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Homogenization of an optimal control problem in a domain with highly oscillating boundary using periodic unfolding method

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Abstract. The method of unfolding is used to study optimal control problem in a domain with oscillating boundary. We consider Neumann condition on the oscillating part of the boundary and the rest is more interesting than the Dirichlet condition. Hence the limit problem consists of two parts, namely in the lower part and upper part with appropriate interface conditions. In this article, we have consider two cost functionals, namely L^2 and Dirichlet cost functional. Interior and boundary unfolding operator are introduced in the process.

1 Introduction

In this article, we plan to study a distributed optimal control problem in an oscillating domain with Neumann condition on the oscillating part. The controls are applied away from the oscillating boundary. This article is a continuation of our earlier work where we have studied [31], [32], Laplacian and stokes problem with Dirichlet boundary conditions. The Neumann problem is more interesting, difficult and produces nice limit problem. Our aim is to use the method of unfolding introduced by

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Cioranescu et. al. in [12] and developed by Damlamian [13], [14], for periodic unfolding. We further refer to the paper by A. Damlamian and K. Peterson [15]. There is also a large amount of literature on the homogenization with oscillating boundaries which has tremendous applications as well. For example (see [1],[3],[4], [5], [8], [9], [10], [18], [19], [31]). Regarding optimal control/ controllability result in domain with oscillating boundary are concerned refer to [16], [27], [28], [31], [32], [33]. In [28], an exact controllability problem has been studied where as in [33] an optimal control problem for a fourth order problem has been investigated. One can look into [21], [22], [29], [30] for homogenization of optimal control and controllability, [7], [11], [20], [35] for general homogenization and [2], [6], [9], [23], [24], [25], [26] for reference in optimal control problems and derivation of optimality systems.

The layout of this paper is as following. After a brief introduction in this section, we go to Section 2 where we describe the required domain and its boundaries. We describe the optimal control problems with respect to two different cost functional in Section 3. One is called as L^2-cost functional and another is called as Dirichlet cost functional. We defined periodic unfolding, boundary unfolding operator and its properties in Section 4. In Section 5, we do the convergence analysis and find the limit optimal control problem for the case of L^2-cost functional. Similarly for Dirichlet cost functional, we derived the homogenized optimal control problem in Section 6.

2 Oscillating Boundary Domain

In this paper, we consider the same domain as in [31]. For the sake of completeness, here we would like to describe the oscillating boundary domain once again. For a small parameter $\varepsilon = \frac{1}{N}$, $N \in \mathbb{Z}^+$, we consider a oscillating domain Ω_ε as given in the Figure 1. We now describe mathematically the domain Ω_ε and its boundaries. Let $L > 0$ and $g : \mathbb{R} \to \mathbb{R}$ be a smooth and periodic function with periodic L. This domain is nearly a two-dimensional rectangular region with oscillating part on one side of the region. One can also see it as a transverse cross section of a three- dimensional slab perpendicular to the plane. The oscillating part is sitting at the top of a rectangular region of the domain.

Let $0 < a < b < L$ and η_ε be the εL-periodic function defined on $[0, \varepsilon L]$ by

$$\eta_\varepsilon(x_1) = \begin{cases} M' & if \;\; x_1 \in (\varepsilon a, \varepsilon b), \\ M & if \; x_1 \in [0, \varepsilon L) \backslash (\varepsilon a, \varepsilon b), \end{cases}$$

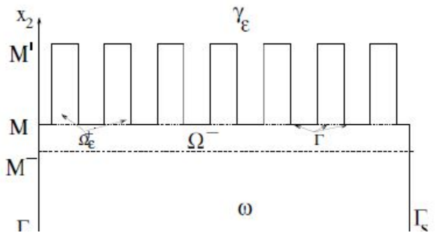
with $M' > M > m$, where m is the maximum value of the smooth function g in $[0, L]$. We can write the domain Ω_ε as $\Omega_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 : \; 0 < x_1 < L, \, g(x_1) < x_2 < \eta_\varepsilon(x_1)\}$. The top boundary of Ω_ε is denoted by γ_ε and defined as $\gamma_\varepsilon = \{(x_1, x_2) : \; x_1 \in [0, L], \, x_2 = \eta_\varepsilon(x_1)\}$. The bottom boundary Γ_b of Ω_ε is defined as $\Gamma_b = \{(x_1, x_2) : \; x_2 = g(x_1), x_1 \in [0, L]\}$. Let Ω_ε^+ is the top part of the domain Ω_ε which is the union of slabs of height $(M' - M)$ and width $\varepsilon(b - a)$. It can defined as

$$\Omega_\varepsilon^+ = \bigcup_{k=0}^{N-1} (k\varepsilon L + \varepsilon a, k\varepsilon L + \varepsilon b) \times (M, M').$$

Denote Ω^- as fixed a part of the domain Ω_ε which is described by

$$\Omega^- = \{(x_1, x_2) : 0 < x_1 < L, \; g(x_1) < x_2 < M\}.$$

The vertical and top boundary of Ω^- denoted by Γ_s and Γ defined as



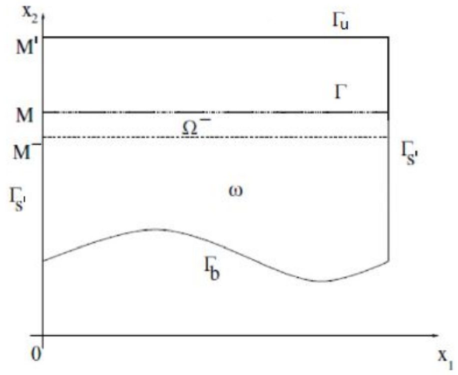


Fig. 2 Ω

$$\Gamma_s = \{(0,x_2) : g(0) \leq x_2 \leq M\} \cup \{(L,x_2) : g(L) \leq x_2 \leq M\}$$

and $\Gamma = \{(x_1,M) : 0 \leq x_1 \leq L\}$ respectively. The common boundary between Ω_ε^+ and Ω^- is denoted by Γ_ε and defined as

$$\Gamma_\varepsilon = \bigcup_{k=0}^{N-1} (k\varepsilon L + \varepsilon a, k\varepsilon L + \varepsilon b).$$

We can also write Ω_ε as $\Omega_\varepsilon = Int\left(\overline{\Omega_\varepsilon^+ \cup \Omega^-}\right)$. Let ω be the sub-domain of Ω^-, In this sub-domain control acts. Without loss of generality, we can consider

$$\omega = \{(x_1,x_2) : 0 < x_1 < L,\ g(x_1) < x_2 < M^-\}$$

where $M > M^- > m$. Our full domain will be denoted by Ω (see Figure 2). Mathematically we can write

$$\Omega = \{(x_1,x_2) : 0 < x_1 < L,\ g(x_1) < x_2 < M'\}.$$

The bottom part of the boundary of Ω is same as Ω_ε. We still denote it by Γ_b. The vertical boundary of Ω is denoted by $\Gamma_{s'}$ and can be written as

$$\Gamma_{s'} = \{(0,x_2) : g(0) \leq x_2 \leq M'\} \cup \{(L,x_2) : g(L) \leq x_2 \leq M'\}.$$

The top boundary of Ω is denoted by $\Gamma_u = \{(x_1,M') : 0 \leq x_1 \leq L\}$. If we denote Ω^+ as $\Omega^+ = \{(x_1,x_2) : 0 < x_1 < L, M < x_2 < M'\}$ then we can write $\Omega = Int\left(\overline{\Omega^+ \cup \Omega^-}\right)$. Let $L^2_{per}(\Omega_\varepsilon) = \{f \in L^2(\Omega_\varepsilon), f(x_1+kL,x_2) = f(x_1,x_2)\ \forall k \in \mathbb{Z}\}$, $H^1_{per}(\Omega_\varepsilon) = \{f :\ f \in H^1(\Omega_\varepsilon), f(x_1+kL,x_2) = f(x_1,x_2)\ \forall k \in \mathbb{Z}\}$. We call function are Γ_s-periodic, which are taking the same value on the both side of Γ_s.

3 Problem description

We consider the following control problem:

$$\begin{cases} -\Delta u_\varepsilon = f + \theta\chi_\omega \quad in\ \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 \ \ on\ \gamma_\varepsilon, u_\varepsilon = h \ \ on\ \Gamma_b, \\ u_\varepsilon \ \ is\ \Gamma_s - periodic. \end{cases} \tag{3.1}$$

Here, $\theta \in L^2(\omega)$ is a control function acting in the sub-domain ω and χ_ω is the characteristic function of ω. We consider source term f in $L^2_{per}(\Omega)$ and h in $H^{1/2}_{per}(\Gamma_b)$. It is known that if $\theta \in L^2_{per}(\omega)$, then (3.1) admits a unique solution u_ε (depending on θ) in $H^1_{per}(\Omega_\varepsilon)$ that satisfies $u_\varepsilon|_{\Gamma_b} = h$. The solution operator is linear and continuous from $L^2_{per}(\Omega) \times L^2_{per}(\omega)$ into $H^1_{per}(\Omega_\varepsilon)$, i.e.

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C(\|f\|_{L^2(\Omega)} + \|\theta\|_{L^2(\omega)} + \|h\|_{H^{1/2}(\Gamma_b)}) \tag{3.2}$$

where $C > 0$ is independent of ε. For regularization parameter $\beta > 0$, let us consider two cost functionals, first one known as L^2-cost functional, more precisely,

$$J_{1,\varepsilon}(u_\varepsilon, \theta) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_{\omega} \theta^2$$

where the desired state $u_d \in H^1_{per}(\Omega_\varepsilon)$ satisfies $u_d|_{\Gamma_b} = h$. With this cost functional, we consider the optimal control problem

$$\inf\{J_{1,\varepsilon}(u_\varepsilon, \theta) | \theta \in L^2(\omega), (u_\varepsilon, \theta)\ obeys\ (3.1)\}. \tag{$P_{1,\varepsilon}$}$$

We also consider the Dirichlet cost functional given by

$$J_{2,\varepsilon}(u_\varepsilon, \theta) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon - \nabla u_d|^2 + \frac{\beta}{2}\int_{\omega} \theta^2$$

with desired state $u_d \in H^1_{per}(\Omega_\varepsilon)$. The corresponding minimization problem is

$$\inf\{J_{2,\varepsilon}(u_\varepsilon, \theta) |\ \theta \in L^2(\omega),\ (u_\varepsilon, \theta)\ obeys\ (3.1)\}. \tag{$P_{2,\varepsilon}$}$$

It is well known that $(P_{1,\varepsilon})$ and $(P_{2,\varepsilon})$ admit unique solution (see [14], [34]).

