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PERIODIC CONTROLS IN AN OSCILLATING DOMAIN: CONTROLS VIA UNFOLDING AND HOMOGENIZATION*

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Abstract. An optimal control problem in a two-dimensional domain with a rapidly oscillating boundary is considered. The main features of this article are on two points, namely, we consider periodic controls in the thin periodic slabs of period $\epsilon > 0$, a small parameter, and height $O(1)$ in the oscillatory part, and the controls are characterized using unfolding operators. We then do a homogenization analysis of the optimal control problems as $\epsilon \to 0$ with L^2 as well as Dirichlet (gradient-type) cost functionals.

Key words. optimal control and optimal solution, homogenization, oscillating boundary, internal periodic control, adjoint system, unfolding operator, boundary unfolding

AMS subject classifications. 35B27, 35B40, 35B37, 49J20, 49K20

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1. Introduction.

In this article, we consider an optimal control problem associated to a boundary value problem in a two-dimensional oscillatory domain Ω_ϵ with oscillating boundary. The domain Ω_ϵ consists of a fixed bottom region Ω^- and an oscillatory (rugose) upper region Ω_ϵ^+ (see Figure 1). We introduce an optimal control problem in Ω_ϵ for the Laplacian operator with controls acting on the oscillatory part Ω_ϵ^+ with periodic controls and with Neumann condition on the oscillating boundary. More precisely, the periodic controls are acting on the periodic slabs of the domain of period $\epsilon > 0$, a small parameter, and height $O(1)$. The choice of such controls are new, and it comes from a fixed region (that is, from a reference cell Λ^+; see Figure 3). The controls coming from a fixed domain are useful in numerics, though we did not carry out any computations in this paper. The aim of the present article is to characterize the controls and then study the limiting analysis (homogenization) of the optimal solution (namely, optimal control and the corresponding state) and the associated adjoint state. This involves the homogenization of the optimality system and proving the limit system is indeed the optimality system corresponding to the limit optimal control problem. In general, the motivations for studying problems defined on oscillatory domains come from the need to understand flows in channels with rugose boundary, heat transmission in winglets, propagation of electromagnetic waves in regions having rough interface, etc. (see, [4], [19]).

In the optimal control problems studied in [34], [35], [36], [37], the authors introduced controls acting away from the oscillating part of the domain. There is a vast amount of literature related to the asymptotic analysis of problems with oscillating

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boundaries; see, for example, [1], [3], [4], [5], [9], [10], [11], [17], [18], [20], [21], [30], [35], [38].

As mentioned earlier, we consider the periodic controls on the oscillatory part together with Neumann condition on the oscillating boundary. Unlike Dirichlet condition, the limit problem is quite different in the case of Neumann problem. One of the main features of the article is the characterization of the optimal control via an unfolding operator. In fact, our analysis leads to a nice relation between optimal control and adjoint state. Later, this relation is exploited to obtain the limit system. Further, we get a distributed control in the upper part of the limit domain (see Figure 2) consisting of fixed upper and lower parts. The limit system also satisfies appropriate interface conditions.

The method of periodic unfolding was introduced in [13] and further developed in [14] (see also Damlamian [15]). The periodic unfolding method adapted to oscillating boundaries can be found in Damlamian and Pettersson [16]. Though several methods, like 2-scale convergence, are available in the literature, we use the method of unfolding to study the problem under consideration since the other methods do not seem to be amenable in this situation. Here, we could give the characterization of optimal control via adjoint system. Hence, we believe the elegant method of unfolding is well suited to the problem in this article.

Indeed, the method of unfolding is well developed and applied to many problems in the literature. In addition to the normal difficulties encountered in an oscillating domain, we also have to deal with the optimality system (a coupled system) involving state, optimal control, adjoint state, and also the associated cost functional, and hence further analysis is required to obtain the limit optimal control problem (see Theorems 5.2 and 6.2).

The layout of the paper is as follows. We describe the configuration of the domain in section 2, which is similar to the domain considered in [16] (also see [35]). In the same section, we also describe the control problems with different settings, namely, with two types of cost functionals. The existence of periodic controls are also presented in the same section. The unfolding operator in the domain as well as unfolding on the boundary are introduced in section 3. The properties of these unfolding operators are recalled from [16]. The optimality systems and adjoint states are given in section 4. The characterization of the optimal control via the unfolding operator is the main result of this section (see Theorems 4.1 and 4.2). This is done both for L^2 and Dirichlet (gradient-type) cost functionals. Appropriate estimates, the convergence analysis, homogenization, and the main result for the L^2 cost functional are presented in section 5, whereas section 6 is devoted to the study of analogous results in the case of the Dirichlet cost functional.

We have studied the problem in the two-dimensional domain, but we believe that it can be carried out for three or more dimensions with appropriate modifications. For example, see [31] for a three-dimensional oscillating domain. For general literature in homogenization, we refer to [7], [8], [12], [22], [40], and the references therein. For some references regarding the homogenization of the optimal control/controllability, the reader can refer to [23], [24], [31], [32], [33]. Also, see Lions [27] for a survey on controllability, stabilizability, etc. See the references [2], [6], [10], [25], [26], [28], [29], [39] for optimal control problems and the derivation of optimality systems.

2. Oscillating boundary domain and problem description. Now, we define the domain Ω_ϵ and limit domain Ω. Let $L > 0$, and for a small parameter $\epsilon = \frac{L}{N}$, $N \in \mathbb{Z}$, we consider an oscillating domain Ω_ϵ as given in the Figure 1 and describe

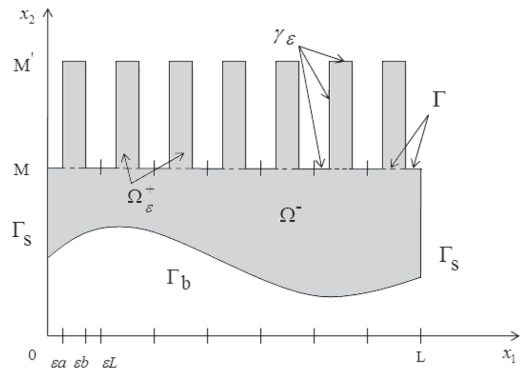

FIG. 1. *The domain* Ω_ϵ.

it below. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth periodic function defined on the interval $[0, L]$ with period L. Let $0 < a < b < L$ and η_ϵ be a periodic function defined on $[0, L]$ with period ϵL, where in the fundamental cell $[0, \epsilon L]$, the function η_ϵ is defined by

$$\eta_\epsilon(x_1) = \begin{cases} M' & \text{if} \quad x_1 \in (\epsilon a, \epsilon b), \\ M & \text{if} \quad x_1 \in [0, \epsilon L) \backslash (\epsilon a, \epsilon b) \end{cases}$$

with $M' > M > m$. Here, m is the maximum value of the smooth function g in $[0, L]$. We can write the domain Ω_ϵ as

$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 : \ 0 < x_1 < L, \, g(x_1) < x_2 < \eta_\epsilon(x_1)\}.$$

The bottom boundary Γ_b of Ω_ϵ is defined as

$$\Gamma_b = \{(x_1, x_2) : \ x_2 = g(x_1), \, x_1 \in [0, L]\}.$$

Let Ω_ϵ^+ be the upper region (rugose) of the domain Ω_ϵ which is the union of slabs of height $(M' - M)$ and width $\epsilon(b - a)$. It can be defined as

$$\Omega_\epsilon^+ = \bigcup_{k=0}^{N-1} (k\epsilon L + \epsilon a, k\epsilon L + \epsilon b) \times (M, M').$$

Denote Ω^-, the fixed bottom region of the domain Ω_ϵ which is described by

$$\Omega^- = \{(x_1, x_2) : 0 < x_1 < L, \ g(x_1) < x_2 < M\}.$$

The vertical and top boundary of Ω^- denoted by Γ_s and Γ, respectively, are defined as

$$\Gamma_s = \{(0, x_2) : \ g(0) \leq x_2 \leq M\} \cup \{(L, x_2) : \ g(L) \leq x_2 \leq M\}$$

and

$$\Gamma = \{(x_1, M) : \ 0 \leq x_1 \leq L\}.$$

The highly oscillating boundary γ_ϵ of Ω_ϵ is given by

$$\gamma_\epsilon = \partial\Omega_\epsilon \setminus \left(\Gamma_b \bigcup \Gamma_s\right),$$

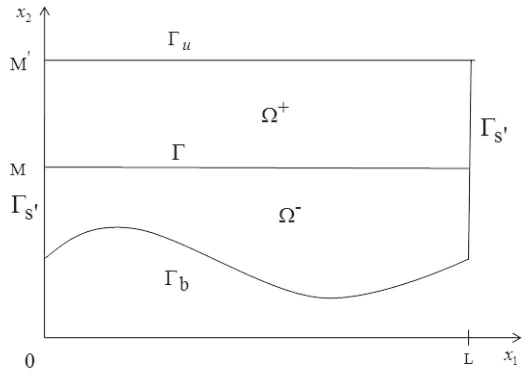

FIG. 2. *The domain* Ω.

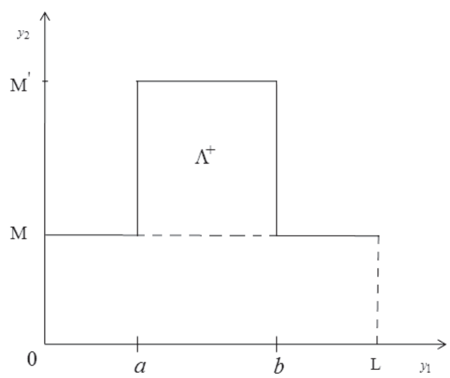

FIG. 3. *The domain* Λ^+.

where $\partial\Omega_\epsilon$ is the boundary of Ω_ϵ. The common boundary between Ω_ϵ^+ and Ω^-, denoted by Γ_ϵ, is defined as

$$\Gamma_\epsilon = \bigcup_{k=0}^{N-1} (k\epsilon L + \epsilon a, k\epsilon L + \epsilon b) \times \{M\}.$$

We can also write Ω_ϵ as $\Omega_\epsilon = Int\ (\overline{\Omega_\epsilon^+ \cup \Omega^-})$. Let $\Omega = \{(x_1, x_2) : 0 < x_1 < L,\ g(x_1) < x_2 < M'\}$ (see Figure 2). The bottom part of the boundary of Ω is the same as that of Ω_ϵ. The vertical boundary of Ω is denoted by $\Gamma_{s'}$ and can be written as

$$\Gamma_{s'} = \{(0, x_2) :\ g(0) \le x_2 \le M'\} \cup \{(L, x_2) :\ g(L) \le x_2 \le M'\}.$$

The top boundary of Ω is given by

$$\Gamma_u = \{(x_1, M') :\ 0 \le x_1 \le L\}.$$

Define Ω^+ as $\Omega^+ = \{(x_1, x_2) :\ 0 < x_1 < L, M < x_2 < M'\}$, and then we can write $\Omega = Int\ (\overline{\Omega^+ \cup \Omega^-})$. Let Λ^+ be the reference cell (see Figure 3), defined as $\Lambda^+ = (a, b) \times (M, M')$.

Let $H^1_{per}(\Omega_\epsilon) = \{f :\ f \in H^1(\Omega_\epsilon),\ f(x_1 + kL, x_2) = f(x_1, x_2)$ for all $k \in \mathbb{Z}\}$. A function defined in Ω_ϵ is called Γ_s-periodic if they take the same value on both side of Γ_s.

