

OSCILLATION CRITERIA FOR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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ABSTRACT. In this article, we give sufficient condition in the form of integral inequalities to establish the oscillatory nature of non linear homogeneous differential equations of the form

$$(r(t)y')' + q(t)y' + p(t)f(y)g(y') = 0, \quad t \geq t_0,$$

where r, q, p, f and g are given data. We do this by separating the two cases f is monotonous and non monotonous.

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1. Introduction

Comparison theorems for solution of ordinary differential equations, distribution of zeros of solutions for large time, oscillatory behavior of zeros of solutions etc. have been extensively studied in the literature. These are very important to understand the qualitative behavior of solutions specially in the non linear equations. In this article, we wish to study the oscillatory behavior of non linear homogeneous differential equations of the form

$$(r(t)y')' + q(t)y' + p(t)f(y)g(y') = 0, \quad t \geq t_0, \quad (1.1)$$

where r, q, p, f and g satisfies various assumptions to be given later. In fact, we give sufficient conditions to establish the oscillatory behavior. Coming to the literature, Wong and Burton [28] have been investigated the oscillatory behavior in connection with that of corresponding linear homogeneous equation

$$y'' + p(t)y = 0, \quad (1.2)$$

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where $p(t)$ is oscillatory.

In [5], Grace and Lalli have considered the behavior of solution of non-linear differential equations

$$y'' + p(t)f(y(t))g(y'(t)) = 0, \quad t \geq t_0, \quad (1.3)$$

where $p \in C([t_0, \infty), [0, \infty))$ and $f, g \in C(\mathbb{R}, \mathbb{R})$ with $yf(y) > 0$ for $y \neq 0$ and $g(y) > 0$ for $y \neq 0$.

The stability, boundedness, and convergence to zero of all solutions of equation (1.3) have been investigated by Burton and Grimmer [1], Graef and Spikes [6], [7], Lalli [13] and Wong and Burton [28].

Rogovchenko [22], established new sufficient conditions which ensure the oscillatory character of equation (1.3). These are different from those of Grace and Lalli [5] and applicable to other class of equations that are not covered by the results of [5]. All the above mentioned oscillation results require the information of p on the entire half-line $[t_0, \infty)$.

However, from Sturm separation theorem, we see that oscillation is only an interval property, that is, if there exist a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \rightarrow \infty$, such that for each i , there exists a solution of (1.2) that has at least two zeros in $[a_i, b_i]$, then every solution of (1.2) is oscillatory.

Ei-Sayed [4], established an interval criterion for oscillation of a forced second-order equation but the result is not sharp because a comparison with equation of constant co-efficient is used in the proof.

In [9], Huang established the following interval criteria for oscillation and non oscillation of the second order linear differential equation (1.2), where $p(t) \geq 0$, $t \in [t_0, \infty)$.

THEOREM 1.1. *If there exists $t_0 > 0$ such that for every $n \in \mathbb{N}$,*

$$\int_{2^n t_0}^{2^{n+1} t_0} p(s) ds \leq \frac{\alpha_0}{2^{n+1} t_0}, \quad (1.4)$$

then every solution of equation (1.2) is non oscillatory, where $\alpha_0 = 3 - 2\sqrt{2}$.

(ii) *If there exist $t_0 > 0$ and $\alpha > \alpha_0$ such that for every $n \in \mathbb{N}$,*

$$\int_{2^n t_0}^{2^{n+1} t_0} p(s) ds \geq \frac{\alpha}{2^n t_0}, \quad (1.5)$$

then every solution of equation (1.2) is oscillatory, where $\alpha_0 = 3 - 2\sqrt{2}$.

As an application, Huang [9] obtained the following corollary.

COROLLARY 1.2.

(i) If

$$\lim_{t \rightarrow \infty} t \int_t^{2t} p(s) \, ds = \alpha < \frac{\alpha_0}{2},$$

then every solution of equation (1.2) is non oscillatory.

(ii) If

$$\lim_{t \rightarrow \infty} t \int_t^{2t} p(s) \, ds = \alpha > \alpha_0,$$

then every solution of equation (1.2) is oscillatory, where $\alpha_0 = 3 - 2\sqrt{2}$.

H u a n g's result do not cover the interval $(\frac{\alpha_0}{2^{n+1}}t_0, \frac{\alpha_0}{2^n}t_0)$, and can be seen by the following example. Take, $p(t) = \gamma/t^2$, with, $\gamma > 0$, a constant, then

$$\lim_{t \rightarrow \infty} t \int_t^{2t} \frac{\gamma}{s^2} \, ds = \frac{\gamma}{2}.$$

This implies that H u a n g's result remains open for $\gamma \in (\alpha_0, 2\alpha_0) = (3 - 2\sqrt{2}, 6 - 4\sqrt{2})$. and hence H u a n g's oscillation criterion is not sharp. In fact, the Euler equation

$$y'' + \frac{\gamma}{t^2}y = 0$$

is oscillatory if $\gamma > \frac{1}{4}$, and non oscillatory if $\gamma \leq \frac{1}{4}$.

L i and A g a r w a l [16], [17], [18], [19] have established new interval oscillation criteria by use of integral average technique for equation (1.3). Moreover, the mentioned oscillation results based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line. They have shown that the obtained results are sharper than some known results which are not covered by known criteria.

We have motivated by the results of R a g o v c h e n k o [22], K o n g [12] and L i and A g a r w a l [18]. In this paper, we consider the oscillatory behavior of nonlinear homogeneous differential equations of the form

$$(r(t)y')' + q(t)y' + p(t)f(y)g(y') = 0, \quad t \geq t_0,$$

where r, q, p, f and g satisfies the following conditions.

(C₁) $r \in C^1([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), \mathbb{R})$

(C₂) $p: [t_0, \infty) \rightarrow [0, \infty)$ is continuous and $p(t) \neq 0$ on any ray $[T_0, \infty)$ for some $T_0 \geq t_0$.

(C₃) $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $yf(y) > 0$ for $y \neq 0$.

(C₄) $g: \mathbb{R} \rightarrow [L, \infty)$ is continuous, where $L > 0$.