Let $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon)$ be the optimal solution to $(P_{1,\varepsilon})$. The following theorem will give us the characterization of $\overline{\theta}_\varepsilon$ with the help of adjoint state $\overline{v}_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ which solves the partial differential equation

$$\begin{cases} -\Delta \overline{v}_\varepsilon = \overline{u}_\varepsilon - u_d \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial \overline{v}_\varepsilon}{\partial \nu} = 0 \ \text{ on } \gamma_\varepsilon,\ \overline{v}_\varepsilon = 0 \ \text{ on } \ \Gamma_b, \\ \overline{v}_\varepsilon \ \text{ is } \Gamma_s - \text{periodic}. \end{cases} \tag{3.3}$$

Theorem 3.1. Let $f \in L^2(\Omega)$, $h \in H^{1/2}(\Gamma_b)$ and $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon)$ be the optimal solution of $(P_{1,\varepsilon})$. Let $\overline{v}_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ solves (3.3), then the optimal control is given by

$$\overline{\theta}_\varepsilon = -\frac{1}{\beta}\overline{v}_\varepsilon \chi_\omega.$$

Conversely, assume that a pair $(\hat{u}_\varepsilon, \hat{v}_\varepsilon) \in H^1_{per}(\Omega_\varepsilon) \times H^1_{per}(\Omega_\varepsilon)$ solves the coupled optimality system

$$\begin{cases} -\Delta \hat{u}_\varepsilon = f - \frac{1}{\beta}\hat{v}_\varepsilon \chi_\omega, -\Delta \hat{v}_\varepsilon = \hat{u}_\varepsilon - u_d \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial \hat{u}_\varepsilon}{\partial \nu} = 0, \dfrac{\partial \hat{v}_\varepsilon}{\partial \nu} = 0 \text{ on } \gamma_\varepsilon, \\ \hat{u}_\varepsilon = h, \hat{v}_\varepsilon = 0 \text{ on } \Gamma_b, \\ \hat{u}_\varepsilon, \hat{v}_\varepsilon \text{ is } \Gamma_s - \text{periodic}. \end{cases} \tag{3.4}$$

Then, the pair $(\hat{u}_\varepsilon, -\frac{1}{\beta}\hat{v}_\varepsilon \chi_\omega)$ is the optimal solution to $(P_{1,\varepsilon})$.

Similarly if $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon)$ optimal solution to the problem $(P_{2,\varepsilon})$ then optimal control $\overline{\theta}_\varepsilon$ will be characterized with the help of adjoint state $\overline{v}_\varepsilon$, that solves the partial differential equations

$$\begin{cases} -\Delta \overline{v}_\varepsilon = -\Delta(\overline{u}_\varepsilon - u_d) \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial \overline{v}_\varepsilon}{\partial \nu} = (\nabla \overline{u}_\varepsilon - \nabla u_d)\cdot \nu \text{ on } \gamma_\varepsilon, \\ \overline{v}_\varepsilon = 0 \text{ on } \Gamma_b, \\ \overline{v}_\varepsilon \text{ is } \Gamma_s - \text{periodic}. \end{cases} \tag{3.5}$$

Theorem 3.2. Let $f \in L^2(\Omega)$, $h \in H^{1/2}(\Gamma_b)$ and $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon)$ be the optimal solution of $(P_{2,\varepsilon})$. Let $\overline{v}_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ solves (3.5), then the optimal control is given by

$$\overline{\theta}_\varepsilon = -\frac{1}{\beta}\overline{v}_\varepsilon \chi_\omega.$$

Conversely, assume that a pair $(\hat{u}_\varepsilon, \hat{v}_\varepsilon) \in H^1_{per}(\Omega_\varepsilon) \times H^1_{per}(\Omega_\varepsilon)$ solves the coupled optimality system

$$\begin{cases} -\Delta \hat{u}_\varepsilon = f - \frac{1}{\beta}\hat{v}_\varepsilon \chi_\omega, -\Delta \hat{v}_\varepsilon = -\Delta(\hat{u}_\varepsilon - u_d) \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial \hat{u}_\varepsilon}{\partial \nu} = 0, \dfrac{\partial \hat{v}_\varepsilon}{\partial \nu} = (\nabla \hat{u}_\varepsilon - \nabla u_d)\cdot \nu \text{ on } \gamma_\varepsilon, \\ \hat{u}_\varepsilon = h, \hat{v}_\varepsilon = 0 \text{ on } \Gamma_b, \\ \hat{u}_\varepsilon, \hat{v}_\varepsilon \text{ is } \Gamma_s - \text{periodic}. \end{cases} \tag{3.6}$$

Then, the pair $(\hat{u}_\varepsilon, -\frac{1}{\beta}\hat{v}_\varepsilon \chi_\omega)$ is the optimal solution to $(P_{2,\varepsilon})$.

4 Unfolding operator and its properties

In this section, we introduce periodic unfolding operator and present some interesting properties (see [15]). Let $[0,L]$ be a reference cell as in Section 2. For $x \in \mathbb{R}$, we write $[x]_L$ as the integer part of x with respect to L, that is $[x]_L = kL$, where k is the largest integer such that $kL \leq x$.

Definition 4.1. (The Unfolding operator) Let $\phi^\varepsilon : \Omega^+ \times (a,b) \to \Omega^+_\varepsilon$ be defined by $x \to \left(\varepsilon \left[\frac{x_1}{\varepsilon}\right]_L + \varepsilon x_3, x_2\right)$. The ε-unfolding of a function $u : \Omega^+_\varepsilon \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \Omega^+ \times (a,b) \to \mathbb{R}$. The operator which maps every function $u : \Omega^+_\varepsilon \to \mathbb{R}$ to its ε unfolding is called the unfolding operator. Let the unfolding operator be denoted by T^ε, i.e.

$$T^\varepsilon : \{u : \Omega^+_\varepsilon \to \mathbb{R}\} \to \{v : \Omega^+ \times (a,b) \to \mathbb{R}\}$$

defined by

$$T^\varepsilon u(x_1,x_2,x_3) = uo\phi^\varepsilon(x_1,x_2,x_3) = u\left(\varepsilon \left[\frac{x_1}{\varepsilon}\right]_L + \varepsilon x_3, x_2\right).$$

If U is an open subset of $\mathbb{R}^2$ containing Ω_ε^+ and u is real valued function on U, $T^\varepsilon u$ will mean T^ε acting on the restriction of u to Ω_ε^+. We would like to state few properties of the unfolding operator T^ε as following.

Proposition 4.1. The unfolding operator T^ε is linear and for functions u,v from $\Omega_\varepsilon^+ \to \mathbb{R}$, we have $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.

Proposition 4.2. Let $u \in L^1(\Omega_\varepsilon^+)$. Then

$$\int_{\Omega^+\times(a,b)} T^\varepsilon u \, dx = L \int_{\Omega_\varepsilon^+} u \, dx$$

Proof.

$$\begin{aligned}\int_{\Omega^+\times(a,b)} T^\varepsilon u \, dx &= \int_{x_2=M}^{M'} \int_{x_3=a}^{b} \sum_{k=0}^{N-1} \int_{x_1=k\varepsilon L}^{(k+1)\varepsilon L} u(k\varepsilon L + \varepsilon x_3, x_2) \, dx_1 dx_2 dx_3 \\ &= L \sum_{k=0}^{N-1} \int_{x_2=M}^{M'} \int_{x_1=k\varepsilon L+\varepsilon a}^{k\varepsilon L+\varepsilon b} u(x_1,x_2) \, dx_1 dx_2 = L \int_{\Omega_\varepsilon^+} u(x) \, dx.\end{aligned}$$

Proposition 4.3. Let $u \in L^2(\Omega_\varepsilon^+)$. Then $T^\varepsilon u \in L^2(\Omega^+ \times (a,b))$ and $\|T^\varepsilon u\|_{L^2(\Omega^+\times(a,b))} = \sqrt{L}\|u\|_{L^2(\Omega_\varepsilon^+)}$

Proof. Proof follows from the above proposition , because $|u|^2 \in L^1(\Omega_\varepsilon^+)$

Proposition 4.4. Let $u \in H^1(\Omega_\varepsilon^+)$. Then $T^\varepsilon u \in L^2(0,L;H^1((M,M') \times (a,b)))$. Moreover

$$\frac{\partial}{\partial x_2} T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial x_2} \; and \; \frac{\partial}{\partial x_3} T^\varepsilon u = \varepsilon T^\varepsilon \frac{\partial u}{\partial x_1}. \tag{4.1}$$

Proof. By definition of $T^\varepsilon u$, it is easy to see (4.1). Now

$$\begin{aligned}\|T^\varepsilon u\|^2_{L^2(0,L;H^1((M,M')\times(a,b)))} &= \int_0^L \|T^\varepsilon u\|^2_{H^1((M,M')\times(a,b)} \, dx_1 \\ &= \int_{\Omega^+\times(a,b)} T^\varepsilon \left(\varepsilon^2 |\frac{\partial u}{\partial x_1}|^2 + |\frac{\partial u}{\partial x_2}|^2 + |u|^2 \right) dx \\ &= L \int_{\Omega_\varepsilon^+} \left(\varepsilon^2 |\frac{\partial u}{\partial x_1}|^2 + |\frac{\partial u}{\partial x_2}|^2 + |u|^2 \right) dx \\ &\leq L\|u\|^2_{H^1(\Omega_\varepsilon)} < \infty.\end{aligned}$$

Proposition 4.5. Let $u \in L^2(\Omega^+)$. Then $T^\varepsilon u \to u$ in $L^2(\Omega^+ \times (a,b))$.

Proof. Its easy to prove. First for $u \in D(\Omega)$ and by density argument follows the result.

Proposition 4.6. Let $u_\varepsilon \to u$ in $L^2(\Omega^+)$. Then $T^\varepsilon u_\varepsilon \to u$ in $L^2(\Omega^+ \times (a,b))$.

Proof. Suppose that $u_\varepsilon \to u$ in $L^2(\Omega^+)$.

$$\begin{aligned}\|T^\varepsilon u_\varepsilon - u\|_{L^2(\Omega^+\times(a,b))} &= \|T^\varepsilon u_\varepsilon - T^\varepsilon u + T^\varepsilon u - u\|_{L^2(\Omega^+\times(a,b))}\\ &\le \|T^\varepsilon u_\varepsilon - T^\varepsilon u\|_{L^2(\Omega^+\times(a,b))} + \|T^\varepsilon u - u\|_{L^2(\Omega^+\times(a,b))}\\ &= \sqrt{L}\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon^+)} + \|T^\varepsilon u - u\|_{L^2(\Omega^+\times(a,b))}\\ &\le \sqrt{L}\|u_\varepsilon - u\|_{L^2(\Omega^+)} + \|T^\varepsilon u - u\|_{L^2(\Omega^+\times(a,b))}\\ &\to 0 \;\; as \;\; \varepsilon \to 0.\end{aligned}$$

Proposition 4.7. Let $u_\varepsilon \in L^2(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega^+ \times (a,b))$. Then

$$\widetilde{u_\varepsilon} \rightharpoonup \frac{1}{L}\int_a^b u\, dx_3$$

weakly in $L^2(\Omega^+)$, where $\widetilde{u_\varepsilon}$ is the extension by 0 outsides Ω_ε^+.

