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Asymptotic analysis and error estimates for an optimal control problem with oscillating boundaries

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Abstract In this article, we consider a distributed optimal control problem associated with the Laplacian in a domain with rapidly oscillating boundary. For simplicity, we consider a rectangular region in $2d$ with oscillations on one part of the boundary. We consider two types of functionals, namely a functional involving the L^2-norm of the state variable and another one involving its H^1-norm. The homogenization of the optimality system is obtained and then we derive appropriate error estimates in both cases.

Keywords Optimal control and optimal solution · Homogenization · Oscillating boundary · Interior control · Adjoint system · Error estimates

Mathematics Subject Classification (2000) 35B27 · 35B40 · 35B37 · 49J20 · 49K20

1 Introduction

In this article, our aim is to consider a distributed optimal control problem associated with the Laplacian with a rapidly oscillating boundary. For simplicity, we consider a rectangular region in a plane ($2d$) with oscillations on one part of the boundary.

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Presently, we consider only a model problem, but it is motivated by real problems modelled by Navier–Stokes/Stokes/Viscous–Moore–Greitzer equations. There are various homogenization problems with oscillating/rough boundaries in the literature. The asymptotic analysis of the solutions of partial differential equations (PDEs) with highly oscillating data in an oscillating boundary arises in many interesting and challenging physical models.

To cite an example, boundary value problems, in particular control or controllability problems involving highly oscillating boundaries or interfaces have various applications in industrial problems such as flows with rough boundaries (rough boundaries can be modelled as oscillating boundaries), rough interface, air flow through compression systems in turbo machines such as jet engine. For example, the last one can be modelled by the Viscous–Moore–Greitzer equation derived from Scaled Navier–Stokes equations (see [9,27,28]). Here the pitch and size of the rotor–stator pair of blades in the engine provides a small parameter compared to the size of the engine which is oscillatory as well as rotating (moving). The motion of the stator and rotor blades in the compressor produces turbulent flow on a fast time scale. When the engine operates close to the optimal parameters, the flow becomes unstable. This model gives motivations to look into control problems described by PDEs of evolution type such as the heat equation or the Navier–Stokes equations. As the problem is quite complicated, we wish to begin with a sample problem of Laplacian with an oscillating boundary and the control region is away from the oscillating regions, though the aim is to consider controls acting on the moving boundaries.

For simplicity, we consider nearly a $2d$ rectangular region with oscillating part on one side of the region to be made precise later. Basically the oscillating part can be viewed as slabs of width $\epsilon > 0$ but of height $O(1)$ fixed to a rectangular region. In fact, the oscillating boundary can be of different types. Basically there are two categories, namely the oscillations with large amplitude (that is $O(1)$) and oscillations with small amplitude (that is $O(\epsilon^\alpha),\ \alpha > 0$). The small amplitude oscillation problems are easier to handle. In this article, we deal with a problem with $O(1)$ amplitude. Such regions are considered in the literature for studying homogenization of PDE problems. We mainly refer to the paper by Amirat et al. [4]. But we do not see much literature regarding optimal control/controllability problems. There are plenty of literature on the asymptotic analysis of problems with oscillating boundaries (see [1,3,6,10–12,15,16,31] and the references therein). For general homogenization, we refer to [8,13,19,33]. Regarding the homogenization of optimal control/ controllability, we cite some of the references as [20,21,29,30]. A few references are concerned with optimal control problems and derivation of optimality systems, one can refer to [2,7,11,14,23,25,26].

2 Notations and problem description

For $\epsilon > 0$, a small parameter, we consider a varying domain Ω_ϵ as in the Fig. 1 which we describe below. Let $L > 0$ and $g : \mathbb{R} \to \mathbb{R}$ be a smooth and periodic function with period L. The smoothness of g is required to establish some regularity results. The graph of g describes the bottom part of the boundary Ω_ϵ, namely

$$\Gamma_b = \{(x_1, x_2) :\ x_2 = g(x_1),\ x_1 \in [0, L] = I\}.$$

Let $0 < a < b < L$ and η_ϵ be the ϵL-periodic function defined on $[0, L]$ by periodic extension of

$$\eta_\epsilon(x_1) = \begin{cases} M' & \text{if } x_1 \in (\epsilon a, \epsilon b), \\ M & \text{if } x_1 \in [0, \epsilon L)\backslash(\epsilon a, \epsilon b), \end{cases}$$

with $M' > M > m$, where

$$m := \max\{|g(x_1)|,\ x_1 \in I\}.$$

The graph of η_ϵ provides the oscillating boundary. One would like to consider moving oscillating domains of the form $\eta(t, \frac{x}{\epsilon})$. In this paper, we do not discuss the analysis in such domains. Define the fixed part of the domain Ω^- as

$$\Omega^- = \{(x_1, x_2) :\ 0 < x_1 < L,\ g(x_1) < x_2 < M\}.$$

Let

$$\Gamma_s = \{(0, x_2) :\ g(0) \leq x_2 \leq M\} \cup \{(L, x_2) :\ g(L) \leq x_2 \leq M\}$$

which is the vertical boundary of Ω^- and

$$\Gamma_u = \{(x_1, M) :\ 0 \leq x_1 \leq L\}$$

is the upper boundary of Ω^-. We, now define Ω_ϵ as

$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 :\ 0 < x_1 < L,\ g(x_1) < x_2 < \eta_\epsilon(x_1)\}$$

which is the domain Ω^- together with small strips of width ϵ and height $(M' - M)$ attached to Ω^- (see Fig. 1). In fact, Ω_ϵ can be viewed as the bidimensional section of a more realistic solid cube in which small slabs are attached to it. The boundary $\partial\Omega_\epsilon$ can be decomposed as

$$\partial\Omega_\epsilon = \Gamma_b \cup \Gamma_s \cup \gamma_\epsilon,$$

where γ_ϵ is the contribution from the periodic strips.

Let $\Omega = \{(x_1, x_2) :\ 0 < x_1 < L,\ g(x_1) < x_2 < M'\}$ be the full domain. Let $\Omega_\epsilon^+ = \{x \in \Omega_\epsilon \mid M < x_2 < M'\}$ denote the top part of Ω_ϵ, with $\Omega_\epsilon = \Omega^- \cup ([0, L] \times \{M\}) \cup \Omega_\epsilon^+$. Let $H^m_{per}(\Omega_\epsilon)$ (respectively $L^2_{per}(\Omega_\epsilon)$) be the subspace in $H^m(\Omega_\epsilon)$ (respectively $L^2(\Omega_\epsilon)$) of functions which are L-periodic with respect to the x_1 variable. In that case we shall say that the functions are Γ_s-periodic because they take same values on both sides of Γ_s. We denote by $\tilde{\Omega}_\epsilon$ the periodic extension of Ω_ϵ in the x_1-direction.

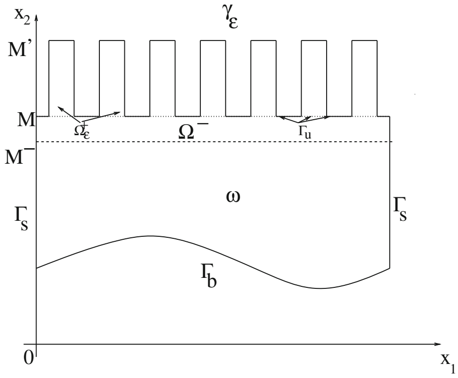

Fig. 1 Ω_ϵ

Let $\omega \subset\subset \Omega^-$ be the subdomain of Ω^- in which the control acts. Without loss of generality, we assume that

$$\omega = \{(x_1, x_2) : \ 0 < x_1 < L, \ g(x_1) < x_2 < M^-\},$$

where $M > M^- > m$.