A function $y: [t_0, t_1] \rightarrow \mathbb{R}$, $t_1 > t_0$ is called a solution of equation (1.1) on $[t_0, t_1]$ if $y(t)$ satisfies equation (1.1) for $t \in [t_0, t_1]$. In the sequel, we assume that the solution of equation (1.1) exists for any $t \geq 0$. A solution $y(t)$ of (1.1) on $[t_0, \infty)$ is called *oscillatory* if it has arbitrary large zeros, otherwise it is called *non oscillatory*.

The outlay of this paper is as follows. In the next Section 2, we consider the case when f is monotonous. We establish a sufficient criteria in the form of an integral inequality for the oscillatory behavior of the solution of the equation (1.1). We also write down the sufficient conditions in more simplified forms in the special cases. In Section 3, we retrace all the results in Section 2, when f is not necessarily monotonous. We present two examples and we prove the oscillatory behavior of the solutions using the previously established results. We also obtained oscillatory solutions of two examples numerically under different initial conditions. In final Section 4, we introduce the concepts of disfocality and disconjugacy and its connection to integral inequalities (*sufficient conditions*). These concepts are already in the literature (see P a r h i and P a n i g r a h i [21])

2. Oscillation with monotonicity of $f(x)$

In this section, we assume that

(C₅) f is monotonous and satisfies the condition $f'(y) \geq \mu > 0$, where μ is a constant.

DEFINITION 2.1. A function $H = H(t, s)$ belongs to a function class X , denoted by $H \in X$ if $H \in C(D, \mathbb{R}_+)$, where $D = \{(t, s) : -\infty < s \leq t < \infty\}$, which satisfies

$$H(t, t) = 0, \quad H(t, s) > 0 \quad \text{for } t > s, \quad (2.1)$$

and has partial derivatives $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial s}$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t, s)H(t, s)^{1/2} \quad \text{and} \quad \frac{\partial H}{\partial s} = -h_2(t, s)H(t, s)^{1/2}, \quad (2.2)$$

where $h_1, h_2 \in L^1_{loc}(D, \mathbb{R})$.

THEOREM 2.2. Suppose that conditions (C₁)–(C₅) hold.

(i) Let y be a positive solution of (1.1) on $[c, b)$. For any $\kappa \in C^1([t_0, \infty), (0, \infty))$, let

$$u(t) = v(t)r(t) \left\{ \frac{y'(t)}{f(y(t))} + \kappa(t) \right\}, \quad t \geq t_0, \quad (2.3)$$

on $[c, b)$. Then for any $H \in X$,

$$\int_c^b H(b, s) \phi(s) \, ds \leq H(b, c) u(c) + \frac{1}{4\mu} \int_c^b r(s) v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)} \sqrt{H(b, s)} \right]^2 \, ds. \quad (2.4)$$

(ii) Similarly, let y be a positive solution of (1.1) on $(a, c]$. For any $\kappa \in C^1([t_0, \infty), (0, \infty))$, define u as above, then for any $H \in X$

$$\int_a^c H(s, a) \phi(s) \, ds \leq -H(c, a) u(c) + \frac{1}{4\mu} \int_a^c r(s) v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)} \sqrt{H(s, a)} \right]^2 \, ds. \quad (2.5)$$

Here v and ϕ are defined as

$$v(t) = \exp \left(-2\mu \int_{t_0}^t \kappa(s) \, ds \right)$$

and

$$\phi(t) = v(t) \{ Lp(t) - q(t)\kappa(t) + \mu r(t)\kappa^2(t) - (r(t)\kappa(t))' \}.$$

Proof. From the equations (1.1) and (2.3) for $s \in [c, b)$, we have

$$\begin{aligned} u'(t) &= -2\mu\kappa(t)v(t)r(t) \left\{ \frac{y'(t)}{f(y(t))} + \kappa(t) \right\} \\ &\quad + v(t) \left\{ \frac{(r(t)y')'}{f(y(t))} - \frac{r(t)f'(y(t))y'^2(t)}{f^2(y(t))} + (r(t)\kappa(t))' \right\} \end{aligned}$$

By using $f'(y) \geq \mu > 0$ and $g(y'(t)) \geq L > 0$, we obtain from the above inequality

$$\begin{aligned} u'(t) &\leq -2\mu\kappa(t)v(t)r(t) \frac{y'(t)}{f(y(t))} - 2\mu\kappa^2(t)v(t)r(t) - \frac{q(t)v(t)y'(t)}{f(y(t))} \\ &\quad - Lv(t)p(t) - \frac{\mu r(t)v(t)y'^2(t)}{f^2(y(t))} + v(t)(r(t)\kappa(t))' \\ &= -\frac{\mu u^2(t)}{r(t)v(t)} - \frac{q(t)}{r(t)} u(t) - v(t)[Lp(t) - q(t)\kappa(t) + \mu r(t)\kappa^2(t) - (r(t)\kappa(t))'] \\ &= -\frac{\mu u^2(t)}{r(t)v(t)} - \frac{q(t)}{r(t)} u(t) - \phi(t). \end{aligned}$$

Thus

$$\phi(t) \leq -u'(t) - \frac{\mu u^2(t)}{r(t)v(t)} - \frac{q(t)}{r(t)} u(t) \quad (2.6)$$

Multiplying (2.6) by $H(t, s)$, integrating it with respect to s from c to t ($c \leq t < b$), and using (2.1) and (2.2), we get

$$\begin{aligned}
 & \int_c^t H(t, s) \phi(s) \, ds \\
 & \leq - \int_c^t H(t, s) u'(s) \, ds - \int_c^t H(t, s) \frac{\mu u^2(s)}{r(s)v(s)} \, ds - \int_c^t H(t, s) \frac{q(s)}{r(s)} u(s) \, ds \\
 & = H(t, c)u(c) - \int_c^t \left\{ h_2(t, s) \sqrt{H}(t, s) u(s) + H(t, s) \frac{q(s)}{r(s)} u(s) + H(t, s) \frac{\mu u^2(s)}{r(s)v(s)} \right\} \, ds \\
 & = H(t, c)u(c) - \int_c^t \left\{ \sqrt{\frac{\mu H(t, s)}{r(s)v(s)}} u(s) + \frac{1}{2} \sqrt{\frac{r(s)v(s)}{\mu}} \left[h_2(t, s) + \frac{q(s)}{r(s)} \sqrt{H}(t, s) \right] \right\}^2 \, ds \\
 & \quad + \frac{1}{4\mu} \int_c^t r(s)v(s) \left[h_2(t, s) + \frac{q(s)}{r(s)} \sqrt{H}(t, s) \right]^2 \, ds \\
 & \leq H(t, c)u(c) + \frac{1}{4\mu} \int_c^t r(s)v(s) \left[h_2(t, s) + \frac{q(s)}{r(s)} \sqrt{H}(t, s) \right]^2 \, ds.
 \end{aligned}$$

Taking $t \rightarrow b^-$, in the above inequality, we obtain (2.4).