We now consider the following control problem:

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f + \theta^\epsilon \chi_{\Omega_\epsilon^+} \ \text{ in } \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial \nu} = 0 \ \text{ on } \gamma_\epsilon, \\ u_\epsilon = h \text{ on } \Gamma_b, \\ u_\epsilon \ \text{ is } \Gamma_s - \text{periodic.} \end{cases} \tag{2.1}$$

Here, $f \in L^2(\Omega)$, the source term and the boundary term $h \in H^{1/2}_{per}(\Gamma_b)$ are given. Further, θ^ϵ is the control function acting on the oscillatory part Ω_ϵ^+ and $\chi_{\Omega_\epsilon^+}$ is the characteristic function of Ω_ϵ^+. One of the attractions of the paper is that we take the control θ^ϵ of the form $\theta^\epsilon(x_1, x_2) = \theta(\frac{x_1}{\epsilon}, x_2)$, where $\theta \in L^2_{per}(\Lambda^+)$ is a control function defined on the reference cell Λ^+. It is known that if $\theta \in L^2_{per}(\Lambda^+)$, then (2.1) is a standard elliptic problem and hence admits a unique solution $u_\epsilon \in H^1_{per}(\Omega_\epsilon)$ (depending on θ) that satisfies $u_\epsilon|_{\Gamma_b} = h$. The solution operator is linear and continuous from $L^2_{per}(\Omega) \times L^2_{per}(\Omega_\epsilon^+)$ into $H^1_{per}(\Omega_\epsilon)$, i.e.,

$$\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \le C(\|f\|_{L^2(\Omega)} + \|\theta^\epsilon\|_{L^2(\Omega_\epsilon^+)} + \|h\|_{H^{1/2}(\Gamma_b)}), \tag{2.2}$$

where $C > 0$ is independent of ϵ. Let us consider two cost functionals, namely, the L^2-cost functional and the Dirichlet cost functional; more precisely,

$$J_{1,\epsilon}(u_\epsilon, \theta) = \frac{1}{2} \int_{\Omega_\epsilon} |u_\epsilon - u_d|^2 + \frac{\beta}{2} \int_{\Omega_\epsilon^+} |\theta^\epsilon|^2,$$
$$J_{2,\epsilon}(u_\epsilon, \theta) = \frac{1}{2} \int_{\Omega_\epsilon} |\nabla u_\epsilon - \nabla u_d|^2 + \frac{\beta}{2} \int_{\Omega_\epsilon^+} |\theta^\epsilon|^2,$$

respectively. Here, $\beta > 0$ is a regularization parameter and $u_\epsilon = u_\epsilon(\theta)$ is the solution of the problem (2.1) corresponding to θ. The desired state is denoted by $u_d \in H^1_{per}(\Omega_\epsilon)$. With these cost functionals, we consider the optimal control problems

$$\inf\{J_{i,\epsilon}(u_\epsilon, \theta) |\ \theta \in L^2(\Lambda^+), (u_\epsilon, \theta^\epsilon) \text{ obeys } (2.1)\} \tag{$P_{i,\epsilon}$}$$

for $i = 1, 2$. Now we prove that $(P_{i,\epsilon})$ for $i = 1, 2$ admit unique solutions. Since $u_\epsilon = u_\epsilon(\theta)$ depends on θ, we can also denote $J_{i,\epsilon}(u_\epsilon, \theta) = J_{i,\epsilon}(\theta)$ when there is no ambiguity.

THEOREM 2.1. *For each $\epsilon > 0$, the minimization problem $(P_{1,\epsilon})$ admits a unique solution.*

Proof. Let $m_\epsilon = \inf_{\theta \in L^2(\Lambda^+)} J_{1,\epsilon}(u_\epsilon, \theta)$. Since $m_\epsilon \le J_{1,\epsilon}(0) < \infty$ and using (2.2), we get $0 \le m_\epsilon < \infty$. Hence, there exists a minimizing sequence $(\theta_{n,\epsilon})_n \in L^2(\Lambda^+)$ such that $\lim_{n\to\infty} J_{1,\epsilon}(u_\epsilon^n, \theta_{n,\epsilon}) = m_\epsilon$. Without loss of generality, we can suppose that $J_{1,\epsilon}(u_\epsilon^n, \theta_{n,\epsilon}) \le J_{1,\epsilon}(u_\epsilon^0, 0)$, which implies $\|\theta_{n,\epsilon}\|_{L^2(\Lambda^+)} \le C$ since $\|\theta_{n,\epsilon}\|_{L^2(\Lambda^+)} = \|\theta^\epsilon_{n,\epsilon}\|_{L^2(\Omega_\epsilon^+)}$. Here, u_ϵ^n, u_ϵ^0 are solutions of (2.1) corresponding to data $\theta^\epsilon_{n,\epsilon}$, $\theta^\epsilon = 0$, respectively. So up to a subsequence $\theta_{n,\epsilon} \rightharpoonup \overline{\theta_\epsilon}$ in $L^2(\Lambda^+)$ as $n \to \infty$. Using the fact that L^2-norm is weakly lower semicontinuous, we have

$$\int_{\Lambda^+} |\overline{\theta_\epsilon}|^2 \le \liminf \int_{\Lambda^+} |\theta_{n,\epsilon}|^2, \tag{2.3}$$

which gives

$$\int_{\Omega_\epsilon^+} |\overline{\theta_\epsilon}^\epsilon|^2 \le \liminf \int_{\Omega_\epsilon^+} |\theta^\epsilon_{n,\epsilon}|^2, \tag{2.4}$$

where $\overline{\theta_\epsilon}^\epsilon(x_1,x_2) = \overline{\theta_\epsilon}(\frac{x_1}{\epsilon},x_2)$ and $\theta^\epsilon_{n,\epsilon} = \theta_{n,\epsilon}(\frac{x_1}{\epsilon},x_2)$. Note that we use the upper script ϵ to represent the periodic oscillations with respect to first variable x_1 and lower script ϵ represent that it comes from the problem. Let us denote $u^n_\epsilon = u_\epsilon(f,\theta^\epsilon_{n,\epsilon})$, the solution of (2.1) corresponding to f, $\theta^\epsilon_{n,\epsilon}$. We know from the L^2 bound of $\theta^\epsilon_{n,\epsilon}$ on Ω^+_ϵ and norm estimate (2.2) that $\|u^n_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C$, which implies $u^n_\epsilon \rightharpoonup \overline{u_\epsilon}$ as $n\to\infty$ in $H^1(\Omega_\epsilon)$.

Claim. We prove $\overline{u}_\epsilon = u_\epsilon(f,\overline{\theta_\epsilon}^\epsilon)$. We know that u^n_ϵ solves the partial differential equation (2.1) for $\theta^\epsilon = \theta^\epsilon_{n,\epsilon}$, and from the variational formulation we get

$$\int_{\Omega_\epsilon} \nabla u^n_\epsilon \cdot \nabla\phi - \int_{\Gamma_b} \frac{\partial u^n_\epsilon}{\partial\nu} h + \int_{\Omega_\epsilon} u^n_\epsilon \phi = \int_{\Omega_\epsilon} f\phi + \int_{\Omega_\epsilon} \theta^\epsilon_{n,\epsilon}\chi_{\Omega^+_\epsilon}\phi. \tag{2.5}$$

To prove our claim, we need to show the following variational formulation:

$$\int_{\Omega_\epsilon} \nabla \overline{u}_\epsilon \cdot \nabla\phi - \int_{\Gamma_b} \frac{\partial \overline{u}_\epsilon}{\partial\nu} h + \int_{\Omega_\epsilon} \overline{u}_\epsilon \phi = \int_{\Omega_\epsilon} f\phi + \int_{\Omega_\epsilon} \chi_{\Omega^+_\epsilon}\overline{\theta_\epsilon}^\epsilon\phi. \tag{2.6}$$

It suffices to prove that

$$\lim_{n\to\infty}\int_{\Omega^+_\epsilon} \theta^\epsilon_{n,\epsilon}\phi = \int_{\Omega^+_\epsilon} \overline{\theta_\epsilon}^\epsilon\phi \;\text{ for } \phi\in L^2(\Omega^+_\epsilon). \tag{2.7}$$

Now compute the limit

$$\begin{aligned}
\lim_{n\to\infty}\int_{\Omega^+_\epsilon} \theta^\epsilon_{n,\epsilon}\phi &= \lim_{n\to\infty}\sum_{k=0}^{N-1}\int_M^{M'}\int_{k\epsilon L+\epsilon a}^{k\epsilon L+\epsilon b} \theta_{n,\epsilon}\left(\frac{x_1}{\epsilon},x_2\right)\phi(x_1,x_2)\,dx_1dx_2\\
&= \epsilon\lim_{n\to\infty}\sum_{k=0}^{N-1}\int_M^{M'}\int_{kL+a}^{kL+b}\theta_{n,\epsilon}(y,x_2)\phi(\epsilon y,x_2)\,dydx_2\\
&= \epsilon\sum_{k=0}^{N-1}\int_M^{M'}\int_{kL+a}^{kL+b}\overline{\theta_\epsilon}(y,x_2)\phi(\epsilon y,x_2)\,dydx_2\\
&= \sum_{k=0}^{N-1}\int_M^{M'}\int_{k\epsilon L+\epsilon a}^{k\epsilon L+\epsilon b}\overline{\theta_\epsilon}\left(\frac{x_1}{\epsilon},x_2\right)\phi(x_1,x_2)\,dx_1dx_2\\
&= \int_{\Omega^+_\epsilon}\overline{\theta_\epsilon}^\epsilon\phi.
\end{aligned}$$

Hence, (2.7) proved. We know that $u_\epsilon(f,\theta^\epsilon_{n,\epsilon}) \rightharpoonup \overline{u}_\epsilon$ in $H^1(\Omega_\epsilon)$, by weakly lower semicontinuity of the L^2-norm, gives

$$\int_{\Omega_\epsilon}|\overline{u}_\epsilon - u_d|^2 \leq \liminf\int_{\Omega_\epsilon}|u_\epsilon(f,\theta^\epsilon_{n,\epsilon}) - u_d)|^2. \tag{2.8}$$

Hence, combining (2.4) and (2.8), we get $J_{1,\epsilon}(\overline{u}_\epsilon,\overline{\theta}_\epsilon) \leq \liminf J_{1,\epsilon}(u_\epsilon(f,\theta_{n,\epsilon}),\theta_{n,\epsilon}) = m_\epsilon$. Therefore, $(\overline{u}_\epsilon,\overline{\theta}_\epsilon)$ is a solution to problem $(P_{1,\epsilon})$. Uniqueness follows from the strict convexity of the L^2-cost functional. ∎

THEOREM 2.2. *For each $\epsilon > 0$, the minimization problem $(P_{2,\epsilon})$ admits a unique solution.*

Proof. The proof is similar to Theorem 2.1. ∎

In the next section, we introduce the unfolding operator and its properties required for our article. Then using these operators, we derive the optimality system and characterize the optimal control using unfolding operators.

3. Unfolding operators and its properties. We define the periodic unfolding operator and some of its properties without proof. The proofs can be found in [16]. For $x \in \mathbb{R}$, we write $[x]_L$ as the integer part of x with respect to L, that is, $[x]_L = kL$, where k is the largest integer such that $kL \leq x$ and $\{x\}_L = x - [x]_L$.