Remark 2.1 We have taken this special domain Ω_ϵ with oscillations of order 1 on one part of the boundary to understand the behavior of optimal control problems. One can indeed consider other type of domains, but we will not discuss it here. □

2.1 Problem description

We consider the following interior optimal control problem, where the control is acting on the sub-domain ω:

$$\begin{cases} -\Delta y_\epsilon = f + \theta\chi_\omega & \text{in } \Omega_\epsilon, \\ y_\epsilon = 0 & \text{on } \gamma_\epsilon, \\ y_\epsilon = u & \text{on } \Gamma_b, \\ y_\epsilon & \text{is } \Gamma_s - \text{periodic}. \end{cases} \tag{2.1}$$

Here $\theta \in L^2(\omega)$ is the control function and χ_ω is the characteristic function of ω. Throughout the paper, we assume that

$$g \in C^1_{per}(\mathbb{R}), \quad u \in H^{1/2}_{per}(\Gamma_b) \quad \text{and } f \in L^2_{per}(\Omega). \tag{2.2}$$

It is well-known that if the Assumptions (2.2) are fulfilled and if $\theta \in L^2_{per}(\omega)$, then Eq. (2.1) admits a unique solution $y_\epsilon = y_\epsilon(\theta) \in H^1_{per}(\Omega_\epsilon)$. We denote $\tilde{y}_\epsilon$ the extension by 0 of y_ϵ to Ω, and thus $\tilde{y}_\epsilon \in H^1_{per}(\Omega)$. The solution operator

$$(f, \theta, u) \longmapsto \tilde{y}_\epsilon$$

is linear and continuous from $L^2_{per}(\Omega) \times L^2_{per}(\omega) \times H^{1/2}_{per}(\Gamma_b)$ into $H^1_{per}(\Omega)$. That is

$$\|\tilde{y}_\epsilon\|_{H^1(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|\theta\|_{L^2(\omega)} + \|u\|_{H^{1/2}(\Gamma_b)}), \tag{2.3}$$

where $C > 0$ is independent of ϵ. Let us consider the following two cost functionals

$$J_{1,\epsilon}(y_\epsilon, \theta) = \frac{1}{2}\int\limits_{\Omega_\epsilon} (y_\epsilon - y_d)^2 + \frac{\beta}{2}\int\limits_{\omega} \theta^2$$

and

$$J_{2,\epsilon}(y_\epsilon, \theta) = \frac{1}{2}\int\limits_{\Omega_\epsilon} |\nabla y_\epsilon - \nabla y_d|^2 + \frac{\beta}{2}\int\limits_{\omega} \theta^2,$$

with $\beta > 0$, y_d is a given desired state belonging to $L^2_{per}(\Omega)$ for $J_{1,\epsilon}$ and to $H^1_{per}(\Omega)$ for $J_{2,\epsilon}$. Since we are going to see that y_ϵ is of order ϵ, i.e. $O(\epsilon)$ in the upper part Ω^+_ϵ, it is reasonable to take supp $y_d \subset \overline{\Omega^-}$. This assumption is assumed throughout the paper.

Associated with these functionals we consider the two optimal control problems

$$\inf\{J_{1,\epsilon}(y_\epsilon, \theta) \mid \theta \in L^2_{per}(\omega),\ (y_\epsilon, \theta) \text{ obeys } (2.1)\}, \tag{$P_{1,\epsilon}$}$$

and

$$\inf\{J_{2,\epsilon}(y_\epsilon, \theta) \mid \theta \in L^2_{per}(\omega),\ (y_\epsilon, \theta) \text{ obeys } (2.1)\}. \tag{$P_{2,\epsilon}$}$$

Remark 2.2 We could be interested in considering much more general elliptic operators and/or general cost functionals. We shall discuss these issues in a later paper. □

For each $\epsilon > 0$, the minimization problem $(P_{1,\epsilon})$ is quite standard and it admits a unique solution $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ (see [7,24,32]). We call $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ the optimal solution, where $\bar{\theta}_\epsilon$ is the optimal control and $\bar{y}_\epsilon$ the optimal state. Further, it can be characterized using the adjoint state (co-state) $\bar{z}_\epsilon$, where $\bar{z}_\epsilon$ solves the adjoint problem

$$\begin{cases} -\Delta \bar{z}_\epsilon = \bar{y}_\epsilon - y_d & \text{in } \Omega_\epsilon, \\ \bar{z}_\epsilon = 0 & \text{on } \gamma_\epsilon \cup \Gamma_b, \\ \bar{z}_\epsilon \in H^1_{per}(\Omega_\epsilon). \end{cases} \tag{2.4}$$

The following theorem is well established.

Theorem 2.3 *Let $f \in L^2(\Omega)$ and $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ be the optimal solution of $(P_{1,\epsilon})$. Let $\bar{z}_\epsilon \in H^1_{per}(\Omega_\epsilon)$ solves (2.4), then the optimal control is given by*

$$\bar{\theta}_\epsilon = -\frac{1}{\beta}\bar{z}_\epsilon \chi_\omega.$$

Conversely, assume that a pair $(\hat{y}_\epsilon, \hat{z}_\epsilon) \in H^1_{per}(\Omega_\epsilon) \times H^1_{per}(\Omega_\epsilon)$ *solves the optimality system*

$$\begin{cases} -\Delta \hat{y}_\epsilon = f - \frac{1}{\beta}\hat{z}_\epsilon \chi_\omega & in\ \Omega_\epsilon,\ \hat{y}_\epsilon = 0 \quad on\ \gamma_\epsilon,\ \ \hat{y}_\epsilon = u \quad on\ \Gamma_b, \\ -\Delta \hat{z}_\epsilon = \hat{y}_\epsilon - y_d & in\ \Omega_\epsilon,\ \hat{z}_\epsilon = 0 \quad on\ \gamma_\epsilon \cup \Gamma_b. \end{cases} \tag{2.5}$$

Then, the pair $(\hat{y}_\epsilon, -\frac{1}{\beta}\hat{z}_\epsilon \chi_\omega)$ *is the optimal solution to* $(P_{1,\epsilon})$.

The first aim of this article is to study the asymptotic behavior of $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ as $\epsilon \to 0$ and obtain the limit equations.

Using the convergence of the optimality system, we in fact show that the minimization problem will converge to a suitable minimization problem. This is done in Sect. 3. The other important aspect of the article is to prove some corrector estimates. We show some H^1- estimates in terms of the L^2- estimates using certain test functions. In fact these test functions were used earlier by other authors for studying homogenization problems (uncontrolled) in such domains, see [4]. This is the content of Sect. 4. Test functions are also introduced in the same section. Finally, we study the analysis of Dirichlet cost functional in Sects. 5 and 6.

3 Homogenization Theorem

3.1 Estimates

Assume that $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ is the optimal solution of $(P_{1,\epsilon})$. With $\theta = 0$, let $y_\epsilon(0)$ be the solution of the problem (2.1), then using the classical weak formulation or (2.3), we get

$$\|y_\epsilon(0)\|_{H^1(\Omega_\epsilon)} \le C. \tag{3.1}$$

Since $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ is a minimal solution, we get

$$\int_{\Omega_\epsilon} (\bar{y}_\epsilon - y_d)^2 + \frac{\beta}{2}\int_\omega \bar{\theta}_\epsilon^2 \le \int_{\Omega_\epsilon} (y_\epsilon(0) - y_d)^2 \le C. \tag{3.2}$$

Thus we have

$$\|\bar{\theta}_\epsilon\|_{L^2(\omega)} \le C, \tag{3.3}$$

$$\|\tilde{\bar{y}}_\epsilon\|_{L^2(\Omega)} \le C, \tag{3.4}$$

where $\tilde{\bar{y}}_\epsilon$ is the extension by 0 to Ω.

Using the weak formulation of the adjoint system (2.4), it follows that

$$\|\bar{z}_\epsilon\|_{H^1(\Omega_\epsilon)} \le C, \tag{3.5}$$

where $C > 0$ is independent of ϵ. Then we have the following theorem:

Theorem 3.1 *Let* $(\bar{y}_\epsilon,\ \bar{\theta}_\epsilon)$ *be the optimal solution of the problem* $(P_{1,\epsilon})$*, then* $\bar{\theta}_\epsilon \in H^1(\omega)$ *and there exists a constant* C *independent of* ϵ *such that*

$$\|\bar{\theta}_\epsilon\|_{H^1(\omega)} \leq C \tag{3.6}$$

and

$$\|\tilde{\bar{y}}_\epsilon\|_{H^1(\Omega)} \leq C. \tag{3.7}$$

Proof The fact $\bar{\theta}_\epsilon \in H^1(\omega)$ and estimate (3.6) follow from the characterization $\bar{\theta}_\epsilon = -\frac{1}{\beta}\bar{z}_\epsilon \chi_\omega$. Estimate (3.7) is then a consequence of the first equation in (2.5). □

Thus, we have, along a subsequence

$$\begin{cases} \tilde{\bar{y}}_\epsilon \rightharpoonup \bar{y}_0 & \text{weakly in } H^1(\Omega) \\ \tilde{\bar{z}}_\epsilon \rightharpoonup \bar{z}_0 & \text{weakly in } H^1(\Omega) \\ \bar{\theta}_\epsilon \rightharpoonup \bar{\theta}_0 & \text{weakly in } H^1(\omega), \end{cases} \tag{3.8}$$

for some $(\bar{y}_0,\ \bar{z}_0,\ \bar{\theta}_0) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\omega)$. In fact, we shall get the strong convergence in $H^1(\Omega)$ for $\tilde{\bar{z}}_\epsilon$ and hence for the control $\bar{\theta}_\epsilon$ in $H^1(\omega)$.

