

Exact internal controllability for a hyperbolic problem in a domain with highly oscillating boundary

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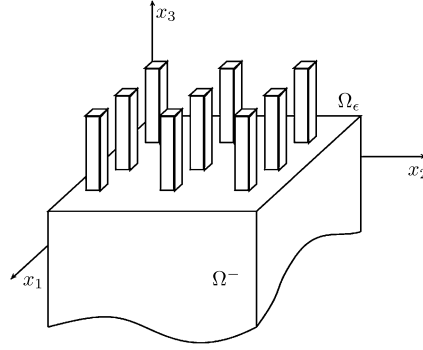
Abstract. In this paper, by using the Hilbert Uniqueness Method (HUM), we study the exact controllability problem described by the wave equation in a three-dimensional horizontal domain bounded at the bottom by a smooth wall and at the top by a rough wall. The latter is assumed to consist in a plane wall covered with periodically distributed asperities whose size depends on a small parameter $\varepsilon > 0$, and with a fixed height. Our aim is to obtain the exact controllability for the homogenized equation. In the process, we study the asymptotic analysis of wave equation in two setups, namely solution by standard weak formulation and solution by transposition method.

Keywords: wave equation, homogenization, oscillating boundary, exact controllability

1. Introduction

In this article, we consider the homogenization of an exact controllability problem described in a domain with oscillating boundary. Such domains appear in a variety of applications. For example, boundary-value problems in domains with highly oscillating boundaries are models for problems in biology and in industrial applications: motion of ciliated micro-organisms, flows over rough walls, electromagnetic waves in a region with a rough interface, structures such as bridges on supports, frameworks of houses, etc. Another interesting application is the air flow through compression systems in turbo machines such as jet engine. For example, such a system is modelled by the Viscous–Moore–Greitzer equation derived from Scaled Navier–Stokes equations (see [4,32,33]). Here the pitch and size of the rotor – stator pair of blades in the engine provides a small parameter compared to the size of the engine which is oscillatory as well as rotating (moving). The motion of the stator and rotor blades in the compressor produces turbulent flow on a fast time scale. When the engine operates close to the optimal parameters, the flow becomes unstable. This model gives motivation to look into control problems described by Partial Differential Equations (PDEs) of evolution type such as heat equation or Navier–Stokes equation.

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Fig. 1. Domain Ω_ε .

The computational calculation of the solution of these problems is very complicated, rather impossible due to singularities of the domain. It is much more delicate for control and controllability problems. Therefore, an asymptotic analysis of boundary value problems in such domains gives the possibility to replace the original problem by the corresponding limit problem defined in a “simpler” domain.

In this paper, we plan to study the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of an exact controllability for a boundary-value problem described by a hyperbolic equation in a domain Ω_ε with oscillating boundary, with homogeneous Dirichlet boundary condition. In order to point out the main difficulties, we consider the wave equation. Our approach to the homogenization for the exact controllability problem, for a hyperbolic equation consists in applying the Hilbert Uniqueness Method (HUM) of J.L. Lions (see [24–26]). By using the method of oscillating test functions of L. Tartar (see [11,38]), we identify the problem satisfied by the limit (u, θ) of the sequence of optimal pairs $\{(u_\varepsilon, \theta_\varepsilon)\}_\varepsilon$, where u_ε and θ_ε denote the state of the system and the exact control respectively.

Notations: Let $S = (0, l_1) \times (0, l_2)$, $\tilde{S} = (a_1, b_1) \times (a_2, b_2)$, with $0 < a_i < b_i < l_i$ ($i = 1, 2$), and let η_ε be the εS -periodic function defined on εS by

$$\eta_\varepsilon(x') = \begin{cases} l_3 & \text{if } x' \in \varepsilon(S \setminus \tilde{S}), \\ l'_3 & \text{if } x' \in \varepsilon\tilde{S}, \end{cases}$$

with $l_3 < l'_3$ and $x' = (x_1, x_2)$ (η_ε is εS -periodic means that η_ε is defined on \mathbb{R}^2 and it is εl_i -periodic with respect to x_i , for $i = 1, 2$). We introduce the domain $\Omega_\varepsilon \subset \mathbf{R}^3$ with highly oscillating boundary (see Fig. 1)

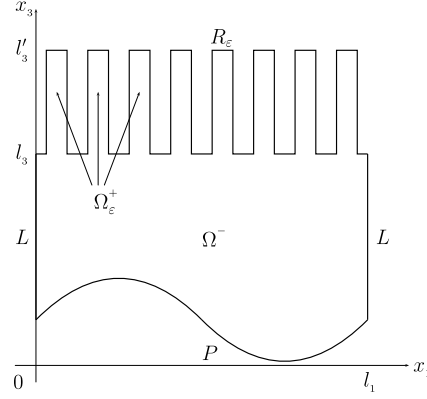
$$\Omega_\varepsilon = \{x = (x', x_3) \in \mathbb{R}^3: x' \in S, b(x') < x_3 < \eta_\varepsilon(x')\},$$

where b is a smooth function on \mathbb{R}^2 , S -periodic and such that $b(x') < l_3$ for every $x' \in \mathbb{R}^2$. Since we assume that $1/\varepsilon \in \mathbb{N}$, the function η_ε is also S -periodic. The domain Ω_ε is bounded at the bottom by the smooth wall

$$P = \{x = (x', x_3) \in \mathbb{R}^3: x' \in S, x_3 = b(x')\}$$

and at the top by the rough wall

$$R_\varepsilon = \partial\Omega_\varepsilon \setminus \overline{(P \cup \{(x', x_3) \in \mathbb{R}^3: x' \in \partial S, b(x') < x_3 < l_3\})}.$$


 Fig. 2. Vertical section of the domain Ω_ε .

Moreover, we set

$$\begin{aligned}\Omega &= \{(x', x_3) \in \mathbb{R}^3: x' \in S, b(x') < x_3 < l'_3\}, \\ \Omega^- &= \{(x', x_3) \in \mathbb{R}^3: x' \in S, b(x') < x_3 < l_3\}, \\ \Omega_\varepsilon^+ &= \{(x', x_3) \in \Omega_\varepsilon: l_3 < x_3 < l'_3\}, \quad \Sigma = S \times \{l_3\}, \quad \Sigma' = S \times \{l'_3\}.\end{aligned}$$

Figure 2 represents a vertical section of the domain Ω_ε .

