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Homogenization of boundary optimal control problems with oscillating boundaries

A. K. Nandakumaran [1*], Ravi Prakash[2]

[1] Department of Mathematics,
Indian Institute of Science, Bangalore, India
E-mail:nands@math.iisc.ernet.in

[2] Department of Mathematics, Indian Institute of Science, Bangalore, India,
E-mail: ravi@math.iisc.ernet.in

* *Corresponding Author*. E-mail address: nands@math.iisc.ernet.in

Abstract. This article is devoted to the study of optimal boundary control problems associated with Laplacian posed on a domain having rapidly oscillating boundary. A rectangular region with oscillations on the top boundary is considered as a domain for simplicity. A control is applied on the regular bottom boundary part, away from the oscillatory one. We discuss both, Dirichlet as well as Neumann boundary control problem. In both of the cases the L^2- cost functional is taken into account. A complete asymptotic analysis of the optimality system is obtained and then we derive appropriate error estimates. Homogenization is quite similar in both and is not very difficulty. But the major contribution of the work in this paper is the error analysis and we need to construct different test functions for Dirichlet and Neumann boundary conditions.

1 Introduction

Optimal control problems whose state equations are described by Partial Differential Equations (PDEs) have applications in various areas and it pursued intensely in the scientific world. The subject is quite matured, but still it has tremendous scope for doing research. See few reference in this direction [6], [7], [13], [19], [21], [22], [29], [30]. If such problems are posed on a highly oscillating media, then an asymptotic analysis (homogenization) is call for. There are plenty of such oscillating

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domains like; composite media, porous media, domains with rough boundaries and so on. The problems defined on domains with oscillating boundaries are rather delicate topics and these are more so with optimal control problems.

The boundary value problems, in particular control or controllability problems involving highly oscillating boundaries or interfaces have various applications in industrial problems such as flows with rough boundaries (rough boundaries can be modelled as oscillating boundaries), rough interface, air flow through compression systems in turbo machines such as jet engine. For example, the last one can be modelled by Viscous- Moore- Schwartz equation derived from Scaled Navier-Stokes equations (see [9], [23], [24]). Here the pitch and size of the rotor - stator pair of blades in the engine provides a small parameter compared to the size of the engine which is oscillatory as well as rotating (moving). The motion of the stator and rotor blades in the compressor produces turbulent flow on a fast time scale. When the engine operates close to the optimal parameters, the flow become unstable. This model motivated to look into control problems described by PDEs of evolution type such as heat, Navier-Stokes equations etc. As the problem is quite complicated, we wish to begin with a sample problem of Laplacian with an oscillating boundary and the control acts on a part of the boundary which is away from the oscillating one, though the aim is to consider controls acting on the moving boundaries. Apart from the boundary control one can also think of distributed control over some region in the domain. The authors recently studied such a homogenization problem for the Laplacian in [27].

As the problem in such generality is extremely difficult, in this article, our aim is to consider an optimal boundary control problem associated with the Laplacian with a rapidly oscillating boundary. For simplicity, we consider nearly a rectangular region with oscillating part on one side of the region to be made precise later. Basically the oscillating part can be viewed as slabs of width $\varepsilon > 0$, fixed to a rectangular region. Such regions and other type of domains with oscillating boundaries are considered in the literature for studying homogenization of PDE problems (see [1], [2], [4], [10], [11], [14], [15], [28]) and the references there in. But we do not see much literature regarding optimal control/ controllability problems in domains with oscillating boundaries, but regarding the general homogenization of optimal control/ controllability with oscillating coefficients and porous domains, we cite some of the references as [17], [18], [25], [26]. For general homogenization, we refer to [8], [12], [16], [30].

The layout of the paper is as follows. In this section, we present the notations and definition of oscillatory domain. In this article, we study the control problem with L^2- *cost functional* with controls acting through the boundary. In fact, we consider two situations, namely Dirichlet boundary control and Neumann boundary control. One can also consider other types of cost functionals which we will not be doing it here. Optimality systems are derived by a usual procedure. This is done in section 2. The section 3 is devoted to the homogenization of the problems under consideration. We first prove that the optimality system converges to a system of equations. We also prove the convergence of the cost functional and optimal control proving thus that the limit system obtained is indeed the optimality system. But the main and interesting contribution of this article is the derivation of error estimates known as corrector results. This is done in section 5, whereas in section 4 some preliminary results are recalled. Though the homogenization follows in a same pattern for Dirichlet as well as Neumann, one requires to construct different test functions for error estimates. Conclusions are available in section 6.

1.1 Notations

We consider the two dimensional domain Ω_ε as in the Figure 1. It consists of two parts. One is related to $\varepsilon > 0$, a small parameter, through the $\varepsilon-$dependent oscillations in its boundary. We denote it by Ω_ε^+. Let us point out the fact that ε is in the form of $\{\frac{1}{n}\}_{n\in\mathbb{Z}^+}$ which will be going towards zero during the analysis in this article. Another part of Ω_ε is a fixed and almost rectangular region Ω^-.

We will first try to understand Ω_ε^+, mathematically. Let $L > 0$ and $0 < a < b < L$. Assume η_ε be the εL-periodic function defined on $[0,L]$ by periodic extension of

$$\eta_\varepsilon(x_1) = \begin{cases} M' & \text{if} \quad x_1 \in (\varepsilon a, \varepsilon b), \\ M & \text{if} \quad x_1 \in [0, \varepsilon L] \setminus (\varepsilon a, \varepsilon b) \end{cases}$$

with $M' > M$. The graph of η_ε provides the oscillating boundary. Now we can write $\Omega_\varepsilon^+ = \{(x_1,x_2) \in \mathbb{R}^2 :\ 0 < x_1 < L,\ M < x_2 < \eta_\varepsilon(x_1)\}$. Now let us move ahead to understand Ω^-, completely. Consider a smooth and periodic function $g : \mathbb{R} \to \mathbb{R}$ with period L such that $m := \max\{|g(x_1)|,\ x_1 \in I\} < M$. The smoothness assumption on g, we demanded, is extremely important for the regularity results, proved in Section 5. Ω^- can be described with the help of g, as $\Omega^- = \{(x_1,x_2) :\ 0 < x_1 < L,\ g(x_1) < x_2 < M\}$. The top boundary Γ_u of Ω^- can be described as $\Gamma_u = \{(x_1,M) :\ 0 \le x_1 \le L\}$. The bottom boundary and the side boundary of Ω^- are $\Gamma_b = \{(x_1,x_2) :\ x_2 = g(x_1),\ x_1 \in I\}$ and

$$\Gamma_s = \{(0,x_2) :\ g(0) \le x_2 \le M\} \cup \{(L,x_2) :\ g(L) \le x_2 \le M\},$$

respectively. Finally, we can write $\Omega_\varepsilon = Interior\{\Omega^- \cup \Omega_\varepsilon^+ \cup \Gamma_u\}$. Here $Interior\{\cdot\}$ is the interior of the set "$\cdot$" in $\mathbb{R}^2$ with respect to lebesgue measure. For the shake of completeness, we need to notify oscillatory boundary part of domain Ω_ε as

$$\gamma_\varepsilon = \{Boundary\{\Omega_\varepsilon\} \setminus \{\Gamma_s \cup \Gamma_b\}\} \cup (0,M) \cup (L,M)$$

and full domain as $\Omega = \{(x_1,x_2) :\ 0 < x_1 < L,\ g(x_1) < x_2 < M'\}$. Here $Boundary\{\cdot\}$ is the boundary of the set "$\cdot$" in $\mathbb{R}^2$ with respect to lebesgue measure.

In a nutshell, Ω_ε can be viewed as the bi-dimensional section of a more realistic solid cube in which a large number of small vertical slabs of small cross section are attached on the top. The boundary $\partial\Omega_\varepsilon$ can be decomposed as $\partial\Omega_\varepsilon = \Gamma_b \cup \Gamma_s \cup \gamma_\varepsilon$, where γ_ε is the contribution from the periodic strips. One may be interested in moving oscillating domains of the form $\eta\left(t, \frac{x}{\varepsilon}\right)$. In this paper, we do not discuss the analysis in such domains.

Let us introduce some convention and also recall some very famous and useful related spaces. On the vertical boundary Γ_s, we always assume periodic conditions throughout the paper. Let $H^m_{per}(\Omega_\varepsilon)$ and $L^2_{per}(\Omega_\varepsilon)$ be, respectively, represents the $H^m(\Omega_\varepsilon)$ and $L^2(\Omega_\varepsilon)$ functions which are Γ_s-periodic, namely it takes same values on both sides of Γ_s. In what follows, we consider Γ_s periodic functions.

Remark 1.1. We have taken this special domain Ω_ε with oscillations of order 1 on one part of the boundary to understand the behavior of boundary optimal control problems in such domains with control applied on the regular boundary, away from the oscillatory one. One can indeed consider other type of domains and control application set up as current one. We believe, at least same homogenization results as like contemporary cases can be proved, but we will not discuss it here. □

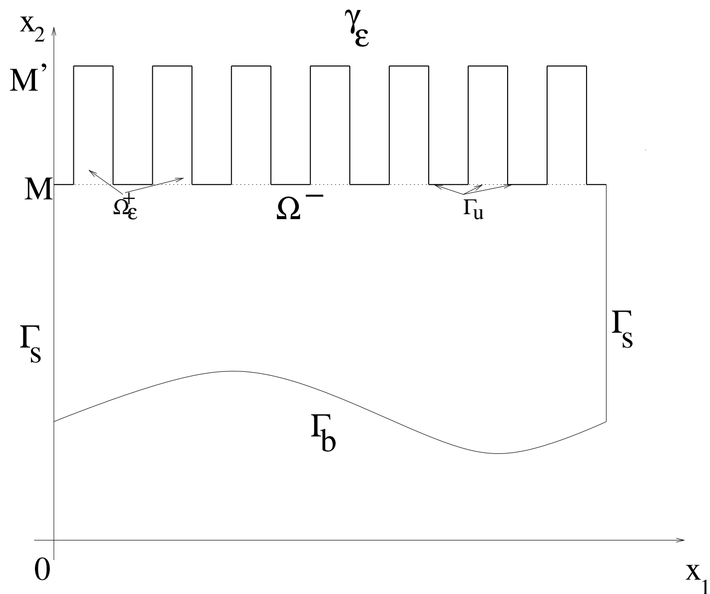

Fig. 1 Ω_ε

2 Problem Description and Optimality System

In this section, we describe problems with Dirichlet boundary control as well as Neumann boundary control. L^2 cost functional is considered in both cases. Appropriate a priori estimates and the corresponding optimality systems are derived using standard theory available in the literature.

2.1 Dirichlet Boundary Control

In this subsection, we consider the Dirichlet boundary optimal control problem, where the control is acting on the lower surface Γ_b such that

$$-\Delta y_\varepsilon = f \text{ in } \Omega_\varepsilon,\ y_\varepsilon = 0 \text{ on } \gamma_\varepsilon,\ y_\varepsilon = u \text{ on } \Gamma_b,\ y_\varepsilon \text{ is } \Gamma_s-\text{periodic}, \tag{2.1}$$

where f is the given source function defined on Ω and $u \in L^2_{per}(\Gamma_b)$ is a Dirichlet control function. Indeed the more general realistic controls are L^2 controls, by trace theorem, we will not have solutions to the above problem using Lax-Milgram in the usual weak formulation. The solution considered here are known in the sense of *transposition*. See [29].

Definition 2.1. A function $y_\varepsilon \in L^2_{per}(\Omega_\varepsilon)$ is a solution to equation (2.1) if and only if

$$\int_{\Omega_\varepsilon} y_\varepsilon \phi = \int_{\Omega_\varepsilon} f \check{y}_\varepsilon - \int_{\Gamma_b} u \frac{\partial \check{y}_\varepsilon}{\partial \nu},\ \text{for all } \phi \in L^2_{per}(\Omega_\varepsilon) \tag{2.2}$$

and $\check{y}_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ is the solution to problem $-\Delta \check{y}_\varepsilon = \phi$ in Ω_ε, $\check{y}_\varepsilon = 0$ on $\gamma_\varepsilon \cup \Gamma_b$.

Consider a smooth function $h \in C^2_{per}(\Omega^-)$ such that $h|_{\Gamma_u} = 0$ and $h|_{\Gamma_b} = 1$. In fact, choose h such that it vanishes in the neighborhood of Γ_u and is identically 1 in the neighborhood of Γ_b. Then the product function $h\check{y}_\varepsilon$ satisfies

$$-\Delta (h\check{y}_\varepsilon) = \varphi \text{ in } \Omega^-,\ h\check{y}_\varepsilon = 0 \text{ on } \Sigma \cup \Gamma_b,\ h\check{y}_\varepsilon \in H^1_{per}\left(\Omega^-\right), \tag{2.3}$$

where $\varphi = -(\Delta h)\phi - \nabla h \cdot \nabla \phi - h(\Delta \phi)$. Regularity results available for the rectangular domain Ω^- provides $h\check{y}_\varepsilon \in H^2_{per}(\Omega^-)$. By classical weak formulation of (2.3) and Poincaré inequality, it follows that $\|h\check{y}_\varepsilon\|_{H^2(\Omega^-)} \leq C$, where C are positive constants independent of ε. Thus $\dfrac{\partial \check{y}_\varepsilon}{\partial \nu} = \dfrac{\partial (h\check{y}_\varepsilon)}{\partial \nu} \in H^{1/2}_{per}(\Gamma_b)$ and by trace theorem

$$\left\| \frac{\partial \check{y}_\varepsilon}{\partial \nu} \right\|_{L^2(\Gamma_b)} = \left\| \frac{\partial (h\check{y}_\varepsilon)}{\partial \nu} \right\|_{L^2(\Gamma_b)} \leq C \|h\check{y}_\varepsilon\|_{H^2(\Omega^-)} \leq C, \tag{2.4}$$

where C is a positive constant independent of ε. So, the term $\displaystyle\int_{\Gamma_b} u \frac{\partial \check{y}_\varepsilon}{\partial \nu}$ introduced in (2.2) is well-defined and hence the Definition 2.1 is meaningful.