Introduce the following problem in Ω^-. Given $\theta \in L^2(\omega)$, let $y \in H^1_{per}(\Omega^-)$ be the solution to

$$\begin{cases} -\Delta y = f + \theta\chi_\omega & \text{in } \Omega^-, \\ y = 0 & \text{on } \Gamma_u, \\ y = u & \text{on } \Gamma_b. \end{cases} \tag{3.9}$$

The limit cost functional J_1 is defined by

$$J_1(y, \theta) = \frac{1}{2}\int_{\Omega^-} (y - y_d)^2 + \frac{\beta}{2}\int_\omega \theta^2. \tag{3.10}$$

Let $(\bar{y},\ \bar{\theta})$ be the solution to the minimization problem

$$\inf\{J_1(y, \theta) \mid \theta \in L^2(\omega)\}, (y, \theta) \text{ obeys } (3.9)\}, \tag{P_1}$$

Then, $\bar{\theta}$ can be characterized by $\bar{\theta} = -\frac{1}{\beta}\bar{z}$, where $\bar{z} \in H^1_{per}(\Omega^-)$ solves

$$\begin{cases} -\Delta \bar{z} = \bar{y} - y_d & \text{in } \Omega^-, \\ \bar{z} = 0 & \text{on } \Gamma_b \cup \Gamma_u. \end{cases} \tag{3.11}$$

Theorem 3.2 *Let* $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ *and* $(\bar{y}, \bar{\theta})$ *be, respectively, the optimal solution of* $(P_{1,\epsilon})$ *and of* (P_1)*. Then*

$$\begin{array}{ll} \bar{\theta}_\epsilon \to \bar{\theta} & \text{strongly in } H^1(\omega), \\ \tilde{\bar{y}}_\epsilon \to \tilde{\bar{y}} & \text{strongly in } H^1(\Omega), \end{array}$$

where

$$\tilde{\bar{y}} = \begin{cases} \bar{y} & \text{in } \Omega^-, \\ 0 & \text{in } \Omega^+. \end{cases}$$

Moreover

$$J_{1,\epsilon}(\bar{y}_\epsilon, \bar{\theta}_\epsilon) \longrightarrow J_1(\bar{y}, \bar{\theta}) \quad \text{when } \epsilon \longrightarrow 0.$$

Proof Restricting the Eqs. (2.1) and (2.4) to Ω^-, and using the convergence (3.8), it is easy to pass to the limit in Ω^- to get

$$\begin{cases} -\Delta \bar{y}_0 = f - \frac{1}{\beta}\bar{z}_0\chi_\omega & \text{in } \Omega^-, \ \bar{y}_0 = u \quad \text{on} \Gamma_b, \\ -\Delta \bar{z}_0 = \bar{y}_0 - y_d & \text{in } \Omega^-, \ \bar{z}_0 = 0 \quad \text{on } \Gamma_b, \\ \bar{y}_0, \bar{z}_0 \in H^1_{per}(\Omega^-). \end{cases} \tag{3.12}$$

Recovering the boundary condition $\bar{y}_0 = \bar{z}_0 = 0$ on Γ_u is quite easy. Let $\chi_{\Omega^+_\epsilon}$ be the characteristic function of $\Omega^+_\epsilon \subset \Omega^+$, then by standard result, we have

$$\chi_{\Omega^+_\epsilon} \rightharpoonup A \quad \text{weakly}^* \text{ in } L^\infty(\Omega^+) \quad \text{where } A = \frac{b-a}{L}.$$

Now passing to the limit in the equation $\tilde{\bar{y}}_\epsilon = \tilde{\bar{y}}_\epsilon \chi_{\Omega^+_\epsilon}$, we see that $\bar{y} = 0$ in Ω^+ since $0 < A < 1$. Similarly $\bar{z}_0 = 0$ in Ω^+. Thus (3.12), together with $\bar{y}_0 = \bar{z}_0 = 0$ on Γ_u, is the optimality system corresponding to the minimization problem (P_1). According to Theorem 2.3, its optimal solution is given by $(\bar{y}_0, -\frac{1}{\beta}\bar{z}_0\chi_\omega)$.
Thus, we have

$$\bar{y} = \bar{y}_0, \quad \bar{z} = \bar{z}_0 \quad \text{and} \quad \bar{\theta} = \bar{\theta}_0 = -\frac{1}{\beta}\bar{z}_0\chi_\omega.$$

Moreover, using the strong convergence $\tilde{\bar{y}}_\epsilon \to \bar{y}$ in $L^2(\Omega^+)$, we can verify that

$$\lim_{\epsilon \to 0} J_{1,\epsilon}(\bar{y}_\epsilon, \bar{\theta}_\epsilon) = J_1(\bar{y}, \bar{\theta}).$$

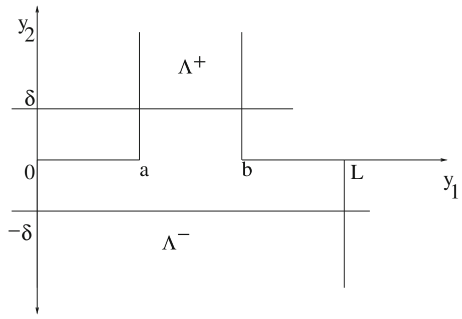

Fig. 2 $\Lambda^{\pm}$

Further,

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} |\nabla z_\epsilon|^2 = \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} (\bar{y}_\epsilon - y_d) z_\epsilon$$
$$= \int_{\Omega^-} (\bar{y} - y_d)\bar{z} = \int_{\Omega^-} |\nabla \bar{z}|^2 .$$

Since $\tilde{\bar{z}}$ is 0 in Ω^+ and $\bar{z} \in H^1(\Omega)$, we get

$$\|\nabla \tilde{\bar{z}}_\epsilon\|^2_{L^2(\Omega)} \longrightarrow \|\nabla \tilde{\bar{z}}\|^2_{L^2(\Omega)}.$$

Thus $\tilde{\bar{z}}_\epsilon \to \tilde{\bar{z}}$ strongly in $H^1(\Omega)$ and in particular $\bar{\theta}_\epsilon = \bar{z}_\epsilon \chi_\omega \to \bar{\theta} = \bar{z}\chi_\omega$ strongly in $H^1(\omega)$. Similarly, we get the strong convergence of $\tilde{\bar{y}}_\epsilon$ and hence the theorem. □

4 Error (Corrector) estimates

The aim of this section is to derive certain error estimates. Indeed, we admit that we are not able to do a complete error analysis, but following the work in [4], we get H^1-estimates of the control (or adjoint state) away from the oscillating boundary in terms of the L^2-estimates of the optimal state. This, in turn, will produce H^1-estimates of the state in terms of L^2-estimates. We need to recall the relevant test functions (see [3–5]).

4.1 Test functions

Let $\Lambda^{\pm}$ be the unbounded domains defined by $\Lambda^+ = (a, b) \times (0, \infty)$ and $\Lambda^- = (0, L) \times (-\infty, 0)$ which in some sense has to be seen as $\frac{1}{\epsilon}$ scaling of $(\epsilon a, \epsilon b) \times (M, M')$ and Ω^- respectively, and then extended up to infinity (see Fig. 2). We denote the variables in the cell domains Λ^+ and Λ^- as $\xi = (\xi_1,\ \xi_2)$. Define the test functions $\psi^\pm$ as

$$\psi^+ \in H^1(\Lambda^+), \quad \psi^- \in H^1_{\text{loc, per}}(\Lambda^-), \quad \nabla\psi^- \in L^2(\Lambda^-), \tag{4.1}$$

satisfying

$$\begin{cases} \Delta\psi^\pm = 0 & \text{in } \Lambda^\pm, \\ \psi^- = 0 & \text{on } (0,a)\cup(b,L)\times\{0\}, \\ \psi^+ = 0 & \text{on } \partial\Lambda^+\backslash(a,b)\times\{0\}, \\ \psi^+ = \psi^- & \text{on } (a,b)\times\{0\} \\ \dfrac{\partial\psi^+}{\partial\xi_2} = \dfrac{\partial\psi^-}{\partial\xi_2} + 1 & \text{on } (a,b)\times\{0\}. \end{cases} \tag{4.2}$$

For $\delta > 0$, define the average of ψ^- along the horizontal line $\xi_2 = -\delta$ as

$$\beta_1 = \beta_1(\delta) = \frac{1}{L}\int_0^L \psi^-(\xi_1, -\delta)dy_1. \tag{4.3}$$

For following result, the reader can refer to [2,3,5,22,24].