Regarding a brief literature, the limit problem of boundary-value problems in domains with highly oscillating boundary, that is when the amplitude of the oscillations is constant with respect to ε , are derived in [1–3, 5–8, 10, 11, 14, 15, 21–23, 29–31, 34]. Optimal control problems in domains with highly oscillating boundary are considered in [16, 17, 19, 20, 35]. Exact controllability in perforated domains is studied in [12, 13]. Approximate controllability for parabolic equation in perforated domains is studied in [18] and [40].

2. Statement of the problem and main result

Throughout the paper, we assume that the function b is Lipschitz-continuous. We observe that

$$\begin{aligned}\chi_{\Omega_\varepsilon^+} &\rightharpoonup \mu = \frac{|\tilde{S}|}{|S|} \text{ weakly } * \text{ in } L^\infty(\Omega^+) \quad \text{and} \\ \chi_{\Omega_\varepsilon^+ \cap \Sigma} &\rightharpoonup \mu = \frac{|\tilde{S}|}{|S|} \text{ weakly } * \text{ in } L^\infty(\Sigma),\end{aligned}\tag{1}$$

where $\chi_{\Omega_\varepsilon^+}$ denotes the characteristic function of Ω_ε^+ , and $|\tilde{S}|$ (resp. $|S|$) denotes the \mathbb{R}^2 -Lebesgue measure of \tilde{S} (resp. S). For each $m \geq 0$, we introduce the spaces

$$\begin{aligned}H_{\text{per}}^m(\Omega_\varepsilon) &= \{v \in H^m(A) \text{ for any bounded open set } A \subset \mathcal{O}_\varepsilon, \\ &\quad v(x + (l_1, 0, 0)) = v(x + (0, l_2, 0)) = v(x) \text{ for a.e. } x \in \mathcal{O}_\varepsilon\},\end{aligned}$$

$$\begin{aligned}\mathcal{V}(\Omega_\varepsilon) &= \{z: z \in H_{\text{per}}^1(\Omega_\varepsilon): z|_{R_\varepsilon \cup P} = 0\}, \\ \mathcal{V}(\Omega) &= \{z: z \in H_{\text{per}}^1(\Omega): z|_{\Sigma' \cup P} = 0\}, \\ \mathcal{V}(\Omega^+) &= \{z: z \in H_{\text{per}}^1(\Omega^+): z|_{\Sigma \cup \Sigma'} = 0\}\end{aligned}$$

and

$$\mathcal{V}(\Omega^-) = \{z: z \in H_{\text{per}}^1(\Omega^-): z|_{\Sigma \cup P} = 0\},$$

where $\mathcal{O}_\varepsilon = \{x = (x', x_3) \in \mathbb{R}^3: x' \in \mathbb{R}^2, b(x') < x_3 < \eta_\varepsilon(x')\}$. Moreover, \tilde{v} will denote the zero-extension to Ω (resp. $]0, T[\times \Omega$) of a function v defined on A (resp. $]0, T[\times A$), with $A \subset \Omega$. Furthermore, v^+ (resp. v^-) denote the restriction of v to Ω^+ (resp. Ω^-), if v is defined on Ω ; the restriction of v to $]0, T[\times \Omega^+$ (resp. $]0, T[\times \Omega^-$) if v is defined on $]0, T[\times \Omega$.

Now, we formulate our exact controllability problem for a hyperbolic equation in Ω_ε . For a control $\theta_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon))$, the state u_ε of the system solves the following problem:

$$\begin{cases} u_\varepsilon'' - \Delta u_\varepsilon = \theta_\varepsilon & \text{in }]0, T[\times \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{in }]0, T[\times (P \cup R_\varepsilon), \\ u_\varepsilon(0) = u_\varepsilon^0, u_\varepsilon'(0) = u_\varepsilon^1 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x'), \end{cases} \quad (2)$$

where $(u_\varepsilon^0, u_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ and $T > 0$. Let us introduce the space

$$\mathcal{W}_\varepsilon = \{v: v \in L^2(0, T; \mathcal{V}(\Omega_\varepsilon)), v' \in L^2(0, T; L^2(\Omega_\varepsilon))\},$$

which is a Banach space with respect to the graph norm defined by

$$\|v\|_{\mathcal{W}_\varepsilon} = \|v\|_{L^2(0, T; \mathcal{V}(\Omega_\varepsilon))} + \|v'\|_{L^2(0, T; L^2(\Omega_\varepsilon))}.$$

It is well known (see [27,28]) that problem (2) admits a unique weak solution $u_\varepsilon = u_\varepsilon(\theta_\varepsilon)$:

$$\begin{cases} u_\varepsilon \in \mathcal{W}_\varepsilon, \\ \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon z h'' + \nabla_x u_\varepsilon \nabla z h \, dx \, dt \\ \quad = \int_0^T \int_{\Omega_\varepsilon} \theta_\varepsilon z h \, dx \, dt \quad \forall z \in \mathcal{V}(\Omega_\varepsilon), \forall h \in C_0^\infty([0, T]), \\ u_\varepsilon(0) = u_\varepsilon^0, u_\varepsilon'(0) = u_\varepsilon^1 \quad \text{in } \Omega_\varepsilon. \end{cases} \quad (3)$$

Remark 2.1. Let us point out that the solution u_ε of problem (2) has more regularity. In fact, we have $u_\varepsilon \in C([0, T]; \mathcal{V}(\Omega_\varepsilon)) \cap C^1([0, T]; L^2(\Omega_\varepsilon))$ and $u_\varepsilon'' \in L^2(0, T; (\mathcal{V}(\Omega_\varepsilon))')$ (see [36,39]).

Definition 2.1 (Exact controllability). We say that system (2) is exactly controllable at time T if for every $(u_\varepsilon^0, u_\varepsilon^1), (v_\varepsilon^0, v_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$, there exists a control $\theta_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon))$ such that the corresponding solution of problem (2) satisfies

$$u_\varepsilon(T) = v_\varepsilon^0, \quad u_\varepsilon'(T) = v_\varepsilon^1.$$

It is well known that for the above linear system, driving the system to any state is equivalent of driving the system to null state and this is known as null controllability. In other words, (2) is *null controllable* if there exists a control $\theta_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon))$ such that $u_\varepsilon(T) = u'_\varepsilon(T) = 0$.