It is well known that for every $f \in L^2_{per}(\Omega)$ and $u \in L^2_{per}(\Gamma_b)$, by transposition method (see [5], [20], [29]), the equation (2.1) admits an unique solution $y_\varepsilon = y_\varepsilon(u) \in L^2_{per}(\Omega_\varepsilon)$. In this paper $\tilde{\cdot}$ will denote the extension of the underlined function "$\cdot$" to whole of Ω by 0. Thus $\tilde{y}_\varepsilon \in L^2_{per}(\Omega)$. Moreover the operator $(f,u) \longmapsto \widetilde{y_\varepsilon}$ is linear and continuous from $L^2_{per}(\Omega) \times L^2_{per}(\Gamma_b)$ into $L^2_{per}(\Omega)$, i.e. $\|\tilde{y}_\varepsilon\|_{L^2(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Gamma_b)}\right)$, where $C > 0$ is independent of ε. For some $\beta > 0$ which also acts as regularization parameter and a given desired state $y_d \in L^2_{per}(\Omega)$, we wish to consider the L^2-cost functional

$$J_{1,\varepsilon}(u) = J_{1,\varepsilon}(y_\varepsilon(u), u) := \frac{1}{2} \int_{\Omega_\varepsilon} (y_\varepsilon - y_d)^2 + \frac{\beta}{2} \int_{\Gamma_b} u^2,$$

where $u \in L^2_{per}(\Gamma_b)$ and $y_\varepsilon = y_\varepsilon(u) \in L^2_{per}(\Omega_\varepsilon)$ is the solution of (2.1) in the sense of (2.2). We wish to analyze the following optimal control problem;

$$\inf\{J_{1,\varepsilon}(y_\varepsilon, u) \,|\, (y_\varepsilon, u) \in L^2_{per}(\Omega_\varepsilon) \times L^2_{per}(\Gamma_b), (y_\varepsilon, u) \text{ satisfies (2.1)}\}. \tag{2.5}$$

For fixed $\varepsilon > 0$, it is a standard quadratic optimal control problem (see [29]) and hence the problem (2.5) admits a unique solution $\overline{u}_\varepsilon$ and the corresponding solution to equation (2.1) is denoted by $\overline{y}_\varepsilon = \overline{y}_\varepsilon(\overline{u}_\varepsilon)$. We call $(\overline{y}_\varepsilon, \overline{u}_\varepsilon)$, the optimal solution, where $\overline{u}_\varepsilon$ and $\overline{y}_\varepsilon$ are known as optimal control and $\overline{y}_\varepsilon$ optimal state corresponding to (2.5), respectively. The characterization of $\overline{u}_\varepsilon$ can also be established as $\overline{u}_\varepsilon = \dfrac{1}{\beta} \dfrac{\partial \bar{z}_\varepsilon}{\partial \nu}$, where $\bar{z}_\varepsilon$, known as adjoint state or co-state, is the solution of the adjoint problem

$$-\Delta \bar{z}_\varepsilon = \overline{y}_\varepsilon - y_d \text{ in } \Omega_\varepsilon,\ \bar{z}_\varepsilon = 0 \text{ on } \gamma_\varepsilon \cup \Gamma_b,\ \bar{z}_\varepsilon \in H^1_{per}(\Omega_\varepsilon). \tag{2.6}$$

Since we will be seeing that y_ε is of order ε i.e. $O(\varepsilon)$ in the upper part Ω^+_ε, it is reasonable to take $supp\ y_d \subset \overline{\Omega^-}$.

Remark 2.1. Boundary optimal control may be quite interesting for much more general elliptic operators and/or general cost functionals. We planned to discuss these issues at a later paper. □

In the current scenario one has the following well established theorem ([5], [20], [29]).

Theorem 2.1. Let $f,\ y_d \in L^2_{per}(\Omega)$ with $supp\ y_d \subset \overline{\Omega^-}$, let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ is the optimal solution to equation (2.5) then $\bar{u}_\varepsilon = \dfrac{1}{\beta} \dfrac{\partial \bar{z}_\varepsilon}{\partial \nu}$, where $\bar{z}_\varepsilon$ is the solution to equation (2.6). Conversely, if a pair $(\hat{y}_\varepsilon, \hat{z}_\varepsilon) \in L^2_{per}(\Omega_\varepsilon) \times H^1_{per}(\Omega_\varepsilon)$ obeys the system

$$\begin{cases} -\Delta \hat{y}_\varepsilon = f \text{ in } \Omega_\varepsilon,\ \hat{y}_\varepsilon = 0 \text{ on } \gamma_\varepsilon,\ \hat{y}_\varepsilon = \dfrac{1}{\beta} \dfrac{\partial \hat{z}_\varepsilon}{\partial \nu} \text{ on } \Gamma_b, \\ -\Delta \hat{z}_\varepsilon = \hat{y}_\varepsilon - y_d \text{ in } \Omega_\varepsilon,\ \hat{z}_\varepsilon = 0 \text{ on } \gamma_\varepsilon \cup \Gamma_b \end{cases} \tag{2.7}$$

then the pair $\left(\hat{y}_\varepsilon, \frac{1}{\beta}\frac{\partial \hat{z}_\varepsilon}{\partial \nu}\right)$ is the optimal solution to problem (2.5).

2.2 Neumann Boundary Control

In this subsection, we introduce the Neumann boundary optimal control problem, where the control is acting on the lower surface Γ_b through the Neumann data as follows

$$-\Delta y_\varepsilon = f \text{ in } \Omega_\varepsilon,\ y_\varepsilon = 0 \text{ on } \gamma_\varepsilon,\ \frac{\partial y_\varepsilon}{\partial \nu} = u \text{ on } \Gamma_b,\ y_\varepsilon \text{ is } \Gamma_s - \text{periodic}, \tag{2.8}$$

where $u \in L^2_{per}(\Gamma_b)$ is a Neumann control function. Though, there may not be much difference in the homogenization analysis, the error estimates seems to be different. It is well known that for every $f \in L^2_{per}(\Omega)$ and $u \in L^2_{per}(\Gamma_b)$, the problem (2.8) admits a unique solution $y_\varepsilon = y_\varepsilon(u_\varepsilon) \in H^1_{per}(\Omega_\varepsilon)$. Note that in this case the solution is defined via standard weak formulation. Notice $\tilde{y}_\varepsilon \in H^1_{per}(\Omega)$. Moreover, the operator $(f, u_\varepsilon) \longmapsto \widetilde{y_\varepsilon}$ is linear and continuous from $L^2_{per}(\Omega) \times L^2_{per}(\Gamma_b)$ into $H^1_{per}(\Omega)$, i.e. $\|\tilde{y}_\varepsilon\|_{H^1(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Gamma_b)}\right)$, where $C > 0$ is independent of ε. For some $\beta > 0$ and a given desired state $y_d \in L^2_{per}(\Omega)$ with $supp\ y_d \subset \overline{\Omega^-}$, we wish to consider the same L^2- cost functional as in Dirichlet case

$$J_{2,\varepsilon}(y_\varepsilon, u) = \frac{1}{2}\int_{\Omega_\varepsilon} (y_\varepsilon - y_d)^2 + \frac{\beta}{2}\int_{\Gamma_b} u^2.$$

The problem is to find $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ which minimizes the cost functional, namely,

$$J_{2,\varepsilon}(\bar{u}_\varepsilon) = J_{2,\varepsilon}(\bar{y}_\varepsilon, \bar{u}_\varepsilon) = \inf_{u \in L^2(\Gamma_b)} J_{2,\varepsilon}(y_\varepsilon, u). \tag{2.9}$$

Here $\bar{y}_\varepsilon,\ y_\varepsilon$, respectively, the solutions corresponding to $\bar{u}_\varepsilon,\ u$ of the problem (2.8).

For each $\varepsilon > 0$, the minimization problem (2.9) is quite standard and it admits a unique solution $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ (see [5], [20], [29])). We call $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$, the optimal solution, where $\bar{u}_\varepsilon$ is the optimal control and $\bar{y}_\varepsilon$, the optimal state. Further, it can be characterized using the adjoint state (co-state) $\bar{z}_\varepsilon$. Let $\bar{z}_\varepsilon$ solves the adjoint problem

$$-\Delta \bar{z}_\varepsilon = \bar{y}_\varepsilon - y_d \text{ in } \Omega_\varepsilon,\ \bar{z}_\varepsilon = 0 \text{ on } \gamma_\varepsilon,\ \frac{\partial \bar{z}_\varepsilon}{\partial \nu} = 0 \text{ on } \Gamma_b,\ \bar{z}_\varepsilon \in H^1_{per}(\Omega_\varepsilon). \tag{2.10}$$

The optimal control is then given by $\overline{u}_\varepsilon = -\frac{1}{\beta}\bar{z}_\varepsilon|_{\Gamma_b}$. The following theorem is well established.

Theorem 2.2. Assume $f \in L^2_{per}(\Omega)$ and $y_d \in L^2_{per}(\Omega)$ with $supp\ y_d \subset \overline{\Omega^-}$. Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ be the optimal solution to equation (2.9) then $\bar{u}_\varepsilon = -\frac{1}{\beta}\bar{z}_\varepsilon|_{\Gamma_b}$, where $\bar{z}_\varepsilon$ is the solution to equation (2.10). Conversely, if a pair $(\hat{y}_\varepsilon, \hat{z}_\varepsilon) \in H^1_{\text{per}}(\Omega_\varepsilon) \times H^1_{\text{per}}(\Omega_\varepsilon)$ obeys the system

$$\begin{cases} -\Delta \hat{y}_\varepsilon = f \ \text{ in } \ \Omega_\varepsilon,\ \hat{y}_\varepsilon = 0 \ \text{ on } \ \gamma_\varepsilon,\ \dfrac{\partial \hat{y}_\varepsilon}{\partial \nu} = -\dfrac{1}{\beta}\hat{z}_\varepsilon \text{ on } \ \Gamma_b, \\ -\Delta \hat{z}_\varepsilon = \hat{y}_\varepsilon - y_d \ \text{ in } \ \Omega_\varepsilon,\ \hat{z}_\varepsilon = 0 \ \text{ on } \ \gamma_\varepsilon,\ \dfrac{\partial \hat{z}_\varepsilon}{\partial \nu} = 0 \text{ on } \ \Gamma_b \end{cases} \tag{2.11}$$

then the pair $\left(\hat{y}_\varepsilon, -\frac{1}{\beta}\hat{z}_\varepsilon|_{\Gamma_b}\right)$ is the optimal solution to problem (2.9).

The first aim of this article is to study the asymptotic behavior of $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ as $\varepsilon \to 0$ and obtain the limit equations for Dirichlet and Neumann boundary control. Using the convergence of the optimality system, we in fact show that the minimization problem will converge to a suitable minimization problem. This is done in Section 3 for both cases of Dirichlet as well as Neumann boundary control. The other important aspect of the article is to prove some corrector estimates. We show some H^1-estimates in terms of the L^2- estimates using certain test functions. The test functions used for deriving corrector results are different for Dirichlet and Neumann boundary control cases.

Remark 2.2. For Neumann boundary control, we took $supp\ y_d \subset \overline{\Omega^-}$ because of the same reason mentioned in Dirichlet boundary control case. Although, the optimality discussion and homogenization analysis have nothing to do with this assumption i.e. optimality and homogenization results are still valid without this assumption.

3 Homogenization Theorem

3.1 Estimates (Dirichlet Boundary Control)

Let $y_\varepsilon = y_\varepsilon(0)$ be the solution of the problem

$$-\Delta y_\varepsilon = f \text{ in } \Omega_\varepsilon,\ y_\varepsilon = 0 \text{ on } \gamma_\varepsilon \cup \Gamma_b,\ y_\varepsilon \in H^1_{per}(\Omega_\varepsilon), \tag{3.1}$$

where $f \in L^2_{per}(\Omega)$. By simply writing the classical weak formulation of equation (3.1), we will have $\|y_\varepsilon(0)\|_{H^1(\Omega_\varepsilon)} \leq C$, with a positive constant C, independent of ε. Since $y_\varepsilon(0)$ is a solution of (2.1) corresponding to the control $u = 0$, using the optimality of $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ for the cost function $J_{1,\varepsilon}(y_\varepsilon, u_\varepsilon)$, we can easily conclude $\|\bar{u}_\varepsilon\|_{L^2(\Gamma_b)} \leq C_1,\ \|\bar{y}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_2$, with positive constants independent of ε, $C_1,\ C_2$. Let $\bar{z}_\varepsilon$ be the solution of problem

$$-\Delta \bar{z}_\varepsilon = \bar{y}_\varepsilon - y_d \text{ in } \Omega_\varepsilon,\ \bar{z}_\varepsilon = 0 \text{ on } \gamma_\varepsilon \cup \Gamma_b,\ \bar{z}_\varepsilon \in H^1_{per}(\Omega_\varepsilon). \tag{3.2}$$

Classical weak formulation of (3.2) and boundedness of $\bar{y}_\varepsilon$ in $L^2(\Omega_\varepsilon)$ yields $\|\tilde{\bar{z}}_\varepsilon\|_{H^1(\Omega)} \leq C$, where C is a positive constant independent of ε. We get more regularity for $\bar{u}_\varepsilon$ and $\bar{y}_\varepsilon$ which is given in the following theorem.

Theorem 3.1. [Regularity]: Let $(\bar{y}_\varepsilon,\ \bar{u}_\varepsilon)$ be the optimal solution of the problem (2.5), then $\bar{u}_\varepsilon \in H^{1/2}_{per}(\Gamma_b),\ \bar{y}_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ and there exists positive constants $C_1,\ C_2$ independent of ε such that $\|\bar{u}_\varepsilon\|_{H^{1/2}(\Gamma_b)} \leq C_1,\ \|\tilde{\bar{y}}_\varepsilon\|_{H^1(\Omega)} \leq C_2$.