To get the second part, again multiply (2.6) by $H(s, t)$, integrate with respect to s from t to c for $t \in (a, c]$. Using (2.1) and (2.2), we get

$$\begin{aligned}
 & \int_t^c H(s, t) \phi(s) \, ds \\
 & \leq - \int_t^c H(s, t) u'(s) \, ds - \int_t^c H(s, t) \frac{\mu u^2(s)}{r(s)v(s)} \, ds - \int_t^c H(s, t) \frac{q(s)}{r(s)} u(s) \, ds \\
 & = - H(c, t)u(c) + \int_t^c \left\{ h_1(s, t) \sqrt{H}(s, t) u(s) - H(s, t) \frac{q(s)}{r(s)} u(s) - H(s, t) \frac{\mu u^2(s)}{r(s)v(s)} \right\} \, ds \\
 & = - H(c, t)u(c) - \int_t^c \left\{ \sqrt{\frac{\mu H(s, t)}{r(s)v(s)}} u(s) - \frac{1}{2} \sqrt{\frac{r(s)v(s)}{\mu}} \left[h_1(s, t) - \frac{q(s)}{r(s)} \sqrt{H}(s, t) \right] \right\}^2 \, ds \\
 & \quad + \frac{1}{4\mu} \int_t^c r(s)v(s) \left[h_1(s, t) - \frac{q(s)}{r(s)} \sqrt{H}(s, t) \right]^2 \, ds \\
 & \leq - H(c, t)u(c) + \frac{1}{4\mu} \int_t^c r(s)v(s) \left[h_1(s, t) - \frac{q(s)}{r(s)} \sqrt{H}(s, t) \right]^2 \, ds.
 \end{aligned}$$

Taking $t \rightarrow a^+$, in the above inequality, we obtain (2.5). The proof of the theorem is complete. \square

THEOREM 2.3. *Suppose that $(C_1) - (C_5)$ hold. Assume that for some $c \in (a, b)$, there exist $H \in X$, $\kappa \in C^1([t_0, \infty), (0, \infty))$, the following inequality is satisfied*

$$\begin{aligned} & \frac{1}{H(c, a)} \int_a^c H(s, a) \phi(s) \, ds + \frac{1}{H(b, c)} \int_c^b H(b, s) \phi(s) \, ds \\ & > \frac{1}{4\mu H(c, a)} \int_a^c r(s) v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)} \sqrt{H}(s, a) \right]^2 \, ds \\ & \quad + \frac{1}{4\mu H(b, c)} \int_c^b r(s) v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)} \sqrt{H}(b, s) \right]^2 \, ds, \quad (2.7) \end{aligned}$$

where u, v, ϕ, h_1, h_2 are defined as earlier. Then every solution of (1.1) has at least one zero in (a, b) .

Proof. If not, then with out loss of generality we may assume that equation (1.1) admits a solution $y(t)$ such that $y(t) > 0$ for $t \in (a, b)$. Let $c \in (a, b)$. Then by using Theorem 2.2, we conclude that (2.4) and (2.5) holds. By dividing (2.4) and (2.5) by $H(b, c)$ and $H(c, a)$ respectively, and adding them, we have

$$\begin{aligned} & \frac{1}{H(c, a)} \int_a^c H(s, a) \phi(s) \, ds + \frac{1}{H(b, c)} \int_c^b H(b, s) \phi(s) \, ds \\ & \leq \frac{1}{4\mu H(c, a)} \int_a^c r(s) v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)} \sqrt{H}(s, a) \right]^2 \, ds \\ & \quad + \frac{1}{4\mu H(b, c)} \int_c^b r(s) v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)} \sqrt{H}(b, s) \right]^2 \, ds, \end{aligned}$$

which is a contradiction to (2.7). Hence the proof of the theorem is complete. \square

Theorems on oscillations

We now derive various oscillatory criteria using the earlier results.

THEOREM 2.4. *Suppose that $(C_1) - (C_5)$ hold. If for each $T \geq t_0$, there exist $H \in X$, $\kappa \in C^1([t_0, \infty), (0, \infty))$ and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and (2.7) holds, then every solution of (1.1) is oscillatory.*

Proof. We consider a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \rightarrow \infty$ as $i \rightarrow \infty$. By the assumption, for each $i \in \mathbb{N}$, there exists a_i, b_i, c_i and such that $T \leq a_i < c_i < b_i$ and (2.7) holds. From Theorem 2.3, every solution $y(t)$ has at least one zero in (a_i, b_i) . As $T_i \rightarrow \infty$, we see that every solution of (1.1) is oscillatory. Hence the theorem. \square

THEOREM 2.5. *Suppose that conditions (C_1) – (C_5) hold. Assume there exist $H \in X$, $\kappa \in C^1([t_0, \infty), (0, \infty))$ such that for any $t_1 \geq t_0$,*

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left\{ H(s, t_1) \phi(s) - \frac{1}{4\mu} r(s) v(s) \left[h_1(s, t_1) - \frac{q(s)}{r(s)} \sqrt{H}(s, t_1) \right]^2 \right\} ds > 0 \quad (2.8)$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left\{ H(t, s) \phi(s) - \frac{1}{4\mu} r(s) v(s) \left[h_2(t, s) + \frac{q(s)}{r(s)} \sqrt{H}(t, s) \right]^2 \right\} ds > 0, \quad (2.9)$$

where $v(t)$ and $\phi(t)$ are defined as in Theorem 2.1. Then every solution of (1.1) is oscillatory.