A constructive method to determine the control θ_ε such that $u_\varepsilon(T, \cdot) = 0$ and $u'_\varepsilon(T, \cdot) = 0$ is the Hilbert Uniqueness Method (HUM) introduced by Lions (see [24,25]). The idea is to build a control as the solution of a transposed problem associated to some initial conditions. These initial conditions are obtained by calculating at zero time the solution of a backward problem. The source term of the backward problem is the unique solution of the transposed problem. The control obtained by HUM is also a energy minimizing control. We briefly outline the HUM procedure. Let $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$ and consider the problem

$$\begin{cases} \varphi_\varepsilon'' - \Delta \varphi_\varepsilon = 0 & \text{in }]0, T[\times \Omega_\varepsilon, \\ \varphi_\varepsilon = 0 & \text{on }]0, T[\times (P \cup R_\varepsilon), \\ \varphi_\varepsilon(0) = \varphi_\varepsilon^0, \varphi'_\varepsilon(0) = \varphi_\varepsilon^1 & \text{a.e. in } \Omega_\varepsilon, \\ \varphi_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases} \quad (4)$$

Since the initial data is in a weak space, one need to apply the so called *transposition method* (see [28]) to obtain a unique solution $\varphi_\varepsilon \in C([0, T]; L^2(\Omega_\varepsilon)) \cap C^1([0, T]; (\mathcal{V}(\Omega_\varepsilon))')$ to the problem (4). Now, let $\psi_\varepsilon \in C([0, T]; \mathcal{V}(\Omega_\varepsilon)) \cap C^1([0, T]; L^2(\Omega_\varepsilon))$ be the unique solution of the backward problem

$$\begin{cases} \psi_\varepsilon'' - \Delta \psi_\varepsilon = -\varphi_\varepsilon & \text{in }]0, T[\times \Omega_\varepsilon, \\ \psi_\varepsilon = 0 & \text{on }]0, T[\times (P \cup R_\varepsilon), \\ \psi_\varepsilon(T) = 0, \psi'_\varepsilon(T) = 0 & \text{in } \Omega_\varepsilon, \\ \psi_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x'), \end{cases} \quad (5)$$

where φ_ε is the solution of the problem (4). The weak formulation of problem (5) (see [39]) is given by

$$\begin{cases} \psi_\varepsilon \in \mathcal{W}_\varepsilon, \\ \int_0^T \int_{\Omega_\varepsilon} \psi_\varepsilon z h'' + \nabla_x \psi_\varepsilon \nabla z h \, dx \, dt \\ = - \int_0^T \int_{\Omega_\varepsilon} \varphi_\varepsilon z h \, dx \, dt \quad \forall z \in \mathcal{V}(\Omega_\varepsilon), \forall h \in C_0^\infty(]0, T[), \\ \psi_\varepsilon(T) = 0, \psi'_\varepsilon(T) = 0 & \text{in } \Omega_\varepsilon. \end{cases} \quad (6)$$

Inspired by HUM, we introduce the linear operator

$$\Lambda_\varepsilon : L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))' \longrightarrow L^2(\Omega_\varepsilon) \times \mathcal{V}(\Omega_\varepsilon)$$

by setting for all $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$,

$$\Lambda_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1) = (-\psi'_\varepsilon(0), \psi_\varepsilon(0)), \quad (7)$$

where ψ_ε is the solution of the problem (5). Moreover, it results that

$$\begin{aligned} \langle \Lambda_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1), (\varphi_\varepsilon^0, \varphi_\varepsilon^1) \rangle &= \langle (-\psi'_\varepsilon(0), \psi_\varepsilon(0)), (\varphi_\varepsilon^0, \varphi_\varepsilon^1) \rangle \\ &= \langle \varphi_\varepsilon^1, \psi_\varepsilon(0) \rangle_{(\mathcal{V}(\Omega_\varepsilon))', \mathcal{V}(\Omega_\varepsilon)} - \int_{\Omega_\varepsilon} \varphi_\varepsilon^0 \psi'_\varepsilon(0) \, dx, \end{aligned} \quad (8)$$

for every $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$.

Remark 2.2. For each $\varepsilon > 0$, the operator Λ_ε is linear, continuous and injective. If Λ_ε is surjective then, we define the control $\theta_\varepsilon \in L^2((0, T) \times \Omega_\varepsilon)$ by $\theta_\varepsilon = -\varphi_\varepsilon$, where φ_ε is the solution of the problem (4) with initial data $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) = \Lambda_\varepsilon^{-1}(-u_\varepsilon^1, u_\varepsilon^0)$. The state is given by $u_\varepsilon = \psi_\varepsilon$, where ψ_ε is the solution of the problem (5). So we obtain the exact controllability in $\mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ at time T for the system (2).

The aim of this paper is to study the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of the sequence of the control pairs $\{(u_\varepsilon, \theta_\varepsilon)\}_\varepsilon$, under the following assumptions:

$$\begin{cases} \tilde{u}_\varepsilon^0 \rightharpoonup u^0, & \text{weakly in } \mathcal{V}(\Omega), \\ \tilde{u}_\varepsilon^1 \rightharpoonup u^1, & \text{weakly in } L^2(\Omega). \end{cases} \quad (9)$$

We now state the main result of this paper.

Theorem 2.1. Assume (9) and let $T > 0$ be the controllability time. Let u_ε be the solution of the controllability problem (2) and θ_ε is the exact control given by HUM. Then, as $\varepsilon \rightarrow 0$

$$\begin{cases} \tilde{u}_\varepsilon \rightharpoonup u, & \text{weakly in } C([0, T]; \mathcal{V}(\Omega)), \\ \tilde{\theta}_\varepsilon \rightharpoonup \theta, & \text{weakly in } L^2(0, T; L^2(\Omega)), \end{cases} \quad (10)$$

where, $u \in L^2(0, T; L^2(\Omega)) \cap L^2(0, T; H_{\text{per}}^1(\Omega^-))$ and θ are respectively the unique solution and the exact control for the homogenized system:

$$\begin{cases} u = 0 & \text{in }]0, T[\times \Omega^+, \\ u'' - \Delta u = \theta & \text{in }]0, T[\times \Omega^-, \\ u = 0 & \text{in }]0, T[\times (\Sigma \cup P), \\ u(0) = u^0, u'(0) = u^1 & \text{in } \Omega^-, \\ u \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases} \quad (11)$$

3. A priori norm-estimates

In this section, we deduce some a priori norm-estimates for the initial conditions $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$ of problem (4), for the control θ_ε and for the corresponding solution u_ε of problem (2). The following lemma provides an explicit formula for the operator Λ_ε .