Proof. Recall $h \in C^2_{per}(\Omega^-)$, introduced in subsection 2.1. The product function $h\bar{z}_\varepsilon$ satisfies

$$-\Delta(h\bar{z}_\varepsilon) = f_\varepsilon \text{ in } \Omega^-,\ h\bar{z}_\varepsilon = 0 \text{ on } \Sigma \cup \Gamma_b,\ h\bar{z}_\varepsilon \in H^1_{per}\left(\Omega^-\right),$$

where $f_\varepsilon = -(\Delta h)\bar{z}_\varepsilon - \nabla h \cdot \nabla \bar{z}_\varepsilon - h(\bar{y}_\varepsilon - y_d)$. Thanks to the uniform boundedness of $\bar{y}_\varepsilon$ and $\tilde{\bar{z}}_\varepsilon$ in the space $L^2(\Omega_\varepsilon)$ and $H^1(\Omega)$ respectively, we can conclude f_ε is uniformly bounded in $L^2_{per}(\Omega^-)$. Hence using regularity for the rectangular domain Ω^- we get $h\bar{z}_\varepsilon \in H^2_{per}(\Omega^-)$ and $\|h\bar{z}_\varepsilon\|_{H^2(\Omega^-)} \leq C\|f_\varepsilon\|_{L^2(\Omega^-)} \leq C$, with a positive constant C, independent of ε. Thus $\dfrac{\partial z_\varepsilon}{\partial \nu} = \dfrac{\partial (hz_\varepsilon)}{\partial \nu} \in H^{1/2}_{per}(\Gamma_b)$ and by trace theorem

$$\|\bar{u}_\varepsilon\|_{H^{1/2}(\Gamma_b)} = \frac{1}{\beta}\left\|\frac{\partial z_\varepsilon}{\partial \nu}\right\|_{H^{1/2}(\Gamma_b)} = \frac{1}{\beta}\left\|\frac{\partial (h z_\varepsilon)}{\partial \nu}\right\|_{H^{1/2}(\Gamma_b)} \leq C\|h\bar{z}_\varepsilon\|_{H^2(\Omega^-)} \leq C_1, \tag{3.3}$$

where C and C_1 is positive constant independent of ε. Now, we can declare the validity of classical weak formulation of equation (2.1) with $u = \bar{u}_\varepsilon$ and the estimate (3.3) will easily provide the required H^1- estimate on $\bar{y}_\varepsilon$.

We can now provide the homogenization theorem. For that, first let us introduce the limit problem.
Limit Problem: Given $f \in L^2_{per}(\Omega)$ and $u \in L^2_{per}(\Gamma_b)$, let $y \in L^2_{per}(\Omega^-)$ solves the problem

$$-\Delta y = f \text{ in } \Omega^-,\ y = 0 \text{ on } \Gamma_u,\ y = u \text{ on } \Gamma_b \tag{3.4}$$

As earlier, we seek the solution in the sense of transposition. Then consider the minimization problem

$$\inf\left\{J_1(y(u),u) : (y(u),u) \in L^2_{per}\left(\Omega^-\right) \times L^2(\Gamma_b) \ \text{ satisfies } (3.4)\right\}, \tag{3.5}$$

where the cost functional is given by

$$J_1(y,u) = \frac{1}{2}\int_{\Omega^-} (y - y_d)^2 + \frac{\beta}{2}\int_{\Gamma_b} u^2. \tag{3.6}$$

The problem (3.5) has a unique solution $(\bar{y},\bar{u}) \in L^2_{per}(\Omega^-) \times L^2_{per}(\Gamma_b)$ and satisfies the system (3.4). In fact, $\bar{u}$ can be characterized using the adjoint system and is given by $\bar{u} = \dfrac{1}{\beta}\dfrac{\partial \bar{z}}{\partial \nu}$ and $\bar{z} \in H^1_{per}(\Omega^-)$ is the solution of the problem

$$-\Delta \bar{z} = \bar{y} - y_d \text{ in } \Omega^-,\ \bar{z} = 0 \text{ on } \Gamma_u \cup \Gamma_b. \tag{3.7}$$

As earlier, one use the regularity of the adjoint system to see that, the optimal solution $(\bar{y},\bar{u}) \in H^1_{per}(\Omega^-) \times H^{1/2}_{per}(\Gamma_b)$. Hence the solution a-posteriori can be defined as a usual weak solution.

Theorem 3.2 (Homogenization). Let $(\bar{y}_\varepsilon,\ \bar{u}_\varepsilon,\ \bar{z}_\varepsilon)$ be the optimal solution of the system (2.1) and (2.6) with respect to the cost function $J_\varepsilon(y_\varepsilon,u_\varepsilon)$, then

$$\bar{u}_\varepsilon \rightharpoonup \bar{u} \text{ weakly in } H^{1/2}_{per}(\Gamma_b),\ \tilde{\bar{y}}_\varepsilon \rightharpoonup \tilde{\bar{y}} \text{ weakly in } H^1_{per}(\Omega),\ \tilde{\bar{z}}_\varepsilon \longrightarrow \tilde{\bar{z}} \text{ strongly in } H^1_{per}(\Omega). \tag{3.8}$$

$(\bar{y},\bar{u})$ solves the minimization problems (3.5), which also means that $J_{1,\varepsilon}(\bar{y}_\varepsilon,\bar{u}_\varepsilon) \longrightarrow J_1(\bar{y},\bar{u})$.

Proof. Theorem 3.1 and the uniform boundedness of $\tilde{\bar{z}}_\varepsilon$ in $H^1(\Omega)$ suggest that there exist $(u_0,y_0,z_0) \in H^{1/2}_{per}(\Gamma_b) \times H^1_{per}(\Omega) \times H^1_{per}(\Omega)$ such that $\bar{u}_\varepsilon \rightharpoonup u_0$ weakly in $H^{1/2}_{per}(\Gamma_b)$ and
$\tilde{\bar{y}}_\varepsilon, \tilde{\bar{z}}_\varepsilon \rightharpoonup y_0, z_0$ weakly in $H^1_{per}(\Omega)$, respectively, upto a subsequence, stll denoted by ε. The limits y_0 and z_0 satisfies (3.4) and (3.7) respectively. Compact embedding of $H^1_{per}(\Omega)$ in $L^2_{per}(\Omega)$ implies $\tilde{\bar{y}}_\varepsilon, \tilde{\bar{z}}_\varepsilon \longrightarrow y_0, z_0$ strongly in $L^2_{per}(\Omega)$ respectively. This implies $y_0 = z_0 = 0$ in Ω^+ because $\chi_{\Omega^+_\varepsilon} \rightharpoonup \frac{b-a}{L}$ weakly* in $L^\infty(\Omega^+)$ and we can write $\tilde{\bar{y}}_\varepsilon = \tilde{\bar{y}}_\varepsilon \chi_{\Omega^+_\varepsilon}$, $\tilde{\bar{z}}_\varepsilon = \tilde{\bar{z}}_\varepsilon \chi_{\Omega^+_\varepsilon}$. But $y_0, z_0 \in H^1_{per}(\Omega)$, hence $y_0 = z_0 = 0$ on Γ_u.

Weak formulation of (2.1) gives us $\int_{\Omega^-} \nabla \tilde{\bar{y}}_\varepsilon \cdot \nabla \varphi = \int_{\Omega^-} f\varphi$ for all smooth function φ with compact support inside Ω^-. So we can say y_0 and u_0 satisfies (3.4). Similarly we can prove z_0 and y_0 satisfies the problem (3.7). By using z_ε and z_0 as a test functions in (3.2) and (3.7), respectively, we will have

$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \mid \nabla \bar{z}_\varepsilon \mid^2 = \int_{\Omega^-} \mid \nabla z_0 \mid^2$. It implies $\tilde{\bar{z}}_\varepsilon \longrightarrow z_0$ strongly in $H^1_{per}(\Omega)$. From here, it is not difficult to see that $u_0 = \dfrac{1}{\beta}\dfrac{\partial z_0}{\partial \nu}$ and hence the limits y_0, u_0 and z_0 satisfies the limit optimality system and also the cost functional converges to the limit cost functional.

Thus by the uniqueness, we indeed see that $y_0 = \tilde{\bar{y}}$, $u_0 = \bar{u}$ and $z_0 = \tilde{\bar{z}}$.

3.2 Estimates (Neumann Boundary Control)

In the case of Neumann boundary control, the homogenization analysis is more or less similar to that of Dirichlet boundary control. Hence we quickly go through this section. Let $y_\varepsilon = y_\varepsilon(0)$ be solution of problem

$$-\Delta y_\varepsilon = f \text{ in } \Omega_\varepsilon,\; y_\varepsilon = 0 \text{ on } \gamma_\varepsilon,\; \frac{\partial y_\varepsilon}{\partial \nu} = 0 \text{ on } \Gamma_b,\; y_\varepsilon \in H^1_{per}(\Omega_\varepsilon), \tag{3.9}$$

where $f \in L^2_{per}(\Omega)$. By simply writing the classical weak formulation of equation (3.9), we will have $\|y_\varepsilon(0)\|_{H^1(\Omega_\varepsilon)} \leq C$, with a positive constant C, independent of ε. This implies $\|\bar{u}_\varepsilon\|_{L^2(\Gamma_b)} \leq C_1$ and $\|\bar{y}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_2$ because of the optimality of $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ for the cost function $J_\varepsilon(y_\varepsilon, u_\varepsilon)$. Here C_1, C_2 are positive constants independent of ε. Let $\bar{z}_\varepsilon$ be the solution of the problem

$$-\Delta \bar{z}_\varepsilon = \bar{y}_\varepsilon - y_d \text{ in } \Omega_\varepsilon,\; \bar{z}_\varepsilon = 0 \text{ on } \gamma_\varepsilon,\; \frac{\partial \bar{z}_\varepsilon}{\partial \nu} = 0 \text{ on } \Gamma_b,\; \bar{z}_\varepsilon \in H^1_{per}(\Omega_\varepsilon). \tag{3.10}$$

Classical weak formulation of (3.10) and uniform L^2-boundedness of $\bar{y}_\varepsilon$ give $\|\tilde{\bar{z}}_\varepsilon\|_{H^1(\Omega)} \leq C$ with C, a positive constant independent of ε. For $\bar{u}_\varepsilon$ and $\bar{y}_\varepsilon$ we get more regularity than we got earlier. Thanks to the uniform H^1-boundedness of $\tilde{\bar{z}}_\varepsilon$, we can conclude by trace theorem $\|\bar{u}_\varepsilon\|_{H^{1/2}(\Gamma_b)} = \dfrac{1}{\beta}\|\bar{z}_\varepsilon\|_{H^{1/2}(\Gamma_b)} \leq C$, where C is positive constant independent of ε. By the classical weak formulation of equation (2.8) and uniform $H^{1/2}$-estimate of $\bar{u}_\varepsilon$, it is easy to see $\|\tilde{\bar{y}}_\varepsilon\|_{H^1(\Omega)} \leq C$, where C is a positive constant independent of ε. □

Now we are in a situation to provide the homogenization theorem.
Limit Problem: Given $f \in L^2_{per}(\Omega)$ and $u \in L^2_{per}(\Gamma_b)$, let $y \in H^1_{per}(\Omega^-)$ solves the problem

$$-\Delta y = f \text{ in } \Omega^-,\; y = 0 \text{ on } \Gamma_u,\; \frac{\partial y}{\partial \nu} = u \text{ on } \Gamma_b \tag{3.11}$$

Consider the minimization problem

$$\inf\left\{J_2(y(u),u) : (y(u),u) \in H^1_{per}\left(\Omega^-\right) \times L^2(\Gamma_b) \;\text{ satisfies }\; (3.11)\right\}, \tag{3.12}$$

where the cost functional is given by

$$J_2(y,u) = \frac{1}{2}\int_{\Omega^-}(y-y_d)^2 + \frac{\beta}{2}\int_{\Gamma_b} u^2. \tag{3.13}$$

The problem (3.12) has a unique solution $(\bar{y},\bar{u}) \in H^1_{per}(\Omega^-) \times L^2_{per}(\Gamma_b)$ and satisfies the system (3.11). In fact, $\bar{u}$ can be characterized using the adjoint system and is given by $\bar{u} = -\dfrac{1}{\beta}\bar{z}$ and $\bar{z} \in H^1_{per}(\Omega^-)$ is the solution of the problem

$$-\Delta \bar{z} = \bar{y} - y_d \text{ in } \Omega^-,\; \bar{z} = 0 \text{ on } \Gamma_u,\; \frac{\partial \bar{z}}{\partial \nu} = 0 \text{ on } \Gamma_b. \tag{3.14}$$

As earlier, one use the regularity of the adjoint system to see that, the optimal solution $(\bar{y},\bar{u}) \in H^1_{per}(\Omega^-) \times H^{1/2}_{per}(\Gamma_b)$. Hence the solution a-posteriori can be defined as a usual weak solution.

Theorem 3.3 (Homogenization). Let $(\bar{y}_\varepsilon,\ \bar{u}_\varepsilon,\ \bar{z}_\varepsilon)$ be the optimal solution of the problem (2.8) and (2.10) with respect to the cost function $J_\varepsilon(y_\varepsilon,u_\varepsilon)$, then

$$\bar{u}_\varepsilon \longrightarrow \bar{u} \text{ strongly in } H^{1/2}_{per}(\Gamma_b)\,,\ \tilde{\bar{y}}_\varepsilon,\ \tilde{\bar{z}}_\varepsilon \longrightarrow \tilde{\bar{y}},\ \tilde{\bar{z}} \text{ strongly in } H^1_{per}(\Omega)\,. \tag{3.15}$$

Moreover, $(\bar{y},\bar{u})$ solves the minimization problems (3.12), which also means that

$$J_{2,\varepsilon}(\bar{y}_\varepsilon,\bar{u}_\varepsilon) \longrightarrow J_2(\bar{y},\bar{u})\,. \tag{3.16}$$

Proof. Above estimates suggest the existence of $(u_0,y_0,z_0) \in H^{1/2}_{per}(\Gamma_b) \times H^1_{per}(\Omega) \times H^1_{per}(\Omega)$ and a subsequence of ε, still denoted by ε, such that

$$\bar{u}_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^{1/2}_{per}(\Gamma_b)\,,\ \tilde{\bar{y}}_\varepsilon,\ \tilde{\bar{z}}_\varepsilon \rightharpoonup y_0,\ z_0 \text{ weakly in } H^1_{per}(\Omega)\,, \tag{3.17}$$

Proceeding the same way as in Dirichlet case, we can prove $y_0 = 0$ in Ω^+, $y_0 = 0$ on Γ_u and $z_0 = 0$ in Ω^+, $z_0 = 0$ on Γ_u. Weak formulation of (2.8) gives us $\int_{\Omega^-} \nabla\tilde{\bar{y}}_\varepsilon \cdot \nabla\varphi = \int_{\Omega^-} f\varphi$ for all smooth function φ with compact support inside Ω^-. So by (3.17) we can say y_0 and u_0 satisfies (3.11). Similarly we can prove z_0 and y_0 satisfy the problem (3.14). By taking $\bar{z}_\varepsilon$ and z_0 as a test functions in (2.10) and (3.14), respectively, and using equation (3.17), give $\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla\bar{z}_\varepsilon|^2 = \int_{\Omega^-} |\nabla z_0|^2$. It implies $\tilde{\bar{z}}_\varepsilon \longrightarrow z_0$ strongly in $H^1(\Omega)$.