Proposition 4.8. Let $u_\varepsilon \in H^1(\Omega_\varepsilon^+)$ for every $\varepsilon > 0$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0,L) \times (a,b); H^1((M,M')))$ Then $\widetilde{u_\varepsilon} \rightharpoonup \frac{1}{L}\int_a^b u\, dx_3$ weakly in $L^2((0,L); H^1((M,M')))$.

Proof. Given that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0,L) \times (a,b); H^1((M,M')))$ implies

$$T^\varepsilon u_\varepsilon \rightharpoonup u \; weakly \; in L^2((0,L) \times (M,M') \times (a,b))$$

and

$$\frac{\partial}{\partial x_2} T^\varepsilon u_\varepsilon \rightharpoonup \frac{\partial u}{\partial x_2} \; in \; L^2((0,L) \times (M,M') \times (a,b))$$

i.e

$$T^\varepsilon \frac{\partial u_\varepsilon}{\partial x_2} \rightharpoonup \frac{\partial u}{\partial x_2} \; in \; L^2((0,L) \times (M,M') \times (a,b)).$$

Using Proposition 4.7 we get $\widetilde{u_\varepsilon} \rightharpoonup \frac{1}{L}\int_a^b u\, dx_3$ in $L^2(\Omega^+)$ and $\widetilde{\frac{\partial u_\varepsilon}{\partial x_2}} \rightharpoonup \frac{1}{L}\int_a^b \frac{\partial u}{\partial x_2}\, dx_3$ in $L^2(\Omega^+)$. But $\frac{\partial \widetilde{u_\varepsilon}}{\partial x_2} = \widetilde{\frac{\partial u_\varepsilon}{\partial x_2}}$. Hence $\tilde{u}_\varepsilon \rightharpoonup \frac{1}{L}\int_a^b u\, dx_3$ weakly in $L^2((0,L); H^1((M,M')))$.

4.1 Unfolding on the boundary:

For our analysis, we also need to unfold the common boundary of Ω_ε^+ and Ω^-. So we define the boundary unfolding operator on Γ_ε.

Definition 4.2. Let $\phi^\varepsilon_{x_2=M} : (0,L) \times (a,b) \to \Gamma_\varepsilon$ be defined by $x \to \left(\varepsilon \left[\frac{x_1}{\varepsilon}\right]_L + \varepsilon x_3\right)$. The ε-unfolding of a function $u : \Gamma_\varepsilon \to \mathbb{R}$ is the function $uo\phi^\varepsilon_{x_2=M} : (0,L) \times (a,b) \to \mathbb{R}$ denoted by T^ε_Γ, that is $T^\varepsilon_\Gamma : \{u : \Gamma_\varepsilon \to \mathbb{R}\} \to \{v : (0,L) \times (a,b) \to \mathbb{R}\}$ by $T^\varepsilon_{x_2=M} u = uo\phi^\varepsilon_{x_2=M} = u\left(\varepsilon \left[\frac{x_1}{\varepsilon}\right]_L + \varepsilon x_3\right)$.

If U is an open subset of $\mathbb{R}^2$ such that $\Gamma_\varepsilon \subset U$ and $u : U \to \mathbb{R}$ then $T^\varepsilon_\Gamma u = T^\varepsilon_\Gamma (u|_{\Gamma_\varepsilon})$

The properties of boundary unfolding are given below without proof.

Proposition 4.9. (i) T^ε_Γ is linear and for functions u, v from $\Gamma_\varepsilon \to \mathbb{R}$, we have $T^\varepsilon_\Gamma(uv) = T^\varepsilon_\Gamma(u) T^\varepsilon_\Gamma(v)$.

(ii) Let $u \in L^2(\Gamma_\varepsilon)$. Then $T_\Gamma^\varepsilon u \in L^2((0,L)\times(a,b))$ and $\|T_\Gamma^\varepsilon u\|_{L^2((0,L)\times(a,b))} = \sqrt{L}\|u\|_{L^2(\Gamma_\varepsilon)}$

(iii) Let $u \in H^1(\Gamma_\varepsilon)$. Then $T_\Gamma^\varepsilon u \in L^2(0,L;H^1((a,b)))$and $\frac{\partial}{\partial x_3}T_\Gamma^\varepsilon u = \varepsilon T_\Gamma^\varepsilon \frac{\partial u}{\partial x_1}$.

(iv) Let $u_\varepsilon \to u$ in $L^2(0,L)$. Then $T_\Gamma^\varepsilon u_\varepsilon \to u$ in $L^2((0,L)\times(a,b))$.

(v) Let u_ε is a sequence in $L^2(\Gamma_\varepsilon)$ such that $T_\Gamma^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0,L)\times(a,b))$. Then $\widetilde{u_\varepsilon} \rightharpoonup \frac{1}{L}\int_a^b u\,dx_3$weakly in $L^2(0,L)$.

5 L^2-cost Functional

5.1 Homogenized System

Consider the space

$$W(\Omega) = \{\psi \in L^2(\Omega):\ \psi|_{\Omega^-} \in H^1(\Omega^-),\ \frac{\partial \psi}{\partial x_2} \in L^2(\Omega)\ and\ \psi|_{\Gamma_b} = h\}$$

and

$$W_0(\Omega) = \{\psi \in L^2(\Omega):\ \psi|_{\Omega^-} \in H^1(\Omega^-),\ \frac{\partial \psi}{\partial x_2} \in L^2(\Omega)\ and\ \psi|_{\Gamma_b} = 0\}.$$

The spaces $W(\Omega)$ and $W_0(\Omega)$ are Hilbert spaces with respect to the norm defined by

$$\|\psi\|_{W(\Omega)}^2 = \|\psi\|_{L^2(\Omega)}^2 + \left\|\frac{\partial \psi}{\partial x_2}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial \psi|_{\Omega^-}}{\partial x_1}\right\|_{L^2(\Omega^-)}^2.$$

We, now define the limit equations. Given $\theta \in L^2(\omega)$ and $h \in H^{1/2}(\Gamma_b)$, consider the partial differential equation

$$\begin{cases} -\dfrac{\partial^2 u^+}{\partial x_2^2} = f & in\ \Omega^+, \\ -\Delta u^- = f + \theta\chi_\omega & in\ \Omega^-, \\ \dfrac{\partial u^+}{\partial \nu} = 0 & on\ \Gamma_u, \\ u^+ = u^-,\ \dfrac{b-a}{L}\dfrac{\partial u^+}{\partial x_2} = \dfrac{\partial u^-}{\partial x_2} & on\ \Gamma, \\ u^- = h & on\ \Gamma_b, \\ u\ \ is\ \Gamma_{s'} - periodic. \end{cases} \tag{5.1}$$

where

$$u(x) = \begin{cases} u^+\ if\ x \in \Omega^+ \\ u^-\ if\ x \in \Omega^- \end{cases} \tag{5.2}$$

The variational formulation of the problem (5.1) is given as: Let $f \in L^2(\Omega)$. Find $u \in W(\Omega)$ such that

$$\frac{b-a}{L}\int_{\Omega^+} \frac{\partial u^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + \int_{\Omega^-} \nabla u^- \cdot \nabla \psi - \int_{\Gamma_b} \frac{\partial u^-}{\partial x_2} h = \frac{b-a}{L}\int_{\Omega^+} f\psi + \int_{\Omega^-} f\psi + \int_\omega \theta\psi \tag{5.3}$$

for all $\psi \in W(\Omega)$. The solution operator of (5.1) is linear and taking $\psi = u$ as a test function in (5.3), we will have the continuity of the solution operator. Moreover,

$$\|u\|_{W(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\theta\|_{L^2(\omega)} + \|h\|_{H^{1/2}(\Gamma_b)}), \tag{5.4}$$

where C is independent of ε. Existence and uniqueness of $u \in W(\Omega)$ as a solution of (5.3) is well known. Now consider the L^2-cost functional J_1 defined by

$$J_1(u,\theta) = \frac{1}{2}\int_{\Omega} \left(\frac{b-a}{L}\chi_{\Omega^+} + \chi_{\Omega^-}\right)|u-u_d|^2 + \frac{\beta}{2}\int_{\omega} \theta^2. \tag{5.5}$$

Associated with this cost functional, we introduce the optimal control problem as

$$\inf\{J_1(u,\theta)|\ \theta \in L^2(\omega),\ (u,\theta)\ \textit{obeys}\ (5.1)\}. \tag{P_1}$$

This problem admits a unique solution say $(\overline{u},\overline{\theta})$. As we characterized earlier, for $\overline{\theta}_\varepsilon$, we now characterize optimal control $\overline{\theta}$ of the problem (P_1) using adjoint state $\overline{v}$, in a similar fashion. The co-state $\overline{v}$ solves the adjoint problem

$$\begin{cases} -\dfrac{\partial^2 \overline{v}^+}{\partial x_2^2} = (\overline{u}^+ - u_d) \quad in\ \Omega^+, \\ -\Delta \overline{v}^- = (\overline{u}^- - u_d) \quad in\ \Omega^-, \\ \dfrac{\partial \overline{v}^+}{\partial \nu} = 0 \quad on\ \Gamma_u, \\ \overline{v}^+ = \overline{v}^-,\ \dfrac{b-a}{L}\dfrac{\partial \overline{v}^+}{\partial x_2} = \dfrac{\partial \overline{v}^-}{\partial x_2}\ on\ \Gamma, \\ \overline{v}^- = 0\ \ on\ \ \Gamma_b, \\ \overline{v}\ \ is\ \Gamma_{s'} - periodic. \end{cases} \tag{5.6}$$

Theorem 5.1. Let $f \in L^2(\Omega)$, $h \in H^{1/2}(\Gamma_b)$ and $(\overline{u},\overline{\theta})$ be the optimal solution of (P_1). Let $\overline{v} \in W(\Omega)$ solves (5.6), then the optimal control is given by

$$\overline{\theta} = -\frac{1}{\beta}\overline{v}\chi_\omega.$$

Conversely, assume that a pair $(\hat{u},\hat{v}) \in W(\Omega) \times W_0(\Omega)$ solves the coupled optimality system

$$\begin{cases} -\dfrac{\partial^2 \hat{u}^+}{\partial x_2^2} = f, -\dfrac{\partial^2 \hat{v}^-}{\partial x_2^2} = (\hat{u}^- - u_d)\ \ in\ \ \Omega^+, \\ -\Delta \hat{u}^- = f - \frac{1}{\beta}\hat{v}^-\chi_\omega, -\Delta \hat{v}^- = (\hat{u} - u_d) \quad in\ \Omega^-, \\ \dfrac{\partial \hat{u}^+}{\partial \nu} = 0,\ \dfrac{\partial \hat{v}^+}{\partial x_2} = 0 \quad on\ \ \Gamma, \\ \hat{u}^+ = \hat{u}^-,\ \dfrac{b-a}{L}\dfrac{\partial \hat{u}^+}{\partial x_2} = \dfrac{\partial \hat{u}^-}{\partial x_2}, \hat{v}^+ = \hat{v}^-,\ \ \dfrac{b-a}{L}\dfrac{\partial \hat{v}^+}{\partial x_2} = \dfrac{\partial \hat{v}^-}{\partial x_2}\ \ on\ \Gamma, \\ \hat{u}^- = h, \hat{v}^- = 0\ \ on\ \ \Gamma_b, \\ \hat{u},\hat{v}\ is\ \Gamma_{s'} - periodic. \end{cases} \tag{5.7}$$

Then, the pair $(\hat{u}, -\frac{1}{\beta}\hat{v}_\varepsilon\chi_\omega)$ is the optimal solution to (P_1).

