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EXISTENCE AND APPROXIMATIONS OF SOLUTIONS TO SOME FRACTIONAL ORDER FUNCTIONAL INTEGRAL EQUATIONS

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ABSTRACT. In this paper we shall study a fractional order functional integral equation. In the first part of the paper, we proved the existence and uniqueness of mild and global solutions in a Banach space. In the second part of the paper, we used the analytic semigroups theory of linear operators and the fixed point method to establish the existence, uniqueness and convergence of approximate solutions of the given problem in a separable Hilbert space. We also proved the existence and convergence of Faedo-Galerkin approximate solution to the given problem. Finally, we give an example.

1. Introduction. We consider the following fractional order evolution equation in a Banach space $(X, \|.\|)$:

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1}(-Au(\theta))\, d\theta + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} f(\theta, u(\theta), u(a(\theta)))\, d\theta, \tag{1.1}$$

where A is a closed linear operator defined on a dense set and $0 < \beta < 1$, $0 < T < \infty$. We assume that $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \geq 0\}$ in X, Γ is the gamma function and $u(0) = u_0 \in X$. The functions f and a satisfy certain conditions to be specified later.

Regarding earlier works on existence and uniqueness of different type of solutions to fractional differential equations we refer to [**1, 8**–**14, 24**] and references cited in these papers.

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For the initial study on existence, uniqueness and stability of various types of solutions of differential and functional differential equations we refer to Balachandran and Chandrasekaran [**3**], Byszewski and Lakshmikantham [**7**], Byszewski and Akca [**6**], Lin and Liu [**18**] and references cited in these papers.

Initial studies concerning existence, uniqueness and finite-time blow-up of solutions for the following equation,

$$\begin{aligned} u'(t) + Au(t) &= g(u(t)), \quad t \geq 0, \\ u(0) &= \phi, \end{aligned} \tag{1.2}$$

have been considered by Heinz and von Wahl [**17**], Murakami [**21**] and Segal [**26**]. Bazley [**4, 12**] has considered the following semilinear wave equation

$$\begin{aligned} u''(t) + Au(t) &= g(u(t)), \quad t \geq 0, \\ u(0) = \phi, &\quad u'(0) = \psi, \end{aligned} \tag{1.3}$$

and has established the uniform convergence of approximate solutions to (1.3) by using the existence results of Heinz and von Wahl [**6**]. Goethel [**22**] has proved the convergence of approximate solutions to the problem (1.2) but assumed g to be defined on the whole of H. Based on the ideas of Bazley [**4, 5**], Miletta [**20**] has proved the convergence of approximate solutions of (1.2).

2. Preliminaries. We note that if $-A$ is the infinitesimal generator of an analytic semigroup, then for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence, without loss of generality, we suppose that

$$||S(t)|| \leq M \text{ for } t \geq 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with domain

$D(A^\alpha)$ being dense in X. We have $X_\kappa \hookrightarrow X_\alpha$ for $0 < \alpha < \kappa$ and the embedding is continuous. For more details on the fractional powers of closed linear operators we refer to Pazy [**23**]. It can be proved easily that $X_\alpha := D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ and it is equivalent to the graph norm of A^α.

We notice that $\mathcal{C}_T = C([0,T], X)$, the set of all continuous functions from $[0,T]$ into X is a Banach space under the supremum norm given by

$$\|\psi\|_T := \sup_{0\le\eta\le T} \|\psi(\eta)\|, \quad \psi \in \mathcal{C}_T.$$

It can also be proved easily that $C_t^\alpha = C([0,t]; X_\alpha)$, for all $t \in [0,T]$, is a Banach space endowed with the supremum norm

$$\|\psi\|_{t,\alpha} := \sup_{0\le\eta\le t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^\alpha.$$

We assume the following conditions:

(**A1**): The nonlinear map $f : [0,T] \times X_\alpha \times X_\alpha \to X$ satisfies:

$$\|f(t, x_1, x_2) - f(s, y_1, y_2)\| \le L(r)[|t-s|^\nu + \sum_{i=1}^{2} \|x_i - y_i\|_\alpha],$$

for all $t, s \in [0,T]$, a fixed ν, $0 < \nu \le 1$, x_i, $y_i \in B_r(X_\alpha)$ for all $i = 1, 2$. Here $L : R_+ \to R_+$ is a nondecreasing function. For any $r > 0$ and Banach space $(Z, \|.\|_Z)$, we define

$$B_r(Z) = \{z \in Z : \|z\|_Z \le r\}.$$

(**A2**): The function $a : [0,T] \to [0,T]$ satisfies the following two conditions:

(i) a satisfies the delay property $a(t) \le t$ for all $t \in [0,T]$;

(ii) The function a is Lipschitz continuous; that is, there exists a positive constant L_a such that

$$|a(t) - a(s)| \le L_a|t-s|, \quad \text{for all } t, s \in [0,T].$$

We define the Riemann-Liouville integral of order $\beta > 0$ by

$$I^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} g(\theta)\, d\theta.$$

By a mild solution of the evolution problem (1.1), we mean a continuous solution u of the following integral equation given below

(2.1)
$$u(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) u_0 d\theta + \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) f(s, u(s), u(a(\theta)))\, d\theta\, ds,$$

where $\zeta_\beta(\theta)$ is the probability density function [**15, 25**]. For the further detail on mild solution, we refer to [**10, 11, 12**].

