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Homogenization of low-cost control problems on perforated domains

T. Muthukumar *, A.K. Nandakumaran

Department of Mathematics, Indian Institute of Science, Bangalore-560012, India

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ABSTRACT

The aim of this paper is to study the asymptotic behaviour of some low-cost control problems in periodically perforated domains with Neumann condition on the boundary of the holes. The optimal control problems considered here are governed by a second order elliptic boundary value problem with oscillating coefficients. It is assumed that the cost of the control is of the same order as that describing the oscillations of the coefficients. The asymptotic analysis of small cost problem is more delicate and need the H-convergence result for weak data. In this connection, an H-convergence result for weak data under some hypotheses is also proved.

1. Introduction

Let $n \geqslant 2$, Ω be a bounded open set in $\mathbb{R}^n$ and $Y = (0,1)^n$ be the reference cell. Let S, the hole, be an open subset of Y and $\bar{S}$ denote the closure of S in $\mathbb{R}^n$. Let $Y' = Y \setminus \bar{S}$ be the material part. If $S + \mathbb{Z}^n = \{x + k \mid x \in S \text{ and } k \in \mathbb{Z}^n\}$, then, for a parameter $\varepsilon > 0$ tending to zero, we set $S_\varepsilon = \varepsilon(S + \mathbb{Z}^n)$. Note that the open set $\mathbb{R}^n \setminus \bar{S_\varepsilon}$ is periodically perforated with the reference periodic cell $\varepsilon Y'$. We define a periodically perforated domain Ω_ε as,

$$\Omega_\varepsilon = \Omega \cap \left(\mathbb{R}^n \setminus \bar{S_\varepsilon}\right).$$

Let χ be the characteristic function of Y' in Y, i.e.,

$$\chi(y) = \begin{cases} 1 & \text{if } y \in Y', \\ 0 & \text{if } y \in S. \end{cases}$$

We extend χ periodically to whole of $\mathbb{R}^n$ and shall denote this extension also by χ. Observe that

$$\Omega_\varepsilon = \left\{ x \in \Omega \;\middle|\; \chi\left(\frac{x}{\varepsilon}\right) = 1 \right\}.$$

Let χ_0 denote the ratio of the measure of Y' to the measure of Y, i.e.,

$$\chi_0 = \frac{|Y'|}{|Y|} = |Y'|.$$

Assume that the constant χ_0 is strictly positive. The holes S_ε may either be isolated or connected, and they may also meet the boundary $\partial\Omega$. We also assume that the open set $\mathbb{R}^n \setminus \bar{S_\varepsilon}$ is connected and that the boundaries are all of Lipschitz type.

* Corresponding author.
E-mail addresses: tmk@math.iisc.ernet.in (T. Muthukumar), nands@math.iisc.ernet.in (A.K. Nandakumaran).

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We remark that the hypothesis on the holes S_ε and the perforated domain Ω_ε are similar to that of [2], and are stated here only for completeness sake.

We shall denote the norm in $L^2(\Omega)$ by $\|\cdot\|_2$ and the norm in $L^2(\Omega_\varepsilon)$ by $\|\cdot\|_{2,\Omega_\varepsilon}$. For a function g defined on Ω_ε, we shall denote by $\tilde{g}$ its extension by zero on the holes $\Omega \cap \bar{S_\varepsilon}$. The symbol C will always denote a generic positive constant independent of ε.

Let a, b, c and d be given constants such that $0 < a \leqslant b$ and $0 < c \leqslant d$. Let $A = A(x, y)$ be an $n \times n$ matrix, Y-periodic in y, such that,

$$a|\xi|^2 \leqslant A(x, y)\xi.\xi \leqslant b|\xi|^2 \quad \text{a.e. in } x \text{ and } y,\ \forall \xi = (\xi_i) \in \mathbb{R}^n.$$

The above assumptions on A implies that the matrix is in $[L^\infty(\Omega \times Y)]^{n\times n}$. We shall, in addition, assume that A is in $C[\Omega; L^\infty_{\mathrm{per}}(Y)]^{n\times n}$, i.e., the entries of the matrix are continuous functions with respect to $x \in \Omega$ and with their values in the Banach space of essentially bounded measurable functions, Y-periodic in y. We remark that the results of this article remain valid if $C[\Omega; L^\infty_{\mathrm{per}}(Y)]^{n\times n}$ is replaced by either $L^\infty[\Omega; C_{\mathrm{per}}(Y)]^{n\times n}$ or $L^\infty_{\mathrm{per}}[Y; C(\bar{\Omega})]^{n\times n}$. The weakest possible assumption on $A(x, y) = (a_{ij}(x, y))$ is that $a_{ij}(x, \frac{x}{\varepsilon})$ is measurable and satisfies

$$\int_\Omega \left| a_{ij}\left(x, \frac{x}{\varepsilon}\right)\right|^2 dx \to \int_\Omega \int_Y \left|a_{ij}(x, y)\right|^2 dy\, dx.$$

Let $B = B(x, y)$ be a matrix with entries from $L^\infty(\Omega \times Y)$, Y-periodic in y satisfying

$$c|\xi|^2 \leqslant B(x, y)\xi.\xi \leqslant d|\xi|^2 \quad \text{a.e. in } x \text{ and } y,\ \forall \xi = (\xi_i) \in \mathbb{R}^n.$$

Also, B satisfies the other hypotheses as that of A. In addition, we assume that B is symmetric. This assumption will not play any role in the homogenization process and is inherited from the optimal control problem.

Let U_ε be a closed convex subset of $L^2(\Omega_\varepsilon)$, called the admissible control set and $f \in L^2(\Omega)$. Given $\theta_\varepsilon \in U_\varepsilon$, let the cost functional J_ε be defined as,

$$J_\varepsilon(\theta_\varepsilon) = F_\varepsilon(u_\varepsilon, \theta_\varepsilon) + \frac{\varepsilon}{2}\|\theta_\varepsilon\|^2_{2,\Omega_\varepsilon} \tag{1.1}$$

where $u_\varepsilon = u_\varepsilon(\theta_\varepsilon) \in H^1(\Omega_\varepsilon)$ is the unique solution of

$$\begin{cases} -\mathrm{div}\left(A\left(x, \dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f + \theta_\varepsilon & \text{in } \Omega_\varepsilon, \\ A\left(x, \dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon.\nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \cap \partial\Omega_\varepsilon, \end{cases} \tag{1.2}$$

and ν is the unit outward normal on $\partial\Omega_\varepsilon \setminus \partial\Omega$.

The aim of this article is to study the limiting behaviour of the system (1.1)–(1.2) in the following two situations:

(a) When $F_\varepsilon(u_\varepsilon, \theta_\varepsilon)$ is a Dirichlet-type integral, i.e.,

$$F_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon(\theta_\varepsilon).\nabla u_\varepsilon(\theta_\varepsilon)\, dx. \tag{1.3}$$

(b) When $F_\varepsilon(u_\varepsilon, \theta_\varepsilon)$ is the L^2-norm of the state, i.e.,

$$F_\varepsilon(u_\varepsilon) = \frac{1}{2}\left\|u_\varepsilon(\theta_\varepsilon)\right\|^2_{2,\Omega_\varepsilon}. \tag{1.4}$$

It is a classical result from the calculus of variations that there exists a unique $\theta^*_\varepsilon \in U_\varepsilon$ such that

$$J_\varepsilon\left(\theta^*_\varepsilon\right) = \min_{\theta_\varepsilon \in U_\varepsilon} J_\varepsilon(\theta_\varepsilon). \tag{1.5}$$

The main results of this paper are: Theorem 2.1, Theorem 3.3, Theorem 3.7 and Theorem 4.4. In this article, we study the situation where the Neumann condition is imposed on the boundary of the holes and the Dirichlet condition on the exterior boundary (cf. (1.2)). In fact, the results will remain valid (locally inside Ω) when the Neumann condition is imposed both on the boundary of holes and on the exterior boundary (cf. [2]). However, the problem with Dirichlet condition on the holes is more delicate and is the topic of a future article.

