quad M[\|h\|_{0,\,\nu} + \|A^{\nu-\beta}\|\|g_n(0, h(-\tau))\|_\beta] + \|A^{\nu-\beta}\|L_g(R) \\
&+ \quad \frac{C_{1+\nu-\beta}L_g(R)T^{\beta-\nu}}{(\beta-\nu)} + \frac{C_\nu L_f(R)T^{1-\nu}}{(1-\nu)} + M\sum_{k=1}^p L_k \\
&= \quad M_0'. \qquad \square
\end{aligned}$$

Theorem 2. *Assume that the conditions (H1)–(H5) are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then, $\{u_n\} \subset B_R(\mathcal{C}_T^\alpha, \bar{h})$ is a Cauchy sequence, and therefore converges to a unique function $u \in B_R(\mathcal{C}_T^\alpha, \bar{h})$.*

Proof. For $n \ge m \ge n_0$ where n_0 is large enough, $n, m, n_0 \in \mathbb{N}$, $t \in [-\tau, 0]$, we have

$$\|u_n(t) - u_m(t)\|_\alpha = \|h(t) - h(t)\|_\alpha = 0. \tag{17}$$

Now for $0 < t_0' < t \le T$, and n, m and n_0 as above, we find

$$\begin{aligned}
&\|u_n(t) - u_m(t)\|_\alpha \\
&\quad\le \|e^{-tA}A^\alpha[g_n(0, \bar{h}(-\tau)) - g_m(0, \bar{h}(-\tau))]\| \\
&\quad\quad + \|A^{\alpha-\beta}\|\|A^\beta g_n(t, u_n(t-\tau)) - A^\beta g_m(t, u_m(t-\tau))\| \\
&\quad\quad + \left(\int_0^{t_0'} + \int_{t_0'}^t\right) \|A^{1+\alpha-\beta}e^{-(t-s)A}\|\|A^\beta g_n(s, u_n(s-\tau)) \\
&\quad\quad - A^\beta g_m(s, u_m(s-\tau))\|ds \\
&\quad\quad + \left(\int_0^{t_0'} + \int_{t_0'}^t\right) \|e^{-(t-s)A}A^\alpha\|\|f_n(s, (u_n)_s) - f_m(s, (u_m)_s)\|ds \\
&\quad\quad + \sum_{0<t_k<t} \|e^{-(t-t_k)A}\|\|I_{k,n}(u_n(t_k)) - I_{k,m}(u_m(t_k))\|_\alpha.
\end{aligned} \tag{18}$$

Also, for $0 < \alpha < \nu < 1$, we have

$$\begin{aligned} \|f_n(s,(u_n)_s) - f_m(s,(u_m)_s)\| &\leq \|f_n(s,(u_n)_s) - f_n(s,(u_m)_s)\| \\ &+ \|f_n(s,(u_m)_s) - f_m(s,(u_m)_s)\| \\ &\leq L_f[\|u_n - u_m\|_{s,\,\alpha} + \frac{M_0}{\lambda_m^{\nu-\alpha}}] \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\|A^\beta g_n(t,u_n(t-\tau)) - A^\beta g_m(t,u_m(t-\tau))\| \\ &\quad \leq \|A^\beta g_n(t,u_n(t-\tau)) - A^\beta g_n(t,u_m(t-\tau))\| \\ &\quad\quad + \|A^\beta g_n(t,u_m(t-\tau)) - A^\beta g_m(t,u_m(t-\tau))\| \\ &\quad \leq L_g[\|u_n(t-\tau) - u_m(t-\tau)\|_\alpha + \frac{M_0}{\lambda_m^{\nu-\alpha}}], \end{aligned} \tag{20}$$

where M_0 is same as in Corollary (2).