Proof. For any $T \geq t_0$, let $a = T$. In (2.8), take $t_1 = a$. Then there exists $c > a$ such that

$$\int_a^c \left\{ H(s, a) \phi(s) - \frac{1}{4\mu} r(s) v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)} \sqrt{H}(s, a) \right]^2 \right\} ds > 0. \quad (2.10)$$

In (2.9), take $t_1 = c$. Then there exist $b > c$ such that

$$\int_c^b \left\{ H(b, s) \phi(s) - \frac{1}{4\mu} r(s) v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)} \sqrt{H}(b, s) \right]^2 \right\} ds > 0. \quad (2.11)$$

Dividing the equations (2.10) and (2.11), by $H(c, a)$ and $H(b, c)$, respectively, and adding, we get

$$\begin{aligned} & \frac{1}{H(c, a)} \int_a^c H(s, a) \phi(s) \, ds + \frac{1}{H(b, c)} \int_c^b H(b, s) \phi(s) \, ds \\ & > \frac{1}{4\mu H(c, a)} \int_a^c r(s) v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)} \sqrt{H}(s, a) \right]^2 \, ds \\ & \quad + \frac{1}{4\mu H(b, c)} \int_c^b r(s) v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)} \sqrt{H}(b, s) \right]^2 \, ds. \end{aligned}$$

Then it follows by Theorem 2.4, that every solution of (1.1) is oscillatory. \square

Remark 2.6. If $q(t) = 0$, $f(y) = y$, $g(y') = 1$, $\kappa(t) = 0$, then Theorems 2.3 and 2.5 reduces to [12, Corollaries 2.2, 2.4].

Consider the special case $H(t, s) = H(t - s)$, then

$$\frac{\partial H}{\partial t} = h_1(t - s) H(t - s)^{1/2} \quad \text{and} \quad \frac{\partial H}{\partial s} = h_2(t - s) H(t - s)^{1/2}.$$

Thus for $H = H(t - s) \in X$, we have $h_1(t - s) = h_2(t - s)$ and denote them by $h(t - s)$. The subclass containing such $H(t - s)$ is denoted by X_0 . Applying Theorem 2.4 to X_0 we have the following result:

THEOREM 2.7. Suppose that conditions (C_1) – (C_5) hold. If for each $T \geq t_0$, there exist $H \in X_0$, $\kappa \in C^1([t_0, \infty), (0, \infty))$ and $a, c \in \mathbb{R}$ such that $T \leq a < c$ and

$$\begin{aligned} & \int_a^c H(s - a) [\phi(s) + \phi(2c - s)] \, ds \\ & > \frac{1}{4\mu} \int_a^c [r(s) v(s) + r(2c - s) v(2c - s)] h^2(s - a) \, ds \\ & \quad + \frac{1}{2\mu} \int_a^c [v(2c - s) q(2c - s) - v(s) q(s)] h(s - a) \sqrt{H}(s - a) \, ds \\ & \quad + \frac{1}{4\mu} \int_a^c \left[\frac{v(s) q^2(s)}{r(s)} + \frac{v(2c - s) q^2(2c - s)}{r(2c - s)} \right] H(s - a) \, ds, \end{aligned} \quad (2.12)$$

where $v(t)$ and $\phi(t)$ are as defined in Theorem 2.1, then every solution of equation (1.1) is oscillatory.

Proof. Let $c = \frac{a+b}{2}$, that is, $b = 2c - a$, then

$$H(b - c) = H(c - a) = H((b - a)/2).$$

For any $\omega \in L^1(a, b)$, we have

$$\begin{aligned} \int_c^b \omega(s) \, ds &= \int_a^c \omega(2c - s) \, ds \\ \int_c^b H(b - s)\phi(s) \, ds &= \int_a^c H(s - a)\phi(2c - s) \, ds \\ \int_c^b v(s)q(s)h(b - s)\sqrt{H}(b - s) \, ds &= \int_a^c v(2c - s)q(2c - s)h(s - a)\sqrt{H}(s - a) \, ds \\ \int_c^b r(s)v(s)h^2(b - s) \, ds &= \int_a^c r(2c - s)v(2c - s)h^2(s - a)\sqrt{H}(s - a) \, ds. \end{aligned}$$

It is easy to see that (2.12) implies that the inequality (2.7). Hence every solution of (1.1) is oscillatory by Theorem 2.4.

For different choices of $H(t, s)$, we will obtain different sufficient conditions for oscillatory behavior of solutions of equation (1.1). Let

$$H(t, s) = (t - s)^\lambda, \quad t \geq s \geq t_0,$$

where $\lambda > 1$ is a constant. Then the sufficient conditions (2.8) and (2.9), respectively can be written as

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t (s - t_1)^\lambda \left[\phi(s) - \frac{1}{4\mu} r(s)v(s) \left(\frac{\lambda}{(s - t_1)} - \frac{q(s)}{r(s)} \right)^2 \right] ds > 0, \quad (2.13)$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t (t - s)^\lambda \left[\phi(s) - \frac{1}{4\mu} r(s)v(s) \left(\frac{\lambda}{(t - s)} + \frac{q(s)}{r(s)} \right)^2 \right] ds > 0. \quad (2.14)$$

□

COROLLARY 2.8. Suppose that (C_1) – (C_5) hold. Assume for each $t_1 \geq t_0$ and for some $\lambda > 1$, $\kappa \in C^1([t_0, \infty), (0, \infty))$, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (s - t_1)^\lambda \left[\phi(s) - \frac{1}{4\mu} r(s)v(s) \left(\frac{\lambda}{(s - t_1)} - \frac{q(s)}{r(s)} \right)^2 \right] ds > 0,$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (t-s)^{\lambda} \left[\phi(s) - \frac{1}{4\mu} r(s) v(s) \left(\frac{\lambda}{(t-s)} + \frac{q(s)}{r(s)} \right)^2 \right] ds > 0,$$

then every solution of (1.1) is oscillatory.

The oscillatory nature of the solutions follows because, the above two conditions implies the conditions (2.13) and (2.14). We, however use the above two conditions to get the following result.