The ε factor appearing in the second term of the right-hand side of the cost functional (1.1) is, usually, in literature called the *cost of the control*. The fact that this is of ε order motivates the name *low-cost* control. The low-cost control problems were first introduced by J. L. Lions (who called it 'cheap' control) in [14,15]. Several variants of low-cost control problems have been studied by Kesavan and Saint Jean Paulin in [12], for non-perforated domains. The study is taken up further in [8,19] for both perforated and non-perforated domains. In the above articles, the problems are considered in the general (non-periodic) case using the method of oscillating test functions, introduced by L. Tartar (cf. [17,18,23]). In spite of these studies, the problem has not been completely settled in its full generality. One major drawback is the lack of information on the convergence of the optimal controls itself, except in some trivial cases (cf. [19]). The problem has only been settled for particular cases of admissible control set.

In [8,19], the state-adjoint system governing the optimal control problem was homogenized for the non-perforated domain with the assumption that the control set was the set of all positive functions (positive cone) of the control space. The perforated case was unsettled, even for the positive cone of the control space. In this paper, using the notion of two-scale convergence (cf. [1,16,20,21]), we show that the state-adjoint system governing the optimal control problem can be homogenized *without any assumptions on the control set* for both perforated and non-perforated domains.

However, we would like to note that the positive cone hypothesis on the admissible control set is not relaxed for the homogenization process of the optimal control problem. This is due to the lack of characterisation of the optimal controls, in general. The case of the positive cone gives a nice characterisation of the optimal controls in terms of the adjoint state which cancels out the product of two weak limits. Such a characterisation is not available, in general.

This article settles the problem, in non-perforated case, left open in [12] and the perforated case left open in [8]. Also, the positive cone hypothesis has been relaxed wherever possible. Moreover, it contains different approach to the results obtained for non-perforated case which easily carry forward to the perforated case. Though this article announces only the results concerning perforated domains, the results remain valid even for non-perforated domains with necessary modifications.

We end this section by recalling some preliminary notions used in the sequel. We first define the sequential notion of K-lower limit in its full generality. A detailed study of this and related notions can be found in [5]. Let X be a topological space and E_n be a sequence of subsets of X.

Definition 1.1. A point $x \in X$ is said to be in the *sequential K-lower limit* of E_n w.r.t. the topology in X if and only if there exists $k \in \mathbb{N}$ and a sequence $\{x_n\}$ converging to x in X such that $x_n \in E_n$, for all $n \geqslant k$.

We now recall the notion of two-scale convergence. We refer to [1,16,20,21] for a detailed study of the same and certain applications.

Definition 1.2. A sequence of functions $\{v_\varepsilon\}$ in $L^2(\Omega)$ is said to *two-scale converge* to a limit $v \in L^2(\Omega \times Y)$ (denoted as $v_\varepsilon \overset{2s}{\rightharpoonup} v$) if

$$\int_\Omega v_\varepsilon \phi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y v(x,y)\phi(x,y)\, dy\, dx, \quad \forall \phi \in L^2\big[\Omega; C_{\mathrm{per}}(Y)\big].$$

The most interesting property of two-scale convergence is the following compactness result.

Theorem 1.3. *For any bounded sequence v_ε in $L^2(\Omega)$, there exist a subsequence and $v \in L^2(\Omega \times Y)$ such that, v_ε two-scale converges to v along the subsequence. Also, if v_ε is bounded in $H^1(\Omega)$, then v is independent of y and is in $H^1(\Omega)$, and there exists $v_1 \in L^2[\Omega; H^1_{\mathrm{per}}(Y)]$ such that, up to a subsequence, ∇v_ε two-scale converges to $\nabla v + \nabla_y v_1$.*

The article is organized as follows: In Section 2, we prove H-convergence for weak data under some assumptions, one of the main results of this article. This is significant by itself and we use this result for our analysis. In this section, we work in the non-perforated set-up to fix ideas. However the results remain valid in the perforated case and are used with necessary modification in the subsequent sections. In Section 3, we study (1.1) where F_ε is the Dirichlet-type integral as given in (1.3). In Section 4, we homogenize the system (1.1)–(1.2) under the situation (1.4).

2. *H*-convergence for weak data

We begin by recalling the notion of H-convergence. Let g_ε be a sequence in $H^{-1}(\Omega)$ which converges to g *strongly* in $H^{-1}(\Omega)$. If v_ε is the solution of

$$\begin{cases} -\operatorname{div}\left(A\left(x, \frac{x}{\varepsilon}\right)\nabla v_\varepsilon\right) = g_\varepsilon & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

then, there exist v_0 and a matrix A_0 such that

$$\left.\begin{aligned} & v_\varepsilon \rightharpoonup v_0 && \text{weakly in } H_0^1(\Omega), \\ & A\left(x, \frac{x}{\varepsilon}\right)\nabla v_\varepsilon \rightharpoonup A_0 \nabla v_0 && \text{weakly in } \left(L^2(\Omega)\right)^n. \end{aligned}\right\} \tag{2.2}$$

Here $v_0 \in H_0^1(\Omega)$ is the unique solution of the homogenized problem

$$\begin{cases} -\operatorname{div}(A_0 \nabla v_0) = g & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Further, the ijth entry of the matrix A_0 is given by

$$(A_0)_{ij} = \int_Y A(x, y)\left[\nabla_y \mu_i(x, y) + e_i\right].\left[\nabla_y \mu_j(x, y) + e_j\right] dy. \tag{2.4}$$

The function μ_i, for $1 \leqslant i \leqslant n$, is the solution of the cell problem

$$\begin{cases} -\operatorname{div}_y\left(A(x, y)\left[\nabla_y \mu_i(x, y) + e_i\right]\right) = 0 & \text{in } Y, \\ y \mapsto \mu_i(x, y) & \text{is } Y\text{-periodic,} \end{cases} \tag{2.5}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

The above result is not true, in general, when g_ε converge *weakly* in $H^{-1}(\Omega)$. However, it was shown in [6] that if, in addition, g_ε are positive distributions of $H^{-1}(\Omega)$, the H-convergence result remains valid. A very general study of the weak converging data was done by L. Tartar and his result can be found in [3]. It involves a homogenized problem defined in a complicated way (involving corrector functions) and which reduces to (2.3) under the strong convergence hypothesis of data in $H^{-1}(\Omega)$.

In this section, we give another hypothesis on the data under which the H-convergence remains valid.