3. Existence of mild solutions. We take

$$\int_0^\infty \theta^{1-\alpha} \zeta_\beta(\theta)\, d\theta = N_1, \tag{3.1}$$

where $\zeta_\beta(\theta)$ is the probability density function [**16**]. For $R > 0$, let $M\|u_0\|_\alpha \leq (R/3)$ and

$$S = \{u : u \in \mathcal{C}_{t_0}^\alpha, \|u\|_{t_0,\alpha} \leq R\}.$$

Choose t_0, $0 < t_0 \leq T$ such that

$$t_0 < \left[\frac{2R}{3} {C_\alpha}^{-1} {N_1}^{-1} (1-\alpha) \{L(R)[T^\nu + 2R] + N_2\}^{-1}\right]^{1/(\beta(1-\alpha))}, \tag{3.2}$$

where C_α is a positive constant depending on α satisfying

$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}, \tag{3.3}$$

for all $t > 0$ and $\|f(0,0,0)\| = N_2$. We have used the above inequality (3.3) throughout the paper.

Theorem 3.1. *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$ with $\|S(t)\| \leq M$, $t \geq 0$ and that $0 \in \rho(-A)$. If the conditions* (**A1**)–(**A2**) *hold and $u_0 \in D(A)$, then equation* (1.1) *has a unique local mild solution. Moreover the mild solution u is uniformly Hölfer continuous.*

Proof. We will establish the existence of a solution u of equation (1.1) on $[0, t_0]$ for some t_0 such that $0 < t_0 \leq T$.

For any $0 < \widetilde{T} \leq T$, we define a mapping F from $\mathcal{C}^{\alpha}_{\widetilde{T}}$ into $\mathcal{C}^{\alpha}_{\widetilde{T}}$ given by

$$(3.4) \quad (F\psi)(t) = \int_0^{\infty} \zeta_\beta(\theta) S(t^\beta \theta) u_0 \, d\theta + \beta \int_0^t \int_0^{\infty} \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) f(s, \psi(s), \psi(a(s))) \, d\theta \, ds.$$

Clearly, F is well defined.

To prove this theorem, first we need to show that $F : S \to S$. For any $\psi \in S$, we have $(F\psi)(0) = u_0$. If $t \in (0, t_0]$, then we have

$$\begin{aligned}
\|(F\psi)(t)\|_\alpha \leq & \int_0^{\infty} \zeta_\beta(\theta) \|S(t^\beta \theta)\| \|A^\alpha u_0\| \, d\theta \\
& + \beta \int_0^t \int_0^{\infty} \theta (t-s)^{\beta-1} \zeta_\beta(\theta) \|A^\alpha S((t-s)^\beta \theta)\| \\
& \qquad \|f(s, \psi(s), \psi(a(s))) - f(0,0,0)\| \, d\theta \, ds \\
& + \beta \int_0^t \int_0^{\infty} \theta (t-s)^{\beta-1} \zeta_\beta(\theta) \|A^\alpha S((t-s)^\beta \theta)\| \\
& \qquad \times \|f(0,0,0)\| \, d\theta \, ds \\
\leq & \, M\|u_0\|_\alpha + N_1 C_\alpha \{L(R)[T^\nu + 2R] + N_2\} \frac{t_0^{\beta(1-\alpha)}}{(1-\alpha)}.
\end{aligned}$$

Hence, from the above inequality, we get $\|F\psi\|_{t_0,\alpha} \leq R$. Therefore, $F : S \to S$. Our next goal is to show that F is a strict contraction mapping on S.

For all $t \in [0, t_0]$ and $\psi_1, \psi_2 \in S$, we have

$$\begin{aligned}
(F\psi_1)(t) - (F\psi_2)(t) = & \, \beta \int_0^t \int_0^{\infty} \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) \\
& [f(s, \psi_1(s), \psi_1(a(s))) - f(s, \psi_2(s), \psi_2(a(s)))] \, d\theta \, ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|(F\psi_1)(t) - (F\psi_2)(t)\|_\alpha \leq & \, \beta \int_0^t \int_0^{\infty} \theta (t-s)^{\beta-1} \zeta_\beta(\theta) \|A^\alpha S((t-s)^\beta \theta)\| \\
& \|f(s, \psi_1(s), \psi_1(a(s))) - f(s, \psi_2(s), \psi_2(a(s)))\| \, d\theta \, ds.
\end{aligned}$$

From condition (**A1**) and inequality (3.2), we get

$$\begin{aligned}\|(F\psi_1)(t) - (F\psi_2)(t)\|_\alpha &\le \frac{1}{R}\frac{C_\alpha}{(1-\alpha)}N_1[L(R)(T^\nu + 2R) + N_2]t_0^{\beta(1-\alpha)}\|\psi_1 - \psi_2\|_{t_0,\alpha}\\ &\le \frac{2}{3}\|\psi_1 - \psi_2\|_{t_0,\alpha},\end{aligned}$$

for all $\psi_1, \psi_2 \in S$. Hence, F is a strict contraction mapping on S and therefore F has a unique fixed point in S.