DEFINITION 3.1 (the unfolding operator). *Let $\phi^\epsilon : \Omega^+ \times (a,b) \to \Omega_\epsilon^+$ be defined by $(x_1, x_2, y_1) \mapsto (\epsilon[\frac{x_1}{\epsilon}]_L + \epsilon y_1, x_2)$. The ϵ-unfolding of a function $u : \Omega_\epsilon^+ \to \mathbb{R}$ is the composite function $u \circ \phi^\epsilon : \Omega^+ \times (a,b) \to \mathbb{R}$. The operator that maps every function $u : \Omega_\epsilon^+ \to \mathbb{R}$ to its ϵ-unfolding is called the unfolding operator. The unfolding operator denoted by T^ϵ, i.e.,*

$$T^\epsilon : \{u : \Omega_\epsilon^+ \to \mathbb{R}\} \to \{T^\epsilon u : \Omega^+ \times (a,b) \to \mathbb{R}\},$$

is defined by

$$T^\epsilon u(x_1, x_2, y_1) = u \circ \phi^\epsilon(x_1, x_2, y_1) = u\left(\epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y_1, x_2\right).$$

If U is an open subset of $\mathbb{R}^2$ containing Ω_ϵ^+ and u is real valued function on U, then $T^\epsilon u$ will mean T^ϵ acting on the restriction of u to Ω_ϵ^+. We now state the following properties of the unfolding operator T^ϵ.

PROPOSITION 3.2.

(i) *T^ϵ is linear.*

(ii) *Let u,v be functions $\Omega_\epsilon^+ \to R$. Then $T^\epsilon(uv) = T^\epsilon(u)T^\epsilon(v)$.*

(iii) *Let $u \in L^1(\Omega_\epsilon^+)$. Then*

$$\int_{\Omega^+ \times (a,b)} T^\epsilon u \, dx = L \int_{\Omega_\epsilon^+} u \, dx.$$

(iv) *Let $u \in L^2(\Omega_\epsilon^+)$. Then $T^\epsilon u \in L^2(\Omega^+ \times (a,b))$ and $\|T^\epsilon u\|_{L^2(\Omega^+ \times (a,b))} = \sqrt{L}\|u\|_{L^2(\Omega_\epsilon^+)}$.*

(v) *Let $u \in H^1(\Omega_\epsilon^+)$. Then $T^\epsilon u \in L^2(0, L; H^1((M, M') \times (a,b)))$. Moreover, $\frac{\partial}{\partial x_2}(T^\epsilon u) = T^\epsilon(\frac{\partial u}{\partial x_2})$ and $\frac{\partial}{\partial y_1}(T^\epsilon u) = \epsilon T^\epsilon(\frac{\partial u}{\partial x_1})$.*

(vi) *Let $u \in L^2(\Omega^+)$. Then $T^\epsilon u \to u$ in $L^2(\Omega^+ \times (a,b))$.*

(vii) *Let $u_\epsilon \to u$ in $L^2(\Omega^+)$. Then $T^\epsilon u_\epsilon \to u$ in $L^2(\Omega^+ \times (a,b))$.*

(viii) *For every ϵ, let $u_\epsilon \in L^2(\Omega_\epsilon^+)$ be such that $T^\epsilon u_\epsilon \rightharpoonup u$ weakly in $L^2(\Omega^+ \times (a,b))$. Then*

$$\widetilde{u}_\epsilon \rightharpoonup \frac{1}{L}\int_a^b u \, dy_1$$

weakly in $L^2(\Omega^+)$.

(ix) *Let $u_\epsilon \in H^1(\Omega_\epsilon^+)$ for every $\epsilon > 0$ be such that $T^\epsilon u_\epsilon \rightharpoonup u$ weakly in $L^2((0,L) \times (a,b); H^1((M, M')))$. Then*

$$\widetilde{u}_\epsilon \rightharpoonup \frac{1}{L}\int_a^b u \, dy_1$$

weakly in $L^2((0,L); H^1((M, M')))$.

We use $\sim$ to represent the extension by 0 to the bigger domain under consideration. We now define boundary unfolding on Γ_ϵ, i.e., on the common boundary of Ω_ϵ^+ and Ω^-.

DEFINITION 3.3. *Let $\phi_{x_2=M}^\epsilon : (0,L) \times (a,b) \to \Gamma_\epsilon$ be defined by $(x_1, y_1) \mapsto (\epsilon[\frac{x_1}{\epsilon}]_L + \epsilon y_1)$. The ϵ-unfolding of a function $u : \Gamma_\epsilon \to \mathbb{R}$ is the function $u \circ \phi_{x_2=M}^\epsilon :$*

$(0,L)\times(a,b)\to\mathbb{R}$ *denoted by* $T^{\epsilon}_{x_2=M}$, *that is,*

$$T^{\epsilon}_{x_2=M}:\{u:\Gamma_\epsilon\to\mathbb{R}\}\to\{T^{\epsilon}_{x_2=M}u:(0,L)\times(a,b)\to\mathbb{R}\}$$

is defined by

$$T^{\epsilon}_{x_2=M}u=u\circ\phi^{\epsilon}_{x_2=M}=u\left(\epsilon\left[\frac{x_1}{\epsilon}\right]_L+\epsilon y_1\right).$$

If U is an open subset of $\mathbb{R}^2$ such that $\Gamma_\epsilon\subset U$ and $u:U\to\mathbb{R}$, then $T^{\epsilon}_{x_2=M}u=T^{\epsilon}_{x_2=M}(u|_{\Gamma_\epsilon})$. The properties of boundary unfolding operators are given below.

PROPOSITION 3.4.

(i) $T^{\epsilon}_{x_2=M}$ *is linear.*

(ii) *Let u,v be functions from* $\Gamma_\epsilon\to R$. *Then* $T^{\epsilon}_{x_2=M}(uv)=T^{\epsilon}_{x_2=M}(u)T^{\epsilon}_{x_2=M}(v)$.

(iii) *Let* $u\in L^2(\Gamma_\epsilon)$. *Then* $T^{\epsilon}_{x_2=M}u\in L^2((0,L)\times(a,b))$ *and, moreover,* $\|T^{\epsilon}_{x_2=M}u\|_{L^2((0,L)\times(a,b))}=\sqrt{L}\|u\|_{L^2(\Gamma_\epsilon)}$.

(iv) *Let* $u\in H^1(\Gamma_\epsilon)$. *Then* $T^{\epsilon}_{x_2=M}u\in L^2(0,L;H^1((a,b)))$ *and* $\frac{\partial}{\partial y_1}(T^{\epsilon}_{x_2=M}u)=\epsilon T^{\epsilon}_{x_2=M}(\frac{\partial u}{\partial x_1})$.

(v) *Let* $u\in L^2(0,L)$. *Then* $T^{\epsilon}_{x_2=M}u\to u$ *in* $L^2((0,L)\times(a,b))$.

(vi) *Suppose that* $u_\epsilon\to u$ *in* $L^2(0,L)$. *Then* $T^{\epsilon}_{x_2=M}u_\epsilon\to u$ *in* $L^2((0,L)\times(a,b))$.

(vii) *Suppose that* u_ϵ *is a sequence in* $L^2(\Gamma_\epsilon)$ *such that* $T^{\epsilon}_{x_2=M}u_\epsilon\rightharpoonup u$ *weakly in* $L^2((0,L)\times(a,b))$. *Then* $\widetilde{u}_\epsilon\rightharpoonup\frac{1}{L}\int_a^b u\,dy_1$ *weakly in* $L^2(0,L)$.

4. Optimality system. Let $(\overline{u}_\epsilon,\overline{\theta}_\epsilon)$ be the optimal solution to the problem $(P_{1,\epsilon})$. In this section, we derive the characterization of $\overline{\theta}_\epsilon$ with the help of unfolding operators and adjoint state $\overline{v}_\epsilon\in H^1_{per}(\Omega_\epsilon)$, which solves

$$\begin{cases}-\Delta\overline{v}_\epsilon+\overline{v}_\epsilon=\overline{u}_\epsilon-u_d \text{ in } \Omega_\epsilon,\\ \dfrac{\partial\overline{v}_\epsilon}{\partial\nu}=0 \quad\text{on } \gamma_\epsilon,\\ \overline{v}_\epsilon=0 \text{ on } \Gamma_b,\\ \overline{v}_\epsilon \text{ is } \Gamma_s-\text{periodic.}\end{cases}\tag{4.1}$$

THEOREM 4.1. *Let* $f\in L^2(\Omega)$, $h\in H^{1/2}(\Gamma_b)$, *and* $(\overline{u}_\epsilon,\overline{\theta}_\epsilon)$ *be the optimal solution of* $(P_{1,\epsilon})$. *Let* $\overline{v}_\epsilon\in H^1_{per}(\Omega_\epsilon)$ *solve* (4.1), *and then the optimal control is given by*

$$\overline{\theta}_\epsilon(y_1,y_2)=-\frac{1}{\beta}\left[\frac{1}{L}\int_0^L T^{\epsilon}\overline{v}_\epsilon(x_1,y_2,y_1)dx_1\right],$$

where T^ϵ *is the unfolding operator as in Definition* 3.1. *Conversely, assume that a pair* $(\hat{u}_\epsilon,\hat{v}_\epsilon)\in H^1_{per}(\Omega_\epsilon)\times H^1_{per}(\Omega_\epsilon)$ *solves the optimality system*

$$\begin{cases}-\Delta\hat{u}_\epsilon+\hat{u}_\epsilon=f+\hat{\theta}^{\epsilon}_\epsilon\chi_{\Omega^+_\epsilon} \text{ in } \Omega_\epsilon,\\ -\Delta\hat{v}_\epsilon+\hat{v}_\epsilon=\hat{u}_\epsilon-u_d \text{ in } \Omega_\epsilon,\\ \dfrac{\partial\hat{u}_\epsilon}{\partial\nu}=0,\ \dfrac{\partial\hat{v}_\epsilon}{\partial\nu}=0 \text{ on } \gamma_\epsilon,\\ \hat{u}_\epsilon=h,\ \hat{v}_\epsilon=0 \text{ on } \Gamma_b,\ \ \hat{u}_\epsilon,\hat{v}_\epsilon \text{ are } \Gamma_s-\text{periodic},\\ \hat{\theta}_\epsilon(y_1,y_2)=-\dfrac{1}{\beta}\left[\dfrac{1}{L}\displaystyle\int_0^L T^\epsilon\hat{v}_\epsilon(x_1,y_2,y_1)dx_1\right].\end{cases}\tag{4.2}$$

Then, the pair $(\hat{u}_\epsilon,\hat{\theta}_\epsilon)$ *is the optimal solution to* $(P_{1,\epsilon})$.