THEOREM 2.9. Suppose that $(C_1) - (C_5)$ hold. If for each $t_1 \geq t_0$ and for some $\lambda > 1$ satisfies the following conditions

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (s-t_1)^{\lambda} \left[\phi(s) - \frac{1}{4\mu} \left(q^2(s) - \frac{2\lambda q(s)}{(s-t_1)} \right) \right] ds > \frac{\lambda^2}{4\mu(\lambda-1)}, \quad (2.15)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (t-s)^{\lambda} \left[\phi(s) - \frac{1}{4\mu} \left(q^2(s) + \frac{2\lambda q(s)}{(t-s)} \right) \right] ds > \frac{\lambda^2}{4\mu(\lambda-1)}, \quad (2.16)$$

then (1.1) is oscillatory.

Proof. We may observe that

$$h_1(t, s) = h_2(t, s) = \lambda(t-s)^{\lambda/2-1}.$$

Note that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{4\mu t^{\lambda-1}} \int_{t_1}^t h_1^2(s-t_1) ds &= \limsup_{t \rightarrow \infty} \frac{1}{4\mu t^{\lambda-1}} \int_{t_1}^t \lambda^2 (s-t_1)^{\lambda-2} ds \\ &= \frac{\lambda^2}{4\mu(\lambda-1)} \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{4\mu t^{\lambda-1}} \int_{t_1}^t h_2^2(t-s) ds &= \limsup_{t \rightarrow \infty} \frac{1}{4\mu t^{\lambda-1}} \int_{t_1}^t \lambda^2 (t-s)^{\lambda-2} ds \\ &= \frac{\lambda^2}{4\mu(\lambda-1)}. \end{aligned} \quad (2.18)$$

From (2.15) and (2.17), we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t \left\{ H(s-t_1) \phi(s) - \frac{1}{4\mu} \left[h_1(s-t_1) - q(s) \sqrt{H}(s-t_1) \right]^2 \right\} ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (s-t_1)^\lambda \left[\phi(s) - \frac{1}{4\mu} \left(q^2(s) + \frac{2\lambda q(s)}{(s-t_1)} \right) \right] ds - \frac{\lambda^2}{4\mu(\lambda-1)} > 0, \end{aligned}$$

that is, (2.8) holds. Similarly, (2.16) and (2.18) implies that (2.9) holds. By Theorem 2.5, every solution of (1.1) is oscillatory. Hence the theorem. \square

More generally, one may consider

$$H(t, s) = [R(t) - R(s)]^\lambda,$$

where λ is a constant and $R(t) = \int_{t_1}^t \frac{1}{r(s)} ds$ and $\lim_{t \rightarrow \infty} R(t) = \infty$.

THEOREM 2.10. *Suppose that (C_1) – (C_5) hold. If for each $t_1 \geq t_0$ and for some $\lambda > 1$, $\kappa \in C^1([t_0, \infty), (0, \infty))$ such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_{t_1}^t [R(s) - R(t_1)]^\lambda \left[\phi(s) - \frac{v(s)}{4\mu r(s)} \left(\frac{\lambda}{R(s) - R(t_1)} - q(s) \right)^2 \right] ds > 0,$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_{t_1}^t [R(t) - R(s)]^\lambda \left[\phi(s) - \frac{v(s)}{4\mu r(s)} \left(\frac{\lambda}{R(t) - R(s)} + q(s) \right)^2 \right] ds > 0,$$

then every solution of (1.1) is oscillatory.

The proof is similar to that of Theorem 2.5 and hence omitted. We also have the following theorem whose proof is similar to Theorem 2.9.

THEOREM 2.11. *Suppose that (C_1) – (C_5) hold. If for each $t_1 \geq t_0$ and for some $\lambda > 1$, $\kappa \in C^1([t_0, \infty), (0, \infty))$ such that*

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_{t_1}^t [R(s) - R(t_1)]^\lambda \left[\phi(s) - \frac{1}{4\mu r(s)} \left(q^2(s) - \frac{2\lambda q(s)}{R(s) - R(t_1)} \right) \right] ds \\ & > \frac{\lambda^2}{4\mu(\lambda^2 - 1)} \end{aligned}$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_{t_1}^t [R(t) - R(s)]^\lambda \left[\phi(s) - \frac{1}{4\mu r(s)} \left(q^2(s) + \frac{2\lambda q(s)}{R(t) - R(s)} \right) \right] ds$$

$$> \frac{\lambda^2}{4\mu(\lambda^2 - 1)},$$

then every solution of (1.1) is oscillatory.

3. Oscillation without monotonicity of $f(x)$

We now consider non monotonous situation. But, we make the following assumption

(C₆) $\frac{f(y)}{y} \geq \mu_1 > 0$, where μ_1 is a constant.

THEOREM 3.1. Suppose that conditions (C₁)–(C₄) and (C₆) hold.

(i) Let y be a solution of (1.1) such that $y(t) > 0$ on $[c, b)$. For any $\kappa \in C^1([t_0, \infty), (0, \infty))$, let

$$\omega(t) = v(t)r(t) \left\{ \frac{y'(t)}{y(t)} + \kappa(t) \right\}, \quad t \geq t_0, \quad (3.1)$$

on $[c, b)$. Then for any $H \in X$, we have

$$\int_c^b H(t, s) \phi(s) ds \leq H(b, c) \omega(c) + \frac{1}{4} \int_c^b r(s) v(s) \left[h_2(b, c) + \frac{q(s)}{r(s)} \sqrt{H}(b, c) \right]^2 ds.$$

(ii) Let y be a solution of (1.1) such that $y(t) > 0$ on $(a, c]$ and ω be as above. Then for any $H \in X$, we have

$$\int_a^c H(s, a) \phi(s) ds \leq -H(c, a) \omega(c) + \frac{1}{4} \int_a^c r(s) v(s) \left[h_1(s, a) + \frac{q(s)}{r(s)} \sqrt{H}(s, a) \right]^2 ds,$$

where

$$v(t) = \exp \left(-2 \int_{t_0}^t \kappa(s) ds \right)$$

and

$$\phi(t) = v(t) \{ L\mu_1 p(t) - q(t)\kappa(t) + r(t)\kappa^2(t) - (r(t)\kappa(t))' \}.$$

Proof. From (1.1) and (3.1), we have

$$\begin{aligned}\omega'(t) = & -2\kappa(t)v(t)r(t)\left\{\frac{y'(t)}{y(t)} + \kappa(t)\right\} \\ & + v(t)\left\{\frac{(r(t)y')'}{y(t)} - \frac{r(t)y'^2(t)}{y^2(t)} + (r(t)\kappa(t))'\right\}.\end{aligned}$$