Hence, for all $t \in [0, t_0]$, we have

$$u(t) = \int_0^\infty \zeta_\beta(\theta)S(t^\beta\theta)u_0\,d\theta + \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)S((t-s)^\beta\theta)f(s,u(s),u(a(s)))\,d\theta\,ds, \tag{3.5}$$

where $u(0) = u_0$. Now we will show that the function u is Hölder continuous on $[0, t_0]$ with respect to the α norm. For any t_1, $t_2 \in [0, t_0]$, where $t_1 < t_2$, we have,

$$\begin{aligned}A^\alpha[u(t_2) - u(t_1)] &= \int_0^\infty \zeta_\beta(\theta)[S(t_2^\beta\theta) - S(t_1^\beta\theta)]A^\alpha u_0\,d\theta\\ &\quad + \beta\int_{t_1}^{t_2}\int_0^\infty \theta(t_2-s)^{\beta-1}\zeta_\beta(\theta)A^\alpha S((t_2-s)^\beta\theta)\\ &\qquad\qquad \times f(s,u(s),u(a(s)))\,d\theta\,ds\\ &\quad + (-\beta)\int_0^{t_1}\int_0^\infty \theta[(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}]\zeta_\beta(\theta)\\ &\qquad\qquad \times A^\alpha S((t_2-s)^\beta\theta)f(s,u(s),u(a(s)))\,d\theta\,ds\\ &\quad + \beta\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{\beta-1}\zeta_\beta(\theta)A^\alpha[S((t_2-s)^\beta\theta)\\ &\qquad\qquad - S((t_1-s)^\beta\theta)]f(s,u(s),u(a(s)))\,d\theta\,ds\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.6}$$

Hence,

$$\|u(t_2) - u(t_1)\|_\alpha \le \|I_1\| + \|I_2\| + \|I_3\| + \|I_4\|. \tag{3.7}$$

We have

$$I_1 = \int_0^\infty \zeta_\beta(\theta)[S(t_2^\beta\theta) - S(t_1^\beta\theta)]A^\alpha u_0 d\theta$$
$$= \int_0^\infty \zeta_\beta(\theta)\left[\int_{t_1}^{t_2} \beta\theta t^{\beta-1}A^\alpha S(t^\beta\theta)Au_0\, dt\right] d\theta.$$

Therefore,

$$\|I_1\| \leq \int_0^\infty \zeta_\beta(\theta)\int_{t_1}^{t_2} \beta\theta t^{\beta-1}\|A^\alpha S(t^\beta\theta)\|\|Au_0\|\, dt\, d\theta$$
$$\leq C_\alpha\beta\int_0^\infty \theta^{1-\alpha}\zeta_\beta(\theta)\int_{t_1}^{t_2} t^{\beta(1-\alpha)-1}\|Au_0\|\, dt\, d\theta.$$

Hence,

$$\begin{aligned} \|I_1\| &\leq \frac{N_1}{(1-\alpha)}C_\alpha\|Au_0\|(t_2^{\beta(1-\alpha)} - t_1^{\beta(1-\alpha)}) \\ &\leq C_\alpha\|Au_0\|N_1\beta(t_1+\kappa(t_2-t_1))^{\beta(1-\alpha)-1}(t_2-t_1) \\ &\leq C_\alpha\|Au_0\|N_1\beta\kappa^{\beta(1-\alpha)-1}(t_2-t_1)^{\beta(1-\alpha)}, \end{aligned} \tag{3.8}$$

where $0 < \kappa < 1$.

Also,

$$\|I_2\| \leq \frac{L_f(R)}{(1-\alpha)}C_\alpha N_1(t_2-t_1)^{\beta(1-\alpha)}, \tag{3.9}$$

$$\|I_3\| \leq \beta N_1 L_f(R)C_\alpha\int_0^{t_1}(t_1-s)^{\lambda-1}[(t_1-s)^{-\lambda\mu}-(t_2-s)^{-\lambda\mu}]\, ds, \tag{3.10}$$

where $L_f(R) = \{L(R)[T^\nu + 2R] + N_2\}$, $\lambda = 1-\beta\alpha$ and $\mu = (1-\beta)/(1-\beta\alpha)$.

Hence, after some calculation we get

$$\|I_3\| \leq \beta N_1 L_f(R)C_\alpha\mu\delta^{\mu-1}(1-c)^{-\lambda(1-\mu)-1}(t_2-t_1)^{\lambda(1-\mu)}, \tag{3.11}$$

where $c = (1-(\mu/\lambda)^{1/(\lambda\mu)})$ and $0 < \delta \leq 1$.