Proof. We know from Theorem 2.1 that $(P_{1,\epsilon})$ admits a unique solution, say, $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$, where $\overline{\theta}_\epsilon$ is the optimal control, $\overline{u}_\epsilon$ is the optimal state, and $\overline{\theta}_\epsilon^\epsilon(x_1, x_2) = \overline{\theta}_\epsilon(\frac{x_1}{\epsilon}, x_2)$. Let $F(\theta) = J_{1,\epsilon}(u_\epsilon(f,\theta),\theta)$. From the optimality condition of $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$, it follows that

$$\frac{1}{\lambda}(F(\overline{\theta}_\epsilon + \lambda\theta) - F(\overline{\theta}_\epsilon)) \geq 0 \quad \forall \lambda > 0 \text{ and } \theta \in L^2(\Lambda^+).$$

Now calculate

$$\begin{aligned} &F(\overline{\theta}_\epsilon + \lambda\theta) - F(\overline{\theta}_\epsilon) \\ &\quad = \frac{1}{2}\int_{\Omega_\epsilon} |u_{\epsilon,\lambda} - u_d|^2 + \frac{\beta}{2}\int_{\Omega_\epsilon^+} |\overline{\theta}_\epsilon^\epsilon + \lambda\theta^\epsilon|^2 - \frac{1}{2}\int_{\Omega_\epsilon} |\overline{u}_\epsilon - u_d|^2 - \frac{\beta}{2}\int_{\Omega_\epsilon^+} |\overline{\theta}_\epsilon^\epsilon|^2 \\ &\quad = \frac{1}{2}\int_{\Omega_\epsilon} (u_{\epsilon,\lambda} - \overline{u}_\epsilon)(u_{\epsilon,\lambda} + \overline{u}_\epsilon - 2u_d) + \frac{\beta}{2}\int_{\Omega_\epsilon^+} (2\lambda\overline{\theta}_\epsilon^\epsilon\theta^\epsilon + \lambda^2\theta^{\epsilon 2}), \end{aligned}$$

where $u_{\epsilon,\lambda} = u_\epsilon(f, \overline{\theta_\epsilon}^\epsilon + \lambda\theta^\epsilon)$. Note that $w_{\epsilon,\lambda} = u_{\epsilon,\lambda} - \overline{u}_\epsilon$ is the solution to the equation

$$\begin{cases} -\Delta w + w = \lambda\theta^\epsilon \chi_{\Omega_\epsilon^+} & \text{in } \Omega_\epsilon, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \gamma_\epsilon, \\ w = 0 & \text{on } \Gamma_b, \\ w \quad \text{is } \Gamma_s - \text{periodic}. \end{cases}$$

Using the continuity of solution operator, we get

$$\|w_{\epsilon,\lambda}\|_{H^1(\Omega_\epsilon)} \leq C|\lambda|\|\theta\|_{L^2(\Lambda^+)}.$$

Thus, $w_{\epsilon,\lambda} \to 0$ in $H^1(\Omega_\epsilon)$, and hence the sequence $(u_{\epsilon,\lambda})_\lambda$ converges to $\overline{u}_\epsilon$ in $H^1(\Omega_\epsilon)$ as $\lambda \to 0$. Set $w_{\theta^\epsilon,\epsilon} = \frac{1}{\lambda}w_{\epsilon,\lambda}$. Notice that $w_{\theta^\epsilon,\epsilon} \in H^1(\Omega_\epsilon)$ is a solution to the equation

$$\begin{cases} -\Delta w + w = \theta^\epsilon \chi_{\Omega_\epsilon^+} & \text{in } \Omega_\epsilon, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \gamma_\epsilon, \\ w = 0 & \text{on } \Gamma_b, \\ w \text{ is } \Gamma_s - \text{periodic}. \end{cases} \tag{4.3}$$

Then, we get $0 \leq \lim_{\lambda\to 0} \frac{1}{\lambda}(F(\overline{\theta}_\epsilon + \lambda\theta) - F(\overline{\theta}_\epsilon)) = \int_{\Omega_\epsilon} (\overline{u}_\epsilon - u_d)w_{\theta^\epsilon,\epsilon} + \beta\int_{\Omega_\epsilon^+} \overline{\theta}_\epsilon^\epsilon\theta^\epsilon$. Thus, $F'(\overline{\theta}_\epsilon)\theta \geq 0$ for all $\theta \in L^2(\Lambda^+)$, which implies that $F'(\overline{\theta}_\epsilon)\theta = 0$ for all $\theta \in L^2(\Lambda^+)$. Hence, for the optimal solution, we get

$$\int_{\Omega_\epsilon} (\overline{u}_\epsilon - u_d)w_{\theta^\epsilon,\epsilon} = -\beta\int_{\Omega_\epsilon^+} \overline{\theta}_\epsilon^\epsilon\theta^\epsilon. \tag{4.4}$$

Since $\overline{v}_\epsilon$ satisfies the system (4.1) and $w_{\theta^\epsilon,\epsilon}$ satisfies (4.3), we have

$$\int_{\Omega_\epsilon} (\overline{u}_\epsilon - u_d)w_{\theta^\epsilon,\epsilon} = \int_{\Omega_\epsilon} \nabla\overline{v}_\epsilon \cdot \nabla w_{\theta^\epsilon,\epsilon} + \int_{\Omega_\epsilon} \overline{v}_\epsilon w_{\theta^\epsilon,\epsilon} = \int_{\Omega_\epsilon^+} \overline{v}_\epsilon\theta^\epsilon. \tag{4.5}$$

Using the unfolding operator,

$$\begin{aligned} \int_{\Omega_\epsilon^+} \overline{\theta}_\epsilon^\epsilon\theta^\epsilon &= \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\epsilon\overline{\theta}_\epsilon^\epsilon T^\epsilon\theta^\epsilon = \frac{1}{L}\int_{\Omega^+\times(a,b)} \overline{\theta}_\epsilon(y_1, y_2)\theta(y_1, y_2) \\ &= \int_{\Lambda^+} \overline{\theta}_\epsilon(y_1, y_2)\theta(y_1, y_2) \end{aligned}$$

and

$$\int_{\Omega_\epsilon^+} \overline{v}_\epsilon \theta^\epsilon = \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\epsilon \overline{v}_\epsilon T^\epsilon \theta^\epsilon = \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\epsilon \overline{v}_\epsilon(x_1,y_2,y_1)\,\theta(y_1,y_2)$$
$$= \int_{\Lambda^+}\left[\frac{1}{L}\int_0^L T^\epsilon \overline{v}_\epsilon(x_1,y_2,y_1)dx_1\right]\theta(y_1,y_2).$$

Hence, from (4.4) and (4.5), it follows that

$$\int_{\Lambda^+} \overline{\theta}_\epsilon(y_1,y_2)\theta(y_1,y_2)$$
$$= -\frac{1}{\beta}\int_{\Lambda^+}\left[\frac{1}{L}\int_0^L T^\epsilon \overline{v}_\epsilon(x_1,y_2,y_1)dx_1\right]\theta(y_1,y_2) \text{ for all } \theta\in L^2(\Lambda^+).$$

This gives the optimal control in terms of the adjoint state via unfolding operator as

$$\overline{\theta}_\epsilon = -\frac{1}{\beta}\left[\frac{1}{L}\int_0^L T^\epsilon \overline{v}_\epsilon(x_1,y_2,y_1)\,dx_1\right]. \tag{4.6}$$

To prove the converse, suppose that $(\hat{u}_\epsilon,\hat{v}_\epsilon)\in H^1(\Omega_\epsilon)\times H^1(\Omega_\epsilon)$ and $\hat{\theta}_\epsilon$ obeys the optimality system (4.2). For every $\theta\in L^2(\Lambda^+)$, we have

$$F(\hat{\theta}_\epsilon+\theta)-F(\hat{\theta}_\epsilon) = \frac{1}{2}\int_{\Omega_\epsilon}|u_{\epsilon,1}-\hat{u}_\epsilon|^2 + \frac{\beta}{2}\int_{\Omega_\epsilon^+}|\hat{\theta}_\epsilon^\epsilon|^2$$
$$+\int_{\Omega_\epsilon}(u_{\epsilon,1}-\hat{u}_\epsilon)(\hat{u}_\epsilon-u_d)+\beta\int_{\Omega_\epsilon^+}\hat{\theta}_\epsilon^\epsilon\theta^\epsilon,$$

where $u_{\epsilon,1}=u_\epsilon(f,\hat{\theta}_\epsilon^\epsilon+\theta^\epsilon)$. Observe that

$$\int_{\Omega_\epsilon}(u_{\epsilon,1}-\hat{u}_\epsilon)(\hat{u}_\epsilon-u_d)$$
$$= \int_{\Omega_\epsilon}\nabla(u_{\epsilon,1}-\hat{u}_\epsilon)\cdot\nabla\hat{v}_\epsilon + \int_{\Omega_\epsilon}(u_{\epsilon,1}-\hat{u}_\epsilon)\hat{v}_\epsilon + \int_{\partial\Omega_\epsilon}\frac{\partial\hat{v}_\epsilon}{\partial\nu}(u_{\epsilon,1}-\hat{u}_\epsilon)$$
$$= \int_{\Omega_\epsilon^+}\hat{v}_\epsilon\theta^\epsilon = \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\epsilon\hat{v}_\epsilon T^\epsilon\theta^\epsilon$$
$$= \int_{\Lambda^+}\left[\frac{1}{L}\int_0^L T^\epsilon\hat{v}_\epsilon\right]\theta = -\beta\int_{\Lambda^+}\hat{\theta}_\epsilon\theta = -\beta\int_{\Omega_\epsilon^+}\hat{\theta}_\epsilon^\epsilon\theta^\epsilon.$$

Hence, $F(\hat{\theta}_\epsilon+\theta)-F(\hat{\theta}_\epsilon)\geq 0$. Thus, $(\hat{u}_\epsilon,\hat{\theta}_\epsilon)$ is the optimal solution to $(P_{1,\epsilon})$. ∎

A similar characterization can also be obtained with the Dirichlet problem, namely, the optimal control problem with the cost functional $J_{2,\epsilon}$. Suppose, $(\overline{u}_\epsilon,\overline{\theta}_\epsilon)$ is the solution to problem $(P_{2,\epsilon})$; then the optimal control $\overline{\theta}_\epsilon$ can be characterized with the help of adjoint state $\overline{v}_\epsilon$, which solves the partial differential equation

$$\begin{cases} -\Delta\overline{v}_\epsilon + \overline{v}_\epsilon = -\Delta(\overline{u}_\epsilon - u_d) \quad \text{in } \Omega_\epsilon, \\ \dfrac{\partial \overline{v}_\epsilon}{\partial\nu} = (\nabla\overline{u}_\epsilon - \nabla u_d)\cdot\nu \ \text{ on } \gamma_\epsilon, \\ \overline{v}_\epsilon = 0 \ \text{ on } \Gamma_b, \ \overline{v}_\epsilon \ \text{ is } \Gamma_s - \text{periodic.} \end{cases} \tag{4.7}$$

Once again, we need to consider $F(\theta) = J_{2,\epsilon}(u_\epsilon(f,\theta),\theta)$, and then as in the above proof, we have to compute $F'(\overline{\theta}_\epsilon)$. By defining the adjoint system as in (4.7), one can simplify the optimality condition $F'(\overline{\theta}_\epsilon)\theta = 0$ for all $\theta \in L^2(\Lambda^+)$. In the process of simplification, we get the second condition in (4.7), which comes from the right-hand side of the first equation. In a similar way, as in the proof of the Theorem 4.1, with appropriate modifications, we arrive at the following theorem.

THEOREM 4.2. *Let $f \in L^2(\Omega)$, $h \in H^{1/2}(\Gamma_b)$, and $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ be the optimal solution of $(P_{2,\epsilon})$. Let $\overline{v}_\epsilon \in H^1_{per}(\Omega_\epsilon)$ solve (4.7), and then the optimal control is given by*

$$\overline{\theta}_\epsilon(y_1,y_2) = -\frac{1}{\beta}\left[\frac{1}{L}\int_0^L T^\epsilon\overline{v}_\epsilon(x_1,y_2,y_1)dx_1\right]. \tag{4.8}$$

Conversely, assume that a pair $(\hat{u}_\epsilon, \hat{v}_\epsilon) \in H^1_{per}(\Omega_\epsilon) \times H^1_{per}(\Omega_\epsilon)$ solves the optimality system

$$\begin{cases} -\Delta\hat{u}_\epsilon + \hat{u}_\epsilon = f + \hat{\theta}^\epsilon_\epsilon \chi_{\Omega^+_\epsilon} \quad \text{in } \Omega_\epsilon, \\ -\Delta\hat{v}_\epsilon + \hat{v}_\epsilon = -\Delta(\hat{u}_\epsilon - u_d) \ \text{ in } \Omega_\epsilon, \\ \dfrac{\partial\hat{u}_\epsilon}{\partial\nu} = 0, \ \dfrac{\partial\hat{v}_\epsilon}{\partial\nu} = (\nabla\hat{u}_\epsilon - \nabla u_d)\cdot\nu \ \text{ on } \gamma_\epsilon, \\ \hat{u}_\epsilon = h, \ \hat{v}_\epsilon = 0 \ \text{ on } \Gamma_b, \ \hat{u}_\epsilon, \ \hat{v}_\epsilon \ \text{ are } \Gamma_s - \text{periodic}, \\ \hat{\theta}_\epsilon(y_1,y_2) = -\dfrac{1}{\beta}\left[\dfrac{1}{L}\displaystyle\int_0^L T^\epsilon\hat{v}_\epsilon(x_1,y_2,y_1)dx_1\right]. \end{cases} \tag{4.9}$$

Then, the pair $(\hat{u}_\epsilon, \hat{\theta}_\epsilon)$ is the optimal solution to $(P_{2,\epsilon})$.