By using $f(y)y \geq \mu_1 > 0$ and $g(y'(t)) \geq L > 0$, we obtain from the above inequality

$$\begin{aligned}\omega'(t) \leq & -2\kappa(t)v(t)r(t)\frac{y'(t)}{y(t)} - 2\kappa^2(t)v(t)r(t) - q(t)v(t)\frac{y'(t)}{y(t)} \\ & - L\mu_1v(t)p(t) - r(t)v(t)\frac{y'^2(t)}{y^2(t)} + v(t)(r(t)\kappa(t))' \\ = & -\frac{\omega^2(t)}{r(t)v(t)} - \frac{q(t)}{r(t)}\omega(t) - v(t)[L\mu_1p(t) - q(t)\kappa(t) + r(t)\kappa^2(t) - (r(t)\kappa(t))'] \\ = & -\frac{\omega^2(t)}{r(t)v(t)} - \frac{q(t)}{r(t)}\omega(t) - \phi(t)\end{aligned}$$

that is,

$$\phi(t) \leq -\omega'(t) - \frac{\omega^2(t)}{r(t)v(t)} - \frac{q(t)}{r(t)}\omega(t)$$

The remaining part of the proof is the same as that of the proof of Theorem 2.1 and hence omitted. \square

Once we have the above theorem, we obtain similar results as in Section 2. For the sake of completeness, we present the results without proof.

THEOREM 3.2. *Suppose that (C_1) – (C_4) and (C_6) hold. Assume that for some $c \in (a, b)$, $H \in X$, $\kappa \in C^1([t_0, \infty), (0, \infty))$, the following inequality is satisfied*

$$\begin{aligned}& \frac{1}{H(c, a)} \int_a^c H(s, a)\phi(s) \, ds + \frac{1}{H(b, c)} \int_c^b H(b, s)\phi(s) \, ds \\ & > \frac{1}{4H(c, a)} \int_a^c r(s)v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)}\sqrt{H}(s, a) \right]^2 \, ds \\ & \quad + \frac{1}{4H(b, c)} \int_c^b r(s)v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)}\sqrt{H}(b, s) \right]^2 \, ds.\end{aligned}\tag{3.2}$$

Then every solution of (1.1) has at least one zero in (a, b) .

THEOREM 3.3. Suppose that conditions (C_1) – (C_4) and (C_6) hold. If for each $T \geq t_0$, there exist $H \in X$, $\kappa \in C^1([t_0, \infty), (0, \infty))$ and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and (3.2) holds, then every solution of (1.1) is oscillatory.

THEOREM 3.4. Suppose that conditions (C_1) – (C_4) and (C_6) hold. Assume there exist $H \in X$, $\kappa \in C^1([t_0, \infty), (0, \infty))$ such that for $t_1 \geq t_0$,

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left\{ H(s, t_1) \phi(s) - \frac{1}{4} r(s) v(s) \left[h_1(s, t_1) - \frac{q(s)}{r(s)} \sqrt{H}(s, t_1) \right]^2 \right\} ds > 0$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left\{ H(t, s) \phi(s) - \frac{1}{4} r(s) v(s) \left[h_2(t, s) + \frac{q(s)}{r(s)} \sqrt{H}(t, s) \right]^2 \right\} ds > 0,$$

then every solution of (1.1) is oscillatory.

COROLLARY 3.5. Suppose that conditions (C_1) – (C_4) and (C_6) hold. If for each $T \geq t_0$ and for some $\lambda > 1$, then there exists a function $H \in X$, $\kappa \in C^1([t_0, \infty), (0, \infty))$, such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (s - t_1)^\lambda \left[\phi(s) - \frac{1}{4} r(s) v(s) \left(\frac{\lambda}{(s - t_1)} - \frac{q(s)}{r(s)} \right)^2 \right] ds > 0$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (t - s)^\lambda \left[\phi(s) - \frac{1}{4} r(s) v(s) \left(\frac{\lambda}{(t - s)} - \frac{q(s)}{r(s)} \right)^2 \right] ds > 0$$

Then every solution of (1.1) is oscillatory.

We complete this section by presenting two examples. We conclude the oscillatory nature of the solutions by employing the previous results.

Example 3.6. Consider the nonlinear homogeneous differential equations

$$(t^2 y')' - 2ty' + \frac{t^2}{(1 + \sin^2 t)(1 + \cos^4 t)} y(1 + y^4)(1 + y'^2) = 0, \quad t \geq 1. \quad (3.3)$$

Here $r(t) = t^2$, $q(t) = -2t$, $p(t) = \frac{t^2}{(1 + \sin^2 t)(1 + \cos^4 t)}$, $f(y) = y(1 + y^4)$ and $g(y') = 1 + y'^2$. We may observe that

$$f'(y) = 1 + 5y^4 \geq 1 = \mu, \quad \text{and} \quad g(y') = 1 + y'^2 \geq 1 = L.$$

If $\lambda = 2$ and we choose $v(t) = \exp\left(-2\mu \int_1^t \kappa(s) ds\right) = \frac{1}{t^2}$, then $\phi(t) = v(t)\{Lp(t) - \kappa(t)q(t) + \mu r(t)\kappa^2(t) - (r(t)\kappa(t))'\} = \frac{1}{(1 + \sin^2 t)(1 + \cos^4 t)} + 2$.

By use of Corollary 2.8, we may show that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (s-t_1)^\lambda \left[\phi(s) - \frac{1}{4\mu} r(s) v(s) \left(\frac{\lambda}{s-t_1} - \frac{q(s)}{r(s)} \right)^2 \right] ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t (s-t_1)^2 \left[\frac{1}{(1+\sin^2 s)(1+\cos^4 s)} + 2 - \frac{1}{2} \left(\frac{1}{s-t_1} + \frac{1}{s} \right)^2 \right] ds \\ &\geq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t \left[(s-t_1)^2 - \frac{1}{2s^2} (2s-t_1)^2 + 2 \right] ds = \infty, \end{aligned}$$

and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (t-s)^\lambda \left[\phi(s) - \frac{1}{4\mu} r(s) v(s) \left(\frac{\lambda}{t-s} - \frac{q(s)}{r(s)} \right)^2 \right] ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t (t-s)^2 \left[\frac{1}{(1+\sin^2 s)(1+\cos^4 s)} + 2 - \left(\frac{1}{t-s} + \frac{1}{s} \right)^2 \right] ds = \infty. \end{aligned}$$

Hence (3.3) is oscillatory by Corollary 1. Moreover, $y = \cos t$ is an oscillatory solution of (3.3).