This along with the characterization of $\bar{u}_\varepsilon$ give $\bar{u}_\varepsilon \longrightarrow u_0$ strongly in $H^{1/2}(\Gamma_b)$ and $u_0 = -\frac{1}{\beta}z_0$. This give the required convergence of $\overline{y}_\varepsilon$ by taking $\overline{y}_\varepsilon$ and y_0 as test functions in (2.8) and (3.11), respectively. Hence the limits y_0, u_0 and z_0 satisfies the limit optimality system and also the cost functional converges to the limit cost functional. Thus by the uniqueness, we indeed see that $y_0 = \tilde{\bar{y}}$, $u_0 = \bar{u}$ and $z_0 = \tilde{\bar{z}}$.

4 Test Functions and Estimates

In this section we will see some test functions and error estimates related to them. These test functions and error estimates are not new. We pick them up from [2]. One can also check [1], [2], [3] and [27] and the reference there in for details and applications.

Let $\Lambda^\pm$ be the unbounded domains defined by $\Lambda^+ = (a,b)\times(0,\infty)$ and $\Lambda^- = (0,L)\times(-\infty,0)$. From figure 2, It is easy to relate these unbounded domains to our domain Ω_ε by ε scaling in x_1-direction and finite restriction in x_1-direction. We denote the variables in the cell domains Λ^+ and Λ^- as $y = (y_1,\ y_2)$. Define the test functions $\psi^\pm$ as $\psi^+ \in H^1(\Lambda^+),\ \psi^- \in H^1_{\text{loc, per}}(\Lambda^-),\ \nabla\psi^- \in L^2(\Lambda^-)$, satisfying

$$\begin{cases} \Delta\psi^\pm = 0 \text{ in } \Lambda^\pm,\ \psi^+ = \psi^- \text{ and } \frac{\partial\psi^+}{\partial y_2} = \frac{\partial\psi^-}{\partial y_2} + 1 \text{ on } (a,b)\times\{0\}, \\ \psi^- = 0 \text{ on } (0,a)\cup(b,L)\times\{0\},\ \psi^+ = 0 \text{ on } \partial\Lambda^+ \setminus (a,b)\times\{0\}. \end{cases} \tag{4.1}$$

For $\delta > 0$, define the average of ψ^- along the horizontal line $y_2 = -\delta$ as $\beta_1 = \beta_1(\delta) = \frac{1}{L}\int_0^L \psi^-(y_1,-\delta)dy_1$. We will borrow the following results from [1], [3].

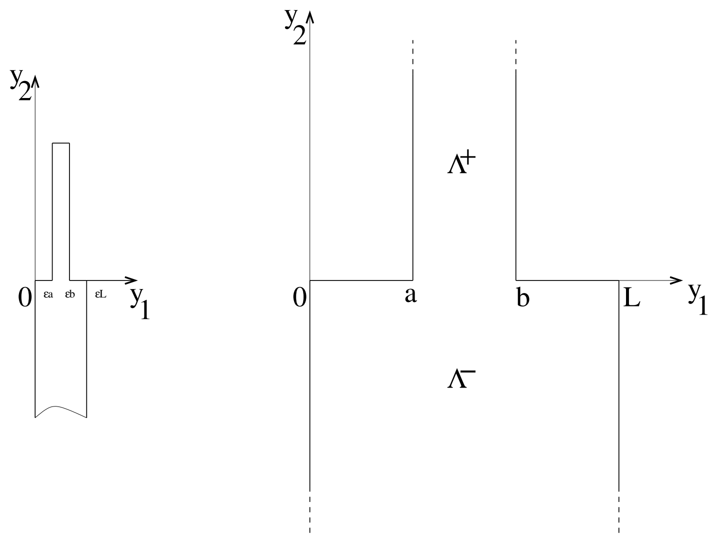

Fig. 2 $\Lambda^{\pm}$

Proposition 4.1. The problem (4.1) admits a unique solution. Further,
(1). $\beta_1(\delta)$ is independent of δ and we denote it by β_1.
(2). For any $\alpha \in \mathbb{N}\times\mathbb{N}$, $\delta > 0$, there are positive constants $C,\ C_{\alpha,\delta}$ such that

$$| \partial^{\alpha}\psi^{+}(y) | \leq C_{\alpha,\delta}\, e^{-Cy_2}, \quad \forall\, y = (y_1,\ y_2) \in (a,b)\times(\delta,\infty) \tag{4.2}$$

and

$$| \partial^{\alpha}(\psi^{-}(y) - \beta_1) | \leq C_{\alpha,\delta}\, e^{Cy_2}, \quad \forall\, y = (y_1,\ y_2) \in (0,L)\times(-\infty,-\delta). \tag{4.3}$$

(3). $\psi^{-} - \beta_1 \in H^1_{\text{per}}(\Lambda^{-})$.

The proof of Proposition 4.1(3) is trivial and can be found in [27] and in other references. □

To obtain the correctors results, we need to redefine the above mentioned test functions, ψ^+ and ψ^-, to whole of $\mathbb{R}^2$. Extend ψ^+ by 0 to $(0,L)\times(0,\infty)$ and then extend periodically to $\mathbb{R}^2_+$ which is again denoted by ψ^+. Similarly the periodic extension of ψ^- to $\mathbb{R}^2_-$ is also denoted by ψ^-. These test functions are used to obtain corrector results. We will see in next section that it is not, however, possible to obtain exact corrector results as in an uncontrolled problem since we have to work with the optimality system with varying right hand side. One can find the upcoming corollary and its proof in [27] and see other references.

Corollary 4.1. The test functions $\psi^{\pm}$ defined by (4.1) satisfies

$$\int_{\Omega_\varepsilon^+} \left|\psi^+\left(\frac{x_1}{\varepsilon},\frac{x_2-M}{\varepsilon}\right)\right|^2 dx \leq C\varepsilon,\ \int_{\Omega^-} \left|\psi^-\left(\frac{x_1}{\varepsilon},\frac{x_2-M}{\varepsilon}\right)-\beta_1\right|^2 dx \leq C\varepsilon \text{ and}$$

$$\int_{\Omega_\varepsilon\setminus B_\varepsilon} \left|\nabla\left(\psi\left(\frac{x_1}{\varepsilon},\frac{x_2-M}{\varepsilon}\right)\right)\right|^2 dx \leq Ce^{-c/\varepsilon},$$

where C, c are positive constants independent of ε and $B_\varepsilon = (0,L)\times(M-\varepsilon, M+\varepsilon)$ is a strip of width 2ε around the upper part Γ_u.

5 Corrector Results

This section is devoted to the study of corrector results for the optimal solutions and the adjoint states of Dirichlet as well as Neumann boundary control problems. Test functions, described in Section 4, will be extensively used in different forms for this analysis. A tremendous amount of results for different cases will be presented here but we will skip the proof of those ones who are of similar type by briefly explaining the key changes. This section contains our major contribution of the article. Denote $\psi_\varepsilon^+(x_1,x_2) = \psi^+\left(\frac{x_1}{\varepsilon}, \frac{x_2-M}{\varepsilon}\right)$ in Ω_ε^+ and $\psi_\varepsilon^-(x_1,x_2) = \psi^-\left(\frac{x_1}{\varepsilon}, \frac{x_2-M}{\varepsilon}\right)$ in Ω^-.

5.1 Optimal state (Dirichlet Boundary Control)

We will start with the corrector for the optimal state of Dirichlet boundary control case. To derive corrector results, we need good regularity for the optimal solution and adjoint state. Hence, we assume the data

$$f \in H^4_{per}\left(\Omega^-\right) \cap L^2_{per}(\Omega),\ g \in H^6_{per}(0,L) \tag{5.1}$$

and $O^- = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R},\ g(x_1) < x_2 < M\right\}$. The extension of $\bar{y}$ to O^- by L- periodicity, is a solution in $H^1\left(O^-\right)$ of the problem

$$-\Delta \bar{y} = f \text{ in } O^-,\ \bar{y} = 0 \text{ on } R \times \{M\},\ \bar{y} = \bar{u} \text{ on } \{(x_1, g(x_1)) : x_1 \in \mathbb{R}\}.$$

Standard regularity results and assumption (5.1) gives

$$\bar{y} \in H^6_{per}\left(\Omega^-\right) \subset C^4\left(\overline{\Omega^-}\right). \tag{5.2}$$

Further, introduce

$$\xi_\varepsilon(x) = \begin{cases} \xi_\varepsilon^+(x) = \dfrac{\partial \bar{y}}{\partial x_2}(x_1,M)\psi_\varepsilon^+(x_1,x_2) & \text{in } \Omega_\varepsilon^+, \\ \xi_\varepsilon^-(x) = \dfrac{\partial \bar{y}}{\partial x_2}(x_1,M)\left(\psi_\varepsilon^-(x_1,x_2) - \beta_1\right) & \text{in } \Omega^-. \end{cases} \tag{5.3}$$

Now we will state our main theorem of this subsection whose proof will come later.

Theorem 5.1. Assume (5.1). Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$, $(\bar{y}, \bar{u})$, respectively, be the optimal solutions of the in-homogenized and homogenized control problems given by (2.5) and (3.5). If ξ_ε is as defined in (5.3), then for small enough $\varepsilon > 0$,

$$\|\bar{y}_\varepsilon - \varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)} + \|\bar{y}_\varepsilon - \bar{y} - \varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)} \leq C\left(\varepsilon + \|\bar{u}_\varepsilon - \bar{u}\|_{H^{1/2}(\Gamma_b)}\right), \tag{5.4}$$

where C is a positive constant independent of ε.

Proof of this theorem needs some preparation. Define ω and ω_ε by

$$\omega = \begin{cases} 0 & \text{in } \Omega^+, \\ \omega^- & \text{in } \Omega^-, \end{cases} \quad \text{and} \quad \omega_\varepsilon = \begin{cases} \omega_\varepsilon^+ & \text{in } \Omega^+, \\ \omega_\varepsilon^- & \text{in } \Omega^-. \end{cases} \tag{5.5}$$

The functions $\omega^- \in H^1_{per}(\Omega^-)$ and $\omega_\varepsilon^+ \in H^1(\Omega_\varepsilon)$, $\omega_\varepsilon^- \in H^1_{per}(\Omega^-)$ solves the following partial differential equations

$$\Delta\omega^- = 0 \text{ in } \Omega^-,\ \omega^- = \beta_1 \frac{\partial \bar{y}}{\partial x_2}(x_1,M) \text{ on } \Gamma_u,\ \omega^- = 0 \text{ on } \Gamma_b. \tag{5.6}$$

and

$$\begin{cases} \Delta\omega_\varepsilon^+ = 0 \ \text{ in } \Omega_\varepsilon^+,\ \ \Delta\omega_\varepsilon^- = 0 \ \text{ in } \Omega^-,\ \omega_\varepsilon^- = -\xi_\varepsilon^- \ \text{ on } \Gamma_b, \\ \omega_\varepsilon^+ = -\xi_\varepsilon^+ \ \text{ on } \gamma_\varepsilon \setminus \Gamma_u,\ \ \omega_\varepsilon^+ = \omega_\varepsilon^- - \beta_1 \dfrac{\partial \bar{y}}{\partial x_2}(x_1,M) \ \text{ on } \Gamma_u \setminus \gamma_\varepsilon, \\ \omega_\varepsilon^- = \beta_1 \dfrac{\partial \bar{y}}{\partial x_2}(x_1,M) \ \text{ on } \gamma_\varepsilon \cap \Gamma_u,\ \ \dfrac{\partial \omega_\varepsilon^+}{\partial x_2} = \dfrac{\partial \omega_\varepsilon^-}{\partial x_2} \ \text{ on } \Gamma_u \setminus \gamma_\varepsilon. \end{cases} \tag{5.7}$$

Let τ_ε be the function defined by

$$\tau_\varepsilon = \begin{cases} \tau_\varepsilon^+ = \bar{y}_\varepsilon - \varepsilon\omega_\varepsilon^+ - \varepsilon\xi_\varepsilon^+ \ \text{ in } \Omega_\varepsilon^+, \\ \tau_\varepsilon^- = \bar{y}_\varepsilon - \bar{y} - \varepsilon\omega_\varepsilon^- - \varepsilon\xi_\varepsilon^- \ \text{ in } \Omega^-. \end{cases} \tag{5.8}$$

Clearly, $\tau_\varepsilon^+ \in H^1(\Omega_\varepsilon^+)$ and $\tau_\varepsilon^- \in H^1(\Omega^-)$. At the interface of Ω_ε^+ and Ω^-, trace of τ_ε^+ and τ_ε^- agrees because of the boundary conditions of $\bar{y}_\varepsilon,\ \bar{y},\ \omega_\varepsilon^\pm$ and $\xi_\varepsilon^\pm$ at $\Omega_\varepsilon^+ \cap \Omega^-$. Hence, $\tau_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$. $\tau_\varepsilon = 0$ on γ_ε. Moreover, $\dfrac{\partial \tau_\varepsilon^+}{\partial x_2} = \dfrac{\partial \tau_\varepsilon^-}{\partial x_2}$ in $H^{-1/2}(\bar{\Omega}_\varepsilon^+ \cap \bar{\Omega}^-)$. In the weak sense, we can write

$$-\Delta\tau_\varepsilon = \begin{cases} f + \varepsilon\Delta\left(\dfrac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\psi_\varepsilon^+ + 2\varepsilon\nabla\left(\dfrac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\cdot\nabla\psi_\varepsilon^+ \text{ in } \ \Omega_\varepsilon^+, \\ \\ \varepsilon\Delta\left(\dfrac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\left(\psi_\varepsilon^- - \beta_1\right) + 2\varepsilon\nabla\left(\dfrac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\cdot\nabla\left(\psi_\varepsilon^- - \beta_1\right) \ \text{ in } \ \Omega^-. \end{cases}$$