5.2 Convergence Analysis

Assume that $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon)$ is the optimal solution of $(P_{1,\varepsilon})$. Let $u_\varepsilon(0)$ be the solution of the problem (3.1) corresponding to $\theta = 0$, then from (3.2) we get

$$\|u_\varepsilon(0)\|_{H^1(\Omega_\varepsilon)} \leq C, \tag{5.8}$$

where $C > 0$ is independent of ε. Using optimality of the solution $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon)$, we get

$$\int_{\Omega_\varepsilon} (\overline{u}_\varepsilon - u_d)^2 + \frac{\beta}{2} \int_{\omega} \overline{\theta}_\varepsilon^2 \leq \int_{\Omega_\varepsilon} (u_\varepsilon(0) - u_d)^2 \leq C. \tag{5.9}$$

Thus we have

$$\|\overline{\theta}_\varepsilon\|_{L^2(\omega)} \leq C \text{ and } \|\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C. \tag{5.10}$$

We know $h \in H^{1/2}_{per}(\Gamma_b)$, by trace map there exist z in $H^1_{per}(\Omega^-)$ such that $z|_{\Gamma_u} = 0$ and $z|_{\Gamma_b} = h$. Let $K = \{\phi \in H^1(\Omega_\varepsilon) : \phi|_{\Gamma_b} = 0\}$ Set $\overline{u}_\varepsilon = \widetilde{z} + \overline{y}_\varepsilon$, where $\widetilde{z}$ is an extension by zero on Ω_ε and $\overline{y}_\varepsilon \in K$ solves the following partial differential equations

$$\begin{cases} -\Delta \overline{y}_\varepsilon = f + \theta\chi_\omega + \Delta\widetilde{z} \quad in\ \Omega_\varepsilon, \\ \dfrac{\partial \overline{y}_\varepsilon}{\partial \nu} = 0 \quad on\ \gamma_\varepsilon, \\ \overline{y}_\varepsilon = 0 \quad on\ \Gamma_b, \\ \overline{y}_\varepsilon \quad is\ \Gamma_s - periodic. \end{cases} \tag{5.11}$$

Then variational formulation of the above problem (5.11). Find $\overline{y}_\varepsilon \in K$ such that

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla\phi = \int_{\Omega_\varepsilon} f\phi - \int_{\Omega_\varepsilon} \nabla\widetilde{z} \cdot \nabla\phi + \int_{\omega} \theta\phi \tag{5.12}$$

for all $\phi \in K$.

We use the following lemma to prove next theorem.

Lemma 5.2. (see [15]) Let, m be a fixed integer, $\{\alpha_n^k\}_n$ for $k = 1, 2, ..., m$ be m bounded sequence of real numbers and α^k be m real numbers. Suppose that $\sum_{k=1}^{m} \alpha_n^k \to \sum_{k=1}^{m} \alpha^k$ and for every $k = 1, 2, ..., m$, $\liminf_{n\to\infty} \alpha_n^k \geq \alpha^k$. Then $\lim_{n\to\infty} \alpha_n^k = \alpha^k$ for every k=1,2,...,m.

We now state and prove the main theorem of this section.

Theorem 5.3 (Main Theorem)**.** Let $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon)$ and $(\overline{u}, \overline{\theta})$ be the optimal solution of $(P_{1,\varepsilon})$ and of (P_1), respectively. Then

$$\begin{aligned} &\overline{\theta}_\varepsilon \rightharpoonup \overline{\theta} \text{ weakly in } H^1(\omega), \\ &\widetilde{\overline{u}_\varepsilon|_{\Omega_\varepsilon^+}} \rightharpoonup \frac{b-a}{L} \overline{u}|_{\Omega^+} \text{ weakly in } L^2(0, L; H^1(M, M')), \\ &\overline{u}_\varepsilon|_{\Omega^-} \to \overline{u}|_{\Omega^-} \text{ strongly in } H^1(\Omega^-), \\ &\widetilde{\overline{v}_\varepsilon|_{\Omega_\varepsilon^+}} \rightharpoonup \frac{b-a}{L} \overline{v}|_{\Omega^+} \text{ weakly in } L^2(0, L; H^1(M, M')), \\ &\overline{v}_\varepsilon|_{\Omega^-} \to \overline{v}|_{\Omega^-} \text{ strongly in } H^1(\Omega^-), \end{aligned}$$

where $\overline{\theta} = -\frac{1}{\beta}\overline{v}\chi_\omega$ and $\overline{v}_\varepsilon$, $\overline{v}$ is the solution of (3.3) and (5.6) respectively. Moreover,

$$J_{1,\varepsilon}(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon) \to J_1(\overline{u}, \overline{\theta}) \text{ as } \varepsilon \to 0.$$

Proof. Step1(boundedness of $\overline{u}_\varepsilon$, $\overline{\theta}_\varepsilon$): We can conclude from the continuity of solution operator and by (5.10) that

$$\|\overline{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C. \tag{5.13}$$

From the weak formulation of the adjoint problem (3.3), we have

$$\|\overline{v}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \tag{5.14}$$

where C is independent of ε. Since $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon)$ is the optimal solution of $(P_{1,\varepsilon})$. By Theorem 3.2 we have $\overline{\theta}_\varepsilon = -\dfrac{1}{\beta}\overline{v}_\varepsilon\chi_\omega$. By estimate (5.14) gives

$$\|\overline{\theta}_\varepsilon\|_{H^1(\omega)} \leq C. \tag{5.15}$$

Eberlein-Šmuljan theorem ensure the existence of subsequence $(\overline{\theta}_\varepsilon)$ (still denote by ε) and a function $\theta_0 \in H^1(\omega)$ such that

$$\overline{\theta}_\varepsilon \rightharpoonup \theta_0 \text{ weakly in } H^1(\omega). \tag{5.16}$$

Let us denote $\overline{u}_\varepsilon{}^+ = \overline{u}_\varepsilon|_{\Omega_\varepsilon^+}$ and $\overline{u}_\varepsilon{}^- = \overline{u}_\varepsilon|_{\Omega^-}$. Using the estimate (5.13), we have the boundedness of $\overline{u}_\varepsilon{}^-$ in the space $H^1(\Omega^-)$. Thus upto a subsequence (still denote by ε)

$$\overline{u}_\varepsilon{}^- \rightharpoonup u_0^- \text{ weakly in } H^1(\Omega^-) \tag{5.17}$$

for some $u_0^- \in H^1(\Omega^-)$. We observe that

$$\left\|T^\varepsilon \overline{u}_\varepsilon{}^+\right\|^2_{L^2(0,L;H^1((M,M')\times(a,b)))} \leq L\, \|\overline{u}_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}. \tag{5.18}$$

Step2 (convergence): The boundedness of the sequence $T^\varepsilon\overline{u}_\varepsilon{}^+$ in $L^2(0,L;H^1((M,M')\times(a,b))$ follows from (5.13) and (5.18). By weak compactness, there exist a subsequence (still denoted by ε) such that

$$T^\varepsilon\overline{u}_\varepsilon{}^+ \rightharpoonup u_0^+ \text{ weakly in } L^2(0,L;H^1((M,M')\times(a,b))), \tag{5.19}$$

which implies

$$T^\varepsilon\overline{u}_\varepsilon{}^+ \rightharpoonup u_0^+, \frac{\partial}{\partial x_2}T^\varepsilon\overline{u}_\varepsilon{}^+ \rightharpoonup \frac{\partial u_0^+}{\partial x_2}, \frac{\partial}{\partial x_3}T^\varepsilon\overline{u}_\varepsilon{}^+ \rightharpoonup \frac{\partial u_0^+}{\partial x_3} \text{ weakly in } L^2(\Omega^+\times(a,b)),$$

That is

$$T^\varepsilon\frac{\partial\overline{u}_\varepsilon{}^+}{\partial x_2} \rightharpoonup \frac{\partial u_0^+}{\partial x_2} \text{ weakly in } L^2(\Omega^+\times(a,b)), \tag{5.20}$$

$$\varepsilon T^\varepsilon\frac{\partial\overline{u}_\varepsilon{}^+}{\partial x_1} \rightharpoonup \frac{\partial u_0^+}{\partial x_3} \text{ weakly in } L^2(\Omega^+\times(a,b)) \tag{5.21}$$

From the Proposition 4.3, we have

$$\begin{aligned}\left\|T^{\varepsilon}\frac{\partial \overline{u}_{\varepsilon}{}^{+}}{\partial x_1}\right\|_{L^2(\Omega^+\times(a,b))} &= \sqrt{L}\left\|\frac{\partial \overline{u}_{\varepsilon}{}^{+}}{\partial x_1}\right\|_{L^2(\Omega_{\varepsilon}^+)}\\ &\leq \sqrt{L}\|\overline{u}_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})}.\end{aligned}$$

Again (5.13) implies the boundedness of the sequence $T^{\varepsilon}\frac{\partial \overline{u}_{\varepsilon}{}^{+}}{\partial x_1}$ in the space $L^2(\Omega^+\times(a,b))$.