Proposition 4.1 *The problem* (4.1) *and* (4.2) *admits a unique solution. Further,*

1. $\beta_1(\delta)$ *is independent of* δ *and we denote it by* β_1.
2. *For any* $\alpha \in \mathbb{N}\times\mathbb{N}$, $\delta > 0$, *there are positive constants* $C,\ C_{\alpha,\delta}$ *such that*

$$|\ \partial^\alpha\psi^+(\xi)\ | \le C_{\alpha,\delta}\, e^{-C\xi_2}, \qquad \forall\ \xi = (\xi_1,\ \xi_2) \in (a,b)\times(\delta,\infty), \tag{4.4}$$

and

$$|\ \partial^\alpha(\psi^-(\xi) - \beta_1)\ | \le C_{\alpha,\delta}\, e^{C\xi_2}, \qquad \forall\ \xi = (\xi_1,\ \xi_2) \in (0,L)\times(-\infty,-\delta). \tag{4.5}$$

□

Corollary 4.2 $\psi^- - \beta_1 \in H^1_{\text{per}}(\Lambda^-)$. □

This is trivial because, by (4.5), we have

$$\begin{aligned}\|\psi^- - \beta_1\|^2_{L^2(\Lambda^-)} &= \|\psi^- - \beta_1\|^2_{L^2((0,L)\times(-1,0))} + \|\psi^- - \beta_1\|^2_{L^2((0,L)\times(-\infty,-1))} \\ &\le C_1 + C_2\int_{-\infty}^{-1} e^{c\xi_2}d\xi_2 \le C.\end{aligned}$$

It is already given that $\nabla(\psi^- - \beta_1) = \nabla\psi^- \in L^2(\Lambda^-)$.

Extend ψ^+ by 0 to $(0, L) \times (0, \infty)$ and then extend periodically to $\mathbb{R}^2_+$ which is again denoted by ψ^+. Similarly the periodic extension of ψ^- to $\mathbb{R}^2_-$ is also denoted by ψ^-. These test functions are used to obtain corrector results. It is not, however, possible to obtain exact corrector results as in an uncontrolled problem since we have to work with the optimality system with varying right hand side.

Corollary 4.3 *The test functions $\psi^\pm$ defined by (4.1) and (4.2) satisfy*

$$\int_{\Omega_\epsilon^+} \left|\psi^+\left(\frac{x_1}{\epsilon}, \frac{x_2 - M}{\epsilon}\right)\right|^2 dx \leq C\epsilon,$$

$$\int_{\Omega^-} \left|\psi^-\left(\frac{x_1}{\epsilon}, \frac{x_2 - M}{\epsilon}\right) - \beta_1\right|^2 dx \leq C\epsilon,$$

$$\int_{\Omega_\epsilon \setminus B_\epsilon} \left|\nabla\left(\psi\left(\frac{x_1}{\epsilon}, \frac{x_2 - M}{\epsilon}\right)\right)\right|^2 dx \leq Ce^{-c/\epsilon},$$

where $B_\epsilon = (0, L) \times (M - \epsilon, M + \epsilon)$ is a strip of width 2ϵ around the upper part Γ_u, C is a positive constant independent of ε and ψ is the function defined by

$$\psi = \begin{cases} \psi^- & \text{in } \Lambda^-, \\ \psi^+ & \text{in } \Lambda^+. \end{cases}$$

Proof Since there are $O(\epsilon^{-1})$ ϵ-cells like $\epsilon\Lambda^+$, by periodicity, we get

$$\begin{aligned}\int_{\Omega_\epsilon^+} \left|\psi^+\left(\frac{x_1}{\epsilon}, \frac{x_2 - M}{\epsilon}\right)\right|^2 dx &\approx \epsilon^{-1} \int_{\epsilon a}^{\epsilon b} \int_M^{M'} \left|\psi^+\left(\frac{x_1}{\epsilon}, \frac{x_2 - M}{\epsilon}\right)\right|^2 dx \\ &\leq C\epsilon \int_a^b \int_0^{\frac{M'-M}{\epsilon}} \left|\psi^+(y)\right|^2 dy \\ &\leq C\epsilon \|\psi^+\|^2_{L^2(\Lambda^+)} \leq C\epsilon.\end{aligned}$$

Similarly, we get the second estimate. Again

$$\begin{aligned}&\int_{\Omega_\epsilon \setminus B_\epsilon} \left|\nabla\left(\psi\left(\frac{x_1}{\epsilon}, \frac{x_2 - M}{\epsilon}\right)\right)\right|^2 dx \\ &\leq C\left[\int_{\Lambda^+ \setminus (a,b)\times(1,\infty)} \left|\nabla\psi^+(\xi)\right|^2 d\xi + \int_{\Lambda^- \setminus (0,L)\times(-1,0)} \left|\nabla\psi^-(\xi)\right|^2 d\xi\right] \leq Ce^{-c/\epsilon},\end{aligned}$$

by (4.4) and (4.5). ◻

4.2 Estimates on co-state and control

To derive corrector estimates, one needs to have more regularity on the solution of the corresponding homogenized problem. This can be obtained with the additional regularity assumptions

$$f \in H^2_{per}(\Omega^-) \cap L^2_{per}(\Omega), \;\; g \in H^6_{per}(0, L) \quad \text{and} \quad y_d \in H^4_{per}(\Omega^-). \tag{4.6}$$

Remark 4.4 To get error estimates on the optimal solution $\bar{y}$, one may require $f \in H^4_{per}(\Omega^-)$, but for the co-state, it is enough to have (4.6). □

Recall that the co-state $\bar{z} \in H^1_{per}(\Omega^-)$ is the solution to

$$\begin{cases} -\Delta \bar{z} = \bar{y} - y_d & \text{in } \Omega^-, \\ \bar{z} = 0 & \text{on } \Gamma_u \cup \Gamma_b, \end{cases} \tag{4.7}$$

where $\bar{y}$ is the solution of equation (3.9) with $\theta = \bar{\theta} = -\frac{1}{\beta}\bar{z}$. Since the control is located in ω, we cannot deduce from the optimality system (3.12) that $\bar{y} \in H^4_{per}(\Omega^-)$ and $\bar{z} \in H^6_{per}(\Omega^-)$. However, using for example a truncation argument, since the dimension is $n = 2$, by the standard regularity (see [17,18]) we can show that

$$\bar{y} \in H^4_{per}(R) \subset C^2(\overline{R}) \quad \text{and} \quad \bar{z} \in H^6_{per}(R) \subset C^4(\overline{R}), \tag{4.8}$$

where R is the band

$$R = \{(x_1, x_2) \mid 0 < x_1 < L, \;\; \frac{M + M^-}{2} < x_2 < M\}.$$

The error estimate is based on the flux of $\bar{z}$, namely $\dfrac{\partial \bar{z}}{\partial x_2}$, across the upper boundary of Ω^-. Introduce $\vartheta \in H^1_{per}(\Omega^-)$, the solution of

$$\begin{cases} \Delta \vartheta = 0 & \text{in } \Omega^-, \\ \vartheta = 0 & \text{on } \Gamma_b, \\ \vartheta = \beta_1 \frac{\partial \bar{z}}{\partial x_2} & \text{on } \Gamma_u. \end{cases} \tag{4.9}$$

Denote by $\tilde{\vartheta}$ the extension by 0 of ϑ to Ω. Since $\dfrac{\partial \bar{z}}{\partial x_2}\Big|_{\Gamma_u} \in H^{9/2}(\Gamma_u)$, we get

$$\vartheta \in H^5_{per}(\Omega^-) \subset C^3(\overline{\Omega^-}). \tag{4.10}$$

Since $\tilde{\vartheta}$ does not belong to $H^1(\Omega)$, we are using it only in the domain $\Omega_\epsilon \backslash B_\epsilon$ (where B_ϵ is defined in Corollary 4.3) and we have the following theorem.

Theorem 4.5 *Let $\bar{z}_\epsilon$ and $\bar{z}$ be respectively the solutions of the inhomogenized and homogenized co-state equation defined in (2.4) and (4.7), let ϑ be the solution of (4.9) and $\bar{y}_\epsilon$ be the optimal state. Then*

$$\|\bar{z}_\epsilon - \tilde{\bar{z}} - \epsilon\tilde{\vartheta}\|_{H^1(\Omega_\epsilon \backslash B_\epsilon)} \leq C\left[\|\bar{y}_\epsilon - \bar{y}\|_{L^2(\Omega^-)} + \epsilon^{3/2}\right], \tag{4.11}$$

where $\tilde{\bar{z}}$ the extension by 0 of $\bar{z}$ to Ω. □

Remark 4.6 We would like to obtain estimate without the term $\|\bar{y}_\epsilon - \tilde{\bar{y}}\|_{L^2(\Omega_\epsilon)}$ on the RHS. We believe so, but a proof is not yet worked out. □

The proof of Theorem 4.5 is very similar to the one of Theorem 4.1 in [4]. However our result is different. It is why we rewrite only the first part of the proof where the additional term $\|\bar{y}_\epsilon - \bar{y}\|_{L^2(\Omega^-)}$ appears. For the other parts, we shall refer to [4].