Integration by parts gives us

$$\begin{aligned} \|\tau_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} &\le C\,\|\nabla\tau_\varepsilon\|^2_{(L^2(\Omega_\varepsilon))^3} \\ &= C\left(-\int_{\Omega_\varepsilon}\Delta\tau_\varepsilon\tau_\varepsilon + \left\langle \frac{\partial\tau_\varepsilon}{\partial\nu},(\bar{u}_\varepsilon - \bar{u})\right\rangle_{H^{-1/2}(\Gamma_b),H^{1/2}(\Gamma_b)}\right) \\ &= C\left(\int_{\Omega_\varepsilon^+} f\tau_\varepsilon - \varepsilon\int_{\Omega_\varepsilon^+}\Delta\left(\frac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\psi_\varepsilon^+\tau_\varepsilon - 2\varepsilon\int_{\Omega_\varepsilon^+}\nabla\left(\frac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\cdot\nabla\tau_\varepsilon\psi_\varepsilon^+ \right. \\ &\quad -\varepsilon\int_{\Omega^-}\Delta\left(\frac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\left(\psi_\varepsilon^- - \beta_1\right)\tau_\varepsilon - 2\varepsilon\int_{\Omega^-}\nabla\left(\frac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\cdot\nabla\tau_\varepsilon\left(\psi_\varepsilon^- - \beta_1\right) \\ &\quad \left. +2\varepsilon\int_{\Gamma_b}\frac{\partial}{\partial\nu}\left(\frac{\partial \bar{y}}{\partial x_2}(x_1,M)\right)\left(\psi_\varepsilon^- - \beta_1\right)\tau_\varepsilon + \left\langle \frac{\partial\tau_\varepsilon}{\partial\nu},(\bar{u}_\varepsilon - \bar{u})\right\rangle_{H^{-1/2}(\Gamma_b),H^{1/2}(\Gamma_b)}\right), \end{aligned}$$

where C is a positive constant independent of ε. Cauchy-Schwartz inequality and (5.2) implies

$$\begin{aligned} \|\tau_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \le C\Big(&\|\tau_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \varepsilon\left\|\psi_\varepsilon^+\right\|_{L^2(\Omega_\varepsilon^+)}\|\tau_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \varepsilon\left\|\psi_\varepsilon^+\right\|_{L^2(\Omega_\varepsilon^+)}\|\nabla\tau_\varepsilon\|_{(L^2(\Omega_\varepsilon^+))^3} \\ &+\varepsilon\left\|\psi_\varepsilon^- - \beta_1\right\|_{L^2(\Omega^-)}\|\tau_\varepsilon\|_{L^2(\Omega^-)} + \varepsilon\left\|\psi_\varepsilon^- - \beta_1\right\|_{L^2(\Omega^-)}\|\nabla\tau_\varepsilon\|_{(L^2(\Omega^-))^3} \\ &+\varepsilon\left\|\psi_\varepsilon^- - \beta_1\right\|_{H^1(\Omega_\varepsilon\setminus B_\varepsilon)}\|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \|\bar{u}_\varepsilon - \bar{u}\|_{H^{1/2}(\Gamma_b)}\|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)}\Big) \end{aligned} \tag{5.9}$$

for $\varepsilon > 0$ small enough. Here again C is a positive constant independent of ε.

$$\|\tau_\varepsilon\|^2_{L^2(\Omega_\varepsilon^+)} = \int_{\Omega_\varepsilon^+} |\tau_\varepsilon|^2 = \sum_{k=0}^{1/\varepsilon-1}\int_M^{M'}\int_{\varepsilon(a+kL)}^{\varepsilon(b+kL)} |\tau_\varepsilon|^2 = \sum_{k=0}^{1/\varepsilon-1}\int_M^{M'}\int_{\varepsilon(a+kL)}^{\varepsilon(b+kL)} \left|\int_{\varepsilon(a+kL)}^{x_1}\frac{\partial\tau_\varepsilon}{\partial t}(t,x_2)\right|^2$$

$$\|\tau_\varepsilon\|^2_{L^2(\Omega_\varepsilon^+)} \le \varepsilon^2(b-a)^2\sum_{k=0}^{1/\varepsilon-1}\int_M^{M'}\int_{\varepsilon(a+kL)}^{\varepsilon(b+kL)}\left|\frac{\partial\tau_\varepsilon}{\partial x_1}\right|^2 \le \varepsilon^2(b-a)^2\|\nabla\tau_\varepsilon\|^2_{(L^2(\Omega_\varepsilon^+))^3} \tag{5.10}$$

Combining (5.9), (5.10) and Corollary 4.1, we get $\|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C\left(\varepsilon + \|\bar{u}_\varepsilon - \bar{u}\|_{H^{1/2}(\Gamma_b)}\right)$, where C is a positive constant independent of ε. Let $\varsigma_\varepsilon = \omega_\varepsilon - \omega$. Note that $\varsigma_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ and

$$\varsigma_\varepsilon = \begin{cases} \varsigma_\varepsilon^+ = \omega_\varepsilon^+ \text{ in } \Omega_\varepsilon^+, \\ \varsigma_\varepsilon^- = \omega_\varepsilon^- - \omega^- \text{ in } \Omega^- \end{cases} \tag{5.11}$$

which solves the partial differential equation

$$\begin{cases} \Delta\varsigma_\varepsilon^+ = 0 \text{ in } \Omega_\varepsilon^+,\ \Delta\varsigma_\varepsilon^- = 0 \text{ in } \Omega^-,\ \varsigma_\varepsilon^- = -\xi_\varepsilon^- \text{ on } \Gamma_b,\ \varsigma_\varepsilon^+ = -\xi_\varepsilon^+ \text{ on } \gamma_\varepsilon\setminus\Gamma_u, \\ \varsigma_\varepsilon^+ = \varsigma_\varepsilon^- \text{ on } \Gamma_u\setminus\gamma_\varepsilon,\ \varsigma_\varepsilon^- = 0 \text{ on } \gamma_\varepsilon\cap\Gamma_u,\ \dfrac{\partial\varsigma_\varepsilon^+}{\partial x_2} = \dfrac{\partial\varsigma_\varepsilon^-}{\partial x_2} + \dfrac{\partial\omega^-}{\partial x_2} \text{ on } \Gamma_u\setminus\gamma_\varepsilon. \end{cases} \tag{5.12}$$

Now, introduce $\varpi_\varepsilon = \varsigma_\varepsilon - \rho_\varepsilon$, where $\rho_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ solves the problem

$$\Delta\rho_\varepsilon = 0 \text{ in } \Omega_\varepsilon,\ \rho_\varepsilon = -\xi_\varepsilon^+ \text{ on } \gamma_\varepsilon\setminus\Gamma_u,\ \rho_\varepsilon = 0 \text{ on } \gamma_\varepsilon\cap\Gamma_u,\ \rho_\varepsilon = -\xi_\varepsilon^- \text{ on } \Gamma_b. \tag{5.13}$$

Integration by parts gives

$$\begin{aligned} \|\varpi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} &\le C\|\nabla\varpi_\varepsilon\|^2_{(L^2(\Omega_\varepsilon))^3} \\ &= C\left(\int_{\Omega_\varepsilon}\nabla\varsigma_\varepsilon\cdot\nabla\varsigma_\varepsilon - \int_{\Omega_\varepsilon}\nabla\varsigma_\varepsilon\cdot\nabla\rho_\varepsilon - \int_{\Omega_\varepsilon}\nabla\varsigma_\varepsilon\cdot\nabla\rho_\varepsilon + \int_{\Omega_\varepsilon}\nabla\rho_\varepsilon\cdot\nabla\rho_\varepsilon\right). \end{aligned}$$

Using the above equations and boundary conditions with some elaborate computation, one can get $\|\varpi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \le C\left\langle \dfrac{\partial\omega^-}{\partial\nu}, \varpi_\varepsilon\right\rangle_{H^{-1/2}(\Gamma_u\setminus\gamma_\varepsilon),H^{1/2}(\Gamma_u\setminus\gamma_\varepsilon)}$. The regularity (5.2) implies that $\omega^- \in H^5_{per}(\Omega^-) \subset C^3\left(\overline{\Omega^-}\right)$. Hence, we can conclude $\|\varpi_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$, where $C > 0$ is a constant independent of ε. Next we prove

$$\|\varsigma_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C. \tag{5.14}$$

If we prove $\|\rho_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$, then we are through because of the H^1 uniform boundedness of ϖ_ε. Construct a function $\psi \in C^2(-\infty,+\infty)$ such that

$$\psi(x_2) = \begin{cases} 1 \ \text{ if } \ \frac{M+3M'}{4} < x_2 \text{ and } \frac{3m+M}{4} > x_2, \\ 0 \ \text{ if } \ \frac{m+3M}{4} < x_2 < \frac{3M+M'}{4}. \end{cases} \tag{5.15}$$

Again, define $\Upsilon_\varepsilon = \psi\xi_\varepsilon$. Notice that $\Upsilon_\varepsilon = \xi_\varepsilon$ on $\gamma_\varepsilon\cup\Gamma_b$ and $\|\Upsilon_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$. Use $(\rho_\varepsilon + \Upsilon_\varepsilon)$ as a test function in (5.13), we will get $\|\rho_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$. Now we have developed enough tools to prove Theorem 5.1.

Proof of Theorem 5.1 :- By triangle inequality, we can write

$$\begin{aligned}
&\|\bar{y}_\varepsilon-\varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)} + \|\bar{y}_\varepsilon-\bar{y}-\varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)}\\
&\qquad\le \|\bar{y}_\varepsilon-\varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)}+\|\bar{y}_\varepsilon-\bar{y}-\varepsilon\omega^- -\varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)}+\varepsilon\|\omega^-\|_{H^1(\Omega^-)}\\
&\qquad= \|\bar{y}_\varepsilon-\tilde{\bar{y}}-\varepsilon\omega-\varepsilon\xi_\varepsilon\|_{H^1(\Omega_\varepsilon)}+\varepsilon\|\omega^-\|_{H^1(\Omega^-)}\\
&\qquad= \|\tau_\varepsilon+\varepsilon(\omega_\varepsilon-\omega)\|_{H^1(\Omega_\varepsilon)}+\varepsilon\|\omega^-\|_{H^1(\Omega^-)}.
\end{aligned}$$

Hence

$$\|\bar{y}_\varepsilon-\varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)}+\|\bar{y}_\varepsilon-\bar{y}-\varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)}\le\|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)}+\varepsilon\|\varsigma_\varepsilon\|_{H^1(\Omega_\varepsilon)}+\varepsilon\|\omega^-\|_{H^1(\Omega^-)}. \tag{5.16}$$

We will have the desired result (5.30), by combining (5.14) and (5.16) with the H^1-estimate of τ_ε in terms of $\bar{u}_\varepsilon$ and $\bar{u}$. □

The recipe, we have prepared for proving Theorem 5.1 will also help to prove the following theorem. We skip the details.

Theorem 5.2. Assume (5.1). Let $(\bar{y}_\varepsilon,\bar{u}_\varepsilon),\ (\bar{y},\bar{u})$ be the optimal solutions of the in-homogenized and homogenized control problems given by (2.5) and (3.5), respectively. If ω is as defined in (5.5) then there exists a positive constant C independent of ε such that

$$\|\tilde{\bar{y}}_\varepsilon-\tilde{\bar{y}}-\varepsilon\omega\|_{H^1(\Omega_\varepsilon\setminus B_\varepsilon)}\le C\left(\varepsilon+\|\bar{u}_\varepsilon-\bar{u}\|_{H^{1/2}(\Gamma_b)}\right), \tag{5.17}$$

where B_ε is a band around the upper boundary as in Corollary 4.1.

5.2 Adjoint state (Dirichlet Boundary Control)

Corrector for adjoint state in Dirichlet boundary control case can be proved similar to Theorem 5.1 but here we are presenting another way, inspired from [2]. Indeed, to derive corrector estimates for the adjoint state in the case of Dirichlet Boundary Control on Γ_b, one need to have regularity on the solution, but it can be achieved with less regularity on the assumptions in comparison to (5.1) on the given data f, g and y_d. We assume that

$$f\in H^2_{per}\left(\Omega^-\right)\cap L^2_{per}\left(\Omega\right),\ g\in H^6_{per}(0,L),\ y_d\in H^4_{per}\left(\Omega^-\right). \tag{5.18}$$

Similar to subsection 5.1, using standard regularity arguments and assumption (5.18), we get

$$\bar{y}\in H^4_{per}\left(\Omega^-\right)\subset C^2\left(\overline{\Omega^-}\right)\text{ and hence }\bar{z}\in H^6_{per}\left(\Omega^-\right)\subset C^4\left(\overline{\Omega^-}\right). \tag{5.19}$$

Let ω now be defined by

$$\omega=\begin{cases}0 & \text{in }\Omega^+,\\ \omega^- & \text{in }\Omega^-,\end{cases} \tag{5.20}$$

where $\omega^-\in H^1_{per}(\Omega^-)$ is the unique solution of the problem

$$\Delta\omega^-=0\text{ in }\Omega^-,\ \omega^-=\beta_1\frac{\partial\bar{z}}{\partial x_2}\text{ on }\Gamma_u,\ \omega^-=0\text{ on }\Gamma_b. \tag{5.21}$$

Theorem 5.3. Assume (5.18). Let $\bar{z}_\varepsilon,\ \bar{z}$, respectively, be the solution of the in-homogenized and homogenized co-states given by (3.2) and (3.7). Again let $\bar{y}_\varepsilon,\ \bar{y}$, respectively, be the optimal state of the in-homogenized and homogenized control problems given by (2.5) and (3.5). If ω be the solution of (5.20), then for small enough $\varepsilon > 0$,

$$\|\tilde{\bar{z}}_\varepsilon - \tilde{\bar{z}} - \varepsilon\omega\|_{H^1(\Omega_\varepsilon \setminus B_\varepsilon)} \leq C\left[\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)} + \varepsilon^{3/2}\right], \tag{5.22}$$

where C is a positive constant independent of ε.