Similarly, it follows that

$$\begin{aligned} &\|I_{k,n}(u_n(t_k)) - I_{k,m}(u_m(t_k))\|_\alpha \\ &\quad \leq \|I_{k,n}(u_n(t_k)) - I_{k,n}(u_m(t_k))\|_\alpha + \|I_{k,n}(u_m(t_k)) - I_{k,m}(u_m(t_k))\|_\alpha \\ &\quad \leq h_k[\|u_n - u_m\|_{t_k,\,\alpha} + \frac{M_0}{\lambda_m^{\nu-\alpha}}]. \end{aligned} \tag{21}$$

Thus from inequalities (19), (20) and (21), the inequality (18) after some adjustment becomes

$$\begin{aligned} \|u_n(t) - u_m(t)\|_\alpha &\leq C_1 + C_1^* t_0^{'} + \frac{C_2}{\lambda_m^{\beta-\alpha}} + \|u_n - u_m\|_{t,\,\alpha}\, C_3 \\ &+ C_4 \int_{t_0^{'}}^{t} (t-s)^{-\alpha} \|u_n - u_m\|_{s,\,\alpha} ds, \end{aligned} \tag{22}$$

where

$$C_1 = M\|A^{\alpha-\beta}\|L_g\|(P^n - P^m)A^\alpha \bar{h}(-\tau)\|,$$

$$C_1^* = 2C_{1+\alpha-\beta}(T - t_0^{'})^{\beta-\alpha-1}L_g(R) + 2C_\alpha(T - t_0^{'})^{-\alpha}L_f(R),$$

$$C_2 = [\|A^{\alpha-\beta}\|L_g + \frac{C_{1+\alpha-\beta}}{\beta-\alpha}T^{\beta-\alpha} + \frac{C_\alpha T^{1-\alpha}}{1-\alpha} + M\sum_{k=1}^{p} h_k]M_0,$$

$$C_3 = \|A^{\alpha-\beta}\|L_g + M\sum_{k=1}^{p} h_k \text{ and } C_4 = C_{1+\alpha-\beta}L_g + C_\alpha L_f.$$

Next we replace t by $t+\theta$ in inequality (22), where $\theta \in [t_0^{'} - t, 0]$, to get

$$\begin{aligned}\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \quad &\leq \quad C_1 + C_1^*.t_0^{'} + \frac{C_2}{\lambda_m^{\beta-\alpha}} + \|u_n - u_m\|_{t,\,\alpha}\, C_3 \\ &+ \quad C_4\int_{t_0^{'}}^{t+\theta} (t+\theta-s)^{-\alpha}\|u_n - u_m\|_{s,\,\alpha} ds.\end{aligned}$$

We put $s - \theta = \gamma$ in the above inequality, to obtain

$$\begin{aligned}\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \quad &\leq \quad C_1 + C_1^*.t_0^{'} + \frac{C_2}{\lambda_m^{\beta-\alpha}} + \|u_n - u_m\|_{t,\,\alpha}\, C_3 \\ &+ \quad C_4\int_{t_0^{'}-\theta}^{t} (t-\gamma)^{-\alpha}\|u_n - u_m\|_{\gamma,\,\alpha} d\gamma \\ &\leq \quad C_1 + C_1^*.t_0^{'} + \frac{C_2}{\lambda_m^{\beta-\alpha}} + \|u_n - u_m\|_{t,\,\alpha}\, C_3 \\ &+ \quad C_4\int_{t_0^{'}}^{t} (t-\gamma)^{-\alpha}\|u_n - u_m\|_{\gamma,\,\alpha} d\gamma.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\sup_{t_0^{'}-t\leq\theta\leq 0} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \\ &\qquad\leq C_1 + C_1^*.t_0^{'} + \frac{C_2}{\lambda_m^{\beta-\alpha}} + \|u_n - u_m\|_{t,\,\alpha}\, C_3 \\ &\qquad+C_4\int_{t_0^{'}}^{t} (t-\gamma)^{-\alpha}\|u_n - u_m\|_{\gamma,\,\alpha} d\gamma. \qquad (23)\end{aligned}$$

Next, we find

$$\begin{aligned}\sup_{-\tau-t\leq\theta\leq 0} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \quad &\leq \quad \sup_{0\leq\theta+t\leq t_0^{'}} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha \\ &+ \quad \sup_{t_0^{'}-t\leq\theta\leq 0} \|u_n(t+\theta) - u_m(t+\theta)\|_\alpha. \qquad (24)\end{aligned}$$

Using (23) and (18) in the above inequality, we get

$$\begin{aligned}&\sup_{-\tau\le t+\theta\le t}\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha\\&\le 2C_1+\{C_1^*+C_5\}.t_0^{'}+\frac{C_2+C_6}{\lambda_m^{\beta-\alpha}}+\|u_n-u_m\|_{t,\,\alpha}\,2C_3\\&+C_4\int_{t_0^{'}}^t(t-\gamma)^{-\alpha}\|u_n-u_m\|_{\gamma,\,\alpha}d\gamma,\end{aligned}$$

where C_5 and C_6 are the other constants.