Lemma 3.1. Let $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$. The following identity holds

$$\langle \Lambda_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1), (\varphi_\varepsilon^0, \varphi_\varepsilon^1) \rangle = \int_0^T \int_{\Omega_\varepsilon} |\varphi_\varepsilon|^2 dx dt. \quad (12)$$

Proof. Multiplying equation in (4) by ψ_ε yields

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega_\varepsilon} (\varphi_\varepsilon'' - \Delta \varphi_\varepsilon) \psi_\varepsilon \, dx \, dt = \int_{\Omega_\varepsilon} (\varphi_\varepsilon'(T) \psi_\varepsilon(T) - \varphi_\varepsilon(T) \psi_\varepsilon'(T)) \, dx \\ &\quad - \int_{\Omega_\varepsilon} (\varphi_\varepsilon'(0) \psi_\varepsilon(0) - \varphi_\varepsilon(0) \psi_\varepsilon'(0)) \, dx + \int_0^T \int_{\Omega_\varepsilon} (\psi_\varepsilon'' - \Delta \psi_\varepsilon) \varphi_\varepsilon \, dx \, dt. \end{aligned}$$

Moreover, by virtue of (5) and (8), identity (12) follows. \square

As we have mentioned above, our first aim will be to prove that the operator A_ε is an isomorphism from $L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$ to $L^2(\Omega_\varepsilon) \times \mathcal{V}(\Omega_\varepsilon)$ for every ε and obtain the estimates independent of ε . This amounts to show the following observability estimate.

Proposition 3.1. *Let us fix $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$. Let φ_ε be the corresponding solution of problem (4). Then, there exists a positive constant C , independent of ε such that*

$$\|\varphi_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}^2 + \|\varphi_\varepsilon^1\|_{(\mathcal{V}(\Omega_\varepsilon))'} \leq C \int_0^T \int_{\Omega_\varepsilon} |\varphi_\varepsilon|^2 \, dx \, dt, \quad (13)$$

$$\|A_\varepsilon^{-1}\|_{\mathcal{L}(L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))', L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))')} \leq C \quad (14)$$

for every ε .

3.1. Proof of Proposition 3.1

Let us establish a preliminary result.

Lemma 3.2. *Let us fix $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ and let φ_ε be the corresponding solution of (4). Then, there exists a positive constant C , independent of ε such that*

$$E(0) \leq C \int_0^T \int_{\Omega_\varepsilon} |\varphi_\varepsilon'|^2 \, dx \, dt \quad (15)$$

for every ε , where $E(0) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon^0|^2 + |\varphi_\varepsilon^1|^2 \, dx$.

Proof. First note that since $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$, we can define the solution φ_ε of (4) by the usual weak formulation. Then, it is easy to see that the energy $E(t) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla_x \varphi_\varepsilon(t)|^2 + |\varphi_\varepsilon'(t)|^2 \, dx$ is conserved, that is

$$E(t) = E(0) \quad \text{for every } t \in [0, T]. \quad (16)$$

Let $\rho(t)$ be the function defined by

$$\rho(t) = t^2(T - t)^2$$

for every $t \in [0, T]$. By choosing $\eta_\varepsilon(x, t) = \rho(t)\varphi_\varepsilon(x, t)$ as a test function in (4) and integrating by parts, we obtain

$$\int_0^T \int_{\Omega_\varepsilon} \rho(t) |\varphi'_\varepsilon|^2 dx dt + \int_0^T \int_{\Omega_\varepsilon} \rho'(t) \varphi_\varepsilon \varphi'_\varepsilon dx dt = \int_0^T \int_{\Omega_\varepsilon} \rho(t) |\nabla_x \varphi_\varepsilon|^2 dx dt. \quad (17)$$

Then, by making use of the Young's inequality, we get

$$\int_0^T \int_{\Omega_\varepsilon} \rho'(t) \varphi_\varepsilon \varphi'_\varepsilon dx dt \leq \gamma \int_0^T \int_{\Omega_\varepsilon} \rho(t) \varphi_\varepsilon^2 dx dt + C(\gamma) \int_0^T \int_{\Omega_\varepsilon} |\varphi'_\varepsilon|^2 dx dt, \quad (18)$$

where $\gamma > 0$ and $C(\gamma) = \frac{1}{4\gamma} \|\frac{(\rho')^2}{\rho}\|_{L^\infty(0,T)}$. Let $\lambda_\varepsilon^0 = \lambda_\varepsilon^0(\Omega_\varepsilon)$ be the first eigenvalue of the Laplacian with eigenvector $v_\varepsilon \in \mathcal{V}(\Omega_\varepsilon)$. That is, $(\lambda_\varepsilon^0, v_\varepsilon)$ satisfies

$$\begin{cases} -\Delta v_\varepsilon = \lambda_\varepsilon^0 v_\varepsilon & \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 & \text{on } P \cup R_\varepsilon, \\ v_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases}$$

Since,

$$\lambda_\varepsilon^0(\Omega_\varepsilon) = \inf \left\{ \frac{\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 dx}{\int_{\Omega_\varepsilon} |v_\varepsilon|^2 dx} : v_\varepsilon \in \mathcal{V}(\Omega_\varepsilon) \right\}. \quad (19)$$

Since $\mathcal{V}(\Omega^-) \hookrightarrow \mathcal{V}(\Omega_\varepsilon) \hookrightarrow \mathcal{V}(\Omega)$, we can estimate $\lambda_\varepsilon^0(\Omega_\varepsilon)$ from below and above, respectively, by the first eigenvalues in Ω^- and Ω . That is

$$\lambda^0(\Omega^-) \geq \lambda_\varepsilon^0(\Omega_\varepsilon) \geq \lambda^0(\Omega). \quad (20)$$