For proving this theorem we need to develop the required tools. From (5.19), we have $\dfrac{\partial \bar{z}}{\partial x_2} \in H^5_{per}(\Omega^-)$ which implies

$$\omega^- \in H^5_{per}\left(\Omega^-\right) \subset C^3\left(\overline{\Omega^-}\right). \tag{5.23}$$

The function ω^- and $\bar{z}$ be extended by L- periodicity to O^-, it follows that $\omega^- \in H^1(O^-)$ is the solution of problem

$$\Delta\omega^- = 0 \text{ in } O^-,\ \omega^- = \beta_1 \frac{\partial \bar{z}}{\partial x_2} \text{ on } R \times \{M\},\ \omega^- = 0 \text{ on } \{(x_1, g(x_1)) : x_1 \in R\}. \tag{5.24}$$

Let τ_ε be the function defined by

$$\tau_\varepsilon = \begin{cases} \tau_\varepsilon^+ = \bar{z}_\varepsilon - \varepsilon\omega_\varepsilon^+ - \varepsilon \dfrac{\partial \bar{z}}{\partial x_2}(x_1, M)\,\psi_\varepsilon^+ \ \text{ in } \Omega_\varepsilon^+, \\ \tau_\varepsilon^- = \bar{z}_\varepsilon - \bar{z} - \varepsilon\omega_\varepsilon^- - \varepsilon \dfrac{\partial \bar{z}}{\partial x_2}(x_1, M)\left(\psi_\varepsilon^- - \beta_1\right) \ \text{ in } \Omega^-. \end{cases} \tag{5.25}$$

and ρ_ε be the function defined by

$$\rho_\varepsilon = \begin{cases} \rho_\varepsilon^+ = \omega_\varepsilon^+ - \varepsilon \dfrac{\partial \omega^-}{\partial x_2}(x_1, M)\,\psi_\varepsilon^+ \ \text{ in } \Omega_\varepsilon^+, \\ \rho_\varepsilon^- = \omega_\varepsilon^- - \omega^- - \varepsilon \dfrac{\partial \omega^-}{\partial x_2}(x_1, M)\,\psi_\varepsilon^- \ \text{ in } \Omega^-. \end{cases} \tag{5.26}$$

where $\omega_\varepsilon^\pm$ are functions in $H^1(\Omega_\varepsilon^+)$ and $H^1_{per}(\Omega^-)$, respectively, satisfying

$$\begin{cases} \Delta\omega_\varepsilon^+ = 0 \text{ in } \Omega_\varepsilon^+,\ \Delta\omega_\varepsilon^- = 0 \text{ in } \Omega^-,\ \omega_\varepsilon^- = 0 \text{ on } \Gamma_b, \\ \omega_\varepsilon^+ = 0 \ \text{ on } \gamma_\varepsilon \setminus \Gamma_u,\ \ \omega_\varepsilon^+ = \omega_\varepsilon^- - \beta_1 \dfrac{\partial \bar{z}}{\partial x_2} \ \text{ on } \Gamma_u \setminus \gamma_\varepsilon, \\ \omega_\varepsilon^- = \beta_1 \dfrac{\partial \bar{z}}{\partial x_2} \ \text{ on } \gamma_\varepsilon \cap \Gamma_u,\ \ \dfrac{\partial \omega_\varepsilon^+}{\partial x_2} = \dfrac{\partial \omega_\varepsilon^-}{\partial x_2} \ \text{ on } \Gamma_u \setminus \gamma_\varepsilon. \end{cases} \tag{5.27}$$

Proposition 5.1. Let τ_ε and ρ_ε are the functions defined in (5.25). Assume (5.18), then for small enough $\varepsilon > 0$, there exist a positive constant C independent of ε such that

$$\|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C\left[\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)} + \varepsilon^{3/2}\right]$$

and

$$\|\rho_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon.$$

Proof: *Estimates on* τ_ε: Since $\tau_\varepsilon^+ \in H^1_{per}(\Omega_\varepsilon^+)$, $\tau_\varepsilon^- \in H^1_{per}(\Omega^-)$ and $\tau_\varepsilon^+ = \tau_\varepsilon^-$ at the interface $\bar{\Omega}_\varepsilon^+ \cap \bar{\Omega}^-$, we get $\tau_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$. In fact, it is easy to see that $\dfrac{\partial \tau_\varepsilon^+}{\partial x_2} = \dfrac{\partial \tau_\varepsilon^-}{\partial x_2}$ on $\bar{\Omega}_\varepsilon^+ \cap \bar{\Omega}^-$. We compute $\Delta \tau_\varepsilon$ in Ω_ε. We have

$$\Delta \tau_\varepsilon = \begin{cases} \Delta \tau_\varepsilon^+ = (\bar{y}_\varepsilon - y_d) - \varepsilon \frac{\partial^3 \bar{z}}{\partial x_1^2 \partial x_2}(x_1, M)\psi_\varepsilon^+ - 2\varepsilon \frac{\partial^2 \bar{z}}{\partial x_1 \partial x_2}(x_1, M)\frac{\partial \psi_\varepsilon^+}{\partial x_1} \text{ in } \Omega_\varepsilon^+ \\ \Delta \tau_\varepsilon^- = (\bar{y}_\varepsilon - \bar{y}) - \varepsilon \frac{\partial^3 \bar{z}}{\partial x_1^2 \partial x_2}(x_1, M)(\psi_\varepsilon^- - \beta_1) - 2\varepsilon \frac{\partial^2 \bar{z}}{\partial x_1 \partial x_2}(x_1, M)\frac{\partial \psi_\varepsilon^-}{\partial x_1} \text{ in } \Omega^-. \end{cases}$$

Further $\tau_\varepsilon \mid_{\Gamma_b} = -\varepsilon \dfrac{\partial \bar{z}}{\partial x_2}(x_1, M)\left(\psi^-\left(\dfrac{x_1}{\varepsilon}, \dfrac{g(x_1) - M}{\varepsilon}\right) - \beta_1\right)$ and $\tau_\varepsilon \mid_{\gamma_\varepsilon \cap (0,L)\times M'} = -\varepsilon \dfrac{\partial \bar{z}}{\partial x_2}(x_1, M)\,\psi^+\left(\dfrac{x_1}{\varepsilon}, \dfrac{M' - M}{\varepsilon}\right)$. We need test functions which vanish on these boundaries to use in the weak formulations. Choose $\phi_1, \phi_2 \in C^2(\mathbb{R}; [0,1])$ such that

$$\phi_1(s) = \begin{cases} 0 \text{ if } s > \frac{m+M}{2}, \\ 1 \text{ if } s < \frac{3m+M}{4}, \end{cases} \quad \phi_2(s) = \begin{cases} 1 \text{ if } s > \frac{M+M'}{2}, \\ 0 \text{ if } s < \frac{3M+M'}{4}. \end{cases}$$

In Ω, define $\tau_\varepsilon^1(x_1, x_2) = -\varepsilon \dfrac{\partial \bar{z}}{\partial x_2}(x_1, M)\left(\psi^-\left(\dfrac{x_1}{\varepsilon}, \dfrac{g(x_1) - M}{\varepsilon}\right) - \beta_1\right)\phi_1(x_2)$ and $\tau_\varepsilon^2(x_1, x_2) = -\varepsilon \dfrac{\partial \bar{z}}{\partial x_2}(x_1, M)\,\psi^+\left(\dfrac{x_1}{\varepsilon}, \dfrac{M' - M}{\varepsilon}\right)\phi_2(x_2)$. Clearly, $\tau_\varepsilon - \tau_\varepsilon^1 - \tau_\varepsilon^2 \in H^1_{per}(\Omega_\varepsilon)$ with $\tau_\varepsilon - \tau_\varepsilon^1 - \tau_\varepsilon^2 = 0$ on the boundary $\gamma_\varepsilon \cup \Gamma_b$. Hence, we can use it as a test function to get

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla \tau_\varepsilon|^2 dx &= \int_{\Omega_\varepsilon} \nabla \tau_\varepsilon \cdot \nabla\left(\tau_\varepsilon - \tau_\varepsilon^1 - \tau_\varepsilon^2\right) dx + \int_{\Omega_\varepsilon} \nabla \tau_\varepsilon \cdot \nabla \tau_\varepsilon^1 dx + \int_{\Omega_\varepsilon} \nabla \tau_\varepsilon \cdot \nabla \tau_\varepsilon^2 dx \\ &= -\int_{\Omega_\varepsilon^+} \Delta \tau_\varepsilon\left(\tau_\varepsilon - \tau_\varepsilon^2\right) dx - \int_{\Omega^-} \Delta \tau_\varepsilon\left(\tau_\varepsilon - \tau_\varepsilon^1\right) dx \\ &\quad + \int_{\Omega_\varepsilon^+} \nabla \tau_\varepsilon \cdot \nabla \tau_\varepsilon^2 dx + \int_{\Omega_\varepsilon^-} \nabla \tau_\varepsilon \cdot \nabla \tau_\varepsilon^1 dx. \end{aligned}$$

Using Proposition 4.1 and the definition of τ_ε, $i = 1,2$, we get $\left|\dfrac{\partial \tau_\varepsilon^1}{\partial x_i}\right| \le C\, e^{-c/\varepsilon}$ in Ω^-, $\left|\dfrac{\partial \tau_\varepsilon^2}{\partial x_i}\right| \le C\, e^{-c/\varepsilon}$ in Ω_ε^+, where C, c are positive constants independent of ε. Now using the expression for $\Delta \tau_\varepsilon$ and Cauchy-Schwartz inequality, we get

$$\begin{aligned} \|\nabla \tau_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le\ & C\Big[\varepsilon \|\psi_\varepsilon^+\|_{L^2(\Omega_\varepsilon^+)} \|\tau_\varepsilon - \tau_\varepsilon^2\|_{L^2(\Omega_\varepsilon^+)} + \varepsilon \|\psi_\varepsilon^- - \beta_1\|_{L^2(\Omega^-)} \|\tau_\varepsilon - \tau_\varepsilon^1\|_{L^2(\Omega^-)} \\ & + \left|\int_{\Omega_\varepsilon^+} (\bar{y}_\varepsilon - y_d)\left(\tau_\varepsilon - \tau_\varepsilon^2\right) + \int_{\Omega^-} (\bar{y}_\varepsilon - \bar{y})\left(\tau_\varepsilon - \tau_\varepsilon^1\right)\right| \\ & + \|\nabla \tau_\varepsilon\|_{L^2(\Omega_\varepsilon)} \left(\|\nabla \tau_\varepsilon^2\|_{L^2(\Omega_\varepsilon^+)} + \|\nabla \tau_\varepsilon^1\|_{L^2(\Omega^-)}\right) \\ & + \varepsilon \|\psi_\varepsilon^+\|_{L^2(\Omega_\varepsilon^+)} \|\nabla\left(\tau_\varepsilon - \tau_\varepsilon^2\right)\|_{L^2(\Omega_\varepsilon^+)} + \varepsilon \|\psi_\varepsilon^- - \beta_1\|_{L^2(\Omega^-)} \|\nabla\left(\tau_\varepsilon - \tau_\varepsilon^1\right)\|_{L^2(\Omega^-)}\Big]. \end{aligned}$$

Applying the estimates of ψ_ε^+, $\psi_\varepsilon^- - \beta_1$ from Proposition 4.1(3), the exponential decay of τ_ε^i for $i = 1,2$ and Poincaré inequality, we get

$$\|\nabla\tau_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C\Big[\left(\varepsilon^{3/2}\right)\|\nabla\tau_\varepsilon\|_{L^2(\Omega_\varepsilon)} + e^{-c/\varepsilon} + \|\bar{y}_\varepsilon - y_d\|_{L^2\left(\Omega_\varepsilon^+\right)}\|\tau_\varepsilon - \tau_\varepsilon^2\|_{L^2\left(\Omega_\varepsilon^+\right)}$$
$$+\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)}\|\nabla\tau_\varepsilon\|_{L^2(\Omega^-)}\Big].$$

Note that Ω_ε^+ consists of ε-strips of length 1 and applying Poincaré inequality in each strip, summing up to have $\|\tau_\varepsilon - \tau_\varepsilon^2\|_{L^2\left(\Omega_\varepsilon^+\right)} \leq C\varepsilon\|\nabla\left(\tau_\varepsilon - \tau_\varepsilon^2\right)\|_{L^2\left(\Omega_\varepsilon^+\right)}$ and since $y_d = 0$ in the strips, then considering y_ε in each strip, using Poincaré inequality we get $\|\bar{y}_\varepsilon\|_{L^2\left(\Omega_\varepsilon^+\right)} \leq C\varepsilon$. In other words the Poincaré constant is of order ε. Thus we have $\varepsilon\|\bar{y}_\varepsilon - y_d\|_{L^2\left(\Omega_\varepsilon^+\right)} \leq C\varepsilon^2$. Hence we get $\|\nabla\tau_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\left[\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)} + \varepsilon^{3/2} + e^{-c/\varepsilon}\right]$.

Estimate on ρ_ε: We can work on an analogous fashion by computing $\Delta\rho_\varepsilon$ and introducing ρ_ε^1 and ρ_ε^2 to get $\|\nabla\rho_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon$. □

Now define

$$\pi_\varepsilon = \begin{cases} \pi_\varepsilon^+ = \varepsilon\dfrac{\partial\bar{z}}{\partial x_2}(x_1,M)\,\psi_\varepsilon^+ + \varepsilon^2\dfrac{\partial\omega^-}{\partial x_2}(x_1,M)\,\psi_\varepsilon^+, \\ \pi_\varepsilon^- = \varepsilon\dfrac{\partial\bar{z}}{\partial x_2}(x_1,M)\,(\psi_\varepsilon^- - \beta_1) + \varepsilon^2\dfrac{\partial\omega^-}{\partial x_2}(x_1,M)\,\psi_\varepsilon^-. \end{cases} \tag{5.28}$$

As far as π_ε is concerned, we can take $\nabla\pi_\varepsilon$ and estimate using (4.1) which is valid only in $\Omega_\varepsilon \setminus B_\varepsilon$. Thus we get $\|\nabla\pi_\varepsilon\|_{L^2(\Omega_\varepsilon\setminus B_\varepsilon)} \leq Ce^{-c/\varepsilon}$, where C, c are positive constants independent of ε. The proof of this statement is same as last discussion of τ_ε, which we are not going to repeat again for π_ε.

Proof of Theorem 5.3 :- Notice that $\bar{z}_\varepsilon - \tilde{\bar{z}} - \varepsilon\omega = \tau_\varepsilon + \varepsilon\rho_\varepsilon + \pi_\varepsilon$ in Ω_ε. Now the proof of the theorem is straightforward because of the availability of required estimates in the above discussion. □

Define

$$\xi_\varepsilon(x) = \begin{cases} \xi_\varepsilon^+(x) = \dfrac{\partial\bar{z}}{\partial x_2}(x_1,M)\psi_\varepsilon^+ \ \text{ in } \Omega_\varepsilon^+, \\ \xi_\varepsilon^-(x) = \dfrac{\partial\bar{z}}{\partial x_2}(x_1,M)\left(\psi_\varepsilon^- - \beta_1\right) \ \text{ in } \Omega^-. \end{cases} \tag{5.29}$$

As we proved Theorem 5.1, one can prove the following result.