Finally, an application of Gronwall's inequality, and the fact that ${t_0}^{'}$ is arbitrary small, gives the required result. This completes the proof of the Theorem. □

Now we shall use Theorems 1 and 2 to prove the following existence and convergence result.

Theorem 3. *Assume that the conditions (H1)–(H5) are satisfied and $h(t)\in D(A)$ for all $t\in[-\tau,0]$. Then, u_n given by equation (15) converges in $B_R(\mathcal{C}_T^\alpha,\bar{h})$ to a unique solution $u\in B_R(\mathcal{C}_T^\alpha,\bar{h})$ of the equation (7).*

Proof. The existence and uniqueness of u on $[0,T]$ is clear from Theorem 2. We only need to prove that u is given by equation (15). We have the following inequality

$$\begin{aligned}\|f(t,P^nu_{nt}))-f(t,u_t)\| &\le L_f\|P^nu_{nt}-u_t\|_{0,\,\alpha}\\&\le L_f[\|(P^n-I)u_{nt}\|_{0,\,\alpha}+\|u_n-u\|_{T,\,\alpha}.\end{aligned}\tag{25}$$

Hence, $\|f(t,P^nu_{nt}))-f(t,u_t)\|\to 0$ as $n\to\infty$. Similarly, we have

$$\|I_{k,n}(u_n(t_k))-I_k(u(t_k))\|_\alpha\to 0\ \text{ as }\ n\to\infty.$$

and

$$\begin{aligned}&\|g(t,P^nu_n(t-\tau)))-g(t,u(t-\tau))\|_\beta\\&\le L_g\|P^nu_n(t-\tau)-u(t-\tau)\|_{0,\,\alpha}\\&\le L_g[\|(P^n-I)u_n(t-\tau)\|_\alpha+\|u_n-u\|_{T,\,\alpha}.\end{aligned}\tag{26}$$

Thus, it follows that

$$\begin{aligned}&\int_0^t\|A^{1+}e^{-(t-s)A}\|\|A^\beta g(s,P^nu_n(s-\tau))-A^\beta g(s,u(s-\tau))\|ds\\&+\int_0^t\|A^\alpha e^{-(t-s)A}\|\|f(s,P^nu_{ns})-f(s,u_s)\|ds\\&+\sum_{0<t_k<t}\|e^{-(t-t_k)A}\|\|I_{k,n}(u_n(t_k))-I_k(u(t_k))\|_\alpha\to 0\text{ when }n\to\infty.\end{aligned}\tag{27}$$

Therefore, u is indeed given by equation (7). □

4. Faedo-Galerkin Approximation of Solutions. From the previous sections it is clear that for any $-\tau \le t \le T$, we have a unique $u \in C_T^\alpha$ satisfying the integral equation (7). Also, there is a unique solution $u_n \in C_T^\alpha$ of the approximate integral equation (15).

Now we shall consider a finite dimensional approximation given by $\bar{u}_n = P^n u_n$ satisfying

$$\bar{u}_n(t) = \begin{cases} P^n h(t), & t \in [-\tau, 0], \\ \\ e^{-tA} P^n[h(0) + g_n(0, \bar{h}(-\tau))] - P^n g_n(t, u_n(t-\tau)) \\ + \int_0^t A e^{-(t-s)A} P^n g_n(s, u_n(s-\tau))\, ds + \int_0^t e^{-(t-s)A} P^n f_n(s, u_{ns}) ds \\ + \sum_{0<t_k<t} e^{-(t-t_k)A} P^n I_{k,n}(u_n(t_k)), \quad t \in [0, T]. \end{cases}$$

If the solution $u(t)$ to equation (7) exits on $-\tau \le t \le T$ then it has the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t) \phi_i, \tag{28}$$

where $\alpha_i(t) = \langle u(t), \phi_i \rangle$ for all $i = 0, 1, 2, 3, \cdots$ and similarly

$$\bar{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t) \phi_i, \tag{29}$$

where $\alpha_i^n(t) = \langle \bar{u}_n(t), \phi_i \rangle$ for all $i = 0, 1, 2, \cdots, n$.