Also, we notice that

(3.12)

$$\|I_4\| \leq \beta N_1 L_f(R) \frac{C_{1+\alpha}}{\alpha} \int_0^{t_1} (t_1 - s)^{\beta-1} [(t_1 - s)^{-\beta\alpha} - (t_2 - s)^{-\beta\alpha}]\, ds$$

$$\leq \beta N_1 L_f(R) \frac{C_{1+\alpha}}{\alpha} {\delta_1}^{\alpha-1} (1 - c_1)^{-\beta(1-\alpha)-1} (t_2 - t_1)^{\beta(1-\alpha)},$$

where $c_1 = (1 - (\alpha/\beta)^{1/(\alpha\beta)})$, $0 < \delta_1 \leq 1$ and $C_{1+\alpha}$ is some positive constant satisfying $\|A^{\alpha+1} S(t)\| \leq C_{1+\alpha} t^{-1-\alpha}$ for all $t > 0$.

Thus the function u satisfies the uniform Hölder condition on $[0, t_0]$. Hence with the help of the conditions (**A1**)–(**A2**), we can show that the map

$$t \longmapsto f(t, u(t), u(a(t)))$$

is uniformly Hölder continuous on $[0, t_0]$. This completes the proof of the theorem. □

4. Approximate solutions and convergence. In this section we assume that $0 < T < \infty$, $0 < \beta < 1$, $0 \leq \alpha < 1$ and X is separable Hilbert space. Also we need an addition conditions on the operator A namely:

(**A3**): A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset X$ into X such that $D(A)$ is dense in X. We assume A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots ,$$

where $\lambda_m \to \infty$ as $m \to \infty$ and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$A\phi_i = \lambda_i \phi_i \text{ and } \langle \phi_i, \phi_j \rangle = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (**A3**) is satisfied, then $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$ in X.

Let X_n denote the finite dimensional subspace of X spanned by $\{\phi_0, \phi_1, \ldots, \phi_n\}$, and let $P^n : X \to X_n$ be the projections associated with the spectral set of the operator A for $n = 0, 1, 2, \ldots$.

We define

$$f_n : [0,T] \times X_\alpha \times X_\alpha \longrightarrow X$$

such that

$$f_n(t,x,y) = f(t, P^n x, P^n y).$$

For $n = 0,1,2,\dots$, we define the maps F_n on S as follows: for $u \in S$ and $t \in [0,T]$

(4.1)

$$(F_n u)(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) u_0 \, d\theta + \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) \times f_n(s, u(s), u(a(s))) \, d\theta \, ds.$$

Theorem 4.1. *Assume the conditions* (**A1**)–(**A3**) *are satisfied. Then, there exist* t_0, $0 < t_0 \le T$ *and a unique* $u_n \in S$ *such that* $F_n u_n = u_n$ *for each* $n = 0,1,2,3,\dots$ *. More precisely,* u_n *satisfies the integral equation*

(4.2)

$$u_n(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) u_0 \, d\theta + \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) \times f_n(s, u_n(s), u_n(a(s))) \, d\theta \, ds,$$

for all $0 \le t \le t_0$*. Moreover,* u_n *is uniformly Hölder continuous on* $[0,t_0]$*, where* $0 < t_0 \le T$*.*

Proof. We can prove this theorem by using a similar technique as in Theorem (3.1). □

Corollary 4.2. *Let* $u_0 \in D(A)$ *and* $0 \le \eta < 1$*. Then,* $u_n(t) \in D(A^\eta)$ *for all* $t \in [0,t_0]$*.*

Proof. As the function u_n is uniformly Hölder continuous on $[0,t_0]$, hence with the help of conditions (**A1**)–(**A2**) we can see that the map

$t \mapsto f(t, P^n u_n(t), P^n u_n(a(t)))$ is uniformly Hölder continuous on $[0, t_0]$; hence, $u_n(t) \in D(A)$. Since $D(A) \subset D(A^\eta)$ and $u_n(t) \in D(A)$, hence our Corollary is proved. □

Corollary 4.3. *Let $u_0 \in D(A)$ and $0 \leq \eta < 1$. Then, there exists a constant M_0 independent of n such that*

$$\|A^\eta u_n\| \leq M_0,$$

for all $t \in [0, t_0]$.

Proof. From equation (4.2), we have

(4.3)
$$\begin{aligned}\|A^\eta u_n(t)\| \leq M\|u_0\|_\eta + \beta \int_0^t \int_0^\infty & \theta(t-s)^{\beta-1}\zeta_\beta(\theta) \\ & \|A^\eta S((t-s)^\beta\theta)\| \|f(s, P^n u_n(s), P^n u_n(a(s)))\| \, d\theta \, ds \\ & \leq M\|u_0\|_\eta + N_1 C_\eta \{L(R)[T^\nu + 2R] + N_2\} \frac{t_0^{\beta(1-\eta)}}{(1-\eta)} \\ & \leq M_0.\end{aligned}$$

This completes the proof of the Corollary. □

Theorem 4.4. *Let $u_0 \in D(A)$ and the assumptions* (**A1**)–(**A3**) *are satisfied. Then, $\{u_n\} \subset S$ is a Cauchy sequence and therefore converges to a unique function $u \in S$.*