5. L^2-cost functional.

Having described the optimality systems in both cases, we now proceed to study the homogenization of optimality systems. After proving the convergence of the optimality system, we see that the optimal solution converges to the optimal solution of the limit system. We start with the study of the homogenization corresponding to the L^2-cost functional.

5.1. Homogenized system.

Consider the spaces

$$V(\Omega) = \left\{\psi \in L^2(\Omega) : \ \frac{\partial\psi|_{\Omega^-}}{\partial x_1} \in L^2(\Omega^-), \ \frac{\partial\psi}{\partial x_2} \in L^2(\Omega)\right\}$$

and

$$V_0(\Omega) = \left\{\psi \in L^2(\Omega) : \ \frac{\partial\psi|_{\Omega^-}}{\partial x_1} \in L^2(\Omega^-), \ \frac{\partial\psi}{\partial x_2} \in L^2(\Omega) \ and \ \psi|_{\Gamma_b} = 0\right\}.$$

These spaces $V(\Omega)$ and $V_0(\Omega)$ are Hilbert spaces with respect to the norm defined by

$$\|\psi\|^2_{V(\Omega)} = \|\psi\|^2_{L^2(\Omega)} + \left\|\frac{\partial\psi}{\partial x_2}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial\psi|_{\Omega^-}}{\partial x_1}\right\|^2_{L^2(\Omega^-)}.$$

For given $f \in L^2(\Omega)$ and $\theta \in L^2(M, M')$, consider the system

$$
\left\{
\begin{aligned}
& -\frac{\partial^2 u^+}{\partial x_2^2} + u^+ = f + \theta \chi_{\Omega^+} \quad \text{in } \Omega^+, \\
& -\Delta u^- + u^- = f \quad \text{in } \Omega^-, \\
& \frac{\partial u^+}{\partial \nu} = 0 \quad \text{on } \Gamma_u, \\
& u^+ = u^-, \quad \frac{b-a}{L}\frac{\partial u^+}{\partial x_2} = \frac{\partial u^-}{\partial x_2} \quad \text{on } \Gamma, \\
& u^- = h \quad \text{on } \Gamma_b, \quad u \text{ is } \Gamma_{s'} - \text{periodic.}
\end{aligned}
\right. \tag{5.1}
$$

Write $u = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-}$. Consider the variational formulation of the problem (5.1) as follows: Let $f \in L^2(\Omega)$. Find $u \in V(\Omega)$, which satisfies $u|_{\Gamma_b} = h$ such that

$$
\left\{
\begin{aligned}
& \frac{b-a}{L}\int_{\Omega^+}\left(\frac{\partial u^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + u^+\psi\right) + \int_{\Omega^-}\left(\nabla u^- \cdot \nabla \psi + u^-\psi\right) - \int_{\Gamma_b}\frac{\partial u^-}{\partial x_2}h \\
& \qquad\qquad = \frac{b-a}{L}\int_{\Omega^+}(f+\theta)\,\psi + \int_{\Omega^-} f\psi
\end{aligned}
\right. \tag{5.2}
$$

for all $\psi \in V(\Omega)$ with $\psi|_{\Gamma_b} = h$. The existence and uniqueness of $u \in V(\Omega)$ that satisfies $u|_{\Gamma_b} = h$ follows in a standard way. The linearity of the solution operator of (5.1) is obvious. If we take $\psi = u$ as a test function in (5.2), we get the continuity of the solution operator. More precisely,

$$
\|u\|_{V(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\theta\|_{L^2(M,M')} + \|h\|_{H^{1/2}(\Gamma_b)}), \tag{5.3}
$$

where $C > 0$ is independent of ϵ. Now consider the L^2-cost functional J_1 defined by

$$
J_1(u,\theta) = \frac{1}{2}\int_\Omega \left(\frac{b-a}{L}\chi_{\Omega^+} + \chi_{\Omega^-}\right)|u-u_d|^2 + \frac{(b-a)\beta}{2}\int_M^{M'} \theta^2. \tag{5.4}
$$

Associated with this cost functional, we introduce the limit optimal control problem as

$$
\inf\{J_1(u,\theta) \mid \theta \in L^2(M,M'), (u,\theta) \text{ obeys } (5.1)\}, \tag{P_1}
$$

which admits a unique optimal solution denoted by $(\overline{u}, \overline{\theta})$. We now characterize the optimal control $\overline{\theta}$ of the problem (P_1) using the adjoint state $\overline{v}$. Let $\overline{v} \in V_0(\Omega)$ solve the adjoint problem

$$
\left\{
\begin{aligned}
& -\frac{\partial^2 \overline{v}^+}{\partial x_2^2} + \overline{v}^+ = (\overline{u}^+ - u_d) \quad \text{in } \Omega^+, \\
& -\Delta \overline{v}^- + \overline{v}^- = (\overline{u}^- - u_d) \quad \text{in } \Omega^-, \\
& \frac{\partial \overline{v}^+}{\partial x_2} = 0 \quad \text{on } \Gamma_u, \\
& \overline{v}^+ = \overline{v}^-, \quad \frac{b-a}{L}\frac{\partial \overline{v}^+}{\partial x_2} = \frac{\partial \overline{v}^-}{\partial x_2} \text{ on } \Gamma, \\
& \overline{v}^- = 0 \quad \text{on } \Gamma_b, \ \overline{v} \text{ is } \Gamma_{s'} - \text{periodic.}
\end{aligned}
\right. \tag{5.5}
$$

THEOREM 5.1. *Let $f \in L^2(\Omega)$, $h \in H^{1/2}(\Gamma_b)$, and $(\overline{u}, \overline{\theta})$ be the optimal solution of (P_1). Let $\overline{v} \in V_0(\Omega)$ solve* (5.5), *and then the optimal control is given by*

$$
\overline{\theta} = -\frac{1}{\beta}\left[\frac{1}{L}\int_0^L \overline{v}\,dx_1\right].
$$

Conversely, assume that a pair $(\hat{u}, \hat{v}) \in V(\Omega) \times V_0(\Omega)$ *solves the optimality system*

$$
\left\{
\begin{array}{l}
-\dfrac{\partial^2 \hat{u}^+}{\partial x_2^2} + \hat{u}^+ = f + \hat{\theta}, \quad -\dfrac{\partial^2 \hat{v}^+}{\partial x_2^2} + \hat{v}^+ = (\hat{u}^+ - u_d) \quad in \ \ \Omega^+, \\
-\Delta \hat{u}^- + \hat{u}^- = f, \quad -\Delta \hat{v}^- + \hat{v}^- = (\hat{u}^- - u_d) \qquad in \ \Omega^-, \\
\dfrac{\partial \hat{u}^+}{\partial x_2} = 0, \quad \dfrac{\partial \hat{v}^+}{\partial x_2} = 0 \quad on \ \ \Gamma_u, \\
\hat{u}^+ = \hat{u}^-, \quad \dfrac{b-a}{L}\dfrac{\partial \hat{u}^+}{\partial x_2} = \dfrac{\partial \hat{u}^-}{\partial x_2}, \quad \hat{v}^+ = \hat{v}^-, \quad \dfrac{b-a}{L}\dfrac{\partial \hat{v}^+}{\partial x_2} = \dfrac{\partial \hat{v}^-}{\partial x_2} \ \ on \ \Gamma, \\
\hat{u}^- = h, \ \ \hat{v}^- = 0 \ \ on \ \ \Gamma_b, \ \ \hat{u}, \hat{v} \ \ are \ \ \Gamma_{s'} - periodic, \\
\hat{\theta} = -\dfrac{1}{\beta}\left[\dfrac{1}{L}\displaystyle\int_0^L \hat{v} dx_1\right].
\end{array}
\right.
\tag{5.6}
$$

Then, the pair $(\hat{u}, \hat{\theta})$ *is the optimal solution to* (P_1).

5.2. Convergence analysis. Assume that $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ is the optimal solution of $(P_{1,\epsilon})$. Let $u_\epsilon(0)$ be the solution of the problem (2.1) corresponding to $\theta = 0$, and then from (2.2) we get

$$
\|u_\epsilon(0)\|_{H^1(\Omega_\epsilon)} \le C, \tag{5.7}
$$

where $C > 0$ is independent of ϵ. Using the optimality of the solution $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$, we get

$$
\int_{\Omega_\epsilon} |\overline{u}_\epsilon - u_d|^2 + \frac{\beta}{2}\int_{\Omega_\epsilon^+} |\overline{\theta}_\epsilon^\epsilon|^2 \le \int_{\Omega_\epsilon} |u_\epsilon(0) - u_d|^2 \le C. \tag{5.8}
$$

Thus, we have

$$
\|\overline{\theta}_\epsilon^\epsilon\|_{L^2(\Omega_\epsilon^+)} = \|\overline{\theta}_\epsilon\|_{L^2(\Lambda^+)} \le C \text{ and } \|\overline{u}_\epsilon\|_{L^2(\Omega_\epsilon)} \le C. \tag{5.9}
$$

From the weak formulation of the adjoint problem (4.1), we have

$$
\|\overline{v}_\epsilon\|_{H^1(\Omega_\epsilon)} \le C, \tag{5.10}
$$

where C is independent of ϵ. Since $h \in H^{1/2}_{per}(\Gamma_b)$, by the standard trace theorem, there exist z in $H^1_{per}(\Omega^-)$ such that $z|_\Gamma = 0$ and $z|_{\Gamma_b} = h$. Let $K = \{\phi \in H^1_{per}(\Omega_\epsilon) : \phi|_{\Gamma_b} = 0\}$. Set $\overline{u}_\epsilon = \widetilde{z} + \overline{y}_\epsilon$, and then $\overline{y}_\epsilon \in K$ solves the problem

$$
\left\{
\begin{array}{l}
-\Delta \overline{y}_\epsilon + \overline{y}_\epsilon = f + \bar{\theta}_\epsilon^{\ \epsilon} \chi_{\Omega_\epsilon^+} + \Delta \widetilde{z} - \widetilde{z} \quad \text{in } \Omega_\epsilon, \\
\dfrac{\partial \overline{y}_\epsilon}{\partial \nu} = 0 \quad \text{on } \gamma_\epsilon, \ \ \overline{y}_\epsilon = 0 \quad \text{on } \Gamma_b, \\
\overline{y}_\epsilon \quad \text{is } \Gamma_s - \text{periodic}.
\end{array}
\right.
\tag{5.11}
$$

The variational formulation of (5.11) is as follows: Find $\overline{y}_\epsilon \in K$ such that, for all $\phi \in K$,

$$
\int_{\Omega_\epsilon} \nabla \overline{y}_\epsilon \cdot \nabla \phi + \int_{\Omega_\epsilon} \overline{y}_\epsilon \phi = \int_{\Omega_\epsilon} f\phi - \int_{\Omega_\epsilon} \nabla \widetilde{z} \cdot \nabla \phi + \int_{\Omega_\epsilon^+} \bar{\theta}_\epsilon^{\ \epsilon} \phi - \int_{\Omega_\epsilon} \widetilde{z}\phi. \tag{5.12}
$$