Note: The above solution $\bar{u}_n(t)$ is known as the *Faedo–Galerkin approximate solution* of equation (1).

As a consequence of Theorems 1 and 2, we have the following result.

Theorem 4. *Suppose that the conditions (H1)–(H5) are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then, there exist functions $\bar{u}_n \in C([-\tau, T]; H_n)$ and $u \in C([-\tau, T]; H)$ satisfying*

$$\bar{u}_n(t) = \begin{cases} P^n h(t), & t \in [-\tau, 0], \\ \\ e^{-tA} P^n[h(0) + g_n(0, \bar{h}(-\tau))] - P^n g_n(t, u_n(t-\tau)) \\ + \int_0^t A e^{-(t-s)A} P^n g_n(s, u_n(s-\tau))\, ds + \int_0^t e^{-(t-s)A} P^n f_n(s, u_{ns}) ds \\ + \sum_{0<t_k<t} e^{-(t-t_k)A} P^n I_{k,n}(u_n(t_k)), \quad t \in [0, T] \end{cases}$$

and

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}[h(0) + g(0, \bar{h}(-\tau))] - g(t, u(t-\tau)) \\ + \int_0^t Ae^{-(t-s)A} g(s, u(s-\tau))\, ds + \int_0^t e^{-(t-s)A} f(s, u_s) ds \\ + \sum_{0<t_k<t} e^{-(t-t_k)A} I_k(u(t_k)), t \in [0,\ T] \end{cases}$$

such that $\bar{u}_n \to u$ *in* $C([-\tau, T]; H)$ *as* $n \to \infty$.

Finally, we shall prove the following result.

Theorem 5. *Let the conditions (H1)–(H5) be satisfied and* $h(t) \in D(A)$ *for all* $t \in [-\tau, 0]$. *Then, the following holds*

$$\lim_{n\to\infty} \sup_{-\tau \le t \le T} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha} \{\alpha_i(t) - \alpha_i^n(t)\}^2 \right] = 0.$$

Proof. We have

$$\begin{aligned} A^\alpha[u(t) - \bar{u}_n(t)] &= A^\alpha \left[\sum_{i=0}^{\infty} \{\alpha_i(t) - \alpha_i^n(t)\}\phi_i \right] \\ &= \sum_{i=0}^{\infty} \lambda_i^\alpha \{\alpha_i(t) - \alpha_i^n(t)\}\phi_i, \end{aligned}$$

where $\alpha_i^n(t) = 0$ for all $i > n$. Thus, it follows that

$$||A^\alpha[u(t) - \bar{u}_n(t)||^2 \ge \sum_{i=0}^{n} \lambda_i^{2\alpha} |\alpha_i(t) - \alpha_i^n(t)|^2.$$

Hence as a consequence of Theorem 4 we get the required result. □

5. Examples. Let $H = L^2(0,1)$ and $\tau > 0$. Consider the following problem

$$\text{(30)} \qquad \begin{cases} \partial_t[w(t,x) + G(t, w(t-\tau, x))] - \partial_x^2 w(t,x) \\ \qquad = F(t, \partial_x w(t-\tau, x)),\ x \in (0,1),\ t > 0, \\ w(t,x) = h_1(t,x),\ \ t \in [-\tau, 0],\ x \in (0,1), \\ w(t,0) = w(t,1) = 0,\ t \in [0,T],\ 0 < T < \infty, \\ \triangle w(t_k, x) = \partial_x \bar{I}_k(\partial_x w(t_k, x)), \\ 0 < t_1 < t_2 < \ldots < t_p \le T < \infty,\ \ p \in \mathbb{N}, \end{cases}$$

where the functions F, G, $\bar{I}_k$, $k = 1, 2, 3, \cdots, p$, and h_1 are real valued sufficiently smooth functions. $\triangle w(t_k, x) = w(t_k^+, x) - w(t_k^-, x)$, where $w(t_k^+, x)$ and $w(t_k^-, x)$ are respectively the right and the left limit of w at $(t, x) = (t_k, x)$.