Theorem 5.4. Assume (5.18). Let $\bar{z}_\varepsilon$, $\bar{z}$, respectively, be the solution of the in-homogenized and homogenized co-states given by (3.2) and (3.7). Again let $\bar{y}_\varepsilon$, $\bar{y}$, respectively, be the optimal state of the in-homogenized and homogenized control problems given by (2.5) and (3.5). If ξ_ε is as defined in (5.29) then for small enough $\varepsilon > 0$

$$\|\bar{z}_\varepsilon - \varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)} + \|\bar{z}_\varepsilon - \bar{z} - \varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)} \leq C\left(\varepsilon + \|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)}\right), \tag{5.30}$$

where C is a positive constant independent of ε.

Corollary 5.1. Assume (5.18). Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$, $(\bar{y}, \bar{u})$, respectively, be the optimal solutions of the in-homogenized and homogenized control problems given by (2.5) and (3.5) then there exists a positive constant C, independent of ε, such that

$$\|\bar{u}_\varepsilon - \bar{u}\|_{H^{-1/2}(\Gamma_b)} \leq C\left(\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)} + \varepsilon\right), \tag{5.31}$$

for ε small enough.

Proof. By trace theorem and Theorem 5.3

$$\begin{aligned} \|\bar{u}_\varepsilon - \bar{u}\|_{H^{-1/2}(\Gamma_b)} &\leq C\|\bar{z}_\varepsilon - \bar{z}\|_{H^1\left((0,l_1)\times\left(g(x_1),\frac{g(x_1)+M}{2}\right)\right)} \\ &\leq C\|\bar{z}_\varepsilon - \bar{z}\|_{H^1(\Omega_\varepsilon\setminus B_\varepsilon)} \leq C\left(\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)} + \varepsilon\right). \end{aligned}$$

5.3 Optimal state (Neumann Boundary Control)

In this subsection, we will deal with the corrector for the optimal state of Neumann boundary control case. The analysis and results look similar to that of Dirichlet. But the main contribution is the construction of appropriate test functions. We skip many of the computations. Let $(\bar{y},\bar{u})$ be the optimal limit solution obtained in Section 3.2 (Theorem 3.3). We assume (5.1). The extension of $\bar{y}$ to O^- by L- periodicity, is a solution in $H^1(O^-)$ of the problem

$$-\Delta\bar{y}=f \text{ in } O^-,\ \bar{y}=0 \text{ on } R\times\{M\},\ \frac{\partial\bar{y}}{\partial\nu}=\bar{u} \text{ on } \{(x_1,g(x_1)) : x_1\in R\}.$$

Standard regularity results and assumption (5.1) gives

$$\bar{y}\in H^6_{per}\left(\Omega^-\right)\subset C^4\left(\overline{\Omega^-}\right). \tag{5.32}$$

Further, introduce

$$\xi_\varepsilon(x)=\begin{cases}\xi_\varepsilon^+(x)=\dfrac{\partial\bar{y}}{\partial x_2}(x_1,M)\psi_\varepsilon^+ \ \text{ in } \Omega_\varepsilon^+,\\ \xi_\varepsilon^-(x)=\dfrac{\partial\bar{y}}{\partial x_2}(x_1,M)\left(\psi_\varepsilon^- -\beta_1\right) \ \text{ in } \Omega^-.\end{cases} \tag{5.33}$$

Now we will state our main theorem of this subsection whose proof will come later.

Theorem 5.5. Assume (5.1). Let $(\bar{y}_\varepsilon,\bar{u}_\varepsilon)$, $(\bar{y},\bar{u})$, respectively, be the optimal solutions of the in-homogenized and homogenized control problems given by (2.9) and (3.12). If ξ_ε is as defined in (5.33) then for small enough $\varepsilon>0$,

$$\|\bar{y}_\varepsilon-\varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)}+\|\bar{y}_\varepsilon-\bar{y}-\varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)}\le C\left(\varepsilon+\|\bar{u}_\varepsilon-\bar{u}\|_{L^2(\Gamma_b)}\right), \tag{5.34}$$

where C is a positive constant independent of small enough ε.

We need to introduce some test functions. Let $\omega^-\in H^1_{per}(\Omega^-)$ is the unique solution of the problem

$$\Delta\omega^-=0 \text{ in } \Omega^-,\ \omega^-=\beta_1\frac{\partial\bar{y}}{\partial x_2}(x_1,M) \text{ on } \Gamma_u,\ \frac{\partial\omega^-}{\partial\nu}=0 \text{ on } \Gamma_b. \tag{5.35}$$

Define $\omega,\ \omega_\varepsilon$ as

$$\omega=\begin{cases}0 & \text{in } \Omega^+,\\ \omega^- & \text{in } \Omega^-,\end{cases}\ \text{and } \omega_\varepsilon=\begin{cases}\omega_\varepsilon^+ & \text{in } \Omega^+,\\ \omega_\varepsilon^- & \text{in } \Omega^-,\end{cases} \tag{5.36}$$

where $\omega_\varepsilon^+\in H^1(\Omega_\varepsilon),\ \omega_\varepsilon^-\in H^1_{per}(\Omega^-)$ also solves the partial differential equation

$$\begin{cases}\Delta\omega_\varepsilon^+=0 \ \text{ in } \Omega_\varepsilon^+,\ \ \Delta\omega_\varepsilon^-=0 \ \text{ in } \Omega^-,\ \dfrac{\partial\omega_\varepsilon^-}{\partial\nu}=-\dfrac{\partial\xi_\varepsilon^-}{\partial\nu} \text{ on } \Gamma_b,\\ \omega_\varepsilon^+=-\xi_\varepsilon^+ \ \text{ on } \gamma_\varepsilon\setminus\Gamma_u,\ \ \omega_\varepsilon^+=\omega_\varepsilon^- -\beta_1\dfrac{\partial\bar{y}}{\partial x_2}(x_1,M) \ \text{ on } \Gamma_u\setminus\gamma_\varepsilon,\\ \omega_\varepsilon^-=\beta_1\dfrac{\partial\bar{y}}{\partial x_2}(x_1,M) \ \text{ on } \gamma_\varepsilon\cap\Gamma_u,\ \dfrac{\partial\omega_\varepsilon^+}{\partial x_2}=\dfrac{\partial\omega_\varepsilon^-}{\partial x_2} \ \text{ on } \Gamma_u\setminus\gamma_\varepsilon.\end{cases} \tag{5.37}$$

Let τ_ε be the function defined by

$$\tau_\varepsilon=\begin{cases}\tau_\varepsilon^+=\bar{y}_\varepsilon-\varepsilon\omega_\varepsilon^+-\varepsilon\xi_\varepsilon^+ \text{ in } \Omega_\varepsilon^+,\\ \tau_\varepsilon^-=\bar{y}_\varepsilon-\bar{y}-\varepsilon\omega_\varepsilon^- -\varepsilon\xi_\varepsilon^- \text{ in } \Omega^-.\end{cases} \tag{5.38}$$

Clearly, $\tau_\varepsilon^+ \in H^1(\Omega_\varepsilon^+)$ and $\tau_\varepsilon^- \in H^1(\Omega^-)$. At the interface of Ω_ε^+ and Ω^-, trace of τ_ε^+ and τ_ε^- agrees because of the boundary conditions of $\bar{y}_\varepsilon$, $\bar{y}$, $\omega_\varepsilon^\pm$ and $\xi_\varepsilon^\pm$ at $\Omega_\varepsilon^+ \cap \Omega^-$. Hence, $\tau_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$. $\tau_\varepsilon = 0$ on γ_ε. Moreover, $\dfrac{\partial \tau_\varepsilon^+}{\partial x_2} = \dfrac{\partial \tau_\varepsilon^-}{\partial x_2}$ as $H^{-1/2}(\bar{\Omega}_\varepsilon^+ \cap \bar{\Omega}^-)$ function. By a suitable computation (we omit here) as in Dirichlet case, we get $\|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C\left(\varepsilon + \|\bar{u}_\varepsilon - \bar{u}\|_{L^2(\Gamma_b)}\right)$, where C is a positive constant independent of ε. Let $\varsigma_\varepsilon = \omega_\varepsilon - \omega$. Note that $\varsigma_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ and solves the partial differential equation

$$\begin{cases} \Delta \varsigma_\varepsilon^+ = 0 \text{ in } \Omega_\varepsilon^+, \ \Delta \varsigma_\varepsilon^- = 0 \text{ in } \Omega^-, \ \dfrac{\partial \varsigma_\varepsilon^-}{\partial \nu} = -\dfrac{\partial \xi_\varepsilon^-}{\partial \nu} \text{ on } \Gamma_b, \\ \varsigma_\varepsilon^+ = -\xi_\varepsilon^+ \text{ on } \gamma_\varepsilon \setminus \Gamma_u, \ \varsigma_\varepsilon^+ = \varsigma_\varepsilon^- \text{ on } \Gamma_u \setminus \gamma_\varepsilon, \\ \varsigma_\varepsilon^- = 0 \text{ on } \gamma_\varepsilon \cap \Gamma_u, \ \dfrac{\partial \varsigma_\varepsilon^+}{\partial x_2} = \dfrac{\partial \varsigma_\varepsilon^-}{\partial x_2} + \dfrac{\partial \omega^-}{\partial x_2} \text{ on } \Gamma_u \setminus \gamma_\varepsilon. \end{cases} \tag{5.39}$$

Here ς_ε^+, ς_ε^- are the restriction of ς_ε to Ω_ε^+, Ω^- respectively. Now, introduce $\varpi_\varepsilon = \varsigma_\varepsilon - \rho_\varepsilon$, where $\rho_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ solves the problem

$$\begin{cases} \Delta \rho_\varepsilon = 0 \text{ in } \Omega_\varepsilon, \ \rho_\varepsilon = -\xi_\varepsilon^+ \text{ on } \gamma_\varepsilon \setminus \Gamma_u, \ \rho_\varepsilon = 0 \text{ on } \gamma_\varepsilon \cap \Gamma_u, \ \dfrac{\partial \rho_\varepsilon}{\partial \nu} = -\dfrac{\partial \xi_\varepsilon^-}{\partial \nu} \text{ on } \Gamma_b. \end{cases} \tag{5.40}$$

Use of integration by parts along with some calculation shows that

$$\begin{aligned} \|\varpi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} &\leq C\|\nabla \varpi_\varepsilon\|^2_{(L^2(\Omega_\varepsilon))^3} \\ &= C\left(\int_{\Omega_\varepsilon} \nabla \varsigma_\varepsilon \cdot \nabla \varsigma_\varepsilon - \int_{\Omega_\varepsilon} \nabla \varsigma_\varepsilon \cdot \nabla \rho_\varepsilon - \int_{\Omega_\varepsilon} \nabla \varsigma_\varepsilon \cdot \nabla \rho_\varepsilon + \int_{\Omega_\varepsilon} \nabla \rho_\varepsilon \cdot \nabla \rho_\varepsilon\right). \end{aligned}$$

Boundary conditions in earlier problems implies $\|\varpi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \leq C\left\langle \dfrac{\partial \omega^-}{\partial \nu}, \varpi_\varepsilon \right\rangle_{H^{-1/2}(\Gamma_u \setminus \gamma_\varepsilon), H^{1/2}(\Gamma_u \setminus \gamma_\varepsilon)}$. Equation (5.2) implies $\omega^- \in H^5_{per}(\Omega^-) \subset C^3\left(\overline{\Omega^-}\right)$. Hence, we can conclude $\|\varpi_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C$, where $C > 0$ is a constant independent of ε. This implies if we prove $\|\rho_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C$, the proof of $\|\varsigma_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C$ is trivial. To do this choose a function $\psi \in C^2(-\infty, +\infty)$. More precisely, let

$$\psi(x_2) = \begin{cases} 1 & \frac{M+3M'}{4} < x_2 \text{ and } \frac{3m+M}{4} > x_2, \\ 0 & \frac{m+3M}{4} < x_2 < \frac{3M+M'}{4}. \end{cases} \tag{5.41}$$

If we define $\Upsilon_\varepsilon = \psi \xi_\varepsilon$, then $\Upsilon_\varepsilon = \xi_\varepsilon$ on γ_ε, $\dfrac{\partial \Upsilon_\varepsilon}{\partial \nu} = \dfrac{\partial \xi_\varepsilon}{\partial \nu}$ on Γ_b and $\|\Upsilon_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C$. Use $(\rho_\varepsilon + \Upsilon_\varepsilon)$ as a test function in (5.40), we will get $\|\rho_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C$. Now we have enough tools developed to prove Theorem 5.5.

Proof of Theorem 5.5 :- By triangle inequality, we can write

$$\begin{aligned} \|\bar{y}_\varepsilon - \varepsilon \xi_\varepsilon^+\|_{H^1(\Omega_\varepsilon^+)} &+ \|\bar{y}_\varepsilon - \bar{y} - \varepsilon \xi_\varepsilon^-\|_{H^1(\Omega^-)} \\ &\leq \|\bar{y}_\varepsilon - \varepsilon \xi_\varepsilon^+\|_{H^1(\Omega_\varepsilon^+)} + \|\bar{y}_\varepsilon - \bar{y} - \varepsilon \omega^- - \varepsilon \xi_\varepsilon^-\|_{H^1(\Omega^-)} + \varepsilon \|\omega^-\|_{H^1(\Omega^-)} \\ &= \|\bar{y}_\varepsilon - \tilde{\bar{y}} - \varepsilon \omega - \varepsilon \xi_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon \|\omega^-\|_{H^1(\Omega^-)} \\ &= \|\tau_\varepsilon + \varepsilon(\omega_\varepsilon - \omega)\|_{H^1(\Omega_\varepsilon)} + \varepsilon \|\omega^-\|_{H^1(\Omega^-)}. \end{aligned}$$

Hence

$$\|\bar{y}_\varepsilon - \varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)} + \|\bar{y}_\varepsilon - \bar{y} - \varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)} \le \|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|\varsigma_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|\omega^-\|_{H^1(\Omega^-)}. \tag{5.42}$$

We will have the desired result (5.34) because we already have all the required estimates. □

The recipe we prepared for proving Theorem 5.5 will also help to prove the following theorem. The details of this result, we are not going to present here.

Theorem 5.6. Assume (5.1). Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$, $(\bar{y}, \bar{u})$ be the optimal solutions of the in-homogenized and homogenized control problems given by (2.9) and (3.12), respectively. If ω is as defined in (5.36) then for small enough $\varepsilon > 0$

$$\|\tilde{\bar{y}}_\varepsilon - \tilde{\bar{y}} - \varepsilon\omega\|_{H^1(\Omega_\varepsilon \setminus B_\varepsilon)} \le C\left(\varepsilon + \|\bar{u}_\varepsilon - \bar{u}\|_{L^2(\Gamma_b)}\right) \tag{5.43}$$

where C is a positive constant independent of ε.