By making use of (17)–(20), we obtain

$$\left(1 - \frac{\gamma}{\lambda^0(\Omega)}\right) \int_0^T \int_{\Omega_\varepsilon} \rho(t) |\nabla_x \varphi_\varepsilon|^2 dx dt \leq \int_0^T \int_{\Omega_\varepsilon} \rho(t) |\varphi'_\varepsilon|^2 dx dt + C(\gamma) \int_0^T \int_{\Omega_\varepsilon} |\varphi'_\varepsilon|^2 dx dt.$$

Thus, there exists a positive constant C , independent of ε such that

$$\int_0^T \int_{\Omega_\varepsilon} \rho(t) |\nabla_x \varphi_\varepsilon|^2 dx dt \leq C \int_0^T \int_{\Omega_\varepsilon} |\varphi'_\varepsilon|^2 dx dt. \quad (21)$$

Multiplying the equation in (16) by $\rho(t)$ and integrating from 0 to T , we obtain

$$\int_0^T \rho(t) dt = \frac{1}{2} \left(\int_0^T \rho(t) \int_{\Omega_\varepsilon} (|\nabla_x \varphi_\varepsilon(t)|^2 + |\varphi'_\varepsilon(t)|^2) dx dt \right). \quad (22)$$

By virtue of (21) and (22), estimate (15) follows. \square

Now, let us prove Proposition 3.1. Let us fix $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$. Let $\pi_\varepsilon \in \mathcal{V}(\Omega_\varepsilon)$ be the unique solution of the problem

$$\begin{cases} -\Delta \pi_\varepsilon = \varphi_\varepsilon^1 & \text{in } \Omega_\varepsilon, \\ \pi_\varepsilon = 0 & \text{on } P \cup R_\varepsilon, \\ \pi_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases}$$

By Poincaré inequality, there exists a positive constant C , independent of ε such that

$$\|\nabla \pi_\varepsilon\|_{L^2} \leq C \|\varphi_\varepsilon^1\|_{(\mathcal{V}(\Omega_\varepsilon))'}. \quad (23)$$

Let φ_ε be the transposition solution of (4) corresponding to the initial data $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$. Then, the function

$$w_\varepsilon(x, t) = \int_0^t \varphi_\varepsilon(x, s) \, ds + \pi_\varepsilon(x)$$

satisfies the problem

$$\begin{cases} w_\varepsilon'' - \Delta w_\varepsilon = 0 & \text{in }]0, T[\times \Omega_\varepsilon, \\ w_\varepsilon = 0 & \text{on }]0, T[\times (P \cup R_\varepsilon), \\ w_\varepsilon(0) = \pi_\varepsilon & \text{in } \Omega_\varepsilon, \\ w_\varepsilon'(0) = \varphi_\varepsilon^0 & \text{in } \Omega_\varepsilon, \\ w_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases}$$

Observe that the solution w_ε is defined by usual weak formulation. Hence by applying Lemma 3.2, we get

$$\int_{\Omega_\varepsilon} |\nabla \pi_\varepsilon|^2 + |\varphi_\varepsilon^0|^2 \, dx \leq C \int_0^T \int_{\Omega_\varepsilon} |w_\varepsilon'|^2 \, dx \, dt. \quad (24)$$

The inequality (13) of Proposition 3.1 is then a direct consequence of (23) and (24).

Now, we prove (14). By making use of Young's inequality, (13) and (12), it follows that

$$\begin{aligned} \|(\varphi_\varepsilon^0, \varphi_\varepsilon^1)\|_{L^2 \times (\mathcal{V}(\Omega_\varepsilon))'}^2 &\leq 2(\|\varphi_\varepsilon^0\|_{L^2}^2 + \|\varphi_\varepsilon^1\|_{(\mathcal{V}(\Omega_\varepsilon))'}^2) \\ &\leq C \int_0^T \int_{\Omega_\varepsilon} |\varphi_\varepsilon|^2 \, dx \, dt \\ &= C \langle \Lambda_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1), (\varphi_\varepsilon^0, \varphi_\varepsilon^1) \rangle \\ &\leq C \|\Lambda_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1)\|_{L^2 \times \mathcal{V}(\Omega_\varepsilon)} \|(\varphi_\varepsilon^0, \varphi_\varepsilon^1)\|_{L^2 \times (\mathcal{V}(\Omega_\varepsilon))'}. \end{aligned} \quad (25)$$

From (25) and taking into account that Λ_ε is an isomorphism, we obtain

$$\begin{aligned} &\|\Lambda_\varepsilon^{-1}\|_{\mathcal{L}(L^2(\Omega_\varepsilon) \times \mathcal{V}(\Omega_\varepsilon), L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))')} \\ &= \sup \left\{ \frac{\|(\varphi_\varepsilon^0, \varphi_\varepsilon^1)\|_{L^2 \times (\mathcal{V}(\Omega_\varepsilon))'}}{\|\Lambda_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1)\|_{L^2 \times \mathcal{V}(\Omega_\varepsilon)}} : (\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))' \right\} \leq C, \end{aligned}$$

from which estimate (14) follows.

We have the following proposition which is a consequence of (14).

Proposition 3.2. *Let $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$ be the initial conditions of problem (4). Then, there exists a constant C , independent of ε , such that*

$$\|(\varphi_\varepsilon^0, \varphi_\varepsilon^1)\|_{L^2 \times \mathcal{V}'} \leq C \quad (26)$$

for every ε .

To describe the limit problem of (4), as $\varepsilon \rightarrow 0$, we introduce the solution $\varphi^- \in L^2(0, T; H_{\text{per}}^1(\Omega^-))$ of the problem

$$\begin{cases} (\varphi^-)'' - \Delta \varphi^- = 0 & \text{in }]0, T[\times \Omega^-, \\ \varphi^- = 0 & \text{in }]0, T[\times (\Sigma \cup P), \\ \varphi^-(0) = \varphi^{-0}, (\varphi^-)'(0) = \varphi^{-1} & \text{in } \Omega^-, \end{cases} \quad (27)$$

where $(\varphi^{-0}, \varphi^{-1}) \in L^2(\Omega^-) \times (\mathcal{V}(\Omega^-))'$, then set

$$\varphi = \begin{cases} 0 & \text{in } \Omega^+, \\ \varphi^- & \text{in } \Omega^-. \end{cases} \quad (28)$$

Proposition 3.3. *Let $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times \mathcal{V}(\Omega_\varepsilon)'$ be as in Proposition 3.2. Let φ_ε and ψ_ε be, respectively, the unique solutions of problems (4) and (5). Then, there exists a constant C , independent of ε , such that*

$$\|\varphi_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))} \leq C, \quad (29)$$

$$\|\varphi_\varepsilon'\|_{L^2(0, T; (\mathcal{V}(\Omega_\varepsilon))')} \leq C, \quad (30)$$

$$\|\psi_\varepsilon\|_{L^\infty(0, T; \mathcal{V}(\Omega_\varepsilon))} \leq C, \quad (31)$$

$$\|\psi_\varepsilon'\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \leq C \quad (32)$$

for every ε .