From (5.21) we have $\frac{\partial u_0^+}{\partial x_3}=0$ and thus u_0^+ is independent of x_3. From the Proposition 4.8 and convergence (5.19), we have

$$\widetilde{\overline{u}_{\varepsilon}{}^{+}} \rightharpoonup \frac{1}{L}\int_a^b u_0^+\,dx_3 = \frac{b-a}{L}u_0^+ \text{ weakly in } L^2(0,L;H^1(M,M')) \tag{5.22}$$

We know that $T^{\varepsilon}\frac{\partial \overline{u}_{\varepsilon}{}^{+}}{\partial x_1}$ is bounded in $L^2(\Omega^+\times(a,b))$. Hence by weak compactness, there is an element $P\in L^2(\Omega^+\times(a,b))$ such that up to subsequence,

$$T^{\varepsilon}\frac{\partial \overline{u}_{\varepsilon}{}^{+}}{\partial x_1} \rightharpoonup P \text{ weakly in } L^2(\Omega^+\times(a,b)). \tag{5.23}$$

Define u_0 as

$$u_0(x)=\begin{cases} u_0^+ \; if\; x\in\Omega^+,\\ u_0^- \; if\; x\in\Omega^-.\end{cases} \tag{5.24}$$

Step3 (Claim the function $u_0\in W(\Omega)$ and satisfies the limit problem (5.3)):

Proof: clearly $u_0\in L^2(\Omega)$ and $u_0^-\in H^1(\Omega^-)$. To prove $u_0\in W(\Omega)$, we need to show $\frac{\partial u_0}{\partial x_2}\in L^2(\Omega)$. Recall that u_0 is independent of x_3 and $\frac{\partial u_0}{\partial x_2}\in L^2(\Omega^+)$ and $\frac{\partial u_0}{\partial x_2}\in L^2(\Omega^-)$. Thus $\frac{\partial u_0}{\partial x_2}\in L^2(\Omega)$ if we prove trace of u_0^+ and u_0^- are equal on Γ_u. Since $\overline{u}_{\varepsilon}{}^{+}|_{\Gamma_{\varepsilon}}=\overline{u}_{\varepsilon}{}^{-}|_{\Gamma_{\varepsilon}}$ implies the equality of trace for the boundary unfolding operator. More precisely we have

$$T_{\Gamma}^{\varepsilon}\left(\overline{u}_{\varepsilon}{}^{+}|_{\Gamma_{\varepsilon}}\right)=T_{\Gamma}^{\varepsilon}\left(\overline{u}_{\varepsilon}{}^{-}|_{\Gamma_{\varepsilon}}\right) \tag{5.25}$$

Observe that $T_{\Gamma}^{\varepsilon}(\overline{u}_{\varepsilon}{}^{+}|_{\Gamma_{\varepsilon}})=(T^{\varepsilon}(\overline{u}_{\varepsilon}{}^{+}))\,|_{x_2=M}$. So, the equation (5.25) becomes

$$\left(T^{\varepsilon}(\overline{u}_{\varepsilon}{}^{+})\right)|_{\Gamma}=T_{\Gamma}^{\varepsilon}\left(\overline{u}_{\varepsilon}{}^{-}|_{\Gamma_{\varepsilon}}\right). \tag{5.26}$$

From the continuity of trace operator we can write

$$\left(T^{\varepsilon}(\overline{u}_{\varepsilon}{}^{+})\right)|_{\Gamma} \rightharpoonup u_0^+|_{\Gamma} \text{ weakly in } L^2((0,L)\times(a,b))$$

and from (5.17), we get

$$\overline{u}_{\varepsilon}{}^{-}|_{\Gamma}\to u^-|_{\Gamma} \text{ strongly in } L^2(0,L).$$

This implies

$$T_{\Gamma}^{\varepsilon}\left(\overline{u}_{\varepsilon}{}^{-}|_{\Gamma}\right)\to u_0^-|_{\Gamma} \text{ in } L^2((0,L)\times(a,b)).$$

Passing to the limit in (5.26) as $\varepsilon \to 0$, we get

$$u_0^+|_\Gamma = u_0^-|_\Gamma \text{ in } L^2(0,L)$$

since u_0^+ and u_0^- are independent on x_3 variable.

Step4 (Identification of the limit P **in** (5.23)): Let $\overline{u}_\varepsilon = \tilde{z} + \overline{y}_\varepsilon$, where $\tilde{z}$ has in Section 5.2 and $\overline{y}_\varepsilon \in H^1_{per}(\Omega_\varepsilon)$ satisfies (5.11) for $\theta = \overline{\theta}_\varepsilon$. We observe that $\overline{u}_\varepsilon{}^+$ is equal to $\overline{y}_\varepsilon|_{\Omega_\varepsilon^+}$, say $\overline{y}_\varepsilon{}^+$. So $\overline{y}_\varepsilon{}^+$ have the same convergence as $\overline{u}_\varepsilon{}^+$, i.e.

$$T^\varepsilon \frac{\partial \overline{y}_\varepsilon{}^+}{\partial x_2} \rightharpoonup \frac{\partial u_0^+}{\partial x_2} \text{ weakly in } L^2(\Omega^+ \times (a,b)) \tag{5.27}$$

$$T^\varepsilon \frac{\partial \overline{y}_\varepsilon{}^+}{\partial x_1} \rightharpoonup P \text{ weakly in } L^2(\Omega^+ \times (a,b)). \tag{5.28}$$

For $\phi \in \mathcal{D}(\Omega^+)$ and $\psi(z) \in C^\infty[0,L)$, choose a test function

$$\phi^\varepsilon(x) = \varepsilon\phi(x)\psi\left(\{\frac{x_1}{\varepsilon}\}\right), \tag{5.29}$$

in such a way that ϕ^ε is continuous on Ω_ε^+ . From the definition of ε-unfolding of ϕ^ε and by Proposition 4.4, we get

$$T^\varepsilon\phi^\varepsilon = \varepsilon\phi\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon x_3, x_2\right)\psi(x_3),$$

$$T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial x_1} = \frac{1}{\varepsilon}\frac{\partial}{\partial x_3}T^\varepsilon\phi^\varepsilon = \varepsilon\frac{\partial\phi}{\partial x_1}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon x_3, x_2\right)\psi(x_3) + \phi\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon x_3, x_2\right)\frac{d\psi}{dz}(x_3),$$

$$T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial x_2} = \varepsilon\frac{\partial\phi}{\partial x_2}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon x_3, x_2\right)\psi(x_3).$$

This equations gives us

$$T^\varepsilon\phi^\varepsilon \to 0 \text{ strongly in } L^2(\Omega^+ \times (a,b)) \tag{5.30}$$

$$T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial x_1} \to \phi(x_1,x_2)\frac{d\psi}{dz}(x_3) \text{ strongly in } L^2(\Omega^+ \times (a,b)) \tag{5.31}$$

$$T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial x_2} \to 0 \text{ strongly in } L^2(\Omega^+ \times (a,b)) \tag{5.32}$$

as $\varepsilon \to 0$. From the variational formulation (5.12) for $\theta = \overline{\theta}_\varepsilon$, we get

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \nabla\overline{y}_\varepsilon \cdot \nabla\widetilde{\phi^\varepsilon} = \lim_{\varepsilon\to 0}\left[\int_{\Omega_\varepsilon} f\widetilde{\phi^\varepsilon} - \int_{\Omega_\varepsilon} \nabla\tilde{w}\cdot\nabla\widetilde{\phi^\varepsilon} + \int_\omega \overline{\theta}_\varepsilon\widetilde{\phi^\varepsilon}\right]. \tag{5.33}$$

Now notice that

$$\int_{\Omega_\varepsilon} \nabla\overline{y}_\varepsilon\cdot\nabla\widetilde{\phi^\varepsilon} = \frac{1}{L}\int_{\Omega^+\times(a,b)} \left(T^\varepsilon\frac{\partial\overline{y}_\varepsilon{}^+}{\partial x_1}T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial x_1} + T^\varepsilon\frac{\partial\overline{y}_\varepsilon{}^+}{\partial x_2}T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial x_2}\right)$$

$$\to \frac{1}{L}\int_{\Omega^+\times(a,b)} P\phi(x_1,x_2)\frac{d\psi}{dz}(x_3) \;\; as\, \varepsilon \to 0. \tag{5.34}$$

and

$$\int_{\Omega_\varepsilon} f\widetilde{\phi^\varepsilon} - \int_{\Omega_\varepsilon} \nabla\widetilde{z}\cdot\nabla\widetilde{\phi^\varepsilon} + \int_{\omega} \overline{\theta}_\varepsilon\widetilde{\phi^\varepsilon} = \int_{\Omega_\varepsilon^+} f\phi^\varepsilon = \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\varepsilon f\, T^\varepsilon\phi^\varepsilon \to 0 \;\; as\; \varepsilon\to 0. \tag{5.35}$$

Combing (5.34), (5.35), from (5.33) we get,

$$\int_{\Omega^+\times(a,b)} P\phi(x_1,x_2)\frac{d\psi}{dz}(x_3) = 0$$

which implies

$$\int_{\Omega^+}\left(\int_a^b P(x_1,x_2,x_3)\frac{d\psi}{dz}(x_3)\,dx_3\right)\phi(x_1,x_2)\,dx_1dx_2 = 0,\ \forall\phi\in\mathcal{D}(\Omega^+)$$

That is

$$\int_a^b P(x_1,x_2,x_3)\frac{d\psi}{dz}(x_3)\,dx_3 = 0\ a.e.\ (x_1,x_2)\in\Omega^+$$

Hence, we get $P=0$ a.e. in $\Omega^+\times(a,b)$ equivalently $T^\varepsilon\frac{\partial u_\varepsilon^+}{\partial x_1} \rightharpoonup 0$ weakly in $L^2(\Omega^+\times(a,b))$. **Step5**: Again taking another test function $\psi\in\{\phi\in C^\infty(\overline{\Omega})|\ \phi|_{\Gamma_b}=h\}$ in the variational formulation of (3.1) for $\theta=\overline{\theta}_\varepsilon$, we get

$$\begin{aligned}\int_{\Omega_\varepsilon}\nabla\overline{u}_\varepsilon\cdot\nabla\psi - \int_{\Gamma_b}\frac{\partial\overline{u}_\varepsilon}{\partial\nu}h \;=\; & \frac{1}{L}\int_{\Omega^+\times(a,b)}\left(T^\varepsilon\frac{\partial\overline{u}_\varepsilon{}^+}{\partial x_1}T^\varepsilon\frac{\partial\psi}{\partial x_1} + T^\varepsilon\frac{\partial\overline{u}_\varepsilon{}^+}{\partial x_2}T^\varepsilon\frac{\partial\psi}{\partial x_2}\right)\\ & + \int_{\Omega^-}\nabla\overline{u}_\varepsilon{}^-\cdot\nabla\psi - \int_{\Gamma_b}\frac{\partial\overline{u}_\varepsilon{}^-}{\partial\nu}h\\ & \to \frac{1}{L}\int_{\Omega^+\times(a,b)}\frac{\partial u_0^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + \int_{\Omega^-}\nabla u_0{}^-\cdot\nabla\psi - \int_{\Gamma_b}\frac{\partial\overline{u}_0{}^-}{\partial\nu}h \end{aligned}\tag{5.36}$$

and

$$\begin{aligned}\int_{\Omega_\varepsilon} f\psi + \int_\omega\overline{\theta}_\varepsilon\psi \;=\; & \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\varepsilon f\,T^\varepsilon\psi + \int_{\Omega^-} f\psi + \int_\omega\overline{\theta}_\varepsilon\psi\\ & \to \frac{1}{L}\int_{\Omega^+\times(a,b)} f\psi + \int_{\Omega^-} f\psi + \int_\omega\theta_0\psi.\end{aligned}\tag{5.37}$$

Hence

$$\frac{1}{L}\int_{\Omega^+\times(a,b)}\frac{\partial u_0^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + \int_{\Omega^-}\nabla u_0{}^-\cdot\nabla\psi - \int_{\Gamma_b}\frac{\partial\overline{u}_0{}^-}{\partial\nu}h = \frac{1}{L}\int_{\Omega^+\times(a,b)} f\psi + \int_{\Omega^-} f\psi + \int_\omega\theta_0\psi$$

which implies

$$\frac{b-a}{L}\int_{\Omega^+}\frac{\partial u_0^+}{\partial x_2}\frac{\partial\psi}{\partial x_2}+\int_{\Omega^-}\nabla u_0^-\cdot\nabla\psi-\int_{\Gamma_b}\frac{\partial\overline{u}_0{}^-}{\partial\nu}h=\frac{b-a}{L}\int_{\Omega^+}f\psi+\int_{\Omega^-}f\psi+\int_{\omega}\theta_0\psi$$

$\forall\psi\in\{\phi\in C^\infty(\overline{\Omega})|\,\phi|_{\Gamma_b}=h\}$ and hence for all ψ in $W(\Omega)$ since $\{\phi\in C^\infty(\overline{\Omega})|\,\phi|_{\Gamma_b}=h\}$ is dense in $W(\Omega)$ (see[17]). Therefore u_0 satisfies the differential equation (5.1) for $\theta=\theta_0$.