5.4 Adjoint state (Neumann Boundary Control)

The corrector for the co-state of Neumann boundary control case will be discussed in this subsection. For that very first step is the demand of some assumption which will give us required regularity of $\bar{y}$. Assume

$$f \in H^2_{per}\left(\Omega^-\right) \cap L^2_{per}\left(\Omega\right),\; g \in H^6_{per}(0, L). \tag{5.44}$$

The extension of $\bar{y}$ to O^- by L- periodicity, is a solution in $H^1\left(O^-\right)$ of problem

$$-\Delta\bar{z} = \bar{y} - y_d \text{ in } O^-,\; \bar{z} = 0 \text{ on } \mathbb{R} \times \{M\},\; \frac{\partial\bar{z}}{\partial\nu} = 0 \text{ on } \{(x_1, g(x_1)) : x_1 \in \mathbb{R}\}.$$

Standard regularity results and assumption (5.44) gives

$$\bar{z} \in H^6_{per}\left(\Omega^-\right) \subset C^4\left(\overline{\Omega^-}\right). \tag{5.45}$$

Further, introduce

$$\xi_\varepsilon(x) = \begin{cases} \xi_\varepsilon^+(x) = \dfrac{\partial\bar{z}}{\partial x_2}(x_1, M)\psi_\varepsilon^+ \text{ in } \Omega_\varepsilon^+, \\ \xi_\varepsilon^-(x) = \dfrac{\partial\bar{z}}{\partial x_2}(x_1, M)\left(\psi_\varepsilon^- - \beta_1\right) \text{ in } \Omega^-. \end{cases} \tag{5.46}$$

Now we will state one of our main theorem of this subsection whose proof will come later.

Theorem 5.7. Assume (5.44). Let $\bar{z}_\varepsilon$, $\bar{z}$, respectively, be the solution of the in-homogenized and homogenized co-states given by (3.10) and (3.14). Again let $\bar{y}_\varepsilon$, $\bar{y}$, respectively, be the optimal state of the in-homogenized and homogenized control problems given by (2.9) and (3.12). If ξ_ε is as defined in (5.46) then for small enough $\varepsilon > 0$,

$$\|\bar{z}_\varepsilon - \varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)} + \|\bar{z}_\varepsilon - \bar{z} - \varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)} \le C\left(\varepsilon + \|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)}\right), \tag{5.47}$$

where C is a positive constant independent of small enough ε.

As in the previous cases, we need to introduce appropriate test function. Define ω and ω_ε by

$$\omega = \begin{cases} 0 & \text{in } \Omega^+, \\ \omega^- & \text{in } \Omega^-, \end{cases} \text{ and } \omega_\varepsilon = \begin{cases} \omega_\varepsilon^+ & \text{in } \Omega^+, \\ \omega_\varepsilon^- & \text{in } \Omega^-. \end{cases} \tag{5.48}$$

Here $\omega^- \in H^1_{per}(\Omega^-)$ is the unique solution of problem

$$\Delta\omega^- = 0 \text{ in } \Omega^-,\ \omega^- = \beta_1 \frac{\partial \bar{z}}{\partial x_2}(x_1, M) \text{ on } \Gamma_u,\ \frac{\partial \omega^-}{\partial \nu} = 0 \text{ on } \Gamma_b. \tag{5.49}$$

and $\omega_\varepsilon^+ \in H^1(\Omega_\varepsilon),\ \omega_\varepsilon^- \in H^1_{per}(\Omega^-)$ solves the partial differential equation

$$\begin{cases} \Delta\omega_\varepsilon^+ = 0 \text{ in } \Omega_\varepsilon^+,\ \Delta\omega_\varepsilon^- = 0 \text{ in } \Omega^-,\ \dfrac{\partial \omega_\varepsilon^-}{\partial \nu} = -\dfrac{\partial \xi_\varepsilon^-}{\partial \nu} \text{ on } \Gamma_b, \\ \omega_\varepsilon^+ = -\xi_\varepsilon^+ \ \text{ on } \gamma_\varepsilon \setminus \Gamma_u,\ \ \omega_\varepsilon^+ = \omega_\varepsilon^- - \beta_1 \dfrac{\partial \bar{y}}{\partial x_2}(x_1, M) \ \text{ on } \Gamma_u \setminus \gamma_\varepsilon, \\ \omega_\varepsilon^- = \beta_1 \dfrac{\partial \bar{y}}{\partial x_2}(x_1, M) \ \text{ on } \gamma_\varepsilon \cap \Gamma_u,\ \ \dfrac{\partial \omega_\varepsilon^+}{\partial x_2} = \dfrac{\partial \omega_\varepsilon^-}{\partial x_2} \ \text{ on } \Gamma_u \setminus \gamma_\varepsilon. \end{cases} \tag{5.50}$$

Let τ_ε be the function defined by

$$\tau_\varepsilon = \begin{cases} \tau_\varepsilon^+ = \bar{z}_\varepsilon - \varepsilon\omega_\varepsilon^+ - \varepsilon\xi_\varepsilon^+ \ \text{ in } \Omega_\varepsilon^+, \\ \tau_\varepsilon^- = \bar{z}_\varepsilon - \bar{z} - \varepsilon\omega_\varepsilon^- - \varepsilon\xi_\varepsilon^- \ \text{ in } \Omega^-. \end{cases} \tag{5.51}$$

Clearly, $\tau_\varepsilon^+ \in H^1(\Omega_\varepsilon^+)$ and $\tau_\varepsilon^- \in H^1(\Omega^-)$. At the interface of Ω_ε^+ and Ω^-, trace of τ_ε^+ and τ_ε^- agrees because of the boundary conditions of $\bar{y}_\varepsilon,\ \bar{y},\ \omega_\varepsilon^\pm$ and $\xi_\varepsilon^\pm$ at $\Omega_\varepsilon^+ \cap \Omega^-$. Hence, $\tau_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$. $\tau_\varepsilon = 0$ on γ_ε and $\dfrac{\partial \tau_\varepsilon}{\partial \nu} = 0$ on Γ_b. Moreover, $\dfrac{\partial \tau_\varepsilon^+}{\partial x_2} = \dfrac{\partial \tau_\varepsilon^-}{\partial x_2}$ as $H^{-1/2}(\bar{\Omega}_\varepsilon^+ \cap \bar{\Omega}^-)$ function. Finally, we get estimate $\|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C\left(\varepsilon + \|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)}\right)$, where C is a positive constant independent of ε. Let $\varsigma_\varepsilon = \omega_\varepsilon - \omega$. Note that $\varsigma_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ and solves the partial differential equation

$$\begin{cases} \Delta\varsigma_\varepsilon^+ = 0 \text{ in } \Omega_\varepsilon^+,\ \Delta\varsigma_\varepsilon^- = 0 \text{ in } \Omega^-,\ \dfrac{\partial \varsigma_\varepsilon^-}{\partial \nu} = -\dfrac{\partial \xi_\varepsilon^-}{\partial \nu} \text{ on } \Gamma_b, \\ \varsigma_\varepsilon^+ = -\xi_\varepsilon^+ \text{ on } \gamma_\varepsilon \setminus \Gamma_u,\ \varsigma_\varepsilon^+ = \varsigma_\varepsilon^- \text{ on } \Gamma_u \setminus \gamma_\varepsilon, \\ \varsigma_\varepsilon^- = 0 \text{ on } \gamma_\varepsilon \cap \Gamma_u,\ \dfrac{\partial \varsigma_\varepsilon^+}{\partial x_2} = \dfrac{\partial \varsigma_\varepsilon^-}{\partial x_2} + \dfrac{\partial \omega^-}{\partial x_2} \text{ on } \Gamma_u \setminus \gamma_\varepsilon. \end{cases} \tag{5.52}$$

Introduce $\varpi_\varepsilon = \varsigma_\varepsilon - \rho_\varepsilon$, where $\rho_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ solves

$$\Delta\rho_\varepsilon = 0 \text{ in } \Omega_\varepsilon,\ \rho_\varepsilon = -\xi_\varepsilon^+ \text{ on } \gamma_\varepsilon \setminus \Gamma_u,\ \rho_\varepsilon = 0 \text{ on } \gamma_\varepsilon \cap \Gamma_u,\ \frac{\partial \rho_\varepsilon}{\partial \nu} = -\frac{\partial \xi_\varepsilon^-}{\partial \nu} \text{ on } \Gamma_b. \tag{5.53}$$

A routine calculation, leads to $\|\varpi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \leq C\left\langle \dfrac{\partial \omega^-}{\partial \nu}, \varpi_\varepsilon \right\rangle_{H^{-1/2}(\Gamma_u \setminus \gamma_\varepsilon), H^{1/2}(\Gamma_u \setminus \gamma_\varepsilon)} \leq C$, since assumption (5.45) implies that $\omega^- \in H^5_{per}(\Omega^-) \subset C^3\left(\overline{\Omega^-}\right)$. Next we prove $\|\varsigma_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C$. It is enough to prove $\|\rho_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C$ then we are through because of the similar arguments given earlier. Construct a function $\psi \in C^2(-\infty, +\infty)$ such that

$$\psi(x_2) = \begin{cases} 1 \ \ \frac{M+3M'}{4} < x_2 \text{ and } \frac{3m+M}{4} > x_2, \\ 0 \ \ \frac{m+3M}{4} < x_2 < \frac{3M+M'}{4}. \end{cases} \tag{5.54}$$

Again, define $\Upsilon_\varepsilon = \psi\xi_\varepsilon$. Notice that $\Upsilon_\varepsilon = \xi_\varepsilon$ on γ_ε, $\dfrac{\partial \Upsilon_\varepsilon}{\partial \nu} = \dfrac{\partial \xi_\varepsilon}{\partial \nu}$ on Γ_b and $\|\Upsilon_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$. Use $(\rho_\varepsilon + \Upsilon_\varepsilon)$ as a test function in (5.53), we will get $\|\rho_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$. Now we have enough tools developed to prove Theorem 5.7.

Proof of Theorem 5.7 :- By triangle inequality, we can write

$$\begin{aligned}
\|\bar{z}_\varepsilon - \varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)} &+ \|\bar{z}_\varepsilon - \bar{z} - \varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)} \\
&\le \|\bar{z}_\varepsilon - \varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)} + \|\bar{z}_\varepsilon - \bar{z} - \varepsilon\omega^- - \varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)} + \varepsilon\|\omega^-\|_{H^1(\Omega^-)} \\
&= \|\bar{z}_\varepsilon - \tilde{\bar{z}} - \varepsilon\omega - \varepsilon\xi_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|\omega^-\|_{H^1(\Omega^-)} \\
&= \|\tau_\varepsilon + \varepsilon(\omega_\varepsilon - \omega)\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|\omega^-\|_{H^1(\Omega^-)}.
\end{aligned}$$

Hence $\|\bar{z}_\varepsilon - \varepsilon\xi_\varepsilon^+\|_{H^1\left(\Omega_\varepsilon^+\right)} + \|\bar{z}_\varepsilon - \bar{z} - \varepsilon\xi_\varepsilon^-\|_{H^1(\Omega^-)} \le \|\tau_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|\varsigma_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|\omega^-\|_{H^1(\Omega^-)}$. Again like earler, we will have the desired result (5.47). □

We can also derive the following result whose proof will not be presented here.

Theorem 5.8. Assume (5.44). Let $\bar{z}_\varepsilon,\ \bar{z}$, respectively, be the solution of the in-homogenized and homogenized co-states given by (3.10) and (3.14). Again let $\bar{y}_\varepsilon,\ \bar{y}$, respectively, be the optimal state of the in-homogenized and homogenized control problems given by (2.9) and (3.12). If ω is as defined in (5.48) then for small enough $\varepsilon > 0$

$$\|\tilde{\bar{z}}_\varepsilon - \tilde{\bar{z}} - \varepsilon\omega\|_{H^1(\Omega_\varepsilon\setminus B_\varepsilon)} \le C\left(\varepsilon + \|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)}\right) \tag{5.55}$$

where C is a positive constant independent of ε.

Corollary 5.2. Assume (5.44). Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon),\ (\bar{y}, \bar{u})$, respectively, be the optimal solutions of the in-homogenized and homogenized control problems given by (2.9) and (3.12). then there exists a positive constant C, independent of ε, such that

$$\|\bar{u}_\varepsilon - \bar{u}\|_{H^{-1/2}(\Gamma_b)} \le C\left(\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)} + \varepsilon\right), \tag{5.56}$$

for ε small enough.

Sketch of the proof:- By trace theorem and Theorem 5.8

$$\begin{aligned}
\|\bar{u}_\varepsilon - \bar{u}\|_{L^2(\Gamma_b)} &\le C\|\bar{z}_\varepsilon - \bar{z}\|_{H^1\left((0,l_1)\times\left(g(x_1),\frac{g(x_1)+M}{2}\right)\right)} \\
&\le C\|\bar{z}_\varepsilon - \bar{z}\|_{H^1(\Omega_\varepsilon\setminus B_\varepsilon)} \le C\left(\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(\Omega^-)} + \varepsilon\right).
\end{aligned}$$

6 Conclusions

In the previous sections, we have considered a rectangular domain with oscillating boundary. Then a controlled Laplacian problem with controls acting on the boundary are introduced together with a L^2-cost functional. There are two types of problems namely one with Dirichlet boundary control and other one with controls acting as Neumann boundary data. Indeed the controls are acting on the boundary where there are no oscillations. Of course, the more interesting problem is to consider controls on the oscillating boundary. But we have not considered this situation in this article.

The homogenization of the problem is relatively easy and follows along the same steps for both the cases. But the main focus of the paper is the derivation of error estimates which depends on various test functions. Though the fundamental ingredient is a function which is available in the literature, one need to cleverly construct different test functions for Dirichlet and Neumann. In each case, we have two types of error estimates (see Theorems 5.1 and 5.2). In Theorem 5.1, we have the error estimate in the full domain Ω_ε^+ where the test functions ξ_ε depends on ε. But in Theorem 5.2, the error estimate is in the domain Ω_ε with a strip is removed from it. The advantage in this situation is that the test function is independent of ε and hence it is computable. Similar results are available in all other situation as in sections 5.2, 5.3 and 5.4.

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