To prove Theorem 4.5, as in [4], we need another class of test functions for the proof. Let $\vartheta_\epsilon^+ \in H^1(\Omega^+)$ and $\vartheta_\epsilon^- \in H^1_{per}(\Omega^-)$ satisfy

$$\begin{cases} \Delta\vartheta_\epsilon^+ = 0 & \text{in } \Omega_\epsilon^+, \\ \Delta\vartheta_\epsilon^- = 0 & \text{in } \Omega^-, \\ \vartheta_\epsilon^+ = 0 & \text{on } \gamma_\epsilon \backslash (\gamma_\epsilon \cap \Gamma_u), \\ \vartheta_\epsilon^- = \beta_1 \dfrac{\partial \bar{z}}{\partial x_2} & \text{on } \gamma_\epsilon \cap \Gamma_u, \\ \vartheta_\epsilon^+ = \vartheta_\epsilon^- - \beta_1 \dfrac{\partial \bar{z}}{\partial x_2} & \text{on } \Gamma_u \backslash (\gamma_\epsilon \cap \Gamma_u), \\ \dfrac{\partial \vartheta_\epsilon^+}{\partial x_2} = \dfrac{\partial \vartheta_\epsilon^-}{\partial x_2} & \text{on } \Gamma_u \backslash (\gamma_\epsilon \cap \Gamma_u). \end{cases} \tag{4.12}$$

Denote $\psi_\epsilon^+ = \psi^+\left(\dfrac{x_1}{\epsilon}, \dfrac{x_2 - M}{\epsilon}\right)$, $\psi_\epsilon^- = \psi^-\left(\dfrac{x_1}{\epsilon}, \dfrac{x_2 - M}{\epsilon}\right)$. Then, we can write

$$\bar{z}_\epsilon - \tilde{\bar{z}} - \epsilon\tilde{\vartheta} = \tau_\epsilon + \epsilon\rho_\epsilon + \pi_\epsilon \quad \text{in} \quad \Omega_\epsilon, \tag{4.13}$$

where

$$\tau_\epsilon = \begin{cases} \tau_\epsilon^+ = \bar{z}_\epsilon - \tilde{\bar{z}} - \epsilon\vartheta_\epsilon^+ - \epsilon\dfrac{\partial\tilde{\bar{z}}}{\partial x_2}(x_1, M)\psi_\epsilon^+ & \text{in } \Omega_\epsilon^+, \\ \tau_\epsilon^- = \bar{z}_\epsilon - \tilde{\bar{z}} - \epsilon\vartheta_\epsilon^- - \epsilon\dfrac{\partial\tilde{\bar{z}}}{\partial x_2}(x_1, M)(\psi_\epsilon^- - \beta_1) & \text{in } \Omega^-, \end{cases} \tag{4.14}$$

$$\rho_\epsilon = \begin{cases} \rho_\epsilon^+ = \vartheta_\epsilon^+ - \epsilon\dfrac{\partial\vartheta}{\partial x_2}(x_1, M)\,\psi_\epsilon^+ & \text{in } \Omega_\epsilon^+, \\ \rho_\epsilon^- = \vartheta_\epsilon^- - \vartheta - \epsilon\dfrac{\partial\vartheta}{\partial x_2}(x_1, M)\psi_\epsilon^- & \text{in } \Omega^-, \end{cases} \tag{4.15}$$

and

$$\pi_\epsilon = \begin{cases} \pi_\epsilon^+ = \epsilon \dfrac{\partial \tilde{\tilde{z}}}{\partial x_2}(x_1, M)\,\psi_\epsilon^+ + \epsilon^2 \dfrac{\partial \vartheta}{\partial x_2}(x_1, M)\,\psi_\epsilon^+, \\ \pi_\epsilon^- = \epsilon \dfrac{\partial \tilde{\tilde{z}}}{\partial x_2}(x_1, M)\left(\psi_\epsilon^- - \beta_1\right) + \epsilon^2 \dfrac{\partial \vartheta}{\partial x_2}(x_1, M)\,\psi_\epsilon^-. \end{cases} \tag{4.16}$$

We need to estimate $\tau_\epsilon,\ \rho_\epsilon$ and π_ϵ.

Proposition 4.7 *Assume the regularity conditions* (4.6). *Then* τ_ϵ *and* ρ_ϵ, *respectively defined by* (4.14) *and* (4.15), *satisfy*

$$\|\tau_\epsilon\|_{H^1(\Omega_\epsilon)} \le C\left[\|\bar{y}_\epsilon - \bar{y}\|_{L^2(\Omega^-)} + \epsilon^{3/2}\right]$$

and

$$\|\rho_\epsilon\|_{H^1(\Omega_\epsilon)} \le C\epsilon.$$

Proof *Estimates on* τ_ϵ: – Since $\tau_\epsilon^+ \in H^1\left(\Omega_\epsilon^+\right),\ \tau_\epsilon^- \in H^1\left(\Omega^-\right)$ and $\tau_\epsilon^+ = \tau_\epsilon^-$ at the interface $\bar{\Omega}_\epsilon^+ \cap \bar{\Omega}^-$, we get $\tau_\epsilon \in H^1_{per}(\Omega_\epsilon)$. Actually, it is easy to see that $\dfrac{\partial \tau_\epsilon^+}{\partial x_2} = \dfrac{\partial \tau_\epsilon^-}{\partial x_2}$ on $\bar{\Omega}_\epsilon^+ \cap \bar{\Omega}^-$. We compute $\Delta\tau_\epsilon$ in Ω_ϵ. In Ω_ϵ^+, we have

$$\Delta\tau_\epsilon = \Delta\tau_\epsilon^+ = (\bar{y}_\epsilon - y_d) - \epsilon \frac{\partial^3 \tilde{\tilde{z}}}{\partial x_1^2 \partial x_2}(x_1, M)\psi_\epsilon^+ - 2\epsilon \frac{\partial^2 \tilde{\tilde{z}}}{\partial x_1 \partial x_2}(x_1, M)\frac{\partial \psi_\epsilon^+}{\partial x_1},$$

and in Ω_ϵ^- we get

$$\Delta\tau_\epsilon = \Delta\tau_\epsilon^- = (\bar{y}_\epsilon - \bar{y}) - \epsilon \frac{\partial^3 \tilde{\tilde{z}}}{\partial x_1^2 \partial x_2}(x_1, M)\left(\psi_\epsilon^- - \beta_1\right) - 2\epsilon \frac{\partial^2 \tilde{\tilde{z}}}{\partial x_1 \partial x_2}(x_1, M)\frac{\partial \psi_\epsilon^-}{\partial x_1}.$$

Further,

$$\tau_\epsilon\,|_{\Gamma_b} = -\epsilon \frac{\partial \tilde{\tilde{z}}}{\partial x_2}(x_1, M)\left(\psi^-\left(\frac{x_1}{\epsilon}, \frac{g(x_1) - M}{\epsilon}\right) - \beta_1\right),$$

$$\tau_\epsilon\,|_{\gamma_\epsilon \cap (0,L)\times M'} = -\epsilon \frac{\partial \tilde{\tilde{z}}}{\partial x_2}(x_1, M)\,\psi^+\left(\frac{x_1}{\epsilon}, \frac{M' - M}{\epsilon}\right).$$

We need test functions which vanish on these boundaries to use in the weak formulations. Choose $\phi_1, \phi_2 \in C^2(\mathbb{R}; [0, 1])$,

$$\phi_1(s) = \begin{cases} 0 & \text{if } s > \frac{m+M}{2}, \\ 1 & \text{if } s < \frac{3m+M}{4}, \end{cases}$$

$$\phi_2(s) = \begin{cases} 1 & \text{if } s > \frac{M+M'}{2}, \\ 0 & \text{if } s < \frac{3M+M'}{4}. \end{cases}$$

In Ω let us define

$$\tau_\epsilon^1(x_1, x_2) = -\epsilon \frac{\partial \tilde{\tilde{z}}}{\partial x_2}(x_1, M) \left(\psi^- \left(\frac{x_1}{\epsilon}, \frac{g(x_1) - M}{\epsilon} \right) - \beta_1 \right) \phi_1(x_2)$$

and

$$\tau_\epsilon^2(x_1, x_2) = -\epsilon \frac{\partial \tilde{\tilde{z}}}{\partial x_2}(x_1, M) \psi^+ \left(\frac{x_1}{\epsilon}, \frac{M' - M}{\epsilon} \right) \phi_2(x_2).$$