Example 3.7. Consider the nonlinear homogeneous differential equation

$$(ty')' - y' + \frac{3t(1+\sin^2 t)}{(1+\cos^2 t)(4+\sin^2 t)} y \left(\frac{1}{3} + \frac{1}{1+y^2} \right) (1+y'^2) = 0, \quad t \geq 1. \quad (3.4)$$

Here $r(t) = t$, $q(t) = -1$, $p(t) = \frac{3t(1+\sin^2 t)}{(4+\sin^2 t)(1+\cos^2 t)}$, $f(y) = \left(\frac{1}{3} + \frac{1}{1+y^2} \right) y$, $g(y') = 1 + y'^2 \geq 1 = L$. As earlier, we take $\lambda = 2$ and $v(t) = \frac{1}{t^2}$. We may observe that

$$\begin{aligned} f'(y) &= \frac{y^4 - y^2 + 4}{(1+y^2)^2} \\ \frac{f(y)}{y} &= \frac{1}{3} + \frac{1}{1+y^2} \geq \frac{1}{3} = \mu_1 \\ g(y') &= 1 + y'^2 \geq 1 = L. \end{aligned}$$

Thus

$$\begin{aligned} \phi(t) &= v(t)[L\mu_1 p(t) - \kappa(t)q(t) + r(t)\kappa^2(t) - (r(t)\kappa(t))'] \\ &= \frac{1+\sin^2 t}{t(4+\sin^2 t)(1+\cos^2 t)} + \frac{2}{t^3}. \end{aligned}$$

Clearly, Theorem 2.9 cannot be applied to equation (3.4). In this case Corollary 3.5 is applicable. By Corollary 3.5, we obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_{t_1}^t (s-t_1)^\lambda \left[\phi(s) - \frac{1}{4} r(s) v(s) \left(\frac{\lambda}{s-t_1} - \frac{q(s)}{r(s)} \right)^2 \right] ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t (s-t_1)^2 \left[\frac{(1+\sin^2 s)}{s(4+\sin^2 s)(1+\cos^2 s)} + \frac{2}{s^3} - \frac{1}{4s^3} \frac{(3s-t_1)^2}{(s-t_1)^2} \right] ds = \infty \end{aligned}$$

and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t (t-s)^\lambda \left[\phi(s) - \frac{1}{4} r(s) v(s) \left(\frac{\lambda}{t-s} - \frac{q(s)}{r(s)} \right)^2 \right] ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t (t-s)^2 \left[\frac{1+\sin^2 s}{s(4+\sin^2 s)(1+\cos^2 s)} + \frac{2}{s^3} - \frac{1}{4s^3} \left(\frac{s+t}{t-s} \right)^2 \right] ds = \infty. \end{aligned}$$

Thus by Corollary 3.5, every solution of equation (3.4) is oscillatory. Moreover, $y = \sin t$ is an oscillatory solution of (3.4).

Numerical results

We have obtained oscillatory solutions of the above equations numerically under different initial conditions. The first three figures represents the solutions and its derivative of the first example and the solutions of the second example is given in the last three figures.

4. Difocality and disconjugacy

In this section, we assume that $y(t)$ is a nontrivial solution of (1.1) with $y(a) = 0 = y(b)$, $y(t) \neq 0$ for $t \in (a, b)$. Then there exists a $c \in (a, b)$ such that $y'(c) = 0$.

DEFINITION 4.1. Equation (1.1) is said to be right-difocal in $[c, b]$ if the solution of (1.1) with $y'(c) = 0$, $y(c) \neq 0$ do not have zero in $[c, b)$.

Equation (1.1) is said to be left-difocal in $(a, c]$ if the solution of (1.1) with $y'(c) = 0$, $y(c) \neq 0$, do not have zeros in $(a, c]$.

Equation (1.1) is said to be disconjugate in (a, b) , if no non trivial solution of (1.1) has more than one zero in (a, b) .

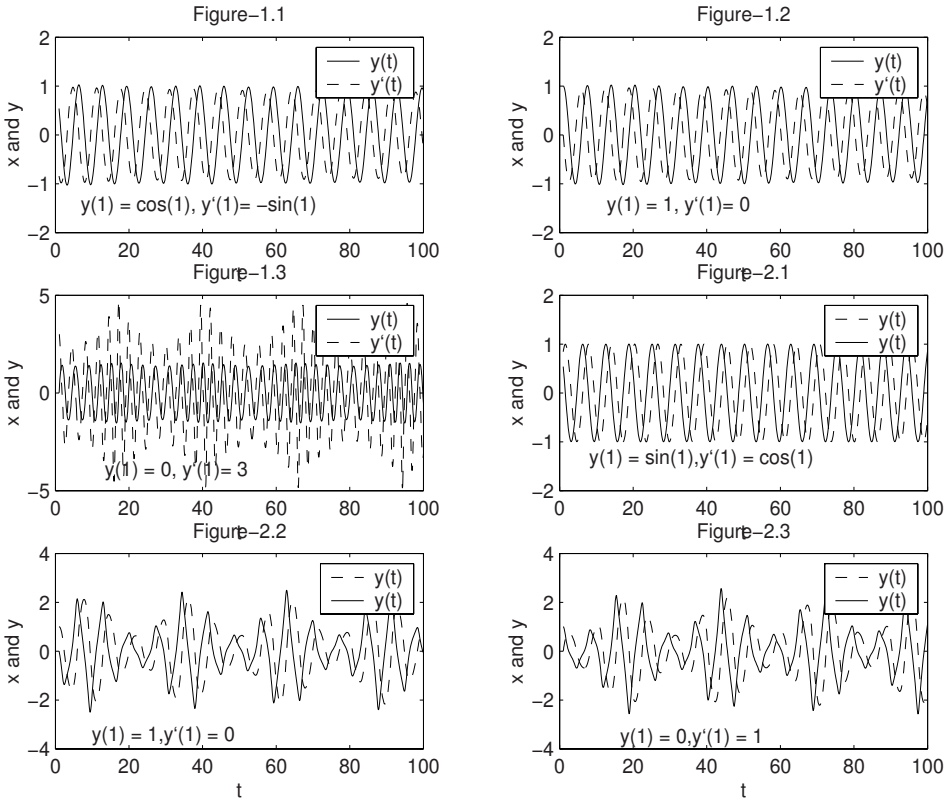


FIGURE 1. Figure for oscillaton

THEOREM 4.2.