THEOREM 5.2 (main theorem). *Let* $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ *and* $(\overline{u}, \overline{\theta})$ *be the optimal solution of* $(P_{1,\epsilon})$ *and of* (P_1)*, respectively. Then*

$$\begin{aligned}
&\overline{\theta}_\epsilon \rightharpoonup \overline{\theta} \text{ weakly in } L^2(\Lambda^+),\\
&\widetilde{\overline{u}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup \frac{b-a}{L}\,\overline{u}|_{\Omega^+} \text{ weakly in } L^2(0,L;H^1(M,M')),\\
&\overline{u}_\epsilon|_{\Omega^-} \rightharpoonup \overline{u}|_{\Omega^-} \text{ weakly in } H^1(\Omega^-),\\
&\widetilde{\overline{v}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup \frac{b-a}{L}\,\overline{v}|_{\Omega^+} \text{ weakly in } L^2(0,L;H^1(M,M')),\\
&\overline{v}_\epsilon|_{\Omega^-} \rightharpoonup \overline{v}|_{\Omega^-} \text{ weakly in } H^1(\Omega^-),
\end{aligned}$$

where $\overline{\theta} = -\frac{1}{\beta}[\frac{1}{L}\int_0^L \overline{v}dx_1]$ *and* $\overline{v}_\epsilon$, $\overline{v}$ *are the solutions of* (4.1) *and* (5.5)*, respectively.*

Proof. We know this from the continuity of the solution operator

$$\|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C(\|f\|_{L^2(\Omega)} + \|\overline{\theta}_\epsilon\|_{L^2(\Lambda^+)} + \|h\|_{H^{1/2}(\Gamma_b)}).$$

Using the estimate (5.9), we have

$$\|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C, \tag{5.13}$$

where C is constant independent of ϵ. Since the sequence $(\overline{\theta}_\epsilon)$ is bounded in $L^2(\Lambda^+)$, by weak compactness, there exists a subsequence (still denote by ϵ) and θ_0 such that

$$\overline{\theta}_\epsilon \rightharpoonup \theta_0 \text{ weakly in } L^2(\Lambda^+). \tag{5.14}$$

Let $\overline{u}_\epsilon{}^+$ and $\overline{u}_\epsilon{}^-$, respectively, be the restriction of $\overline{u}_\epsilon$ to Ω_ϵ^+ and Ω^-.

Step 1 (claim). We prove that the sequence $T^\epsilon \overline{u}_\epsilon{}^+$ is bounded in the space $L^2(0,L;H^1((M,M')\times(a,b)))$ and satisfies the following: there exists $u_0^+ \in L^2(0,L;H^1((M,M')\times(a,b)))$ such that

$$T^\epsilon \overline{u}_\epsilon{}^+ \rightharpoonup u_0^+ \text{ weakly in } L^2(\Omega^+ \times (a,b)), \tag{5.15}$$

$$\widetilde{\overline{u}_\epsilon{}^+} \rightharpoonup \frac{b-a}{L} u_0^+ \text{ weakly in } L^2(0,L;H^1(M,M')). \tag{5.16}$$

Moreover, u_0^+ is independent of the third variable y_1. All convergence takes place along a subsequence, but at the end, by uniqueness, we get the convergence of the entire sequence.

Proof of the claim. We have

$$\begin{aligned}
&\left\|T^\epsilon \overline{u}_\epsilon{}^+\right\|^2_{L^2(0,L;H^1((M,M')\times(a,b)))}\\
&= \int_0^L \left\|T^\epsilon \overline{u}_\epsilon{}^+(x_1)\right\|^2_{H^1((M,M')\times(a,b))}\, dx_1\\
&= \int_{\Omega^+\times(a,b)} \left(\epsilon^2 T^\epsilon \left|\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}\right|^2 + T^\epsilon \left|\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_2}\right|^2 + T^\epsilon \left|\overline{u}_\epsilon{}^+\right|^2\right) dx\\
&= \int_{\Omega^+\times(a,b)} T^\epsilon \left(\epsilon^2 \left|\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}\right|^2 + \left|\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_2}\right|^2 + \left|\overline{u}_\epsilon{}^+\right|^2\right) dx\\
&= L\int_{\Omega_\epsilon^+} \left(\epsilon^2 \left|\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}\right|^2 + \left|\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_2}\right|^2 + \left|\overline{u}_\epsilon{}^+\right|^2\right) dx\\
&\leq L \left\|\overline{u}_\epsilon\right\|^2_{H^1(\Omega_\epsilon)}.
\end{aligned} \tag{5.17}$$

The boundedness of the sequence $T^\epsilon \overline{u}_\epsilon{}^+$ in $L^2(0, L; H^1((M, M') \times (a, b)))$ follows from (5.13) and (5.17). Hence, (5.15), which in turn implies

$$T^\epsilon \overline{u}_\epsilon{}^+ \rightharpoonup u_0^+ \text{ weakly in } L^2(\Omega^+ \times (a, b)), \tag{5.18}$$

$$\frac{\partial}{\partial x_2}\left(T^\epsilon \overline{u}_\epsilon{}^+\right) \rightharpoonup \frac{\partial u_0^+}{\partial x_2} \text{ weakly in } L^2(\Omega^+ \times (a, b)), \tag{5.19}$$

$$\frac{\partial}{\partial y_1}\left(T^\epsilon \overline{u}_\epsilon{}^+\right) \rightharpoonup \frac{\partial u_0^+}{\partial y_1} \text{ weakly in } L^2(\Omega^+ \times (a, b)). \tag{5.20}$$

From Proposition 3.2(v), it follows that

$$T^\epsilon\left(\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_2}\right) \rightharpoonup \frac{\partial u_0^+}{\partial x_2} \text{ weakly in } L^2(\Omega^+ \times (a, b)), \tag{5.21}$$

$$\epsilon T^\epsilon\left(\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}\right) \rightharpoonup \frac{\partial u_0^+}{\partial y_1} \text{ weakly in } L^2(\Omega^+ \times (a, b)). \tag{5.22}$$

Again from Proposition 3.2(iv), we have

$$\left\| T^\epsilon \frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1} \right\|_{L^2(\Omega^+ \times (a,b))} = \sqrt{L} \left\| \frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1} \right\|_{L^2(\Omega_\epsilon^+)} \le \sqrt{L} \|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)},$$

which implies the boundedness of the sequence $T^\epsilon(\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1})$ in the space $L^2(\Omega^+ \times (a, b))$ from (5.13). Thus, from (5.22), it follows that $\frac{\partial u_0^+}{\partial y_1} = 0$, and hence u_0^+ is independent of y_1. Further,

$$\widetilde{\overline{u}_\epsilon{}^+} \rightharpoonup \frac{1}{L}\int_a^b u_0^+ dy_1 = \frac{b-a}{L} u_0^+ \text{ weakly in } L^2(0, L; H^1(M, M')). \tag{5.23}$$

This completes the proof of the claim.

Since $\overline{u}_\epsilon{}^-$ is bounded in $H^1(\Omega^-)$ by (5.13), up to a subsequence (still denoted by ϵ), we get

$$\overline{u}_\epsilon{}^- \rightharpoonup u_0^- \text{ weakly in } H^1(\Omega^-). \tag{5.24}$$

Define u_0 as

$$u_0(x) = \begin{cases} u_0^+ & \text{if } x \in \Omega^+, \\ u_0^- & \text{if } x \in \Omega^-. \end{cases} \tag{5.25}$$

Step 2. We claim that $u_0 \in V(\Omega)$ and trace of u_0^- on Γ_b is h.

The continuity of the trace map and the convergence of $\overline{u}_\epsilon{}^-$ to u_0^- weakly in $H^1(\Omega^-)$ implies that the restriction of u_0^- on Γ_b is h. Since $u_0 \in L^2(\Omega)$ and $\frac{\partial u_0}{\partial x_1} \in L^2(\Omega^-)$, it is enough to prove that $\frac{\partial u_0}{\partial x_2} \in L^2(\Omega)$ for the first part of the claim. This will be achieved if we prove that the trace of u_0^+ and u_0^- are equal on Γ since u_0 is independent of y_1, $\frac{\partial u_0}{\partial x_2} \in L^2(\Omega^+)$, and $\frac{\partial u_0}{\partial x_2} \in L^2(\Omega^-)$. Since $\overline{u}_\epsilon{}^+|_{\Gamma_\epsilon} = \overline{u}_\epsilon{}^-|_{\Gamma_\epsilon}$ implies the equality of traces for the boundary unfolding operator, that is, $T^\epsilon_{x_2=M}\left(\overline{u}_\epsilon{}^+|_{\Gamma_\epsilon}\right) = T^\epsilon_{x_2=M}\left(\overline{u}_\epsilon{}^-|_{\Gamma_\epsilon}\right)$, i.e.,

$$\left(T^\epsilon(\overline{u}_\epsilon{}^+)\right)|_{x_2=M} = T^\epsilon_{x_2=M}\left(\overline{u}_\epsilon{}^-|_{\Gamma_\epsilon}\right). \tag{5.26}$$

From the continuity of the trace operator, we get

$$\left(T^\epsilon(\overline{u}_\epsilon{}^+)\right)|_{x_2=M} \rightharpoonup u_0^+|_{x_2=M} \text{ weakly in } L^2((0,L)\times(a,b)),$$

and from (5.24), we get

$$\overline{u}_\epsilon{}^-|_{x_2=M} \to u_0^-|_{x_2=M} \text{ strongly in } L^2(0,L).$$

This implies

$$T^\epsilon_{x_2=M}\left(\overline{u}_\epsilon{}^-|_{x_2=M}\right) \to u_0^-|_{x_2=M} \text{ in } L^2((0,L)\times(a,b)).$$

Passing to the limit in (5.26) as $\epsilon \to 0$, we get

$$u_0^+|_{x_2=M} = u_0^-|_{x_2=M} \text{ in } L^2(0,L),$$

since u_0^+ and u_0^- are independent on y_1 variable. This proves Step 2.