Proof. The proof follows by Proposition 3.2 and by Theorem 4.2 and Lemma 3.6 in Chapter 1 of [24]. \square

4. Homogenization of wave equation in domain with oscillating boundary

In this section, we prove two homogenization results, namely one for the wave equation with regular initial data to obtain the limit equation corresponding to the solution ψ_ε , where the solution is defined via the standard weak formulation. This is done in the next subsection. Secondly, we also study the homogenization of the wave with weak data whose solution is defined by the method of transposition. This is necessary to obtain the homogenized equation corresponding to φ_ε .

4.1. Homogenization with regular data

Let us consider the problem

$$\begin{cases} y_\varepsilon'' - \Delta y_\varepsilon = f_\varepsilon & \text{in }]0, T[\times \Omega_\varepsilon, \\ y_\varepsilon = 0 & \text{in }]0, T[\times (P \cup R_\varepsilon), \\ y_\varepsilon(0) = y_\varepsilon^0, y_\varepsilon'(0) = y_\varepsilon^1 & \text{in } \Omega_\varepsilon, \\ y_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x'), \end{cases} \quad (33)$$

where $f_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon))$ and $(y_\varepsilon^0, y_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$. It is well known (see [27]) that problem (33) admits a unique weak solution y_ε :

$$\begin{cases} y_\varepsilon \in \mathcal{W}_\varepsilon, \\ \int_0^T \int_{\Omega_\varepsilon} y_\varepsilon z h'' + \nabla_x y_\varepsilon \nabla_x z h \, dx \, dt \\ \quad = \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon z h \, dx \, dt \quad \forall z \in \mathcal{V}(\Omega_\varepsilon), \forall h \in C_0^\infty([0, T]), \\ y_\varepsilon(0) = y_\varepsilon^0, y_\varepsilon'(0) = y_\varepsilon^1 \quad \text{in } \Omega_\varepsilon. \end{cases} \quad (34)$$

Now, we recall the following result.

Lemma 4.1. *The solution y_ε of problem (34) satisfies the following estimate:*

$$\begin{aligned} & \|y_\varepsilon\|_{L^\infty(0, T; \mathcal{V}(\Omega_\varepsilon))} + \|y_\varepsilon'\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \\ & \leq D(\|y_\varepsilon^0\|_{\mathcal{V}(\Omega_\varepsilon)} + \|y_\varepsilon^1\|_{L^2(\Omega_\varepsilon)} + \|f_\varepsilon\|_{L^2(0, T; L^2(\Omega_\varepsilon))}), \end{aligned} \quad (35)$$

where D is a positive constant depending on T . Moreover it holds that $y_\varepsilon \in C([0, T]; \mathcal{V}(\Omega_\varepsilon)) \cap C^1([0, T]; L^2(\Omega_\varepsilon))$.

Proof. See [27], Chapter 3, Remark 8.2, Theorem 8.2 and Lemma 8.3. \square

As far as the weak formulation of the problem (33) is concerned, we prefer to use the following form which is equivalent to the usual one (see [19], Proposition 3.4):

$$\begin{cases} y_\varepsilon \in \mathcal{W}_\varepsilon, \\ \text{(i)} \quad \int_0^T \langle y_\varepsilon''(t, \cdot), \psi(t, \cdot) \rangle_{(H^1(\Omega_\varepsilon))', H^1(\Omega_\varepsilon)} \, dt + \int_0^T \int_{\Omega_\varepsilon} \nabla_x y_\varepsilon \nabla_x \psi \, dx \, dt \\ \quad = \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon \psi \, dx \, dt \quad \forall \psi \in L^2(0, T; \mathcal{V}(\Omega_\varepsilon)), \\ \text{(ii)} \quad y_\varepsilon(0) = y_\varepsilon^0, \quad y_\varepsilon'(0) = y_\varepsilon^1 \quad \text{in } \Omega_\varepsilon. \end{cases} \quad (36)$$

The aim of this section is to study the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of the sequence of solutions $(y_\varepsilon)_\varepsilon$, under the following assumptions:

$$\begin{cases} \tilde{y}_\varepsilon^0 \rightharpoonup y^0 & \text{weakly in } H^1(\Omega), \\ \tilde{y}_\varepsilon^1 \rightharpoonup y^1 & \text{weakly in } L^2(\Omega), \\ \tilde{f}_\varepsilon \rightharpoonup f & \text{weakly in } L^2(0, T; L^2(\Omega)). \end{cases} \quad (37)$$

The above convergence together with Lemma 4.1 gives the following proposition.

Proposition 4.1. *Assume (37). Let y_ε be the solution of problem (33). Then, there exists a constant C , independent of ε , such that*

$$\|y_\varepsilon\|_{L^\infty(0, T; \mathcal{V}(\Omega_\varepsilon))} \leq C, \quad (38)$$

$$\|y'_\varepsilon\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \leq C \quad (39)$$

for every ε .

Now, we give the homogenization of the wave equation (33).