Then clearly, $\tau_\epsilon - \tau_\epsilon^1 - \tau_\epsilon^2 \in H^1_{per}(\Omega_\epsilon)$ with $\tau_\epsilon - \tau_\epsilon^1 - \tau_\epsilon^2 = 0$ on the boundary $\gamma_\epsilon \cup \Gamma_b$. Hence, we can use it as a test function to get

$$\begin{aligned} \int_{\Omega_\epsilon} |\nabla \tau_\epsilon|^2 dx &= \int_{\Omega_\epsilon} \nabla \tau_\epsilon \cdot \nabla \left(\tau_\epsilon - \tau_\epsilon^1 - \tau_\epsilon^2 \right) dx \\ &\quad + \int_{\Omega_\epsilon} \nabla \tau_\epsilon \cdot \nabla \tau_\epsilon^1 dx + \int_{\Omega_\epsilon} \nabla \tau_\epsilon \cdot \nabla \tau_\epsilon^2 dx \\ &= - \int_{\Omega_\epsilon^+} \Delta \tau_\epsilon \left(\tau_\epsilon - \tau_\epsilon^2 \right) dx - \int_{\Omega^-} \Delta \tau_\epsilon \left(\tau_\epsilon - \tau_\epsilon^1 \right) dx \\ &\quad + \int_{\Omega_\epsilon^+} \nabla \tau_\epsilon \cdot \nabla \tau_\epsilon^2 dx + \int_{\Omega_\epsilon^-} \nabla \tau_\epsilon \cdot \nabla \tau_\epsilon^1 dx. \end{aligned}$$

Using Proposition 4.1, from the definition of τ_ϵ, for $i = 1, 2$, we get

$$\begin{cases} \left| \dfrac{\partial \tau_\epsilon^1}{\partial x_i} \right| \le C\, e^{-c/\epsilon} \quad \text{in } \Omega^- \\ \text{and} \\ \left| \dfrac{\partial \tau_\epsilon^2}{\partial x_i} \right| \le C\, e^{-c/\epsilon} \quad \text{in } \Omega_\epsilon^+. \end{cases} \tag{4.17}$$

Now using the expression for $\Delta\tau_\epsilon$ and the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
&\|\nabla\tau_\epsilon\|^2_{L^2(\Omega_\epsilon)} \\
&\quad \le C\Big[\epsilon\|\psi_\epsilon^+\|_{L^2(\Omega_\epsilon^+)}\|\tau_\epsilon-\tau_\epsilon^2\|_{L^2(\Omega_\epsilon^+)}+\epsilon\|\psi_\epsilon^- - \beta_1\|_{L^2(\Omega^-)}\|\tau_\epsilon-\tau_\epsilon^1\|_{L^2(\Omega^-)} \\
&\quad + \left|\int\limits_{\Omega_\epsilon^+}(y_\epsilon-y_d)(\tau_\epsilon-\tau_\epsilon^2)+\int\limits_{\Omega^-}(\bar{y}_\epsilon-\bar{y})(\tau_\epsilon-\tau_\epsilon^1)\right| \\
&\quad + \|\nabla\tau_\epsilon\|_{L^2(\Omega_\epsilon)}\left(\|\nabla\tau_\epsilon^2\|_{L^2(\Omega_\epsilon^+)}+\|\nabla\tau_\epsilon^1\|_{L^2(\Omega^-)}\right). \\
&\quad + \epsilon\|\psi_\epsilon^+\|_{L^2(\Omega_\epsilon^+)}\|\nabla(\tau_\epsilon-\tau_\epsilon^2)\|_{L^2(\Omega_\epsilon^+)}+\epsilon\|\psi_\epsilon^- - \beta_1\|_{L^2(\Omega^-)}\|\nabla(\tau_\epsilon-\tau_\epsilon^1)\|_{L^2(\Omega^-)}\Big].
\end{aligned}
$$

Applying the estimates of $\psi_\epsilon^+,\ \psi_\epsilon^- - \beta_1$ in Proposition 4.1, the inequalities in (4.17) and Poincaré inequality, we get

$$
\begin{aligned}
\|\nabla\tau_\epsilon\|^2_{L^2(\Omega_\epsilon)} \le C\Big[&(\epsilon^{3/2})\|\nabla\tau_\epsilon\|_{L^2(\Omega_\epsilon)}+e^{-c/\epsilon}+\|y_\epsilon-y_d\|_{L^2(\Omega_\epsilon^+)}\|\tau_\epsilon-\tau_\epsilon^2\|_{L^2(\Omega_\epsilon^+)} \\
&+\|\bar{y}_\epsilon-\bar{y}\|_{L^2(\Omega^-)}\|\nabla\tau_\epsilon\|_{L^2(\Omega^-)}\Big].
\end{aligned}
$$

Note that Ω_ϵ^+ consists of ϵ-strips of length 1 and applying Poincaré inequality in each strip, summing up to have

$$\|\tau_\epsilon-\tau_\epsilon^2\|_{L^2(\Omega_\epsilon^+)} \le C\epsilon\|\nabla(\tau_\epsilon-\tau_\epsilon^2)\|_{L^2(\Omega_\epsilon^+)}.$$

Considering y_ϵ in each strip and using Poincaré inequality, we obtain

$$\|y_\epsilon\|_{L^2(\Omega_\epsilon^+)} \le C\epsilon.$$

In other words the Poincaré constant is of order ϵ (see [4]). Since $y_d = 0$ in the strips, we have

$$\epsilon\|y_\epsilon-y_d\|_{L^2(\Omega_\epsilon^+)} \le C\epsilon^2.$$

Hence we get

$$\|\nabla\tau_\epsilon\|_{L^2(\Omega_\epsilon)} \le C[\|\bar{y}_\epsilon-\bar{y}\|_{L^2(\Omega^-)}+\epsilon^{3/2}+e^{-c/\epsilon}]. \tag{4.18}$$

Estimate on ρ_ϵ: We can work on an analogous fashion, by computing $\Delta\rho_\epsilon$ and introducing ρ_ϵ^1 and ρ_ϵ^2, to get

$$\|\nabla\rho_\epsilon\|_{L^2(\Omega_\epsilon)} \le C\epsilon. \tag{4.19}$$

We will not present the details here and we refer to [4]. Let us mention that (4.10) is needed to prove (4.19).

Estimate on π_ϵ: The estimate π_ϵ may be deduced from definition π_ϵ in (4.16) and from the regularity results stated in (4.8). We get

$$\|\nabla \pi_\epsilon\|_{L^2(\Omega_\epsilon \setminus B_\epsilon)} \leq C e^{-c/\epsilon}. \tag{4.20}$$

Combining (4.18), (4.19), (4.20), we get (4.11) and the proof of Theorem 4.5 is complete. □

Remark 4.8 One can also get H^1-error estimates for the optimal solution in terms of the adjoint state. For example, with $f = 0$, one can prove in an analogous fashion that

$$\|\bar{y}_\epsilon - \tilde{\bar{y}} - \epsilon\vartheta\|_{H^1(\Omega_\epsilon \setminus B_\epsilon)} \leq C\left[\|\bar{z}_\epsilon - \bar{z}\|_{L^2(\omega)} + \epsilon^{\frac{3}{2}}\right].$$

Indeed, ϑ is defined via the flux $\dfrac{\partial \bar{y}}{\partial x_2}$ in Ω^-. Our aim is eventually to get the above estimates without the first term on the right hand side, which is not successful so far. □

5 Dirichlet cost functional

One can do an analysis by considering the Dirichlet cost functional $J_{2,\epsilon}$ defined in Sect. 2.1. For convenience, we recall the definition of the control problem

$$\inf\{J_{2,\epsilon}(y_\epsilon, \theta) \mid \theta \in L^2(\omega),\ (y_\epsilon, \theta) \text{ obeys } (5.1)\}. \tag{$P_{2,\epsilon}$}$$

where

$$J_{2,\epsilon}(y_\epsilon, \theta) = \frac{1}{2}\int\limits_{\Omega_\epsilon} |\nabla y_\epsilon - \nabla y_d|^2 + \frac{\beta}{2}\int\limits_{\omega} \theta^2,$$

with $\beta > 0$, $y_d \in H^1(\Omega)$ and

$$\begin{cases} -\Delta y_\epsilon = f + \theta\chi_\omega & \text{in } \Omega_\epsilon, \\ y_\epsilon = 0 \quad \text{on } \gamma_\epsilon,\ y_\epsilon = u & \text{on } \Gamma_b, \\ y_\epsilon & \text{is } \Gamma_s - \text{periodic}. \end{cases} \tag{5.1}$$