Theorem 2.1. *Let $\alpha < 1$ be a fixed real number. Let $v_\varepsilon \in H_0^1(\Omega)$ be the weak solution of* (2.1)*, where $g_\varepsilon \in L^2(\Omega)$ is such that $g_\varepsilon \rightharpoonup g$ weakly in $H^{-1}(\Omega)$ and $\varepsilon^\alpha g_\varepsilon$ is bounded in $L^2(\Omega)$. Then, there exist v_0 and a matrix A_0 such that* (2.2) *is satisfied, where $v_0 \in H_0^1(\Omega)$ is the unique solution of* (2.3) *and A_0 is given by* (2.4) *and* (2.5)*.*

Proof. Let $w_\varepsilon \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta w_\varepsilon = g_\varepsilon & \text{in } \Omega, \\ w_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Then it is easy to see that w_ε is bounded in $H_0^1(\Omega)$. In fact,

$$\|w_\varepsilon\|_{H_0^1(\Omega)} \leqslant \|g_\varepsilon\|_{H^{-1}(\Omega)}.$$

Thus, it follows from the two-scale convergence theory, that there exist $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2[\Omega; H^1_{\mathrm{per}}(Y)/\mathbb{R}]$ such that, up to a subsequence,

$$\left.\begin{aligned} & w_\varepsilon \overset{2s}{\rightharpoonup} w_0, \\ & \nabla w_\varepsilon \overset{2s}{\rightharpoonup} \nabla w_0(x) + \nabla_y w_1(x, y). \end{aligned}\right\} \tag{2.7}$$

Further, we see that the homogenized equation of (2.6) is

$$\begin{cases} -\Delta w_0 = g & \text{in } \Omega, \\ w_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

By choosing $\phi_1 \in \mathcal{D}(\Omega)$, $\phi_2 \in C^\infty_{\mathrm{per}}(Y)$ and setting

$$\phi\left(x, \frac{x}{\varepsilon}\right) = \varepsilon \phi_1(x)\phi_2\left(\frac{x}{\varepsilon}\right) \in \mathcal{D}\left[\Omega; C^\infty_{\mathrm{per}}(Y)\right]$$

as a two-scale test function for (2.6), we have

$$\varepsilon \int_\Omega \nabla w_\varepsilon.\nabla \phi_1 \phi_2\left(\frac{x}{\varepsilon}\right) dx + \int_\Omega \nabla w_\varepsilon.\nabla_y \phi_2\left(\frac{x}{\varepsilon}\right)\phi_1\, dx = \varepsilon^{1-\alpha} \int_\Omega \varepsilon^\alpha g_\varepsilon \phi_1(x)\phi_2\left(\frac{x}{\varepsilon}\right) dx.$$

By passing to the two-scale limit, we get

$$\int_\Omega \int_Y \big[\nabla w_0 + \nabla_y w_1(x,y)\big].\nabla_y \phi_2(y)\phi_1(x)\,dx\,dy = 0.$$

Since $\int_Y \nabla_y \phi_2(y)\,dy = 0$, we have for every $\phi_1 \in \mathcal{D}(\Omega)$,

$$\int_\Omega \phi_1(x)\bigg[\int_Y \nabla_y w_1(x,y).\nabla_y \phi_2(y)\,dy\bigg]dx = 0.$$

Thus, for almost every $x \in \Omega$,

$$\int_Y \nabla_y w_1(x,y).\nabla_y \phi_2(y)\,dy = 0,$$

which implies $\Delta_y w_1(x,y) = 0$, where w_1 is Y-periodic. Thus, w_1 is a function independent of y and $\nabla_y w_1 = 0$. Therefore in (2.7) we, in fact, have

$$\nabla w_\varepsilon \overset{2s}{\rightharpoonup} \nabla w_0(x).$$

Observe that the solution v_ε of (2.1) is bounded in $H_0^1(\Omega)$. Thus, there exist $v_0 \in H_0^1(\Omega)$ and $v_1 \in L^2[\Omega; H^1_{\mathrm{per}}(Y)/\mathbb{R}]$ such that, up to a subsequence,

$$\left.\begin{aligned} &v_\varepsilon \overset{2s}{\rightharpoonup} v_0, \\ &\nabla v_\varepsilon \overset{2s}{\rightharpoonup} \nabla v_0(x) + \nabla_y v_1(x,y). \end{aligned}\right\} \tag{2.9}$$

Now an usual two-scale analysis of (2.1) by replacing g_ε with Eq. (2.6) and by choosing $\phi(x, \frac{x}{\varepsilon}) = \phi_1(x) + \varepsilon\phi_2(x, \frac{x}{\varepsilon})$ as the two-scale test function will yield,

$$\int_\Omega \int_Y \big\{A(x,y)\big[\nabla v_0 + \nabla_y v_1(x,y)\big] - \nabla w_0\big\}.\big\{\nabla\phi_1(x) + \nabla_y \phi_2(x,y)\big\}\,dx\,dy = 0.$$

Then, the usual arguments of decoupling and Eq. (2.8) together will yield (2.3) and the convergences (2.2). □

3. Dirichlet-type integral of state in cost functional

Let U_ε be a closed convex subset of $L^2(\Omega_\varepsilon)$. The optimal control problem is to minimise in U_ε, the cost functional J_ε given as,

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon.\nabla u_\varepsilon\,dx + \frac{\varepsilon}{2}\|\theta_\varepsilon\|^2_{2,\Omega_\varepsilon}, \quad \forall \theta_\varepsilon \in U_\varepsilon, \tag{3.1}$$

where $u_\varepsilon \in H^1(\Omega_\varepsilon)$ is the unique solution of

$$\begin{cases} -\mathrm{div}\left(A\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f + \theta_\varepsilon & \text{in } \Omega_\varepsilon, \\ A\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon.\nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \cap \partial\Omega_\varepsilon, \end{cases} \tag{3.2}$$

and ν is the unit outward normal on $\partial\Omega_\varepsilon \setminus \partial\Omega$.

Let $\theta^*_\varepsilon \in U_\varepsilon$ be the unique optimal control and u^*_ε be the state corresponding to θ^*_ε (cf. (3.12)). The solution space of (3.2) is given as,

$$V_\varepsilon = \big\{u \in H^1(\Omega_\varepsilon) \;\big|\; u = 0 \text{ on } \partial\Omega \cap \partial\Omega_\varepsilon\big\}$$

and $u^*_\varepsilon \in V_\varepsilon$. We now state a Poincaré inequality result of Ω_ε, proved in [2].

Lemma 3.1. *(See [2, Lemma A.4].) There exists a positive constant C, independent of ε, such that*

$$\|v\|_{2,\Omega_\varepsilon} \leqslant C\|\nabla v\|_{2,\Omega_\varepsilon}, \quad \forall v \in V_\varepsilon.$$

The above lemma gives an equivalent norm on V_ε as, $\|v\|_{V_\varepsilon} = \|\nabla v\|_{2,\Omega_\varepsilon}$.

We introduce the adjoint optimal state associated to the optimal control problem. Let $p_\varepsilon^* \in V_\varepsilon$ be the solution of,

$$\begin{cases} -\operatorname{div}\left({}^t A\left(x, \frac{x}{\varepsilon}\right)\nabla p_\varepsilon^*\right) = -\operatorname{div}\left(B\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\right) & \text{in } \Omega_\varepsilon, \\ \left({}^t A\left(x, \frac{x}{\varepsilon}\right)\nabla p_\varepsilon^* - B\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\right).\nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \\ p_\varepsilon^* = 0 & \text{on } \partial\Omega \cap \partial\Omega_\varepsilon. \end{cases} \tag{3.3}$$

Then the optimality condition, in terms of the adjoint optimal state, is

$$\int_{\Omega_\varepsilon} \left(p_\varepsilon^* + \varepsilon\theta_\varepsilon^*\right)\left(\theta_\varepsilon - \theta_\varepsilon^*\right) dx \geqslant 0, \quad \forall \theta_\varepsilon \in U_\varepsilon. \tag{3.4}$$

Note that the symmetry hypothesis on B comes in hand only to derive the optimality condition (3.4).

In the rest of the section we will consider control sets, U_ε, satisfying the following hypothesis:

(H1) There exists $\eta_\varepsilon \in U_\varepsilon$ such that $\{\widetilde{\eta_\varepsilon}\}$ is bounded in $L^2(\Omega)$.