We define an operator A, as follows,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = H^2(0,1) \cap H_0^1(0,1). \tag{31}$$

Here, clearly the operator A is self–adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$.

The problem (30) can be written as the following abstract equation in $H = L^2(0,1)$:

$$\begin{aligned} \frac{d}{dt}[u(t) + g(t, u(t-\tau))] + Au(t) &= f(t, u_t) \quad t > 0,\ t \neq t_k, \\ u(t) &= h(t), \quad t \in [-\tau, 0], \\ \triangle u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \ldots, p, \end{aligned} \tag{32}$$

where $u(t) = w(t, .)$, that is $u(t)(x) = w(t, x)$, $u_t(\theta)(x) = w(t + \theta, x)$, $t \in [0, T]$, $\theta \in [-\tau, 0]$, $x \in (0, 1)$, the operator A is as in equation (31), $I_k(u(t_k))(x) = \partial_x \bar{I}_k(\partial_x w(t_k, x))$, the function $f : [0, T] \times C_0^{1/2} \to H$, is given by,

$$f(t, \psi)(x) = F(t, \partial_x \psi(-\tau)(x)) \tag{33}$$

and $g : [0, T] \times H_{1/2} \to H_\beta$, is given by,

$$g(t, \psi)(x) = G(t, \partial_x \psi(x)). \tag{34}$$

It can be verified that the assumptions (H1)–(H5) are satisfied, and therefore, our results can be employed to obtain approximate solutions and their convergence.

REFERENCES

[1] M. Benchohra, J. Henderson, S. K. Ntouyas, and A. Ouahab, Impulsive functional differential equations with variable times and infinite delay, *Int. J. Appl. Math. Sci.,* **2** (2005), # 1, 130–148.

[2] M. Benchohra, and A. Ouahabi, Some uniqueness results for impulsive semilinear neutral functional differential equations, *Georgian Math. J.,* **9** (2002), # 3, 423–430.

[3] B. C. Dhage, and S. K. Ntouyas, Existence results for impulsive neutral functional differential inclusions, *Topol. Methods Nonlinear Anal.,* **25** (2005), # 2, 349–361.

[4] V. Laxshmikantham , D. D. Bainov and P. S. Simenov, *Theory of impulsive differential Equations*, World Scientific Pub. Co., Singapore, 1989.
[5] S. K. Ntouyas, Existence results for impulsive partial neutral functional differential inclusions. *Electron. J. Differential Equations*, 2005, # 30, 1–11.
[6] E. Heinz, and W. von Wahl, Zn einem Satz von F.W. Browder über nichtlineare Wellengleichungen, *Math. Z.*, **141** (1974), 33–45.
[7] H. Murakami, On linear ordinary and evolution equations, *Funkcial. Ekvac.*, **9** (1966), 151–162.
[8] I. Segal, Nonlinear semigroups, *Ann. Math.*, 78 (1963), 339–364.
[9] N. Bazley, Approximation of wave equations with reproducing nonlinearities, *Nonlinear Analysis*, **3** (1979), 539–546.
[10] N. Bazley, Global convergence of Faedo–Galerkin approximations to nonlinear wave equations, *Nonlinear Analysis*, **4** (1980), 503–507.
[11] R. Goethel, Faedo-Galerkin approximation in equations of evolution, *Math. Meth. in the Appl. Sci.*, **6** (1984), 41–54.
[12] P. D. Miletta, , Approximation of solutions to evolution equations, *Math. Meth. Appl. Sci.*, **17** (1994), 753–763.
[13] M. Muslim, Approximation of Solutions to Nonlocal History-valued Neutral Functional Differential Equations, *Computer and Mathematic with Applications*, **51** (2006), 537–550..
[14] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.