The problem $(P_{2,\epsilon})$ has a unique solution $(\bar{\theta}_\epsilon, \bar{y}_\epsilon)$ which is characterized by the optimality system (see [23,24,32])

$$\begin{cases} \bar{z}_\epsilon,\ \bar{y}_\epsilon \in H^1_{per}(\Omega_\epsilon), \\ -\Delta \bar{y}_\epsilon = f + \bar{\theta}_\epsilon \chi_\omega & \text{in } \Omega_\epsilon, \\ -\Delta \bar{z}_\epsilon = -\Delta\left(\bar{y}_\epsilon - y_d\right) & \text{in } \Omega_\epsilon, \\ y_\epsilon = 0 & \text{on } \gamma_\epsilon, \\ y_\epsilon = u & \text{on } \Gamma_b, \\ \bar{z}_\epsilon = 0 & \text{on } \gamma_\epsilon \cup \Gamma_b, \\ \bar{\theta}_\epsilon = -\frac{1}{\beta}\bar{z}_\epsilon. \end{cases} \tag{5.2}$$

Remark 5.1 It is possible to consider more general oscillating cost functionals of the form

$$\frac{1}{2}\int\limits_{\Omega_\epsilon} B\left(\frac{x}{\epsilon}\right)\nabla y_\epsilon \cdot \nabla y_\epsilon + \frac{\beta}{2}\int\limits_{\omega} \theta^2.$$

The analysis of (5.2) will be more delicate. We will not attempt to do it in this paper. □

Since the Dirichlet integral $J_{2,\epsilon}$ contains derivative terms, it is easy to get the H^1-estimate straight from the functional. Considering the solution $y_\epsilon = y_\epsilon(0)$ with $\theta = 0$ of the Eq. (5.1) and using the fact that $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ is optimal, we get

$$J_{2,\epsilon}\left(\bar{y}_\epsilon, \bar{\theta}_\epsilon\right) \le J_{2,\epsilon}\left(y_\epsilon, 0\right).$$

This yields

$$\|\nabla \bar{y}_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \|\bar{\theta}_\epsilon\|^2_{L^2(\omega)} \le C.$$

Thus, we have

$$\|\tilde{\bar{y}}_\epsilon\|_{H^1(\Omega)} \le C, \qquad \|\bar{\theta}_\epsilon\|_{L^2(\omega)} \le C, \tag{5.3}$$

where $\tilde{\bar{y}}_\epsilon$ is the extension of $\bar{y}_\epsilon$ by zero to Ω. Now using the variational formulation of the second equation in (5.2), we see that

$$\|\tilde{\bar{z}}_\epsilon\|_{H^1(\Omega)} \le C. \tag{5.4}$$

Thus, we may deduce the following convergence as

$$\begin{cases} \tilde{\bar{y}}_\epsilon \rightharpoonup \bar{y}_0 & \text{weakly in } H^1(\Omega), \\ \tilde{\bar{z}}_\epsilon \rightharpoonup \bar{z}_0 & \text{weakly in } H^1(\Omega), \\ \bar{\theta}_\epsilon \rightharpoonup \bar{\theta}_0 & \text{weakly in } H^1(\omega). \end{cases} \tag{5.5}$$

In what follows, we will get the strong convergence of these sequences in H^1. Again by a similar argument as in the previous section, we have

$$\bar{y}_0 = 0, \quad \bar{z}_0 = 0 \quad \text{in } \Omega^+.$$

Passing to the limit in (5.2), which is straightforward, we obtain

$$\begin{cases} -\Delta \bar{y}_0 = f + \bar{\theta}_0 \chi_\omega & \text{in } \Omega^- \\ -\Delta \bar{z}_0 = -\Delta (\bar{y}_0 - y_d) & \text{in } \Omega^- \\ \bar{y}_0 = 0 & \text{on } \Gamma_u \\ \bar{y}_0 = u & \text{on } \Gamma_b \\ \bar{z}_0 = 0 & \text{on } \Gamma_u \cup \Gamma_b \\ \bar{\theta}_0 = -\frac{1}{\beta_1} \bar{z}_0. \end{cases} \tag{5.6}$$

System (5.6) is the optimality system of the limit minimization problem

$$\inf\{J_2(y,\theta) \mid \theta \in L^2(\omega)\}, \ (y,\theta) \text{ obeys } (5.7)\}, \tag{P_2}$$

with

$$J_2(y,\theta) = \frac{1}{2} \int_{\Omega^-} |\nabla (y - y_d)|^2 + \frac{\beta}{2} \int_{\omega} \theta^2,$$

and

$$\begin{cases} y \in H^1_{per}(\Omega^-), \\ -\Delta y = f + \theta \chi_\omega & \text{in } \Omega^-, \\ y = 0 & \text{on } \Gamma_u, \\ y = u & \text{on } \Gamma_b. \end{cases} \tag{5.7}$$

Hence, $(\bar{y}_0, \bar{\theta}_0, \bar{z}_0) = (\bar{y}, \bar{\theta}, \bar{z})$, where $(\bar{y}, \bar{\theta})$ i the solution to problem (P_2) and $\bar{z}$ is the corresponding adjoint state.

Actually, we have the following convergence theorem.

Theorem 5.2 *Let $(\bar{y}_\epsilon, \bar{\theta}_\epsilon)$ be the optimal solution of problem $(P_{2,\epsilon})$ and $(\bar{y}, \bar{\theta})$ be that of problem (P_2). Then*

$$\begin{cases} \tilde{\bar{y}}_\epsilon \to \tilde{\bar{y}} & \text{strongly in } H^1(\Omega), \\ \bar{\theta}_\epsilon \to \bar{\theta} & \text{strongly in } H^1(\omega). \end{cases} \tag{5.8}$$

Further,

$$J_{2,\epsilon}(\bar{\theta}_\epsilon) \longrightarrow J_2(\bar{\theta}) \quad \text{when } \epsilon \longrightarrow 0. \tag{5.9}$$

Proof Since we have the convergence in (5.5), to complete the theorem, it is enough to show that

$$\|\nabla\tilde{\bar{y}}_\epsilon\|^2_{L^2(\Omega)} \longrightarrow \|\nabla\tilde{\bar{y}}\|^2_{L^2(\Omega)}$$

and

$$\|\nabla\tilde{\bar{z}}_\epsilon\|^2_{L^2(\Omega)} \longrightarrow \|\nabla\tilde{\bar{z}}\|^2_{L^2(\Omega)}.$$

Notice that, by trace theorem, we can conclude from (5.5) and (5.2), that

$$\frac{\partial \bar{y}_\epsilon}{\partial \nu} \rightharpoonup \frac{\partial \bar{y}}{\partial \nu} \quad \text{weakly in } H^{-1/2}(\Gamma_b)\,. \tag{5.10}$$

Weak formulation of (5.2) and (5.6) satisfied by $\bar{y}_\epsilon$ and $\bar{y}$, respectively, along with convergence (5.5) and (5.10), give the following

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} |\nabla\bar{y}_\epsilon|^2 &= \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} f\,\bar{y}_\epsilon + \lim_{\epsilon\to 0}\int_\omega \bar{\theta}_\epsilon\,\bar{y}_\epsilon + \lim_{\epsilon\to 0}\left\langle\frac{\partial\bar{y}_\epsilon}{\partial\nu}, u\right\rangle_{H^{-1/2}(\Gamma_b),H^{1/2}(\Gamma_b)}\\ &= \int_{\Omega^-} f\,\bar{y} + \int_\omega \bar{\theta}\,\bar{y} + \left\langle\frac{\partial\bar{y}}{\partial\nu}, u\right\rangle_{H^{-1/2}(\Gamma_b),H^{1/2}(\Gamma_b)}\\ &= \int_{\Omega^-} |\nabla\bar{y}|^2\,.\end{aligned}$$

In the same way, using the weak formulation of (5.2) and (5.6) satisfied by $\bar{z}_\epsilon$ and $\bar{z}$, respectively, with (5.5) and the strong convergence of $\nabla\tilde{\bar{y}}_\epsilon \longrightarrow \nabla\tilde{\bar{y}}$ in $L^2(\Omega)$, we can show that

$$\|\nabla\tilde{\bar{z}}_\epsilon\|^2_{L^2(\Omega)} \longrightarrow \|\nabla\tilde{\bar{z}}\|^2_{L^2(\Omega)},$$

which by the characterization of $\bar{\theta}_\epsilon$ and $\bar{\theta}$ gives (5.8). Finally

$$\begin{aligned}J_{2,\epsilon}(\bar{y}_\epsilon,\bar{\theta}_\epsilon) &= \frac{1}{2}\int_{\Omega_\epsilon} |\nabla(\bar{y}_\epsilon - y_d)|^2 + \frac{\beta}{2}\int_\omega \bar{\theta}_\epsilon^{\,2}\\ &\to \frac{1}{2}\int_{\Omega^-} |\nabla(\bar{y} - y_d)|^2 + \frac{\beta}{2}\int_\omega \bar{\theta}^{\,2} = J_2(\bar{y},\bar{\theta}),\end{aligned}$$

and the proof is complete. □

6 Corrector for Dirichlet cost functional

In this section, we derive corrector (error) estimates similar to the one in Sect. 4, but for the case of Dirichlet cost functional studied in Sect. 5. Indeed, there is a change in the estimate since the cost functional and hence the adjoint equation are different. We will not present the complete proof as it follows in similar lines as in Sect. 4, but sketch the important steps.