The motivation for this hypothesis will be clear in next lemma. We shall now show that the admissible sets satisfying (H1) are in abundance. Let U be a closed convex subset of $L^2(\Omega)$ and U_ε be the set of all elements of U restricted to Ω_ε. Observe that U_ε is a closed convex subset of $L^2(\Omega_\varepsilon)$. Let $\theta \in U$, then by choosing $\eta_\varepsilon = \theta|_{\Omega_\varepsilon}$ we note that U_ε satisfies (H1), since $\|\eta_\varepsilon\|_{2,\Omega_\varepsilon}^2 \leqslant \|\theta\|_2^2$.

Lemma 3.2. *Let the admissible set U_ε satisfy the hypothesis* (H1)*, then $\{u_\varepsilon^*\}$ and $\{p_\varepsilon^*\}$ are bounded in V_ε. Also, there exists $\theta^* \in H^{-1}(\Omega)$ such that, for a subsequence,*

$$\widetilde{\theta_\varepsilon^*} \rightharpoonup \theta^* \quad \text{weakly in } H^{-1}(\Omega) \tag{3.5}$$

and

$$\varepsilon^{1/2}\widetilde{\theta_\varepsilon^*} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega). \tag{3.6}$$

Proof. Let v_ε be the state corresponding to the control $\theta_\varepsilon = \eta_\varepsilon$ in (3.2). Then, it is clear, from the hypotheses on $A(x, \frac{x}{\varepsilon})$ that v_ε is bounded uniformly with respect to ε in V_ε, since,

$$\|v_\varepsilon\|_{V_\varepsilon} \leqslant \frac{C}{a}\left\|\chi\left(\frac{x}{\varepsilon}\right)f + \widetilde{\eta_\varepsilon}\right\|_2.$$

Hence, it follows from (1.5) that

$$J_\varepsilon\left(\theta_\varepsilon^*\right) \leqslant J_\varepsilon(\eta_\varepsilon).$$

Therefore, from (3.1) we obtain that $\{u_\varepsilon^*\}$ and $\{\varepsilon^{\frac{1}{2}}\theta_\varepsilon^*\}$ are bounded in V_ε and $L^2(\Omega_\varepsilon)$, respectively. Moreover, from (3.3), $\{p_\varepsilon^*\}$ is bounded in V_ε. For $v \in H_0^1(\Omega)$, consider

$$\left|\int_\Omega \widetilde{\theta_\varepsilon^*} v\, dx\right| = \left|\int_{\Omega_\varepsilon} \theta_\varepsilon^* v\, dx\right| = \left|\int_\Omega A_\varepsilon \widetilde{\nabla u_\varepsilon^*}.\nabla v\, dx - \int_\Omega \chi\left(\frac{x}{\varepsilon}\right) f v\, dx\right| \leqslant \left(b\left\|u_\varepsilon^*\right\|_{V_\varepsilon} + C\left\|\chi\left(\frac{x}{\varepsilon}\right)f\right\|_2\right)\|v\|_{H_0^1(\Omega)}.$$

Therefore $\{\widetilde{\theta_\varepsilon^*}\}$ is bounded in $H^{-1}(\Omega)$ and thus there exists $\theta^* \in H^{-1}(\Omega)$ such that, up to a subsequence, (3.5) holds and hence (3.6) holds. $\square$

Our objective is to know whether θ^* is an optimal control of a problem similar to (3.1)–(3.2). We deduce from the *a priori* estimates obtained in Lemma 3.2 that there exist $u^*, p^* \in H_0^1(\Omega)$ and $u_1, p_1 \in L^2[\Omega; H_{\text{per}}^1(Y')/\mathbb{R}]$ such that, up to a subsequence,

$$\begin{cases} \widetilde{u_\varepsilon^*} \overset{2s}{\rightharpoonup} u^*(x)\chi(y), \\ \widetilde{\nabla u_\varepsilon^*} \overset{2s}{\rightharpoonup} \chi(y)\left[\nabla u^*(x) + \nabla_y u_1(x, y)\right], \\ \widetilde{p_\varepsilon^*} \overset{2s}{\rightharpoonup} p^*(x)\chi(y), \\ \widetilde{\nabla p_\varepsilon^*} \overset{2s}{\rightharpoonup} \chi(y)\left[\nabla p^*(x) + \nabla_y p_1(x, y)\right]. \end{cases}$$

The above results can be derived from the two-scale convergence analysis which we skip (cf. [1]).

Observe that the right-hand side of both (3.2) (written for $\theta_\varepsilon = \theta_\varepsilon^*$) and (3.3) involve functions whose extensions to Ω are only weakly compact in $H^{-1}(\Omega)$. A homogenization result is not available, in general, for such equations.

We shall now define some cell problems which will be used in the sequel to identify the limit problem. For $1 \leqslant i \leqslant n$, let the function $\mu_i \in H^1_{\mathrm{per}}(Y')/\mathbb{R}$ be the solution of the cell problem

$$\begin{cases} -\mathrm{div}_y\big(A(x,y)\big[\nabla_y \mu_i(x,y) + e_i\big]\big) = 0 & \text{in } Y', \\ A(x,y)\big[\nabla_y \mu_i(x,y) + e_i\big].\nu = 0 & \text{on } \partial Y' \setminus \partial Y, \\ y \mapsto \mu_i(x,y) & \text{is } Y\text{-periodic}. \end{cases} \tag{3.7}$$

A note of caution that we are using the same notation μ_i as in (2.5). But in the rest of the article μ_i will always denote the solution of (3.7). The distinction is clear if one keeps in mind that we are dealing with the perforated situation in this article.

Let $\omega_i \in H^1_{\mathrm{per}}(Y')/\mathbb{R}$ be the solution of the adjoint cell problem

$$\begin{cases} -\mathrm{div}_y\big({}^tA(x,y)\big[\nabla_y \omega_i(x,y) + e_i\big]\big) = 0 & \text{in } Y', \\ {}^tA(x,y)\big[\nabla_y \omega_i(x,y) + e_i\big].\nu = 0 & \text{on } \partial Y' \setminus \partial Y, \\ y \mapsto \omega_i(x,y) & \text{is } Y\text{-periodic}, \end{cases} \tag{3.8}$$

and let $\psi_i \in H^1_{\mathrm{per}}(Y')/\mathbb{R}$ be the solution of the cell problem

$$\begin{cases} -\mathrm{div}_y\big({}^tA(x,y)\nabla_y \psi_i(x,y) + B(x,y)\{\nabla_y \mu_i + e_i\}\big) = 0 & \text{in } Y', \\ \big[{}^tA(x,y)\nabla_y \psi_i(x,y) + B(x,y)\{e_i + \nabla_y \mu_i\}\big].\nu = 0 & \text{on } \partial Y' \setminus \partial Y, \\ y \mapsto \psi_i(x,y) & \text{is } Y\text{-periodic}. \end{cases} \tag{3.9}$$

We now provide the homogenization theorem for the state and adjoint-state equations. The non-perforated analogue of the following theorem was proved in [8,19] under the hypothesis that the admissible control set is the positive cone of $L^2(\Omega)$. The theorem given below relaxes this hypothesis on the admissible control set, thus proving the result in its full generality both for fixed domains and varying domains with Neumann condition on the holes.