Similarly, we find the following convergence for the adjoint state $\overline{v}_\varepsilon$ describe in (3.3).

$$\begin{aligned}
&T^\varepsilon\left(\overline{v}_\varepsilon|_{(}\Omega_\varepsilon^+\right)\rightharpoonup v_0|_{\Omega^+}\text{ weakly in }L^{(}\Omega^+\times(a,b))\\
&T^\varepsilon\left(\frac{\partial\overline{v}_\varepsilon|_{\Omega_\varepsilon^+}}{\partial x_1}\right)\rightharpoonup 0,\ T^\varepsilon\left(\frac{\partial\overline{v}_\varepsilon|_{\Omega_\varepsilon^+}}{\partial x_2}\right)\rightharpoonup\frac{\partial v_0|_{\Omega^+}}{\partial x_2}\text{ weakly in }L^{(}\Omega^+\times(a,b))\\
&\widetilde{\overline{v}_\varepsilon|_{\Omega_\varepsilon^+}}\rightharpoonup\frac{b-a}{L}v_0|_{\Omega^+}\text{ weakly in }L^2(0,L;H^1(M,M'))\\
&\overline{v}_\varepsilon|_{\Omega^-}\rightharpoonup v_0|_{\Omega^-}\text{ weakly in }H^1(\Omega^-)
\end{aligned}$$

where $v_0\in W_0(\Omega)$ satisfies (5.6), when $\overline{u}$ is replaced by u_0.

Regarding the optimal control, we have $\overline{\theta}_\varepsilon=-\frac{1}{\beta}\overline{v}_\varepsilon\chi_\omega$ and the convergence $\overline{\theta}_\varepsilon\to\theta_0$ in $L^2(\omega)$, $\overline{v}_\varepsilon|_{\Omega^-}\rightharpoonup v_0|_{\Omega^-}$ in $H^1(\Omega^-)$. Implies that

$$\theta_0=-\frac{1}{\beta}v_0\chi_\omega. \tag{5.38}$$

Thus $(u_0,v_0.\theta_0)$ satisfies the optimality system corresponding to the minimization problem (P_1). According to Theorem 5.1, its optimal solution is given by $(u_0,-\frac{1}{\beta}v_0\chi_\omega)$. Thus, by uniqueness, we have

$$\overline{u}=u_0,\ \ \overline{v}=v_0\ \text{ and }\overline{\theta}=\theta_0=-\frac{1}{\beta}v_0\chi_\omega.$$

Step 6 (Claim: $\lim_{\varepsilon\to0}J_{1,\varepsilon}(\overline{u}_\varepsilon,\overline{\theta}_\varepsilon)=J_1(\overline{u},\overline{\theta})$**)**: To prove this, let $\alpha_\varepsilon^1=\left\|T^\varepsilon\frac{\partial\overline{u}_\varepsilon{}^+}{\partial x_1}\right\|^2_{L^2(\Omega^+\times(a,b))}$, $\alpha_\varepsilon^2=\left\|T^\varepsilon\frac{\partial\overline{u}_\varepsilon{}^+}{\partial x_2}\right\|^2_{L^2(\Omega^+\times(a,b))}$, $\alpha_\varepsilon^3=L\left\|\frac{\partial\overline{u}_\varepsilon{}^-}{\partial x_1}\right\|^2_{L^2(\Omega^-)}$, $\alpha_\varepsilon^4=L\left\|\frac{\partial\overline{u}_\varepsilon{}^-}{\partial x_2}\right\|^2_{L^2(\Omega^-)}$ and $\alpha^1=\|P\|^2_{L^2(\Omega^+\times(a,b)}=0$, $\alpha^2=\left\|\frac{\partial\overline{u}^-}{\partial x_2}\right\|^2_{L^2(\Omega^+\times(a,b)}$, $\alpha^3=L\left\|\frac{\partial\overline{u}^-}{\partial x_1}\right\|^2_{L^2(\Omega^-)}$, $\alpha^4=L\left\|\frac{\partial\overline{u}^-}{\partial x_2}\right\|^2_{L^2(\Omega^-)}$. Then, with the help of Proposition 4.3, we get

$$
\begin{aligned}
\sum_{k=1}^{4} \alpha_\varepsilon^k &= L\left(\int\limits_{\Omega_\varepsilon^+} |\nabla \overline{u}_\varepsilon{}^+|^2 + \int\limits_{\Omega^-} |\nabla \overline{u}_\varepsilon{}^-|^2\right) \\
&= L\left(\int\limits_{\Omega_\varepsilon^+} f\overline{u}_\varepsilon{}^+ + \int\limits_{\Omega^-} f\overline{u}_\varepsilon{}^- + \int\limits_{\omega} \theta_\varepsilon \overline{u}_\varepsilon{}^- - \int\limits_{\Gamma_b} \frac{\partial \overline{u}_\varepsilon{}^-}{\partial \nu}\overline{u}_\varepsilon{}^-\right) \\
&\to L\left(\frac{b-a}{L}\int\limits_{\Omega^+} f\overline{u}^+ + \int\limits_{\Omega^-} f\overline{u}^- + \int\limits_{\omega} \overline{\theta}\overline{u}^- - \int\limits_{\Gamma_b} \frac{\partial \overline{u}^-}{\partial \nu}\overline{u}^-\right) \quad \text{as } \varepsilon \to 0 \\
&= L\left(\frac{b-a}{L}\int\limits_{\Omega^+} \left|\frac{\partial \overline{u}^+}{\partial x_2}\right|^2 + \int\limits_{\Omega^-} |\nabla \overline{u}^-|^2\right) \\
&= \sum_{k=1}^{4} \alpha^k.
\end{aligned}
\tag{5.39}
$$

Thus, $\sum_{k=1}^{4} \alpha_\varepsilon^k \to \sum_{k=1}^{4} \alpha^k$ as $\varepsilon \to 0$. By weak lower semi-continuity, $\liminf_{\varepsilon\to 0^+} \alpha_\varepsilon^k \geq \alpha^k$ for every $k = 1,2,3,4$. Hence by Lemma 5.2, we conclude

$$
T^\varepsilon \frac{\partial \overline{u}_\varepsilon{}^+}{\partial x_1} \to 0,\ T^\varepsilon \frac{\partial \overline{u}_\varepsilon{}^+}{\partial x_2} \to \frac{\partial \overline{u}^+}{\partial x_2} \text{ strongly in } L^2(\Omega^+ \times (a,b)) \tag{5.40}
$$

$$
\frac{\partial \overline{u}_\varepsilon{}^-}{\partial x_1} \to \frac{\partial \overline{u}^-}{\partial x_1},\ \frac{\partial \overline{u}_\varepsilon{}^-}{\partial x_2} \to \frac{\partial \overline{u}^-}{\partial x_2} \text{ strongly in } L^2(\Omega^-). \tag{5.41}
$$

Therefore (5.17) and (5.41) gives

$$
\overline{u}_\varepsilon|_{\Omega^-} \to \overline{u}|_{\Omega^-} \text{ strongly in } H^1(\Omega^-). \tag{5.42}
$$

Similarly, one can prove

$$
\overline{v}_\varepsilon|_{\Omega^-} \to \overline{v}|_{\Omega^-} \text{ strongly in } H^1(\Omega^-). \tag{5.43}
$$

Choosing $\phi = \overline{u}_\varepsilon - u_d$ in the variational formulation of the problem (3.3) gives

$$
\begin{aligned}
\int\limits_{\Omega_\varepsilon} |\overline{u}_\varepsilon - u_d|^2 &= \int\limits_{\Omega_\varepsilon} \nabla \overline{v}_\varepsilon \cdot \nabla(\overline{u}_\varepsilon - u_d) - \int\limits_{\Gamma_b} \frac{\partial \overline{v}_\varepsilon}{\partial \nu} h \\
&\to \frac{b-a}{L}\int\limits_{\Omega^+} \frac{\partial \overline{v}^+}{\partial x_2} \cdot \left(\frac{\partial \overline{u}^+}{\partial x_2} - \frac{\partial \overline{u}^+}{\partial x_2}\right) + \int\limits_{\Omega^-} \nabla \overline{v}^- \cdot \nabla(\overline{u}^- - u_d) - \int\limits_{\Gamma_b} \frac{\partial \overline{v}^-}{\partial \nu} h \\
&= \int\limits_{\Omega} \left(\frac{b-a}{L}\chi_{\Omega^+} + \chi_{\Omega^-}\right) |\overline{u} - u_d|^2.
\end{aligned}
\tag{5.44}
$$

Therefore using (5.44) we get

$$\lim_{\varepsilon\to 0} J_{1,\varepsilon}(\overline{u}_\varepsilon,\overline{\theta}_\varepsilon) = \frac{1}{2}\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\overline{u}_\varepsilon - u_d|^2 + \frac{\beta}{2}\lim_{\varepsilon\to 0}\int_{\omega}\overline{\theta}_\varepsilon^2$$
$$= \frac{1}{2}\int_{\Omega}\left(\frac{b-a}{L}\chi_{\Omega^+} + \chi_{\Omega^-}\right)|\overline{u}-u_d|^2 + \frac{\beta}{2}\int_{\omega}\overline{\theta}^2$$
$$= J_1(\overline{u},\overline{\theta}).$$

This completes the proof of the theorem.

6 Dirichlet Cost Functional

Recall the Dirichlet cost functional

$$J_{2,\varepsilon}(u_\varepsilon,\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon - \nabla u_d|^2 + \frac{\beta}{2}\int_{\omega}\theta^2$$

given in the Section 3. We now prove the analogous results as in the previous section corresponding to the cost functional.