Now $T^\epsilon \frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}$ is bounded in $L^2(\Omega^+\times(a,b))$, and hence there is an element $P \in L^2(\Omega^+\times(a,b))$ such that

$$T^\epsilon \frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1} \rightharpoonup P \text{ weakly in } L^2(\Omega^+\times(a,b)). \tag{5.27}$$

Step 3 (claim). The limit $P = 0$. To prove the claim, recall $\overline{u}_\epsilon = \tilde{z} + \overline{y}_\epsilon$ from (5.11). We observe that $\overline{u}_\epsilon{}^+$ is equal to $\overline{y}_\epsilon|_{\Omega_\epsilon^+}$, say, $\overline{y}_\epsilon{}^+$. So $\overline{y}_\epsilon{}^+$ have the same limit as $\overline{u}_\epsilon{}^+$, i.e.,

$$T^\epsilon \frac{\partial \overline{y}_\epsilon{}^+}{\partial x_2} \rightharpoonup \frac{\partial u_0^+}{\partial x_2} \text{ weakly in } L^2(\Omega^+\times(a,b)), \tag{5.28}$$

$$T^\epsilon \frac{\partial \overline{y}_\epsilon{}^+}{\partial x_1} \rightharpoonup P \text{ weakly in } L^2(\Omega^+\times(a,b)). \tag{5.29}$$

Let $\phi \in \mathcal{D}(\Omega^+)$ and $\eta \in C^\infty[0,L)$ be arbitrary and let $\psi = \eta'$. Now choose the test function

$$\phi^\epsilon(x) = \epsilon\phi(x)\psi\left(\left\{\frac{x_1}{\epsilon}\right\}_L\right).$$

Note that ϕ^ϵ is continuous in each strip of Ω_ϵ^+ which are disjoint and hence continuous on Ω_ϵ^+. From the definition of ϵ-unfolding of ϕ^ϵ and by Proposition 3.2, we get

$$\begin{aligned}
T^\epsilon\phi^\epsilon &= \epsilon\phi\left(\epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y_1, x_2\right)\psi(y_1),\\
T^\epsilon\left(\frac{\partial\phi^\epsilon}{\partial x_1}\right) &= \frac{1}{\epsilon}\frac{\partial}{\partial y_1}(T^\epsilon\phi^\epsilon),\\
&= \epsilon\frac{\partial\phi}{\partial x_1}\left(\epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y_1, x_2\right)\psi(y_1) + \phi\left(\epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y_1, x_2\right)\psi'(y_1),\\
T^\epsilon\left(\frac{\partial\phi^\epsilon}{\partial x_2}\right) &= \epsilon\frac{\partial\phi}{\partial x_2}\left(\epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y_1, x_2\right)\psi(y_1).
\end{aligned}$$

On convergence, we get

$$T^\epsilon \phi^\epsilon \to 0 \text{ in } L^2(\Omega^+ \times (a,b)), \tag{5.30}$$

$$T^\epsilon \frac{\partial \phi^\epsilon}{\partial x_1} \to \phi(x_1,x_2)\psi'(y_1) \text{ in } L^2(\Omega^+ \times (a,b)), \tag{5.31}$$

$$T^\epsilon \frac{\partial \phi^\epsilon}{\partial x_2} \to 0 \text{ in } L^2(\Omega^+ \times (a,b)), \tag{5.32}$$

as $\epsilon \to 0$. From the variational formulation (5.12), we get

$$\begin{aligned} \lim_{\epsilon\to 0}\left[\int_{\Omega_\epsilon} \nabla \overline{y}_\epsilon \cdot \nabla \widetilde{\phi^\epsilon} + \int_{\Omega_\epsilon} \overline{y}_\epsilon \widetilde{\phi^\epsilon}\right] &= \lim_{\epsilon\to 0}\left[\int_{\Omega_\epsilon} f\widetilde{\phi^\epsilon} - \int_{\Omega_\epsilon} \nabla \tilde{z}\cdot\nabla\widetilde{\phi^\epsilon}\right] \\ &\quad + \lim_{\epsilon\to 0}\left[\int_{\Omega_\epsilon^+} \overline{\theta}^\epsilon_\epsilon \widetilde{\phi^\epsilon} - \int_{\Omega_\epsilon} \widetilde{z}\widetilde{\phi^\epsilon}\right]. \end{aligned} \tag{5.33}$$

Now notice

$$\begin{aligned} \int_{\Omega_\epsilon} \nabla \overline{y}_\epsilon \cdot \nabla \widetilde{\phi^\epsilon} + \int_{\Omega_\epsilon} \overline{y}_\epsilon \widetilde{\phi^\epsilon} &= \int_{\Omega_\epsilon^+} \nabla {\overline{y}_\epsilon}^+ \cdot \nabla \phi^\epsilon + \int_{\Omega_\epsilon^+} {\overline{y}_\epsilon}^+ \phi^\epsilon \\ &= \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\epsilon \frac{\partial {\overline{y}_\epsilon}^+}{\partial x_1} T^\epsilon \frac{\partial \phi^\epsilon}{\partial x_1} + T^\epsilon \frac{\partial {\overline{y}_\epsilon}^+}{\partial x_2} T^\epsilon \frac{\partial \phi^\epsilon}{\partial x_2} \\ &\quad + \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\epsilon {\overline{y}_\epsilon}^+ T^\epsilon \phi^\epsilon. \end{aligned} \tag{5.34}$$

Hence,

$$\lim_{\epsilon\to 0}\left[\int_{\Omega_\epsilon} \nabla \overline{y}_\epsilon \cdot \nabla \widetilde{\phi^\epsilon} + \int_{\Omega_\epsilon} \overline{y}_\epsilon \widetilde{\phi^\epsilon}\right] = \frac{1}{L}\int_{\Omega^+\times(a,b)} P\,\phi(x_1,x_2)\psi'(y_1) \tag{5.35}$$

and

$$\begin{aligned} &\int_{\Omega_\epsilon} f\widetilde{\phi^\epsilon} - \int_{\Omega_\epsilon} \nabla\widetilde{z}\cdot\nabla\widetilde{\phi^\epsilon} + \int_{\Omega_\epsilon^+} {\overline{\theta}_\epsilon}^\epsilon \widetilde{\phi^\epsilon} - \int_{\Omega_\epsilon} \widetilde{z}\widetilde{\phi^\epsilon} \\ &\quad = \int_{\Omega_\epsilon^+} f\phi^\epsilon + \int_{\Omega_\epsilon^+} {\overline{\theta}_\epsilon}^\epsilon \phi^\epsilon \\ &\quad = \frac{1}{L}\int_{\Omega^+\times(a,b)} \left(T^\epsilon f\, T^\epsilon\phi^\epsilon + T^\epsilon {\overline{\theta}_\epsilon}^\epsilon T^\epsilon \phi^\epsilon\right) \\ &\quad \to 0, \end{aligned} \tag{5.36}$$

as $\epsilon \to 0$. Combining (5.35), (5.36), from (5.33) we get

$$\int_{\Omega^+\times(a,b)} P\,\phi(x_1,x_2)\eta(y_1) = 0. \tag{5.37}$$

Since ϕ and η are arbitrary, we get $P = 0$ a.e. $(x_1,x_2)\in\Omega^+$, $y_1 \in (a,b)$ and hence the claim in Step 3.

Step 4. Now take a test function $\psi \in C^\infty(\overline{\Omega})$ such that $\psi|_{\Gamma_b} = h$ in the variational formulation of (2.1) for $\theta = \overline{\theta}_\epsilon$. Now as $\epsilon \to 0$, the left-hand side of (2.1) becomes

$$
\begin{aligned}
\int_{\Omega_\epsilon} \nabla \overline{u}_\epsilon \cdot \nabla \psi + \overline{u}_\epsilon \psi - \int_{\Gamma_b} \frac{\partial \overline{u}_\epsilon}{\partial \nu} h = {} & \frac{1}{L} \int_{\Omega^+ \times (a,b)} T^\epsilon \left(\frac{\partial {\overline{u}_\epsilon}^+}{\partial x_1} \right) T^\epsilon \left(\frac{\partial \psi}{\partial x_1} \right) \\
& + \frac{1}{L} \int_{\Omega^+ \times (a,b)} T^\epsilon \left(\frac{\partial {\overline{u}_\epsilon}^+}{\partial x_2} \right) T^\epsilon \left(\frac{\partial \psi}{\partial x_2} \right) \\
& + \frac{1}{L} \int_{\Omega^+ \times (a,b)} T^\epsilon {\overline{u}_\epsilon}^+ T^\epsilon \psi + \int_{\Omega^-} \nabla {\overline{u}_\epsilon}^- \cdot \nabla \psi \\
& + \int_{\Omega^-} {\overline{u}_\epsilon}^- \psi - \int_{\Gamma_b} \frac{\partial {\overline{u}_\epsilon}^-}{\partial \nu} h \\
\to {} & \frac{1}{L} \int_{\Omega^+ \times (a,b)} \left(\frac{\partial u_0^+}{\partial x_2} \frac{\partial \psi}{\partial x_2} + u_0^+ \psi \right) \\
& + \int_{\Omega^-} \nabla {u_0}^- \cdot \nabla \psi + u_0^- \psi - \int_{\Gamma_b} \frac{\partial {u_0}^-}{\partial \nu} h
\end{aligned} \tag{5.38}
$$

and the right-hand side of (2.1) becomes

$$
\begin{aligned}
\int_{\Omega_\epsilon} f \psi + \int_{\Omega_\epsilon^+} \overline{\theta}_\epsilon^\epsilon \psi &= \int_{\Omega_\epsilon^+} f \psi + \int_{\Omega^-} f \psi + \int_{\Omega_\epsilon^+} \overline{\theta}_\epsilon^\epsilon \psi \\
&= \frac{1}{L} \int_{\Omega^+ \times (a,b)} T^\epsilon f \, T^\epsilon \psi + \int_{\Omega^-} f \psi + \frac{1}{L} \int_{\Omega^+ \times (a,b)} T^\epsilon \overline{\theta}_\epsilon^\epsilon T^\epsilon \psi \\
&\to \frac{1}{L} \int_{\Omega^+ \times (a,b)} f \psi + \int_{\Omega^-} f \psi + \frac{1}{L} \int_{\Omega^+ \times (a,b)} \theta_0 \psi.
\end{aligned} \tag{5.39}
$$

Hence,

$$
\left\{
\begin{aligned}
& \frac{1}{L} \int_{\Omega^+ \times (a,b)} \left(\frac{\partial u_0^+}{\partial x_2} \frac{\partial \psi}{\partial x_2} + u_0^+ \psi \right) + \int_{\Omega^-} \left(\nabla {u_0}^- \cdot \nabla \psi + u_0^- \psi \right) - \int_{\Gamma_b} \frac{\partial {u_0}^-}{\partial \nu} h \\
& \qquad = \frac{1}{L} \int_{\Omega^+ \times (a,b)} (f + \theta_0) \psi + \int_{\Omega^-} f \psi,
\end{aligned}
\right.
$$

which implies

$$
\left\{
\begin{aligned}
& \frac{b-a}{L} \int_{\Omega^+} \left(\frac{\partial u_0^+}{\partial x_2} \frac{\partial \psi}{\partial x_2} + u_0^+ \psi \right) + \int_{\Omega^-} \left(\nabla u_0^- \cdot \nabla \psi + u_0^- \psi \right) - \int_{\Gamma_b} \frac{\partial {u_0}^-}{\partial \nu} h \\
& \qquad = \frac{b-a}{L} \int_{\Omega^+} (f + \theta_0) \psi + \int_{\Omega^-} f \psi
\end{aligned}
\right.
$$

for all $\psi \in C^\infty(\overline{\Omega})$ with $\psi|_{\Gamma_b} = h$ and hence true for all ψ in $V(\Omega)$ by density. Therefore, u_0 satisfies the differential equation (5.1) for $\theta = \theta_0$ or, equivalently, (5.2).

Similarly, we find the following convergence for the adjoint state $\overline{v}_\epsilon$ described in (4.1):

$$\begin{aligned}
T^\epsilon\left(\overline{v}_\epsilon|_{\Omega_\epsilon^+}\right) &\rightharpoonup v_0|_{\Omega^+} \text{ weakly in } L^2(\Omega^+\times(a,b)),\\
T^\epsilon\left(\frac{\partial \overline{v}_\epsilon|_{\Omega_\epsilon^+}}{\partial x_1}\right) &\rightharpoonup 0 \text{ weakly in } L^2(\Omega^+\times(a,b)),\\
T^\epsilon\left(\frac{\partial \overline{v}_\epsilon|_{\Omega_\epsilon^+}}{\partial x_2}\right) &\rightharpoonup \frac{\partial v_0|_{\Omega^+}}{\partial x_2} \text{weakly in } L^2(\Omega^+\times(a,b)),\\
\widetilde{\overline{v}_\epsilon|_{\Omega_\epsilon^+}} &\rightharpoonup \frac{b-a}{L}\, v_0|_{\Omega^+} \text{ weakly in } L^2(0,L;H^1(M,M')),\\
\overline{v}_\epsilon|_{\Omega^-} &\rightharpoonup v_0|_{\Omega^-} \text{ weakly in } H^1(\Omega^-),
\end{aligned}$$

where $v_0 \in V_0(\Omega)$ satisfies (5.5) for $\overline{u} = u_0$. From the optimality condition, $\overline{\theta}_\epsilon(y_1, y_2) = -\frac{1}{\beta}[\frac{1}{L}\int_0^L T^\epsilon \overline{v}_\epsilon(x_1, y_2, y_1)dx_1]$, and the convergence $\overline{\theta}_\epsilon \rightharpoonup \theta_0$ in $L^2(\Lambda^+)$, we get

$$\theta_0 = -\frac{1}{\beta}\left[\frac{1}{L}\int_0^L \overline{v}_0 dx_1\right]. \tag{5.40}$$

Therefore, we get the optimality system corresponding to the minimization problem (P_1). According to Theorem 5.1, the optimal solution is given by (u_0, θ_0). Thus, by uniqueness, we have

$$\overline{u} = u_0, \ \ \overline{v} = v_0, \ \ \text{and } \overline{\theta} = \theta_0.$$

This completes the proof. ∎

6. Dirichlet cost functional. In this section, we derive analogous results with the Dirichlet cost functional $J_{2,\epsilon}$. All the details are not given, as many of the arguments are similar to the previous section.