Let the homogenized matrix A_0 be defined as,

$$(A_0)_{ij} = \int_{Y'} A(x,y)\big[\nabla_y \mu_i(x,y) + e_i\big].\big[\nabla_y \mu_j(x,y) + e_j\big]\,dy \tag{3.10}$$

and the transpose of A_0, ${}^tA_0$, is given by a similar equation written for the function ω_i (cf. (4.9)). Let $B^\sharp$ be given as,

$$\big(B^\sharp\big)e_i = \int_{Y'} \big\{B(x,y)\big[\nabla_y \mu_i(x,y) + e_i\big] + {}^tA(x,y)\nabla_y \psi_i(x,y)\big\}\,dy. \tag{3.11}$$

Theorem 3.3. *Let U_ε satisfy the hypothesis* (H1) *and let u_ε^* be the solution of*

$$\begin{cases} -\mathrm{div}\left(A\left(x, \dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\right) = f + \theta_\varepsilon^* & \text{in } \Omega_\varepsilon, \\ A\left(x, \dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon^*.\nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \\ u_\varepsilon^* = 0 & \text{on } \partial\Omega \cap \partial\Omega_\varepsilon, \end{cases} \tag{3.12}$$

and p_ε^* be the solution of (3.3). *Then, $\chi_0 u^*$ and $\chi_0 p^*$ are the weak limits of $\widetilde{u_\varepsilon^*}$ and $\widetilde{p_\varepsilon^*}$, respectively, in $H_0^1(\Omega)$ where $(u^*, p^*) \in H_0^1(\Omega) \times H_0^1(\Omega)$ solves*

$$\begin{cases} -\mathrm{div}\big(A_0 \nabla u^*\big) = \chi_0 f + \theta^* & \text{in } \Omega, \\ -\mathrm{div}\big({}^tA_0 \nabla p^* - B^\sharp \nabla u^*\big) = 0 & \text{in } \Omega, \\ u^* = p^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

The control θ^* is as obtained in (3.5).

Proof. Since U_ε satisfies (H1), the results of Lemma 3.2 remain valid. Let $\rho_\varepsilon^* \in V_\varepsilon$ be the solution of

$$\begin{cases} -\Delta \rho_\varepsilon^* = \theta_\varepsilon^* & \text{in } \Omega_\varepsilon, \\ \nabla \rho_\varepsilon^*.\nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \\ \rho_\varepsilon^* = 0 & \text{on } \partial\Omega \cap \partial\Omega_\varepsilon. \end{cases} \tag{3.14}$$

Then,

$$\|\rho_\varepsilon^*\|_{V_\varepsilon}^2 \leqslant \|\theta_\varepsilon^*\|_{H^{-1}(\Omega)} \|P_\varepsilon \rho_\varepsilon^*\|_{H_0^1(\Omega)},$$

where P_ε is an extension operator (cf. [4, Lemma 1]) from V_ε to $H_0^1(\Omega)$ such that $\|\nabla(P_\varepsilon v)\|_2 \leqslant C\|\nabla v\|_{2,\Omega_\varepsilon}$, for all $v \in V_\varepsilon$, and the constant C is independent of ε. Thus, $\widetilde{\nabla\rho_\varepsilon^*}$ is bounded in $(L^2(\Omega))^n$. Also, by Lemma 3.1, $\widetilde{\rho_\varepsilon^*}$ is bounded in $L^2(\Omega)$. Thus, by the two-scale convergence theory, there exist $\rho^* \in H_0^1(\Omega)$ and $\rho_1 \in L^2[\Omega; H^1_{\mathrm{per}}(Y')/\mathbb{R}]$ such that, up to a subsequence,

$$\begin{cases} \widetilde{\rho_\varepsilon^*} \overset{2S}{\rightharpoonup} \rho^*(x)\chi(y), \\ \widetilde{\nabla\rho_\varepsilon^*} \overset{2s}{\rightharpoonup} \chi(y)\big[\nabla\rho^*(x) + \nabla_y\rho_1(x,y)\big]. \end{cases} \tag{3.15}$$

It is easy to see that the homogenized equation of (3.14) is

$$\begin{cases} -\Delta\rho^* = \dfrac{1}{\chi_0}\theta^* & \text{in } \Omega, \\ \rho^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Arguing as in the proof of Theorem 2.1 with the obvious modification for perforated domain, we have the first equation of (3.13) as the homogenized state equation. We also have that,

$$u_1(x,y) = \sum_{i=1}^n \mu_i(x,y)\frac{\partial u^*}{\partial x_i}(x).$$

We shall now show that the second equation of (3.13) is the homogenized equation corresponding to adjoint equation (3.3). In the case of (3.3), $g_\varepsilon = -\mathrm{div}(B(x,\frac{x}{\varepsilon})\nabla u_\varepsilon^*)$ may not satisfy the hypotheses of Theorem 2.1, but here we obtain the limit equation using the divergence form of g_ε.

We choose $\phi(x,\frac{x}{\varepsilon}) = \phi_1(x) + \varepsilon\phi_2(x,\frac{x}{\varepsilon})$ as a two-scale test function in (3.3), and passing to the limit, we have

$$\begin{aligned} &\int_\Omega\int_Y {}^tA(x,y)\chi(y)\big\{\nabla p^* + \nabla_y p_1(x,y)\big\}.\big[\nabla\phi_1(x) + \nabla_y\phi_2(x,y)\big]\,dy\,dx \\ &\quad = \int_\Omega\int_Y B(x,y)\chi(y)\big\{\nabla u^* + \nabla_y u_1(x,y)\big\}.\big[\nabla\phi_1(x) + \nabla_y\phi_2(x,y)\big]\,dy\,dx. \end{aligned}$$

By choosing $\phi_1 = 0$ and the cell problems (3.8) and (3.9), we deduce that

$$p_1 = \sum_{i=1}^n\left[\psi_i(x,y)\frac{\partial u^*}{\partial x_i}(x) + \omega_i(x,y)\frac{\partial p^*}{\partial x_i}(x)\right]$$

and by choosing $\phi_2 = 0$, we get the second equation of (3.13). □

Remark 3.4. Let u_ε be the solution of (3.2) and if $\widetilde{\theta_\varepsilon} \rightharpoonup \theta$ *strongly* in $H^{-1}(\Omega)$, then

$$\int_{\Omega_\varepsilon} B_\varepsilon\nabla u_\varepsilon.\nabla u_\varepsilon\,dx \to \int_\Omega B^\sharp\nabla u_0.\nabla u_0\,dx \tag{3.17}$$

where $B^\sharp$ is as given in (3.11) and $u_0 = u_0(\theta) \in H_0^1(\Omega)$ is the unique solution of

$$\begin{cases} -\mathrm{div}(A_0\nabla u_0) = \chi_0 f + \theta & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

A_0 is given as in (3.10). This result is, in a sense, the 'generalised' energy convergence. This fact has been observed in [9,11]. We, however, remark that (3.17) is not true, in general, when $\widetilde{\theta_\varepsilon} \rightharpoonup \theta$ *weakly* in $H^{-1}(\Omega)$. A one dimensional example establishing this fact for the non-perforated domain can be found in [19].

Remark 3.5. If B_0 is the homogenized limit of the sequence $B(x,\frac{x}{\varepsilon})$, then, in general, $B^\sharp \neq B_0$. For more such interesting properties of $B^\sharp$ and its applications we refer to [7,9–11,13,22].

Now that we have identified the limit state equation, it now remains for us to find the limit cost functional and to prove that θ^* is its optimal control.

Using p_ε^* as a test function in (3.12) and u_ε^* as a test function in (3.3), one can rewrite the cost functional $J_\varepsilon(\theta_\varepsilon^*)$ as,

$$J_\varepsilon(\theta_\varepsilon^*) = \frac{1}{2}\int_{\Omega_\varepsilon} (f + \theta_\varepsilon^*) p_\varepsilon^* \, dx + \frac{\varepsilon}{2} \|\theta_\varepsilon^*\|_{2,\Omega_\varepsilon}^2. \tag{3.18}$$

The above functional (3.18) involves product of two weakly converging sequence, namely, $\int_{\Omega_\varepsilon} \theta_\varepsilon^* p_\varepsilon^* \, dx$, whose limit is unknown in general. This makes the problem difficult to tackle in a general admissible control set U_ε.