6.1 Homogenized System

Define the limit Dirichlet cost functional J_2 as

$$J_2(u,\theta) = \frac{1}{2}\int_{\Omega^+}\frac{b-a}{L}\left|\frac{\partial u|_{\Omega^+}}{\partial x_2} - \frac{\partial u_d}{\partial x_2}\right|^2 + \frac{1}{2}\int_{\Omega^-}|\nabla u|_{\Omega^-} - \nabla u_d|^2 + \frac{\beta}{2}\int_{\omega}\theta^2. \tag{6.1}$$

Define the optimal control problem as follows: Find $(\overline{u},\overline{\theta})$ such that

$$J_2(\overline{u},\overline{\theta}) = \inf\{J_2(u,\theta)|\ \theta\in L^2(\omega),\ (u,\theta)\ \textit{obeys}\ (5.1)\} \tag{P_2}$$

The problem (P_2) has a unique solution $(\overline{u},\overline{\theta})$. Then the optimal control $\overline{\theta}$ characterized using adjoint state $\overline{v}$, given by

$$\begin{cases} -\dfrac{\partial^2\overline{v}^+}{\partial x_2^2} = -\dfrac{\partial^2}{\partial x_2^2}(\overline{u}^+ - u_d) \quad in\ \Omega^+, \\ -\Delta\overline{v}^- = -\Delta(\overline{u}^- - u_d) \quad in\ \Omega^-, \\ \dfrac{\partial\overline{v}^+}{\partial\nu} = (\nabla\overline{u}^+ - \nabla u_d)\cdot\nu \ \ on\ \ \Gamma_u, \\ \overline{v}^+ = \overline{v}^-,\ \ \dfrac{b-a}{L}\dfrac{\partial\overline{v}^+}{\partial x_2} = \dfrac{\partial\overline{v}^-}{\partial x_2}\ on\ \Gamma, \\ \overline{v}^- = 0 \quad on\ \ \Gamma_b, \\ \overline{v}\ is\ \Gamma_{s'} - periodic. \end{cases} \tag{6.2}$$

The following theorem is standard and can be proved using classical methodology (see [14], [34]).

Theorem 6.1. Let $f \in L^2(\Omega)$, $h \in H^{1/2}(\Gamma_b)$ and $(\overline{u},\overline{\theta})$ be the optimal solution of (P_2). Let $\overline{v} \in W(\Omega)$ solves (6.2), then the optimal control is given by

$$\overline{\theta} = -\frac{1}{\beta}\overline{v}\chi_{\omega}.$$

Conversely, assume that a pair $(\hat{u},\hat{v}) \in W(\Omega)\times W_0(\Omega)$ solves the optimality system

$$\begin{cases} -\dfrac{\partial^2 \hat{u}^+}{\partial x_2^2} = f, -\dfrac{\partial^2 \hat{v}^+}{\partial x_2^2} = -\dfrac{\partial^2}{\partial x_2^2}(\hat{u}^+ - u_d) \ \ in\ \Omega^+, \\ -\Delta \hat{u}^- = f - \dfrac{1}{\beta}\hat{v}^-\chi_{\omega}, -\Delta \hat{v}^- = -\Delta(\hat{u}^- - u_d) \ \ in\ \Omega^-, \\ \dfrac{\partial \hat{u}^+}{\partial \nu} = 0, \dfrac{\partial \hat{v}^+}{\partial \nu} = (\nabla \overline{u}^+ - \nabla u_d)\cdot \nu \ \ on\ \ \Gamma_u, \\ \hat{u}^+ = \hat{u}^-,\ \dfrac{b-a}{L}\dfrac{\partial \hat{u}^+}{\partial x_2} = \dfrac{\partial \hat{u}^-}{\partial x_2}, \hat{v}^+ = \hat{v}^-,\ \dfrac{b-a}{L}\dfrac{\partial \hat{v}^+}{\partial x_2} = \dfrac{\partial \hat{v}^-}{\partial x_2} \ \ on\ \Gamma, \\ \hat{u}^- = h, \hat{v}^- = 0 \ \ on\ \ \Gamma_b, \\ \hat{u},\hat{v} \ \ is\ \Gamma_{s'} - periodic, \end{cases} \tag{6.3}$$

Then, the pair $(\hat{u}, -\frac{1}{\beta}\hat{v}\chi_{\omega})$ is the optimal solution to (P_2).

6.2 Convergence Analysis

Assume that $(\overline{u}_\varepsilon,\overline{\theta}_\varepsilon)$ is the optimal solution of $(P_{2,\varepsilon})$. Let $u_\varepsilon(0)$ be the solution of the problem (3.1) corresponding to $\theta = 0$, then from (3.2) we get

$$\|u_\varepsilon(0)\|_{H^1(\Omega_\varepsilon)} \leq C, \tag{6.4}$$

where $C > 0$ is independent of ε. Using optimality of the solution $(\overline{u}_\varepsilon,\overline{\theta}_\varepsilon)$, we get

$$\int_{\Omega_\varepsilon} |\nabla(\overline{u}_\varepsilon - u_d)|^2 + \frac{\beta}{2}\int_{\omega} \overline{\theta}_\varepsilon^2 \leq \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon(0) - u_d)|^2 \leq C. \tag{6.5}$$

Thus, as previously, we have

$$\|\overline{\theta}_\varepsilon\|_{L^2(\omega)} \leq C,\ \|\overline{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \text{ and } \|\overline{v}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C. \tag{6.6}$$

where $\overline{v}_\varepsilon$ solves adjoint problem. The variational formulation of $\overline{v}_\varepsilon$ is given by:
Find $\overline{v}_\varepsilon \in \{v \in H^1(\Omega_\varepsilon) :\ v|_{\Gamma_b} = 0\}$ such that

$$\int_{\Omega_\varepsilon} \nabla \overline{v}_\varepsilon \cdot \nabla\phi\, dx = \int_{\Omega_\varepsilon} \nabla \overline{u}_\varepsilon \cdot \nabla\phi\, dx \tag{6.7}$$

for all $\phi \in H^1(\Omega_\varepsilon)$ that satisfies $\phi|_{\Gamma_b} = 0$. We now state the main theorem of this section.

Theorem 6.2 (Main Theorem)**.** Let $(\overline{u}_\varepsilon,\overline{\theta}_\varepsilon)$ and $(\overline{u},\overline{\theta})$ be the optimal solution of $(P_{2,\varepsilon})$ and (P_2), respectively. Then

$$\begin{aligned}&\overline{\theta}_\varepsilon \rightharpoonup \overline{\theta} \text{ weakly in } H^1(\omega),\\ &\widetilde{\overline{u}_\varepsilon|_{\Omega_\varepsilon^+}} \rightharpoonup \frac{b-a}{L}\overline{u}|_{\Omega^+},\ \widetilde{\overline{v}_\varepsilon|_{\Omega_\varepsilon^+}} \rightharpoonup \frac{b-a}{L}\overline{v}|_{\Omega^+} \text{ weakly in } L^2(0,L;H^1(M,M')),\\ &\overline{u}_\varepsilon|_{\Omega^-} \to \overline{u}|_{\Omega^-},\ \overline{v}_\varepsilon|_{\Omega^-} \to \overline{v}|_{\Omega^-} \text{ strongly in } H^1(\Omega^-),\end{aligned}$$

where $\overline{\theta} = -\frac{1}{\beta}\overline{v}\chi_\omega$ and $\overline{v}_\varepsilon$, $\overline{v}$ is the solution of (3.5) and (6.2) respectively. Moreover

$$J_{2,\varepsilon}(\overline{u}_\varepsilon,\overline{\theta}_\varepsilon) \to J_2(\overline{u},\overline{\theta}) \text{ when } \varepsilon \to 0.$$

Proof. We only sketch of the proof. Since $(\overline{u}_\varepsilon,\overline{\theta}_\varepsilon)$ is the optimal solution of $(P_{2,\varepsilon})$. By Theorem 3.2 we have $\overline{\theta}_\varepsilon = -\frac{1}{\beta}\overline{v}_\varepsilon\chi_\omega$. By estimate (6.6) gives

$$\|\overline{\theta}_\varepsilon\|_{H^1(\omega)} \leq C. \tag{6.8}$$

Thus, $\overline{\theta}_\varepsilon \rightharpoonup \theta_0$ weakly in $H^1(\omega)$ along a subsequence for θ_0. The convergence of $\overline{u}_\varepsilon$ will take place in similar fashion as we did in Theorem 5.3. Here we elaborate briefly the technique used to prove the convergence of $\overline{v}_\varepsilon$. Let us denote $\overline{v}_\varepsilon{}^+$ is the restriction of $\overline{v}_\varepsilon$ in Ω_ε^+ and $\overline{v}_\varepsilon{}^-$ is the restriction of $\overline{v}_\varepsilon$ in Ω^-. Now

$$\left\|T^\varepsilon\overline{v}_\varepsilon{}^+\right\|^2_{L^2(0,L;H^1((M,M')\times(a,b))} \leq L\,\|\overline{v}_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}.$$

So, the sequence $T^\varepsilon\overline{v}_\varepsilon{}^+$ is bounded in $L^2(0,L;H((M,M')\times(a,b))$ follows from (6.6). By weak compactness, there exist a subsequence (still denoted by ε) such that

$$T^\varepsilon\overline{v}_\varepsilon{}^+ \rightharpoonup v_0^+ \text{ weakly in } L^2(0,L;H^1((M,M')\times(a,b))) \tag{6.9}$$

which implies

$$T^\varepsilon\overline{v}_\varepsilon{}^+ \rightharpoonup v_0^+ \text{ weakly in } L^2(\Omega^+\times(a,b)), \tag{6.10}$$

$$T^\varepsilon\frac{\partial\overline{v}_\varepsilon{}^+}{\partial x_2} \rightharpoonup \frac{\partial v_0^+}{\partial x_2} \text{ weakly in } L^2(\Omega^+\times(a,b)) \tag{6.11}$$

$$\varepsilon T^\varepsilon\frac{\partial\overline{v}_\varepsilon{}^+}{\partial x_1} \rightharpoonup \frac{\partial v_0^+}{\partial x_3} \text{ weakly in } L^2(\Omega^+\times(a,b)). \tag{6.12}$$

The boundedness of the sequence $T^\varepsilon\frac{\partial\overline{v}_\varepsilon{}^+}{\partial x_1}$ in $L^2(\Omega^+\times(a,b))$ follow from Proposition 4.3 and (6.6). From (6.12) we have $\frac{\partial v_0^+}{\partial x_3} = 0$. Thus with the help of Proposition 4.8, convergence (6.12) and independents of v_0^+ from x_3 variable, we conclude that

$$\widetilde{\overline{v}_\varepsilon{}^+} \rightharpoonup \frac{b-a}{L}v_0^+ \textit{ weakly in } L^2(0,L;H^1(M,M')). \tag{6.13}$$

Since $T^\varepsilon \dfrac{\partial \overline{v}_\varepsilon{}^+}{\partial x_1}$ is bounded in $L^2(\Omega^+ \times (a,b))$, by weak compactness there is an element $R \in L^2(\Omega^+ \times (a,b))$ such that upto subsequence (still denoted by ε),

$$T^\varepsilon \frac{\partial \overline{v}_\varepsilon{}^+}{\partial x_1} \rightharpoonup R \text{ weakly } \text{in} L^2(\Omega^+ \times (a,b)). \tag{6.14}$$