Theorem 4.1. *Assume (37). Let y_ε be the solution of the problem (33). Then, we have*

$$\begin{cases} \tilde{y}_\varepsilon \rightharpoonup 0 & \text{weakly } * \text{ in } L^\infty(0, T; \mathcal{V}(\Omega^+)), \\ y_\varepsilon \rightharpoonup y & \text{weakly } * \text{ in } L^\infty(0, T; \mathcal{V}(\Omega^-)), \end{cases} \quad (40)$$

where $y \in L^2(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_{\text{per}}(\Omega^-))$ is the unique solution of the following problem:

$$\begin{cases} y = 0 & \text{in }]0, T[\times \Omega^+, \\ y'' - \Delta y = f & \text{in }]0, T[\times \Omega^-, \\ y = 0 & \text{on }]0, T[\times (P \cup \Sigma), \\ y(0) = y^0, y'(0) = y^1 & \text{in } \Omega^-. \end{cases}$$

Proof. See [19]. \square

4.2. Homogenization for the transposition solution

Let $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) = \Lambda_\varepsilon^{-1}(-u_\varepsilon^1, u_\varepsilon^0) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$ be the initial conditions of the problem (4) which eventually gives the optimal control. Then by Proposition 3.2, it follows that

$$\begin{cases} \varphi_\varepsilon^0 \rightharpoonup \varphi^0 & \text{weakly in } L^2(\Omega^-), \\ \varphi_\varepsilon^1 \rightharpoonup \varphi^1 & \text{weakly in } (\mathcal{V}(\Omega^-))'. \end{cases} \quad (41)$$

Proposition 4.2. *Let φ_ε be the unique solution of problem (4) corresponding to the initial data given above. Then, there exists a subsequence of $\{\varphi_\varepsilon\}$, still denoted by $\{\varphi_\varepsilon\}$ such that as $\varepsilon \rightarrow 0$*

$$\tilde{\varphi}_\varepsilon \rightharpoonup \varphi \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \quad (42)$$

where φ is solution of the problem

$$\begin{cases} \varphi = 0 & \text{in }]0, T[\times \Omega^+, \\ \varphi'' - \Delta\varphi = 0 & \text{in }]0, T[\times \Omega^-, \\ \varphi = 0 & \text{on }]0, T[\times (\Sigma \cup P), \\ \varphi(0) = \varphi^0, \varphi'(0) = \varphi^1 & \text{in } \Omega^-, \\ \varphi \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases} \quad (43)$$

Proof. Estimate (29) provides the existence of a subsequence of $\{\varphi_\varepsilon\}$, still denoted by $\{\varphi_\varepsilon\}$, $\varphi \in L^2(0, T; L^2(\Omega))$ such that

$$\tilde{\varphi}_\varepsilon \rightharpoonup \varphi \quad \text{weakly in } L^2(0, T; L^2(\Omega)). \quad (44)$$

Let $\xi_\varepsilon \in \mathcal{V}(\Omega_\varepsilon)$ be the unique solution of problem

$$\begin{cases} -\Delta\xi_\varepsilon = \varphi_\varepsilon^1 & \text{in } \Omega_\varepsilon, \\ \xi_\varepsilon = 0 & \text{on } P \cup R_\varepsilon, \\ \xi_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases} \quad (45)$$

Let us consider the function

$$\sigma_\varepsilon(x, t) = \int_0^t \varphi_\varepsilon(x, s) \, ds + \xi_\varepsilon(x), \quad (46)$$

which satisfies the problem

$$\begin{cases} \sigma_\varepsilon'' - \Delta\sigma_\varepsilon = 0 & \text{in }]0, T[\times \Omega_\varepsilon, \\ \sigma_\varepsilon = 0 & \text{on }]0, T[\times (P \cup R_\varepsilon), \\ \sigma_\varepsilon(0) = \xi_\varepsilon, \sigma_\varepsilon'(0) = \varphi_\varepsilon^0 & \text{in } \Omega_\varepsilon, \\ \sigma_\varepsilon \text{ } S\text{-periodic} & \text{(with respect to } x') \end{cases} \quad (47)$$

with $(\xi_\varepsilon, \varphi_\varepsilon^0) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$. Moreover, by (45) and (26), there exists a positive constant C , independent of ε such that

$$\|\xi_\varepsilon\|_{\mathcal{V}(\Omega_\varepsilon)} \leq C.$$

Consequently, up to a subsequence, we have

$$\tilde{\xi}_\varepsilon \rightharpoonup \xi = \begin{cases} 0 & \text{weakly in } H_0^1(\Omega^+), \\ \xi^- & \text{weakly in } \mathcal{V}(\Omega^-). \end{cases} \quad (48)$$

Then, by (48) and (44), passing to the limit in weak formulation of problem (45), we have

$$\Delta\xi^- = \varphi^1 \quad \text{in } \Omega^-.$$

Now, applying Theorem 4.1 to problem (47), it results that

$$\begin{cases} \tilde{\sigma}_\varepsilon \rightharpoonup 0 & \text{weakly in } L^2(0, T; L^2(\Omega^+)), \\ \sigma_\varepsilon \rightharpoonup \sigma & \text{weakly in } L^2(0, T; \mathcal{V}(\Omega^-)). \end{cases} \quad (49)$$

And σ is the solution of the homogenized system:

$$\begin{cases} \sigma = 0 & \text{in }]0, T[\times \Omega^+, \\ \sigma'' - \Delta\sigma = 0 & \text{in }]0, T[\times \Omega^-, \\ \sigma = 0 & \text{on }]0, T[\times (\Sigma \cup P), \\ \sigma(0) = \xi^-, \sigma'(0) = \varphi^0 & \text{in } \Omega^-, \\ \sigma \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases}$$

Moreover, by regularity results for hyperbolic equation, we have

$$\sigma \in C([0, T]; \mathcal{V}(\Omega^-)) \cap C^1([0, T]; L^2(\Omega^-)) \cap C^2([0, T]; (\mathcal{V}(\Omega^-))').$$

It follows then that

$$\sigma''(0) = \varphi^1$$

since $\sigma''(0) = \Delta\sigma(0) = \Delta\xi^-$. Then the function $\sigma' = W$ satisfies the problem

$$\begin{cases} W = 0 & \text{in }]0, T[\times \Omega^+, \\ W'' - \Delta W = 0 & \text{in }]0, T[\times \Omega^-, \\ W = 0 & \text{on }]0, T[\times (\Sigma \cup P), \\ W(0) = \varphi^0, W'(0) = \varphi^1 & \text{in } \Omega^-, \\ W \text{ } S\text{-periodic} & \text{(with respect to } x'). \end{cases}$$