We could identify the limit cost functional only in the situation when the control set is the set of all positive elements of $L^2(\Omega_\varepsilon)$, i.e., the positive cone of $L^2(\Omega_\varepsilon)$. More precisely, let $U_\varepsilon = \{\theta \in L^2(\Omega) \mid \theta \geqslant 0 \text{ a.e. in } \Omega_\varepsilon\}$ be the closed convex subset of $L^2(\Omega)$. Observe that the elements of U_ε restricted to Ω_ε is the set of all positive elements of $L^2(\Omega_\varepsilon)$. Let U' be the sequential K-lower limit of U_ε with respect to the strong topology in $L^2(\Omega)$. We shall now establish that U' is non-empty. If U denotes the positive cone of $L^2(\Omega)$, i.e., $U = \{\theta \in L^2(\Omega) \mid \theta \geqslant 0 \text{ a.e. in } \Omega\}$, then U is a subset of U_ε for all ε. Thus, U is a subset of U' and hence U' is non-empty.

Let V' be the strong closure of U' in $H^{-1}(\Omega)$. Since U' is convex, V' is same as the weak closure of U' in $H^{-1}(\Omega)$.

Remark 3.6. If V is the strong closure of U in $H^{-1}(\Omega)$ then V is the set of all positive distributions of $H^{-1}(\Omega)$. In other words, V is the positive cone of $H^{-1}(\Omega)$. We refer to [19] for a proof of this fact. Also, observe that V is a subset of V'. Since U is convex, V is same as the weak closure of U in $H^{-1}(\Omega)$.

Theorem 3.7. *Let $U_\varepsilon = \{\theta \in L^2(\Omega) \mid \theta \geqslant 0 \text{ a.e. in } \Omega_\varepsilon\}$. Then*

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\int_\Omega B^\sharp \nabla u^* . \nabla u^* \, dx - \frac{1}{2}\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \tag{3.19}$$

where θ^* is as obtained in Lemma 3.2, u^*, p^* are as obtained in (3.13) and $B^\sharp$ is given by (3.11). Also, $p^* \in U$ and hence in V'.

Further, if $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$, then θ^* minimizes

$$J(\theta) = \frac{1}{2}\int_\Omega B^\sharp \nabla u . \nabla u \, dx \tag{3.20}$$

over V', where $u = u(\theta)$ solves

$$\begin{cases} -\operatorname{div}(A_0 \nabla u) = \chi_0 f + \theta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.21}$$

and the convergences (3.5) and (3.6) hold for the entire sequence.

Proof. Observe that $0 \in U_\varepsilon$ for all ε. Thus, the set of all elements of U_ε restricted to Ω_ε (the positive cone of $L^2(\Omega_\varepsilon)$) satisfies the hypothesis (H1). Hence the results of Lemma 3.2 and Theorem 3.3 remain valid.

Since the admissible set is now the positive cone of $L^2(\Omega_\varepsilon)$, the optimality condition (3.4) implies that $\varepsilon\theta_\varepsilon^* = (p_\varepsilon^*)^-$. This equality will in turn imply that

$$\int_{\Omega_\varepsilon} \theta_\varepsilon^* p_\varepsilon^* \, dx = -\varepsilon \|\theta_\varepsilon^*\|_{2,\Omega_\varepsilon}^2. \tag{3.22}$$

Thus, $J_\varepsilon(\theta_\varepsilon^*)$ (cf. (3.18)) reduces to

$$J_\varepsilon(\theta_\varepsilon^*) = \frac{1}{2}\int_{\Omega_\varepsilon} f p_\varepsilon^* \, dx$$

and passing to the limit will yield

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\int_\Omega \chi_0 f p^* \, dx.$$

Using p^* and u^* as test functions in the state and adjoint-state equation of (3.13), respectively, we have

$$\int_\Omega \chi_0 f p^* \, dx = \int_\Omega B^\sharp \nabla u^* . \nabla u^* \, dx - \langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Thus, (3.19) holds true.

Using (3.22), we rewrite the optimality condition (3.4) as,

$$\int_{\Omega_\varepsilon} p_\varepsilon^* \theta \, dx + \int_{\Omega_\varepsilon} \big(\varepsilon^{\frac{1}{2}} \theta_\varepsilon^*\big)\big(\varepsilon^{\frac{1}{2}} \theta\big) \, dx \geqslant 0, \quad \forall \theta \in U_\varepsilon.$$

In particular, the above inequality holds for all $\theta \in U$. Thus, passing to the limit, we deduce that

$$\int_{\Omega} \chi_0 p^* \theta \, dx \geqslant 0, \quad \forall \theta \in U.$$

This implies $p^* \geqslant 0$, i.e., $p^* \in U$ and hence $p^* \in V'$.

For $\theta \in V'$, let J be defined as in (3.20) where u solves (3.21). Recall that θ_ε^* is the minimiser of J_ε over U_ε, i.e.,

$$J_\varepsilon\big(\theta_\varepsilon^*\big) \leqslant J_\varepsilon(\theta), \quad \forall \theta \in U_\varepsilon. \tag{3.23}$$

Given $\theta \in U'$, there exists $\delta > 0$ and a sequence $\theta_\varepsilon \to \theta$ strongly in $L^2(\Omega)$ such that $\theta_\varepsilon \in U_\varepsilon$ for all $\varepsilon < \delta$. Observe that

$$\|\theta_\varepsilon\|_{2,\Omega_\varepsilon}^2 \leqslant \|\theta_\varepsilon\|_2^2, \quad \forall \varepsilon < \delta,$$

and hence

$$\|\widetilde{\theta_\varepsilon}\|_2^2 \leqslant \|\theta_\varepsilon\|_2^2, \quad \forall \varepsilon < \delta.$$

Thus $\varepsilon \|\theta_\varepsilon\|_{2,\Omega_\varepsilon}^2 \to 0$. Thus, passing to the limit in (3.23) (also using Remark 3.4), we deduce

$$\frac{1}{2} \int_{\Omega} B^\sharp \nabla u^* . \nabla u^* \, dx - \frac{1}{2} \langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \leqslant J(\theta), \quad \forall \theta \in U'.$$

Since V' is the strong closure of U' in $H^{-1}(\Omega)$, we actually have (using Remark 3.4)

$$J\big(\theta^*\big) - \frac{1}{2} \langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \leqslant J(\theta), \quad \forall \theta \in V'.$$

Observe that $\widetilde{\theta_\varepsilon^*} \in U \subset U_\varepsilon$. Thus, the weak limit of $\widetilde{\theta_\varepsilon^*}$ in $H^{-1}(\Omega)$ is in $V \subset V'$ by Remark 3.6. Therefore, if $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$, then θ^* minimises J over V'. The strict convexity of J would then imply the uniqueness of θ^* and thus the convergences (3.5) and (3.6) hold for the entire sequence. □

Remark 3.8. We note that though we have identified the possible limit cost functional as (3.20), the possible limit admissible set as V' and the possible homogenized state equation as (3.21), we are unable to prove that θ^* minimises J in V' in general. However, we prove θ^* is a minimiser of J in V' under the hypothesis that $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$. Considering the fact that $\int_{\Omega_\varepsilon} \theta_\varepsilon^* p_\varepsilon^* \, dx \leqslant 0$ (cf. (3.22)) and $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geqslant 0$, one would expect that $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$. However, we have no means of proving this hypothesis.