Also the sequence $\overline{v}_\varepsilon{}^-$ is bounded in $H^1(\Omega^-)$, follows from estimate of $\|\overline{v}_\varepsilon\|_{H^1(\Omega_\varepsilon)}$, $\overline{v}_\varepsilon{}^-$ is bounded in $H^1(\Omega^-)$. Thus upto a subsequence (still denoted by ε)

$$\overline{v}_\varepsilon{}^- \rightharpoonup v_0^- \text{ weakly in } H^1(\Omega^-). \tag{6.15}$$

Define v_0 as,

$$v_0(x) = \begin{cases} v_0^+ \; if \; x \in \Omega^+, \\ v_0^- \; if \; x \in \Omega^-. \end{cases} \tag{6.16}$$

As we proved earlier $u_0 \in W(\Omega)$, one can show $v_0 \in W_0(\Omega)$. Now our claim is that v_0 satisfies the limit problem (6.2). We choosing the same test function ϕ^ε described in (5.29). From L.H.S of (6.7), we have

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla \overline{v}_\varepsilon \cdot \nabla \widetilde{\phi^\varepsilon} &= \frac{1}{L} \int_{\Omega^+ \times (a,b)} \left(T^\varepsilon \frac{\partial \overline{v}_\varepsilon{}^+}{\partial x_1} T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_1} + T^\varepsilon \frac{\partial \overline{v}_\varepsilon{}^+}{\partial x_2} T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_2} \right) \\ &\to \frac{1}{L} \int_{\Omega^+ \times (a,b)} R \phi(x_1,x_2) \frac{d\psi}{dz}(x_3) \;\; as \; \varepsilon \to 0 \end{aligned} \tag{6.17}$$

and from R.H.S of (6.7), we get

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla (\overline{u}_\varepsilon - u_d) \cdot \nabla \widetilde{\phi^\varepsilon} &= \frac{1}{L} \int_{\Omega^+ \times (a,b)} \left(T^\varepsilon \frac{\partial (\overline{u}_\varepsilon{}^+ - u_d)}{\partial x_1} T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_1} + T^\varepsilon \frac{\partial (\overline{u}_\varepsilon{}^+ - u_d)}{\partial x_2} T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_2} \right) \\ &\to -\frac{1}{L} \int_{\Omega^+ \times (a,b)} \frac{\partial u_d}{\partial x_1} \cdot \phi(x_1,x_2) \frac{d\psi}{dz}(x_3) \;\; as \; \varepsilon \to 0. \end{aligned} \tag{6.18}$$

As $\varepsilon \to 0$ in (6.7), (6.17) and (6.18) give us,

$$\begin{aligned} &\int_{\Omega^+ \times (a,b)} \left(R + \frac{\partial u_d}{\partial x_1} \right) \phi(x_1,x_2) \frac{d\psi}{dz}(x_3) = 0 \\ &\Rightarrow \int_{\Omega^+} \left(\int_a^b \left(R(x_1,x_2,x_3) + \frac{\partial u_d}{\partial x_1} \right) \frac{d\psi}{dz}(x_3)\, dx_3 \right) \phi(x_1,x_2)\, dx_1 dx_2 = 0 \; \forall \phi \in \mathcal{D}(\Omega^+) \\ &\Rightarrow \int_a^b \left(R(x_1,x_2,x_3) + \frac{\partial u_d}{\partial x_1} \right) \frac{d\psi}{dz}(x_3)\, dx_3 = 0 \; a.e. \; (x_1,x_2) \in \Omega^+ \\ &\Rightarrow R = -\frac{\partial u_d}{\partial x_1} \; texta.e. \; in \; \Omega^+ \times (a,b). \end{aligned} \tag{6.19}$$

equivalently $T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1} \rightharpoonup -\frac{\partial u_d}{\partial x_1}$ weakly in $L^2(\Omega^+ \times (a,b))$. Taking $\psi \in \{\phi \in C^\infty(\overline{\Omega}) |\ \phi|_{\Gamma_b} = 0\}$ in the L.H.S of (6.7) gives

$$\int_{\Omega_\varepsilon} \nabla \overline{v}_\varepsilon \cdot \nabla \psi = \frac{1}{L} \int_{\Omega^+ \times (a,b)} \left(T^\varepsilon \frac{\partial \overline{v}_\varepsilon^+}{\partial x_1} T^\varepsilon \frac{\partial \psi}{\partial x_1} + T^\varepsilon \frac{\partial \overline{v}_\varepsilon^+}{\partial x_2} T^\varepsilon \frac{\partial \psi}{\partial x_2} \right) + \int_{\Omega^-} \nabla \overline{v}_\varepsilon^- \cdot \nabla \psi$$

$$\to -\frac{1}{L} \int_{\Omega^+ \times (a,b)} \frac{\partial u_d}{\partial x_1} \frac{\partial \psi}{\partial x_1} + \frac{1}{L} \int_{\Omega^+ \times (a,b)} \frac{\partial v_0^+}{\partial x_2} \frac{\partial \psi}{\partial x_2} + \int_{\Omega^-} \nabla v_0^- \cdot \nabla \psi \tag{6.20}$$

and from the R.H.S of (6.7), we can write

$$\int_{\Omega_\varepsilon} \nabla(\overline{u}_\varepsilon - u_d) \cdot \nabla \psi = \frac{1}{L} \int_{\Omega^+ \times (a,b)} \left(T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1} T^\varepsilon \frac{\partial \psi}{\partial x_1} + T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_2} T^\varepsilon \frac{\partial \psi}{\partial x_2} \right) + \int_{\Omega^-} \nabla \overline{u}_\varepsilon^- \cdot \nabla \psi$$

$$\to \frac{1}{L} \int_{\Omega^+ \times (a,b)} \frac{\partial u_d}{\partial x_1} \frac{\partial \psi}{\partial x_1} + \frac{1}{L} \int_{\Omega^+ \times (a,b)} \frac{\partial u_0^+}{\partial x_2} \frac{\partial \psi}{\partial x_2} + \int_{\Omega^-} \nabla u_0^- \cdot \nabla \psi \tag{6.21}$$

Hence

$$\frac{1}{L} \int_{\Omega^+ \times (a,b)} \frac{\partial v_0^+}{\partial x_2} \frac{\partial \psi}{\partial x_2} + \int_{\Omega^-} \nabla v_0^- \cdot \nabla \psi = \frac{1}{L} \int_{\Omega^+ \times (a,b)} \frac{\partial (u_0^+ - u_d)}{\partial x_2} \frac{\partial \psi}{\partial x_2} + \int_{\Omega^-} \nabla (u_0^- - u_d) \cdot \nabla \psi$$

which gives

$$\frac{b-a}{L} \int_{\Omega^+} \frac{\partial v_0^+}{\partial x_2} \frac{\partial \psi}{\partial x_2} + \int_{\Omega^-} \nabla v_0^- \cdot \nabla \psi = \frac{b-a}{L} \int_{\Omega^+} \frac{\partial (u_0^+ - u_d)}{\partial x_2} \frac{\partial \psi}{\partial x_2} + \int_{\Omega^-} \nabla (u_0^- - u_d) \cdot \nabla \psi$$

$\forall \psi \in \{\phi \in C^\infty(\overline{\Omega}) |\ \phi|_{\Gamma_b} = 0\}$. Density argument tell us that the above argument is true for all $\psi \in W_0(\Omega)$ since $\{\phi \in C^\infty(\overline{\Omega}) |\ \phi|_{\Gamma_b} = 0\}$ is dense in $W_0(\Omega)$ (see[17]). Therefore $v_0 \in W_0(\Omega)$ satisfies the following problem

$$\begin{cases} -\dfrac{\partial^2 v_0}{\partial x_2^2} = -\dfrac{\partial^2 (u_0{}^+ - u_d)}{\partial x_2^2} & in\ \Omega^+, \\ -\Delta v_0 = -\Delta (u_0{}^- - u_d) \quad in\ \Omega^-, \\ \dfrac{\partial v_0}{\partial \nu} = (\nabla \overline{u}^+ - \nabla u_d) \cdot \nu \ \ on\ \Gamma_u, \\ v_0{}^+ = v_0{}^-, \ \ \dfrac{b-a}{L} \dfrac{\partial v_0{}^+}{\partial x_2} = \dfrac{\partial v_0{}^-}{\partial x_2} \ \ on\ \Gamma, \\ v_0 = 0 \qquad on\ \Gamma_b, \\ v_0\ is\ \Gamma_{s'} - periodic. \end{cases} \tag{6.22}$$

We have $\overline{\theta}_\varepsilon = -\frac{1}{\beta} \overline{v}_\varepsilon \chi_\omega$ and the convergence result $\overline{\theta}_\varepsilon \to \theta_0$ strongly in $L^2(\omega)$, $\overline{v}_\varepsilon|_{\Omega^-} \rightharpoonup v_0|_{\Omega^-}$ weakly in $H^1(\Omega^-)$. Hence we get

$$\theta_0 = -\frac{1}{\beta} v_0 \chi_\omega. \tag{6.23}$$

Therefore (u_0, v_0, θ_0) satisfies the optimality system corresponding to the minimization problem (P_2). According to Theorem 6.1, its optimal solution is given by $(u_0, -\frac{1}{\beta} v_0 \chi_\omega)$.
Thus, by uniqueness we have

$$\overline{u} = u_0, \ \ \overline{v} = v_0 \ \text{ and } \overline{\theta} = \theta_0 = -\frac{1}{\beta} v_0 \chi_\omega.$$

As we proved strong convergence of $\overline{u}_\varepsilon{}^+$ in $H^1(\Omega^-)$, similarly we get

$$\overline{v}_\varepsilon|_{\Omega^-} \to \overline{v}|_{\Omega^-} \text{ strongly in } H^1(\Omega^-). \tag{6.24}$$

Also

$$\begin{aligned} \lim_{\varepsilon \to 0} J_{2,\varepsilon}(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon) &= \frac{1}{2} \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla(\overline{u}_\varepsilon - u_d)|^2 + \frac{\beta}{2} \lim_{\varepsilon \to 0} \int_\omega \overline{\theta}_\varepsilon^2 \\ &= \frac{1}{2} \left[\frac{b-a}{L} \int_{\Omega^+} \left| \frac{\partial \overline{u}|_{\Omega^+}}{\partial x_2} - \frac{\partial u_d}{\partial x_2} \right|^2 + \int_{\Omega^-} |\nabla(\overline{u}|_{\Omega^-} - u_d)|^2 \right] + \frac{\beta}{2} \int_\omega \overline{\theta}^2 \\ &= J_2(\overline{u}, \overline{\theta}). \end{aligned} \tag{6.25}$$

Therefore, we have $\lim_{\varepsilon \to 0} J_{2,\varepsilon}(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon) = J_2(\overline{u}, \overline{\theta})$. Hence the theorem.

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