Proof. Let $n \geq m \geq n_0$, where n_0 is large enough and $n, m, n_0 \in \mathbf{N}$. Hence from Theorem 4.1 we have

$$\begin{aligned} u_n(t) = & \int_0^\infty \zeta_\beta(\theta) S(t^\beta\theta) u_0 \, d\theta \\ & + \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta) S((t-s)^\beta\theta) \\ & \qquad \times f_n(s, u_n(s), u_n(a(s))) \, d\theta \, ds. \end{aligned} \tag{4.4}$$

For $t \in [0, t_0]$, we have

$$\begin{aligned}(4.5)\quad &\|A^\alpha(u_n(t) - u_m(t))\| \\ &\leq \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\| \\ &\quad \|f_n(s, u_n(s), u_n(a(s))) - f_m(s, u_m(s), u_m(a(s)))\| \, d\theta \, ds.\end{aligned}$$

For $n \geq m$ and $\alpha < \eta$, we have

$$\begin{aligned}(4.6)\quad &\|f_n(s, u_n(s), u_n(a(s))) - f_m(s, u_m(s), u_m(a(s)))\| \\ &\leq \|f_n(s, u_n(s), u_n(a(s))) - f_n(s, u_m(s), u_m(a(s)))\| \\ &\quad + \|f_n(s, u_m(s), u_m(a(s))) - f_m(s, u_m(s), u_m(a(s)))\| \\ &\leq 2L(R)\|u_n - u_m\|_{s,\alpha} + L(R)[\|A^{\alpha-\eta}(P^n - P^m)A^\eta u_m(s)\| \\ &\quad + \|A^{\alpha-\eta}(P^n - P^m)A^\eta u_m(a(s))\|].\end{aligned}$$

Now we calculate the above inequality (4.6) as follows.

Let $m < n$. Then $X_m \subset X_n$. Let $X_m^\perp$ be the orthogonal complement of X_m for all $m = 0, 1, 2, \ldots$; then $X_m^\perp \supset X_n^\perp$. We can write $X = X_m \oplus X_m^\perp = X_n \oplus X_n^\perp$.

Let $z \in X$ be an arbitrary element. Then, we can write $z = z_m + y_m$, where $z_m \in X_m$ and $y_m \in X_m^\perp$. Then, $P^m z = z_m \in X_m$. We can see easily that $y_m \in X_m^\perp \Rightarrow y_m = \Sigma_{i=m+1}^n a_i\phi_i + y'_m$, where $y'_m \in X_n^\perp$. Let, $z'_m = \Sigma_{i=m+1}^n a_i\phi_i$.

Hence, $z = z_m + z'_m + y'_m$ and $P^n z = z_m + z'_m$.

Therefore,

$$P^n z - P^m z = z'_m = \Sigma_{i=m+1}^n a_i\phi_i.$$

If, $z = \Sigma_{i=1}^\infty a_i\phi_i$, then $\|z\|^2 = \Sigma_{i=1}^\infty |a_i|^2$.

Since, $A^{\alpha-\eta}\phi_i = \lambda_i^{\alpha-\eta}\phi_i$ [**19**]. Hence, we have

$$\begin{aligned}\|A^{\alpha-\eta}(P^n - P^m)z\|^2 &= \langle A^{\alpha-\eta}(P^n - P^m)z,\ A^{\alpha-\eta}(P^n - P^m)z\rangle \\ &= \langle \Sigma_{i=m+1}^n a_i A^{\alpha-\eta}\phi_i,\ \Sigma_{j=m+1}^n a_j A^{\alpha-\eta}\phi_j\rangle \\ &= \langle \Sigma_{i=m+1}^n a_i \lambda_i^{\alpha-\eta}\phi_i,\ \Sigma_{j=m+1}^n a_j \lambda_j^{\alpha-\eta}\phi_j\rangle \\ &= \Sigma_{i,j=m+1}^n a_i a_j \lambda_i^{\alpha-\eta}\lambda_j^{\alpha-\eta}\langle\phi_i, \phi_j\rangle \\ &\leq \lambda_{m+1}^{2(\alpha-\eta)}(\Sigma_{i=m+1}^n |a_i|^2) \\ &\leq \frac{1}{\lambda_m^{2(\eta-\alpha)}}\|z\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|A^{\alpha-\eta}(P^n - P^m)A^\eta u_m(s)\| &\le \frac{1}{\lambda_m^{(\eta-\alpha)}}\|A^\eta u_m(s)\| \\ &\le \frac{1}{\lambda_m^{(\eta-\alpha)}}M_0,\end{aligned} \tag{4.7}$$

where M_0 is the same as in Corollary 4.3. Similarly, we can see that

$$\|(P^n - P^m)u_m(a(s)))\|_\alpha \le \frac{1}{\lambda_m^{\eta-\alpha}}M_0. \tag{4.8}$$