Here W is defined in the sense of transposition. By (46), we get

$$\tilde{\sigma}'_\varepsilon = \widetilde{\varphi}_\varepsilon.$$

Moreover, by definition of distributional derivative, one has

$$\int_0^T \int_{\Omega_\varepsilon} \sigma''_\varepsilon z h \, dx \, dt = \int_0^T \int_{\Omega} \tilde{\sigma}_\varepsilon z h'' \, dx \, dt \quad (50)$$

for every $h \in \mathcal{D}((0, T))$. Passing to the limit in (50) as $\varepsilon \rightarrow 0$, using (49) the right-hand side converges to

$$\int_0^T \int_{\Omega} \sigma z h'' \, dx \, dt = - \int_0^T \int_{\Omega} \sigma' z h' \, dx \, dt. \quad (51)$$

Concerning the left-hand side, we have

$$\int_0^T \int_{\Omega_\varepsilon} \sigma_\varepsilon'' z h \, dx \, dt = - \int_0^T \int_{\Omega_\varepsilon} \tilde{\sigma}_\varepsilon' z h' \, dx \, dt = - \int_0^T \int_{\Omega_\varepsilon} \widetilde{\varphi}_\varepsilon z h' \, dx \, dt. \quad (52)$$

Finally, passing to the limit in (50) as $\varepsilon \rightarrow 0$, by (51), (52) and (42), we have

$$W = \varphi.$$

It remains to show that $\varphi = 0$ a.e. in $]0, T[\times \Omega^+$. Since the solution is defined via transposition, we cannot apply Theorem 3 in [37] as in the previous theorem because we do not have the evolution triplet $H_0^1(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon) \subset H^{-1}(\Omega_\varepsilon)$. But, we can deduce it by the following argument. From (44), (48) and (49), by passing to the limit in (46), we get

$$\sigma(x, t) = \int_0^t \varphi(x, s) \, ds + \xi(x).$$

Since $\sigma = 0$ a.e. in $]0, T[\times \Omega^+$ and $\xi = 0$ a.e. in Ω^+ , we get

$$\int_0^t \varphi(x, s) \, ds = 0$$

a.e. in $]0, T[\times \Omega^+$. This shows that $\varphi = 0$ a.e. in $]0, T[\times \Omega^+$. Since the problem (43) admits a unique solution, the convergence (42) holds true for the whole sequence. Proof is complete. \square

5. Proof of Theorem 2.1

Let us consider $(u_\varepsilon^0, u_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$. Let $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$ be the unique solution of equation:

$$A_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1) = (u_\varepsilon^1, u_\varepsilon^0). \quad (53)$$

Let us pose

$$\theta_\varepsilon = -\varphi_\varepsilon, \quad (54)$$

where φ_ε is the unique solution of the problem (4) with initial conditions $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$. By (7), it follows that

$$(-\psi_\varepsilon'(0), \psi_\varepsilon(0)) = (-u_\varepsilon^1, u_\varepsilon^0),$$

where ψ_ε is the unique solution of problem (5). By uniqueness theorem of the solution of problem (2), we obtain

$$u_\varepsilon = \psi_\varepsilon. \quad (55)$$

By final condition of problem (5), (55) and from the continuity of solution, we have

$$u_\varepsilon(T) = 0, \quad u'_\varepsilon(T) = 0.$$

And so, θ_ε is the exact control for system (2). Moreover, by propositions (3.3), (54) and (55), we have that, up to a subsequence,

$$\begin{cases} \tilde{\theta}_\varepsilon \rightharpoonup \theta & \text{weakly in } L^2(0, T; L^2(\Omega)), \\ \tilde{u}_\varepsilon \rightharpoonup u & \text{weakly } * \text{ in } L^\infty(0, T; H^1(\Omega)), \\ \tilde{u}'_\varepsilon \rightharpoonup u' & \text{weakly } * \text{ in } L^2(0, T; L^2(\Omega)), \end{cases} \quad (56)$$

with

$$\theta = \begin{cases} 0 & \text{in }]0, T[\times \Omega^+, \\ -\varphi & \text{in }]0, T[\times \Omega^-. \end{cases} \quad (57)$$

By applying Theorem 4.1 to problem (2) with $f_\varepsilon = \theta_\varepsilon$ and from convergence (56), we obtain

$$\begin{cases} \tilde{u}_\varepsilon \rightharpoonup 0 & \text{weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^+)), \\ u_\varepsilon \rightharpoonup u & \text{weakly } * \text{ in } L^\infty(0, T; H^1(\Omega^-)), \end{cases}$$

where u is the unique solution of the problem (11), with θ given by (57).

Finally, from (55) and (42) by applying Theorem 4.1 to problem (5) with $f_\varepsilon = -\varphi_\varepsilon$, we obtain the limit problem for $\psi = u$ as

$$\begin{cases} \psi = 0 & \text{in }]0, T[\times \Omega^+, \\ \psi'' - \Delta\psi = -\varphi & \text{in }]0, T[\times \Omega^-, \\ \psi = 0 & \text{on }]0, T[\times (P \cup \Sigma), \\ \psi(T) = 0, \quad \psi'(T) = 0 & \text{in } \Omega^-, \end{cases} \quad (58)$$

where $\psi \in C(0, T; \mathcal{V}(\Omega^-)) \cap C^1(0, T; L^2(\Omega^-))$. Moreover, by (55), we have that

$$u(T) = u'(T) = 0.$$

Now we can apply HUM to the wave equation in the domain Ω^- . Define an operator Λ

$$\Lambda: L^2(\Omega^-) \times (\mathcal{V}(\Omega^-))' \rightarrow L^2(\Omega^-) \times \mathcal{V}(\Omega^-)$$

by setting, for all $(\varphi^0, \varphi^1) \in L^2(\Omega^-) \times (\mathcal{V}(\Omega^-))'$,

$$\Lambda(\varphi^0, \varphi^1) = (-\psi'(0), \psi(0)),$$

where ψ is the solution of the problem (58). Note that the operator Λ is an isomorphism by HUM. Let (u^0, u^1) be the initial condition for the problem (11). From the convergence of $u_\varepsilon = \psi_\varepsilon$, it follows that $u^0 = \psi(0)$ and $u^1 = \psi'(0)$. Thus

$$\Lambda(\varphi^0, \varphi^1) = (-u^1, u^0).$$

Hence the limit problem is indeed the exact controllability problem. In other words, $\theta = -\varphi$ is the exact limit control. Further, in the whole analysis the solutions are defined uniquely, we obtain that the whole sequences $(\tilde{\theta}_\varepsilon)$ and (\tilde{u}_ε) converge. The proof of Theorem 2.1 is complete.

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