Let $\bar{y}, \bar{z}$ and $\bar{\theta}$ be the limit optimal solution obtained in Sect. 5. Define $\vartheta \in H^1_{per}(\Omega^-)$ via the flux $\frac{\partial(\bar{y}-\bar{z})}{\partial x_2}$ in Ω^- which solves the following problem

$$\begin{cases} \Delta\vartheta = 0 & \text{in } \Omega^- \\ \vartheta = 0 & \text{on } \Gamma_b \\ \vartheta = \beta_1 \dfrac{\partial(\bar{y}-\bar{z})}{\partial x_2} & \text{on } \Gamma_u. \end{cases} \tag{6.1}$$

We want to estimate of ϑ in $C^3(\overline{\Omega^-})$. For that, we must have $\bar{y}-\bar{z} \in H^6_{per}(R) \subset C^4(\overline{R})$. Thus we assume that

$$g \in H^6_{per}(0, L) \quad \text{and} \quad y_d \in H^6_{per}(\Omega^-). \tag{6.2}$$

We have the following theorem.

Theorem 6.1 *Assume (6.2) and ϑ solves (6.1). Let $\bar{y}_\epsilon$, $\bar{\theta}_\epsilon$ are the optimal solution and $\bar{z}_\epsilon$ is the adjoint states corresponding to the problem $(P_{2,\epsilon})$. Then, there exists a positive constant C, independent of ϵ, such that*

$$\|(\bar{y}_\epsilon - \tilde{\bar{y}}) - (\tilde{\bar{z}}_\epsilon - \tilde{\bar{z}}) - \epsilon\tilde{\vartheta}\|_{H^1(\Omega_\epsilon\setminus B_\epsilon)} \leq C\epsilon^{3/2}. \tag{6.3}$$

for ϵ small enough, where $B_\epsilon = (0, L) \times (M-\epsilon, M+\epsilon)$. Here $\tilde{\vartheta}$ is the extension to Ω of ϑ by zero.

Sketch of the proof. As in Sect. 4.2, we define $\vartheta^+_\epsilon \in H^1(\Omega^+)$ and $\vartheta^-_\epsilon \in H^1_{per}(\Omega^-)$ which satisfy

$$\begin{cases} \Delta\vartheta^+_\epsilon = 0 & \text{in } \Omega^+_\epsilon, \\ \Delta\vartheta^-_\epsilon = 0 & \text{in } \Omega^-, \\ \vartheta^+_\epsilon = 0 & \text{on } \gamma_\epsilon\setminus(\gamma_\epsilon\cap\Gamma_u), \\ \vartheta^-_\epsilon = \beta_1 \dfrac{\partial(\bar{y}-\bar{z})}{\partial x_2} & \text{on } \gamma_\epsilon\cap\Gamma_u, \\ \vartheta^+_\epsilon = \vartheta^-_\epsilon - \beta_1 \dfrac{\partial(\bar{y}-\bar{z})}{\partial x_2} & \text{on } \Gamma_u\setminus(\gamma_\epsilon\cap\Gamma_u), \\ \dfrac{\partial\vartheta^+_\epsilon}{\partial x_2} = \dfrac{\partial\vartheta^-_\epsilon}{\partial x_2} & \text{on } \Gamma_u\setminus(\gamma_\epsilon\cap\Gamma_u). \end{cases} \tag{6.4}$$

Then, we may write

$$(\bar{y}_\epsilon - \tilde{\bar{y}}) - (\bar{z}_\epsilon - \tilde{\bar{z}}) - \epsilon\tilde{\vartheta} = \tau_\epsilon + \epsilon\rho_\epsilon + \pi_\epsilon \quad \text{in} \quad \Omega_\epsilon, \tag{6.5}$$

where

$$\tau_\epsilon = \begin{cases} \tau_\epsilon^+ = \bar{y}_\epsilon - \bar{z}_\epsilon - \epsilon\vartheta_\epsilon^+ - \epsilon\dfrac{\partial\,(\bar{y}-\bar{z})}{\partial x_2}\,(x_1, M)\,\psi_\epsilon^+ & \text{in } \Omega_\epsilon^+, \\ \tau_\epsilon^- = (\bar{y}_\epsilon - \bar{y}) - (\bar{z}_\epsilon - \bar{z}) - \epsilon\vartheta_\epsilon^- - \epsilon\dfrac{\partial\,(\bar{y}-\bar{z})}{\partial x_2}(x_1, M)\left(\psi_\epsilon^- - \beta_1\right) & \text{in } \Omega^-, \end{cases}$$

$$\rho_\epsilon = \begin{cases} \rho_\epsilon^+ = \vartheta_\epsilon^+ - \epsilon\dfrac{\partial\vartheta}{\partial x_2}(x_1, M)\psi_\epsilon^+ & \text{in } \Omega_\epsilon^+, \\ \rho_\epsilon^- = \vartheta_\epsilon^- - \vartheta - \epsilon\dfrac{\partial\vartheta}{\partial x_2}(x_1, M)\psi_\epsilon^- & \text{in } \Omega^-, \end{cases}$$

and

$$\pi_\epsilon = \begin{cases} \pi_\epsilon^+ = \epsilon\dfrac{\partial\,(\bar{y}-\bar{z})}{\partial x_2}\,(x_1, M)\,\psi_\epsilon^+ + \epsilon^2\dfrac{\partial\vartheta}{\partial x_2}\,(x_1, M)\,\psi_\epsilon^+, \\ \pi_\epsilon^- = \epsilon\dfrac{\partial\,(\bar{y}-\bar{z})}{\partial x_2}\,(x_1, M)\left(\psi_\epsilon^- - \beta_1\right) + \epsilon^2\dfrac{\partial\vartheta}{\partial x_2}\,(x_1, M)\,\psi_\epsilon^-. \end{cases}$$

Hence, in Ω_ϵ^+, we have

$$\Delta\tau_\epsilon = \Delta\tau_\epsilon^+ = -\epsilon\frac{\partial^3\,(\bar{y}-\bar{z})}{\partial x_1^2\partial x_2}(x_1, M)\psi_\epsilon^+ - 2\epsilon\frac{\partial^2\,(\bar{y}-\bar{z})}{\partial x_1\partial x_2}(x_1, M)\frac{\partial\psi_\epsilon^+}{\partial x_1},$$

and in Ω_ϵ^- we get

$$\Delta\tau_\epsilon = \Delta\tau_\epsilon^- = -\epsilon\frac{\partial^3\,(\bar{y}-\bar{z})}{\partial x_1^2\partial x_2}(x_1, M)\left(\psi_\epsilon^- - \beta_1\right) - 2\epsilon\frac{\partial^2\,(\bar{y}-\bar{z})}{\partial x_1\partial x_2}(x_1, M)\frac{\partial\psi_\epsilon^-}{\partial x_1}.$$

Similar to the computation as in the proof of Proposition 4.7 (compare $\Delta\tau_\epsilon$ in the proposition), we can derive the estimate

$$\|\tau_\epsilon\|_{H^1(\Omega_\epsilon)} \le C\epsilon^{3/2}. \tag{6.6}$$

Similarly

$$\|\rho_\epsilon\|_{H^1(\Omega_\epsilon)} \le C\epsilon. \tag{6.7}$$

and

$$\|\pi_\epsilon\|_{H^1(\Omega_\epsilon\setminus B_\epsilon)} \le Ce^{-c/\epsilon}. \tag{6.8}$$

We can conclude the proof by combining Eqs. (6.5)–(6.8). □

Remark 6.2 One can also get an error estimate for the state. This is similar to the one in Sect. 4. For example, with $f = 0$, one can prove in an analogous fashion that

$$\|\bar{y}_\epsilon - \bar{y} - \epsilon\tilde{\vartheta}\|_{H^1(\Omega_\epsilon\setminus B_\epsilon)} \le C\left[\|\bar{z}_\epsilon - \bar{z}\|_{L^2(\omega)} + \epsilon^{\frac{3}{2}}\right].$$

Obviously here ϑ is defined via the flux $\dfrac{\partial\bar{y}}{\partial x_2}$ in Ω^-. Our aim is eventually to get the above estimates without the first term on the right hand side, which is not successful so far. □

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