(i) Assume that equation(1.1) is right disfocal in $[c, b)$. Then for any $H \in X$ and $C^1([t_0, \infty), (0, \infty))$, and $\kappa \in C^1[t_0, \infty)$

$$\int_c^b H(b, s) \phi(s) ds \leq H(b, c) v(c) r(c) \kappa(c) + \frac{1}{4\mu} \int_c^b r(s) v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)} \sqrt{H}(b, s) \right]^2 ds, \quad (4.1)$$

where

$$v(t) = \exp \left(-2\mu \int_{t_0}^t \kappa(s) ds \right)$$

and

$$\phi(t) = v(t)\{Lp(t) - q(t)\kappa(t) + \mu r(t)\kappa^2(t) - (r(t)\kappa(t))'\}.$$

(ii) Assume that equation (1.1) is left disfocal in $(a, c]$. Then for any $H \in X$ and $\kappa \in C^1([t_0, \infty), (0, \infty))$,

$$\begin{aligned} & \int_a^c H(s, a)\phi(s) \, ds \\ & \leq -H(c, a)v(c)r(c)\kappa(c) + \frac{1}{4\mu} \int_a^c r(s)v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)}\sqrt{H}(s, a) \right]^2 \, ds. \end{aligned} \quad (4.2)$$

(iii) Assume that equation (1.1) is disconjugate on (a, b) . Then for any $H \in X$ there exists $c \in (a, b)$ such that (4.1) or (4.2) hold.

Proof.

(i) Let y be a solution of (1.1) satisfying $y'(c) = 0$ and $y(c) > 0$. Since (1.1) is right disfocal in $[c, b)$, then $y(t) > 0$ on $[c, b)$. So that $u(t)$ defined by (2.3) exists on $[c, b)$ and $u(c) = v(c)r(c)\kappa(c)$. Then required inequalities follows from Theorem 2.2.

(ii) Let y be a solution of (1.1) satisfying $y'(c) = 0$ and $y(c) > 0$. Since (1.1) is left disfocal in $(a, c]$, then $y(t) > 0$ on $(a, c]$. So that $u(t)$ defined by (2.3) exists on $(a, c]$ and $u(c) = v(c)r(c)\kappa(c)$. Then the required inequality follows from Theorem 2.2.

(iii) Suppose that (1.1) is disconjugate on (a, b) . Let $y(t)$ be a solution of (1.1) with $y'(c) = 0$, $y(c) \neq 0$, where $c \in (a, b)$. Then $y(t)$ has at most one zero in $(a, c]$ or one zero in $[c, b)$. Hence (1.1) is left disfocal in $(a, c]$ or right disfocal in $[c, b)$. Hence (4.1) or (4.2) holds. \square

Remark 4.3.

(i) If for some $H \in X$ and $\kappa \in C^1([t_0, \infty), (0, \infty))$ such that

$$\begin{aligned} & \int_c^b H(b, s)\phi(s) \, ds \\ & > H(b, c)v(c)r(c)\kappa(c) + \frac{1}{4\mu} \int_c^b r(s)v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)}\sqrt{H}(b, s) \right]^2 \, ds, \end{aligned}$$

then (1.1) is not right disfocal in $[c, b)$.

(ii) If for some $H \in X$ and $\kappa \in C^1([t_0, \infty), (0, \infty))$ such that

$$\begin{aligned} & \int_a^c H(s, a) \phi(s) \, ds \\ & > -H(c, a) v(c) r(c) \kappa(c) + \frac{1}{4\mu} \int_a^c r(s) v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)} \sqrt{H}(s, a) \right]^2 \, ds, \end{aligned}$$

then (1.1) is not left disfocal in $(a, c]$.

(iii) If for some $H \in X$ and $\kappa \in C^1([t_0, \infty), (0, \infty))$ such that

$$\begin{aligned} & \int_a^c H(s, a) \phi(s) \, ds \\ & > -H(c, a) v(c) r(c) \kappa(c) + \frac{1}{4\mu} \int_a^c r(s) v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)} \sqrt{H}(s, a) \right]^2 \, ds, \end{aligned}$$

and

$$\begin{aligned} & \int_c^b H(b, s) \phi(s) \, ds \\ & > H(b, c) v(c) r(c) \kappa(c) + \frac{1}{4\mu} \int_c^b r(s) v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)} \sqrt{H}(b, s) \right]^2 \, ds, \end{aligned}$$

holds. Then (1.1) is not disconjugate in (a, b) .

THEOREM 4.4. Assume that for some $H \in X$, there exists $c \in (a, b)$ such that

$$\begin{aligned} & \int_a^c H(s, a) \phi(s) \, ds \\ & > -H(c, a) v(c) r(c) \kappa(c) + \frac{1}{4\mu} \int_a^c r(s) v(s) \left[h_1(s, a) - \frac{q(s)}{r(s)} \sqrt{H}(s, a) \right]^2 \, ds \end{aligned}$$

and

$$\begin{aligned} & \int_c^b H(b, s) \phi(s) \, ds \\ & > H(b, c) v(c) r(c) \kappa(c) + \frac{1}{4\mu} \int_c^b r(s) v(s) \left[h_2(b, s) + \frac{q(s)}{r(s)} \sqrt{H}(b, s) \right]^2 \, ds \end{aligned}$$

then every solution of equation (1.1) has at least two zeros in (a, b) .

P r o o f. If above two inequalities hold, then the equation (1.1) is not left disfocal in $(a, c]$ and the equation (1.1) is not right disfocal in $[c, b)$, which implies that is

not disconjugate in (a, b) . Hence there exists a solution with two zeros in (a, b) . Hence the proof of the theorem is complete. \square

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