Therefore, the inequality (4.6) becomes

$$\begin{aligned}&\|f_n(s, u_n(s), u_n(a(s))) - f_m(s, u_m(s), u_m(a(s)))\| \\ &\qquad \le 2L(R)\|u_n - u_m\|_{s,\alpha} + 2L(R)\frac{M_0}{\lambda_m^{\eta-\alpha}}.\end{aligned} \tag{4.9}$$

We use inequality (4.9) in inequality (4.5) and get

$$\begin{aligned}&\|A^\alpha[u_n(t) - u_m(t)]\| \\ &\quad \le 2L(R)\beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\| \\ &\qquad\qquad \times \|u_n - u_m\|_{s,\alpha}\, d\theta\, ds \\ &\quad + 2L(R)\frac{M_0}{\lambda_m^{\eta-\alpha}}\beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta) \\ &\qquad\qquad \times \|A^\alpha S((t-s)^\beta\theta)\|\, d\theta\, ds.\end{aligned} \tag{4.10}$$

The first integral of the above inequality (4.10) is calculated as follows:

$$\begin{aligned}&2L(R)\beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\|u_n - u_m\|_{s,\alpha}\, d\theta\, ds \\ &\qquad \le \beta C_\alpha 2L(R)N_1 \int_0^t (t-s)^{\beta(1-\alpha)-1}\|u_n - u_m\|_{s,\alpha}\, ds.\end{aligned} \tag{4.11}$$

Second integral of the above inequality (4.10) is calculated as:

$$\begin{aligned}&2L(R)\frac{M_0}{\lambda_m^{\eta-\alpha}}\beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\, d\theta\, ds \\ &\qquad \le \frac{C_\alpha 2L(R)M_0 N_1}{\lambda_m^{\eta-\alpha}(1-\alpha)}t_0^{\beta(1-\alpha)}.\end{aligned} \tag{4.12}$$

We put the values of these integrals in the inequality (4.10) to get

(4.13)

$$\|u_n - u_m\|_{t,\alpha} \leq 2\frac{C_\alpha L(R) M_0 N_1}{\lambda_m^{\eta-\alpha}(1-\alpha)} t_0^{\beta(1-\alpha)} + 2\beta C_\alpha L(R) N_1 \int_0^t (t-s)^{\beta(1-\alpha)-1} \|u_n - u_m\|_{s,\alpha}\, ds.$$

Application of Gronwall's inequality will give the required result. This completes the proof of the theorem. □

With the help of Theorem 4.1 and Theorem 4.2 we have the following existence and convergence result.

Theorem 4.5. *Assume the conditions* (**A1**)–(**A3**) *are satisfied. Then, there exists* t_0*,* $0 < t_0 \leq T$ *such that* u_n *given by equation* (4.2) *converges in* S *to a mild solution* $u \in S$ *of the equation* (2.1).

Proof. The existence of u on $[0, t_0]$ is clear from Theorem 4.4. We only need to prove that u is given by equation (2.1). We have the following inequality

$$\begin{aligned} (4.14) \quad & \|f(t, P^n u_n(t), P^n u_n(a(t))) - f(t, u(t), u(a(t))\| \\ & \leq L(R)[\|P^n u_n(t) - u(t)\|_\alpha + \|P^n u_n(a(t)) - u(a(t))\|_\alpha] \\ & \leq L(R)[\|(P^n - I)u_n(t)\|_\alpha + \|u_n(t) - u(t)\|_\alpha \\ & \quad + \|(P^n - I)u_n(a(t))\|_\alpha + \|u_n(a(t)) - u(a(t))\|_\alpha]. \end{aligned}$$

Hence, $\|f(t, P^n u_n(t), P^n u_n(a(t))) - f(t, u(t), u(a(t))\| \to 0$ as $n \to \infty$.

Thus we have

$$(4.15) \quad \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) \|f(t, P^n u_n(t), P^n u_n(a(t))) - f(t, u(t), u(a(t))\|\, d\theta ds \to 0 \text{ when } n \to \infty.$$

Hence from the above inequality we can see that u is given by equation (2.1). This completes the proof of the theorem. □

The next theorem is all about the uniqueness and global existence of u.

Theorem 4.6. *Suppose that $0 \in \rho(-A)$, $-A$ generates the analytic semigroup $S(t)$ with $\|S(t)\| \leq M$, for $t \geq 0$, $u_0 \in D(A)$, the condition (**A2**) holds and the function $f : [0,\infty[\times X_\alpha \times X_\alpha \to X$ satisfies the condition (**A1**). If there is a continuous nondecreasing real valued function $k(t)$ such that*

(4.16)

$$\|f(t,x_1,x_2)\| \leq k(t)(1+\sum_{i=1}^{2}\|x_i\|_\alpha) \text{ for } t \geq 0, \quad x_i \in X_\alpha,\ i=1,2,$$

then the mild solution u is unique and exists globally.

Proof. We can continue u as long as $\|u(t)\|_\alpha$ stays bounded. It is therefore sufficient to show that if u exists on $[0,T[$, then $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$.