In the next section, we shall study an optimal control problem with a different cost functional (cf. (1.4)) where the difficulties noted in the above remark are overcome for the positive cone situation.

4. L^2-norm of state in the cost functional

Let U_ε be a closed convex subset of $L^2(\Omega_\varepsilon)$. The optimal control problem is to minimise the cost functional J_ε given as,

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2} \|u_\varepsilon\|_{2,\Omega_\varepsilon}^2 + \frac{\varepsilon}{2} \|\theta_\varepsilon\|_{2,\Omega_\varepsilon}^2, \tag{4.1}$$

for all $\theta_\varepsilon \in U_\varepsilon$, where $u_\varepsilon \in H^1(\Omega_\varepsilon)$ is the unique solution of (3.2). From classical theory, there exists a unique $\theta_\varepsilon^* \in U_\varepsilon$ which minimises J_ε in U_ε. Let $u_\varepsilon^* \in V_\varepsilon$ be the state corresponding to θ_ε^*.

We now introduce the adjoint optimal state associated to the optimal control problem. Let $p_\varepsilon^* \in V_\varepsilon$ be the solution of,

$$\begin{cases} -\mathrm{div}\left({}^t A\left(x, \dfrac{x}{\varepsilon}\right) \nabla p_\varepsilon^*\right) = u_\varepsilon^* & \text{in } \Omega_\varepsilon, \\ {}^t A\left(x, \dfrac{x}{\varepsilon}\right) \nabla p_\varepsilon^* . \nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \\ p_\varepsilon^* = 0 & \text{on } \partial\Omega \cap \partial\Omega_\varepsilon. \end{cases} \tag{4.2}$$

Then the optimality condition is as given in (3.4).

Lemma 4.1. *Let U_ε satisfy the hypothesis* (H1), *then $\{p^*_\varepsilon\}$ is bounded in V_ε. Also, $\{u^*_\varepsilon\}$ and $\{\varepsilon^{1/2}\theta^*_\varepsilon\}$ are bounded in $L^2(\Omega_\varepsilon)$. Thus, there exist $u_0(x,y) \in L^2(\Omega \times Y)$, $p^* \in H^1_0(\Omega)$ and $p_1 \in L^2[\Omega; H^1_{\mathrm{per}}(Y')/\mathbb{R}]$ such that, for a subsequence,*

$$\varepsilon^{1/2}\widetilde{\theta^*_\varepsilon} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega), \tag{4.3}$$

$$\widetilde{u^*_\varepsilon} \overset{2s}{\rightharpoonup} u_0(x,y), \tag{4.4}$$

$$\widetilde{p^*_\varepsilon} \overset{2s}{\rightharpoonup} p^*(x)\chi(y), \tag{4.5}$$

and

$$\widetilde{\nabla p^*_\varepsilon} \overset{2s}{\rightharpoonup} \chi(y)\big[\nabla p^*(x) + \nabla_y p_1(x,y)\big]. \tag{4.6}$$

The limit p^* solves

$$\begin{cases} -\operatorname{div}\big({}^t A_0 \nabla p^*\big) = \displaystyle\int_{Y'} u_0(x,y)\,dy & \text{in } \Omega, \\ p^* = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

and

$$p_1 = \sum_{i=1}^n \omega_i(x,y)\frac{\partial p^*}{\partial x_i}(x). \tag{4.8}$$

The entries of ${}^t A_0$ are given as

$$\big({}^t A_0\big)_{ij} = \int_{Y'} {}^t A(x,y)\big[\nabla_y \omega_i(x,y) + e_i\big].\big[\nabla_y \omega_j(x,y) + e_j\big]\,dy \tag{4.9}$$

where $\omega_i(x,y)$ is the solution to the cell problem (3.8).

Proof. Let v_ε be the state corresponding to the control η_ε, given by the hypothesis (H1), in (3.2). Then, as argued in Lemma 3.2, v_ε is bounded uniformly with respect to ε in V_ε. Hence, it follows from (1.5) that $\{u^*_\varepsilon\}$ and $\{\varepsilon^{\frac{1}{2}}\theta^*_\varepsilon\}$ are both bounded in $L^2(\Omega_\varepsilon)$. Moreover, from (4.2), $\{p^*_\varepsilon\}$ is bounded in V_ε. Thus there exist $\theta' \in L^2(\Omega)$, $u_0(x,y) \in L^2(\Omega \times Y)$, $p^* \in H^1_0(\Omega)$ and $p_1 \in L^2[\Omega; H^1_{\mathrm{per}}(Y')/\mathbb{R}]$ such that, for a subsequence,

$$\varepsilon^{1/2}\widetilde{\theta^*_\varepsilon} \rightharpoonup \theta' \quad \text{weakly in } L^2(\Omega),$$

and (4.4)–(4.6) are satisfied.

Now an usual two-scale analysis of (4.2) will yield (4.7) and (4.8). It now only remains to prove that $\theta' = 0$.

Let $q_\varepsilon \in V_\varepsilon$ be the solution of

$$\begin{cases} -\operatorname{div}\left(A\left(x, \dfrac{x}{\varepsilon}\right)\nabla q_\varepsilon\right) = \varepsilon^{\frac{1}{2}}\big(f + \theta^*_\varepsilon\big) & \text{in } \Omega_\varepsilon, \\ A\left(x, \dfrac{x}{\varepsilon}\right)\nabla q_\varepsilon.\nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \\ q_\varepsilon = 0 & \text{on } \partial\Omega \cap \partial\Omega_\varepsilon. \end{cases} \tag{4.10}$$

By linearity, $q_\varepsilon = \varepsilon^{\frac{1}{2}}u^*_\varepsilon$. Since, $\varepsilon^{\frac{1}{2}}\theta^*_\varepsilon$ is bounded in $L^2(\Omega_\varepsilon)$, q_ε is bounded in V_ε and thus, by Lemma 3.1, q_ε is bounded in $L^2(\Omega_\varepsilon)$. This together with the fact that $\widetilde{u^*_\varepsilon}$ is bounded in $L^2(\Omega)$ will imply that, for a subsequence,

$$\widetilde{q_\varepsilon} \to 0 \quad \text{strongly in } L^2(\Omega).$$

Thus, the homogenized equation of (4.10) is the zero equation, which implies that $\theta' = 0$. □

Remark 4.2. In the above lemma, we conclude that $\theta' = 0$ which cannot be done, in general. This argument is also valid in Lemma 3.2. However, there the estimate on the optimal controls was enough to conclude this, which is not available in the present situation.

Remark 4.3. The above lemma is the counterpart of Lemma 3.2 for the problem considered in this section. In contrast to the problem in Section 3, here we do not have *a priori* bound on $\widetilde{\theta_\varepsilon^*}$ even in $H^{-1}(\Omega)$ (as in Lemma 3.2). Hence, we also do not have an estimate of u_ε^* in V_ε. Hence the homogenization of the state equation is not achieved as in Theorem 3.3. In other words, one is unable to address this problem for an arbitrary admissible set U_ε, in general. However, in the next theorem we shall study the positive cone case which will offer some estimates on the optimal controls. Also, in the above lemma, we have homogenized the adjoint-state equation (as in Theorem 3.3). In the next theorem we homogenize the state equation, as well, for the positive cone case.