For $t \in [0,T[$, we have

(4.17)

$$A^\alpha u(t) = \int_0^\infty \zeta_\beta(\theta) A^\alpha S(t^\beta\theta)u_0 d\theta + \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)A^\alpha S((t-s)^\beta\theta)f(s,u(s),u(a(s)))\,d\theta\,ds.$$

From the above equation, we get

$$\|u(t)\|_\alpha \leq M\|u_0\|_\alpha + \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\| \|f(s,u(s),u(a(s)))\|\,d\theta\,ds.$$

Hence

$$\|u\|_{t,\alpha} \leq C_1 + C_2\int_0^t (t-s)^{\beta(1-\alpha)-1}\|u\|_{s,\alpha}\,ds,$$

where $C_1 = M\|u(0)\|_\alpha + (k(T)N_2C_\alpha T^{\beta(1-\alpha)})/(1-\alpha)$ and $C_2 = 2k(T)\beta N_2 C_\alpha$.

Hence, by Lemma 6.7 [**23**, Chapter 5], we get the result.

To complete the proof of the theorem we only need to show that u is unique on the whole interval. Let u_1 and u_2 be two solutions of the given fractional differential equation (1.1). Then, by a similar argument as above, we see that

$$\|u_1-u_2\|_{t,\alpha} \leq 2L(R)\beta N_1 C_\alpha\int_0^t (t-s)^{\beta(1-\alpha)-1}\|u_1-u_2\|_{s,\alpha}\,ds.$$

Hence again by Lemma 6.7 [**23**, Chapter 5], we see that the solution u is unique. This completes the proof of the theorem. □

5. Faedo-Galerkin approximations. We know from the previous sections that, for any $0 < t_0 \leq T$, we have a unique $u \in C^{\alpha}_{t_0}$ satisfying the integral equation

(5.1)
$$u(t) = \int_0^{\infty} \zeta_\beta(\theta) S(t^\beta \theta) u_0 \, d\theta + \beta \int_0^t \int_0^{\infty} \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) f(s, u(s), u(a(\theta))) \, d\theta \, ds.$$

Also, there is a unique solution $u_n \in C^{\alpha}_{t_0}$ of the approximate integral equation

$$u_n(t) = \int_0^{\infty} \zeta_\beta(\theta) S(t^\beta \theta) u_0 \, d\theta + \beta \int_0^t \int_0^{\infty} \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) \times f_n(s, u_n(s), u_n(a(s))) \, d\theta \, ds. \tag{5.2}$$

Faedo-Galerkin approximation is given by $\overline{u}_n = P^n u_n$ satisfying

$$\overline{u}_n(t) = \int_0^{\infty} \zeta_\beta(\theta) S(t^\beta \theta) P^n u_0 \, d\theta + \beta \int_0^t \int_0^{\infty} \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) P^n \times f_n(s, u_n(s), u_n(a(s))) \, d\theta \, ds, \tag{5.3}$$

where f_n is defined as earlier. Note that u_n is in a finite dimensional space. If the solution $u(t)$ to equation (5.1) exists on $0 \leq t \leq t_0$, then it has the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t) \phi_i, \tag{5.4}$$

where $\alpha_i(t) = \langle u(t), \phi_i \rangle$ for all $i = 0, 1, 2, 3 \ldots$, and

$$\overline{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t) \phi_i, \tag{5.5}$$

where $\alpha_i^n(t) = \langle \overline{u}_n(t), \phi_i \rangle$ for all $i = 0, 1, 2, 3, \dots$.

As a consequence of Theorem 4.1 and Theorem 4.4, we have the following existence and uniqueness result.

Theorem 5.1. *Suppose that the conditions* (**A1**)–(**A3**) *are satisfied and* $u_0 \in D(A)$. *Then there exist functions* $\overline{u}_n \in C([0, t_0]; X_n)$ *and* $u \in C([0, t_0]; X)$ *satisfying*

$$\begin{aligned}\overline{u}_n(t) = &\int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) P^n u_0 \, d\theta \\ &+ \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) P^n \\ &\qquad \times f_n(s, u_n(s), u_n(a(s))) \, d\theta \, ds\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}u(t) = &\int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) u_0 \, d\theta \\ &+ \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) f(s, u(s), u(a(\theta))) \, d\theta \, ds\end{aligned} \tag{5.7}$$

such that $\overline{u}_n \to u$ *in* $C([0, t_0]; X)$ *as* $n \to \infty$, *where* f_n *is same as defined earlier.*

Theorem 5.2. *If the conditions* (**A1**)–(**A3**) *are satisfied and* $u_0 \in D(A)$, *then for any* $0 < t_0 \leq T$, *we have*

$$\lim_{n\to\infty} \sup_{0 \leq t \leq t_0} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha} \{\alpha_i(t) - \alpha_i^n(t)\}^2 \right] = 0.$$

Proof. We have

$$\begin{aligned}A^\alpha[u(t) - \bar{u}_n(t)] &= A^\alpha \left[\sum_{i=0}^{\infty} \{\alpha_i(t) - \alpha_i^n(t)\} \phi_i \right] \\ &= \sum_{i=0}^{\infty} \lambda_i^\alpha \{\alpha_i(t) - \alpha_i^n(t)\} \phi_i.\end{aligned} \tag{5.8}$$

Thus, we have

$$\|A^{\alpha}[u(t) - \bar{u}_n(t)\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha} |\alpha_i(t) - \alpha_i^n(t)|^2. \tag{5.9}$$

Hence, as a consequence of Theorem 5.1, we get the required result. □

6. Example. Let $X = L^2((0,1);\mathbf{R})$. We consider the following fractional order integral equation

$$\begin{aligned} w(t,x) &= w(0,x) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} (\partial_x^2 w(\theta,x))\, d\theta \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} F(\theta, \partial_x w(\theta,x), \partial_x w(a(\theta),x))\, d\theta, \\ & w(t,0) = w(t,1) = 0,\ t \in [0,T],\ 0 < T < \infty, \end{aligned} \tag{6.1}$$

where F is a given sufficiently smooth function satisfies the Hölder condition.

We define an operator A

$$Au = -u'' \text{ with } u \in D(A) = H^2(0,1) \cap H_0^1(0,1). \tag{6.2}$$

Here clearly the operator A is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$.

We take $\alpha = 1/2$, $D(A^{1/2})$ is a Banach space with norm

$$\|x\|_{1/2} := \|A^{1/2} x\|, \quad x \in D(A^{1/2}),$$

and we denote this space by $X_{1/2}$.

We observe some properties of the operators A and $A^{1/2}$ defined by (6.2) (cf. [**5, 15**] for more details). For $u \in D(A)$ and $\lambda \in \mathbf{R}$, with $Au = -u'' = \lambda u$, we have $\langle Au, u\rangle = \langle \lambda u, u\rangle$; that is,

$$\langle -u'', u\rangle = |u'|_{L^2}^2 = \lambda |u|_{L^2}^2$$

so $\lambda > 0$. A solution u of $Au = \lambda u$ is of the form

$$u(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

and the conditions $u(0) = u(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2$, $n \in \mathbf{N}$. Thus, for each $n \in \mathbf{N}$, the corresponding solution is given by

$$u_n(x) = D\sin(\sqrt{\lambda_n}x). \tag{6.3}$$

We have $\langle u_n, u_m\rangle = 0$, for $n \neq m$ and $\langle u_n, u_n\rangle = 1$ and hence $D = \sqrt{2}$. For $u \in D(A)$, there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x) = \sum_{n\in\mathbf{N}} \alpha_n u_n(x), \quad \sum_{n\in\mathbf{N}} (\alpha_n)^2 < +\infty \text{ and } \sum_{n\in\mathbf{N}} (\lambda_n)^2(\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbf{N}} \sqrt{\lambda_n}\ \alpha_n\ u_n(x)$$

with $u \in D(A^{1/2})$; that is, $\sum_{n\in\mathbf{N}} \lambda_n(\alpha_n)^2 < +\infty$.

The semigroup $S(t)$ is given by the following expressions

$$S(t)u = \sum_{n=1}^{\infty} \exp(n^2 t)\langle u, u_m\rangle u_m,$$

where, $\{u_m\}$, $m = 1, 2, 3, 4, \dots$ is the orthogonal set of eigenfunctions of A given by equation (6.3).

Thus, the equation (6.1) can be reformulated as the following abstract equation in $X = L^2((0,1); \mathbf{R})$:

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-\theta)^{\beta-1}(-Au(\theta))\,d\theta \\ &\quad + \frac{1}{\Gamma(\beta)}\int_0^t (t-\theta)^{\beta-1} f(\theta, u(\theta), u(a(\theta)))\,d\theta, \end{aligned} \tag{6.4}$$

where, $u(t) = w(t,.)$, that is, $u(t)(x) = w(t,x)$, $t \in [0,T]$, $x \in (0,1)$, $u(0) = u_0$ and the function $f : [0,T] \times X_{1/2} \times X_{1/2} \to X$ is given by

$$f(t, u(t), u(a(t)))(x) = F(t, \partial_x w(t,x), \partial_x w(a(t),x)). \tag{6.5}$$

In particular, we can take

$$f(t, u_1, u_2) = f_o(t) + b(t)\sum_{i=1}^{2} \|u_i'\| u_i',$$

where the functions $f_0 : [0,T] \to X$ and $b : [0,T] \to R$ are Hölder continuous.

For the function a we can take

(i) $a(t) = kt$, where $t \in [0,T]$ and $0 < k \leq 1$.

(ii) $a(t) = kt^n$ for $t \in I = [0,1]$, $k \in (0,1]$ and $n \in \mathbf{N}$;

(iii) $a(t) = k \sin t$ for $t \in I = [0,(\pi/2)]$ and $k \in (0,1]$.

It may be verified that all the assumptions of Theorem (3.1) are satisfied which ensures the existence of solutions of (6.4) as well as that of (6.1). Thus, all the results of the Sections 3, 4 and 5 can be applied to the problems (6.4) and (6.5).

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