Let $U_\varepsilon = \{\theta \in L^2(\Omega) \mid \theta \geqslant 0 \text{ a.e. in } \Omega_\varepsilon\}$ be the closed convex subset of $L^2(\Omega)$. The elements of U_ε restricted to Ω_ε is the set of all positive elements of $L^2(\Omega_\varepsilon)$. Let U' be the sequential K-lower limit of U_ε with respect to the strong topology in $L^2(\Omega)$. U' is non-empty. Let V' be the strong closure of U' in $H^{-1}(\Omega)$. Let U denote the positive cone of $L^2(\Omega)$. Let V be the strong closure of U in $H^{-1}(\Omega)$.

Theorem 4.4. *Let $U_\varepsilon = \{\theta \in L^2(\Omega) \mid \theta \geqslant 0 \text{ a.e. in } \Omega_\varepsilon\}$. Then $\{u_\varepsilon^*\}$ and $\{\widetilde{\theta_\varepsilon^*}\}$ are bounded in V_ε and $H^{-1}(\Omega)$, respectively, and there exists $\theta^* \in H^{-1}(\Omega)$ such that*

$$\widetilde{\theta_\varepsilon^*} \rightharpoonup \theta^* \quad \text{weakly in } H^{-1}(\Omega). \tag{4.11}$$

Also, $u_0(x,y) = \chi(y)u^*(x)$ where $u^* \in H_0^1(\Omega)$ solves

$$\begin{cases} -\operatorname{div}\big(A_0\nabla u^*\big) = \chi_0 f + \theta^* & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

Moreover, $p^* \in U$ and $\langle \theta^*, p^*\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$. Further, θ^* minimises

$$J(\theta) = \frac{1}{2}\int_\Omega \chi_0 u^2\,dx$$

over V', where $u = u(\theta)$ solves (3.21) *and*

$$J_\varepsilon\big(\theta_\varepsilon^*\big) \to J\big(\theta^*\big). \tag{4.13}$$

Proof. Observe that U_ε satisfies the hypothesis (H1), since $0 \in U_\varepsilon$ for all ε. Therefore the results of Lemma 4.1 remain valid. Also, the optimality condition (3.4) implies that $\varepsilon\theta_\varepsilon^* = (p_\varepsilon^*)^-$ and thus (3.22) holds. An argument similar to the one in Theorem 3.7 will show that $p^* \in U$.

Using u_ε^* as a test function in the weak form of the state equation satisfied by u_ε^* (cf. (3.12)), we have

$$\int_{\Omega_\varepsilon} A\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*.\nabla u_\varepsilon^*\,dx = \int_{\Omega_\varepsilon}\big(f + \theta_\varepsilon^*\big)u_\varepsilon^*\,dx = \int_{\Omega_\varepsilon} f u_\varepsilon^*\,dx + \varepsilon^{-1}\int_{\Omega_\varepsilon}\big(p_\varepsilon^*\big)^- u_\varepsilon^*\,dx.$$

Now, using $(p_\varepsilon^*)^-$ as a test function in the weak form of the adjoint equation (4.2), we have

$$\int_{\Omega_\varepsilon}\big(p_\varepsilon^*\big)^- u_\varepsilon^*\,dx = \int_{\Omega_\varepsilon} {}^tA\left(x, \frac{x}{\varepsilon}\right)\nabla\big(p_\varepsilon^*\big)^-.\nabla p_\varepsilon^*\,dx = -\int_{\Omega_\varepsilon} {}^tA\left(x, \frac{x}{\varepsilon}\right)\nabla\big(p_\varepsilon^*\big)^-.\nabla\big(p_\varepsilon^*\big)^-\,dx$$

and hence we derive the equality,

$$\int_{\Omega_\varepsilon} A\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*.\nabla u_\varepsilon^*\,dx + \varepsilon^{-1}\int_{\Omega_\varepsilon} A\left(x, \frac{x}{\varepsilon}\right)\nabla\big(p_\varepsilon^*\big)^-.\nabla\big(p_\varepsilon^*\big)^-\,dx = \int_{\Omega_\varepsilon} f u_\varepsilon^*\,dx. \tag{4.14}$$

Since $\{u_\varepsilon^*\}$ is bounded in $L^2(\Omega_\varepsilon)$, we deduce from (4.14) that $\{u_\varepsilon^*\}$ and $\{\varepsilon^{-1/2}(p_\varepsilon^*)^-\}$ are bounded in V_ε. Therefore, there exist $u^* \in H_0^1(\Omega)$ and $u_1 \in L^2[\Omega; H^1_{\mathrm{per}}(Y')/\mathbb{R}]$ such that $u_0(x,y) = \chi(y)u^*(x)$ and, up to a subsequence,

$$\widetilde{\nabla u_\varepsilon^*} \overset{2s}{\rightharpoonup} \chi(y)\big[\nabla u^*(x) + \nabla_y u_1(x,y)\big].$$

Now, arguing as in Lemma 3.2, we show that $\{\widetilde{\theta_\varepsilon^*}\}$ is bounded in $H^{-1}(\Omega)$ and thus there exists $\theta^* \in H^{-1}(\Omega)$ such that (4.11) holds, for a subsequence. Now, arguing as in the proof of Theorem 3.3, we show that u^* solves (4.12).

Moreover, we have (cf. [2, Theorem A.1])

$$\left.\begin{aligned} &\big\|u_\varepsilon^*\big\|_{2,\Omega_\varepsilon}^2 \to \chi_0\big\|u^*\big\|_2^2, \\ &\big\|\varepsilon^{-1/2}\big(p_\varepsilon^*\big)^-\big\|_{2,\Omega_\varepsilon} = \big\|\varepsilon^{1/2}\theta_\varepsilon^*\big\|_{2,\Omega_\varepsilon} \to 0. \end{aligned}\right\}$$

Thus, passing to the limit in $J_\varepsilon(\theta_\varepsilon^*)$ we have (4.13).

On the other hand, using p^*_ε as a test function in (3.12) and u^*_ε as a test function in (3.3), one can rewrite the cost functional $J_\varepsilon(\theta^*_\varepsilon)$ as in (3.18). Now, using (3.22), $J_\varepsilon(\theta^*_\varepsilon)$ reduces to

$$J_\varepsilon(\theta^*_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} f p^*_\varepsilon\,dx$$

and passing to the limit will yield

$$J_\varepsilon(\theta^*_\varepsilon) \to \frac{1}{2}\int_{\Omega} \chi_0 f p^*\,dx.$$

Using p^* and u^* as test functions in (4.12) and (4.7), respectively, we have

$$\int_{\Omega} \chi_0 f p^*\,dx = J(\theta^*) - \frac{1}{2}\langle \theta^*, p^*\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

Therefore, (4.13) implies that $\langle \theta^*, p^*\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = 0$. Now an argument analogous to the one in the proof of Theorem 3.7 will show that θ^* minimises J in V'. The strict convexity of J would then imply the uniqueness of θ^* and thus the convergences (4.11) and (4.3) hold for the entire sequence. $\square$

Remark 4.5. The above theorem differs from Theorem 3.7 in the sense that we now have the fact that

$$\int_{\Omega_\varepsilon} \theta^*_\varepsilon p^*_\varepsilon\,dx = -\varepsilon \|\theta^*_\varepsilon\|^2_{2,\Omega_\varepsilon} \to 0.$$

In other words, we have identified the limit of the product of the two weakly converging sequence. This was possible due to (4.14) and similar equality is not available for the problem in Section 3.

5. Conclusion

We have studied the asymptotic behaviour of some low-cost control problems left open in [12,19]. The results are proved for perforated domains but they remain valid for non-perforated domains as well, with necessary modification. We prove an H-convergence result for weak data (cf. Theorem 2.1) which forms the backbone of the article. It would be interesting to see if the difficulties discussed in Remark 3.8 could be overcome.

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