6.1. Homogenized system. The limit cost functional J_2 is described as

$$J_2(u,\theta) = \frac{1}{2}\int_{\Omega^+} \frac{b-a}{L}\left|\frac{\partial u|_{\Omega^+}}{\partial x_2} - \frac{\partial u_d}{\partial x_2}\right|^2 + \frac{1}{2}\int_{\Omega^-} |\nabla u|_{\Omega^-} - \nabla u_d|^2 + \frac{\beta(b-a)}{2}\int_M^{M'} \theta^2.$$

The limit optimal control problem is given by

$$\inf\{J_2(u,\theta)|\ \theta \in\ L^2(M,M'),\ (u,\theta) \text{ obeys } (5.1)\}. \tag{P_2}$$

It has a unique solution, say, $(\overline{u}, \overline{\theta})$. The adjoint state $\overline{v}$ solves the problem

$$\begin{cases}
-\dfrac{\partial^2 \overline{v}^+}{\partial x_2^2} + \overline{v}^+ = -\dfrac{\partial^2}{\partial x_2^2}(\overline{u}^+ - u_d) \quad \text{in } \Omega^+,\\
-\Delta \overline{v}^- + \overline{v}^- = -\Delta(\overline{u}^- - u_d) \quad \text{in } \Omega^-,\\
\dfrac{\partial \overline{v}^+}{\partial x_2} = (\nabla \overline{u}^+ - \nabla u_d)\cdot \nu \quad \text{on } \Gamma_u,\\
\overline{v}^+ = \overline{v}^-, \quad \dfrac{b-a}{L}\dfrac{\partial \overline{v}^+}{\partial x_2} = \dfrac{\partial \overline{v}^-}{\partial x_2} \text{ on } \Gamma,\\
\overline{v}^- = 0 \quad \text{on } \Gamma_b, \quad \overline{v} \text{ is } \Gamma_{s'} - \text{periodic}.
\end{cases} \tag{6.1}$$

THEOREM 6.1. *Let $f \in L^2(\Omega)$, $h \in H^{1/2}(\Gamma_b)$, and $(\overline{u}, \overline{\theta})$ be the optimal solution of (P_2). Let $\overline{v} \in V_0(\Omega)$ solve (6.1), and then the optimal control is given by*

$$\overline{\theta} = -\frac{1}{\beta}\left[\frac{1}{L}\int_0^L \overline{v}dx_1\right].$$

Conversely, assume that a pair $(\hat{u}, \hat{v}) \in V(\Omega) \times V_0(\Omega)$ solves the optimality system

$$\begin{cases} -\dfrac{\partial^2 \hat{u}^+}{\partial x_2^2} + \hat{u}^+ = f, \quad -\dfrac{\partial^2 \hat{v}^+}{\partial x_2^2} + \hat{v}^+ = -\dfrac{\partial^2}{\partial x_2^2}(\hat{u}^+ - u_d) \ \text{ in } \Omega^+, \\ -\Delta \hat{u}^- + \hat{u}^- = f, \quad -\Delta \hat{v}^- + \hat{v}^- = -\Delta(\hat{u}^- - u_d) \ \text{ in } \Omega^-, \\ \dfrac{\partial \hat{u}^+}{\partial \nu} = 0, \quad \dfrac{\partial \hat{v}^+}{\partial \nu} = (\nabla \hat{u}^+ - \nabla u_d)\cdot \nu \ \text{ on } \Gamma_u, \\ \hat{u}^+ = \hat{u}^-, \quad \dfrac{b-a}{L}\dfrac{\partial \hat{u}^+}{\partial x_2} = \dfrac{\partial \hat{u}^-}{\partial x_2}, \quad \hat{v}^+ = \hat{v}^-, \quad \dfrac{b-a}{L}\dfrac{\partial \hat{v}^+}{\partial x_2} = \dfrac{\partial \hat{v}^-}{\partial x_2} \ \text{ on } \Gamma, \\ \hat{u}^- = h, \quad \hat{v}^- = 0 \ \text{ on } \Gamma_b, \quad \hat{u}, \hat{v} \ \text{ are } \Gamma_{s'} - \text{periodic}, \\ \hat{\theta} = -\dfrac{1}{\beta}\left[\dfrac{1}{L}\displaystyle\int_0^L \hat{v}dx_1\right]. \end{cases} \tag{6.2}$$

Then, the pair $(\hat{u}, \hat{\theta})$ is the optimal solution to (P_2).

6.2. Convergence analysis. Assume that $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ is the optimal solution of $(P_{2,\epsilon})$. Let $u_\epsilon(0)$ be the solution of the problem (2.1) corresponding to $\theta = 0$, and then from (2.2) we get

$$\|u_\epsilon(0)\|_{H^1(\Omega_\epsilon)} \leq C, \tag{6.3}$$

where $C > 0$ is independent of ϵ. Using the optimality of the solution $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$, we get

$$\int_{\Omega_\epsilon} |\nabla(\overline{u}_\epsilon - u_d)|^2 + \frac{\beta}{2}\int_{\Omega_\epsilon^+} |\overline{\theta}_\epsilon^\epsilon|^2 \leq \int_{\Omega_\epsilon} |\nabla(u_\epsilon(0) - u_d)|^2 \leq C. \tag{6.4}$$

Thus, we have

$$\|\overline{\theta}_\epsilon\|_{L^2(\Lambda^+)} = \|\overline{\theta}_\epsilon^\epsilon\|_{L^2(\Omega_\epsilon^+)} \leq C \ \text{ and } \ \|\nabla \overline{u}_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C. \tag{6.5}$$

The variational formulation of the adjoint problem (4.7) is the following:

Find $\overline{v}_\epsilon \in \{v \in H^1(\Omega_\epsilon) : v|_{\Gamma_b} = 0\}$ such that

$$\int_{\Omega_\epsilon} \nabla \overline{v}_\epsilon \cdot \nabla \phi \ + \int_{\Omega_\epsilon} \overline{v}_\epsilon \phi = \int_{\Omega_\epsilon} \nabla(\overline{u}_\epsilon - u_d)\cdot \nabla \phi \tag{6.6}$$

for all $\phi \in H^1(\Omega_\epsilon)$ that satisfies $\phi|_{\Gamma_b} = 0$.

THEOREM 6.2 (main theorem). *Let $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ and $(\overline{u}, \overline{\theta})$ be the optimal solution of $(P_{2,\epsilon})$ and (P_2), respectively. Then*

$$\begin{aligned} &\overline{\theta}_\epsilon \rightharpoonup \overline{\theta} \ \text{weakly in } L^2(\Lambda^+), \\ &\widetilde{\overline{u}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup \frac{b-a}{L}\overline{u}|_{\Omega^+} \ \text{weakly in } L^2(0, L; H^1(M, M')), \\ &\overline{u}_\epsilon|_{\Omega^-} \rightharpoonup \overline{u}|_{\Omega^-} \ \text{weakly in } H^1(\Omega^-), \\ &\widetilde{\overline{v}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup \frac{b-a}{L}\overline{v}|_{\Omega^+} \ \text{weakly in } L^2(0, L; H^1(M, M')), \\ &\overline{v}_\epsilon|_{\Omega^-} \rightharpoonup \overline{v}|_{\Omega^-} \ \text{weakly in } H^1(\Omega^-), \end{aligned}$$

where $\overline{\theta} = -\frac{1}{\beta}[\frac{1}{L}\int_0^L \overline{v}dx_1]$ and $\overline{v}_\epsilon$, $\overline{v}$ is the solution of (4.7) and (6.1), respectively.

Proof. In a similar fashion as in the previous section, we deduce from (6.5) that

$$\|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C \text{ and } \|\overline{v}_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C \tag{6.7}$$

and the convergence

$$\widetilde{\overline{u}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup \frac{b-a}{L}\overline{u}|_{\Omega^+} \text{ weakly in } L^2(0,L;H^1(M,M')), \tag{6.8}$$

$$\overline{u}_\epsilon|_{\Omega^-} \rightharpoonup \overline{u}|_{\Omega^-} \text{ weakly in } H^1(\Omega^-). \tag{6.9}$$

Further, from (6.5), we see that

$$\overline{\theta}_\epsilon \rightharpoonup \theta_0 \text{ weakly in } L^2(\Lambda^+). \tag{6.10}$$

Let $\overline{v}_\epsilon{}^+$ be the restriction of $\overline{v}_\epsilon$ in Ω_ϵ^+ and $\overline{v}_\epsilon{}^-$ be the restriction of $\overline{v}_\epsilon$ in Ω^-. Now

$$\left\|T^\epsilon \overline{v}_\epsilon{}^+\right\|^2_{L^2(0,L;H^1((M,M')\times(a,b))} \leq L\left\|\overline{v}_\epsilon\right\|^2_{H^1(\Omega_\epsilon)}.$$

So, the sequence $T^\epsilon\overline{v}_\epsilon{}^+$ is bounded in $L^2(0,L;H((M,M')\times(a,b)))$, and hence there exists a subsequence (still denoted by ϵ) such that

$$T^\epsilon\overline{v}_\epsilon{}^+ \rightharpoonup v_0^+ \text{ weakly in } L^2(0,L;H^1((M,M')\times(a,b))), \tag{6.11}$$

which implies

$$T^\epsilon\overline{v}_\epsilon{}^+ \rightharpoonup v_0^+,\ \ T^\epsilon\left(\frac{\partial\overline{v}_\epsilon{}^+}{\partial x_2}\right) \rightharpoonup \frac{\partial v_0^+}{\partial x_2},\ \ \epsilon T^\epsilon\left(\frac{\partial\overline{v}_\epsilon{}^+}{\partial x_1}\right) \rightharpoonup \frac{\partial v_0^+}{\partial y_1} \tag{6.12}$$

weakly in $L^2(\Omega^+\times(a,b))$. From Proposition 3.2(iv) and (6.7), it follows that $T^\epsilon(\frac{\partial\overline{v}_\epsilon{}^+}{\partial x_1})$ is bounded in $L^2(\Omega^+\times(a,b))$. Then from (6.12), we get $\frac{\partial v_0^+}{\partial y_1} = 0$. With the help of Proposition 3.2(ix) and convergence (6.12), we conclude that

$$\widetilde{\overline{v}_\epsilon{}^+} \rightharpoonup \frac{1}{L}\int_a^b v_0^+ \text{ weakly in } L^2(0,L;H^1(M,M')). \tag{6.13}$$

Since v_0^+ is independent of y_1 variable, we have $\int_a^b v_0^+\,dy_1 = (b-a)v_0^+$ and $\int_a^b \frac{\partial v_0^+}{\partial x_2}\,dy_1 = (b-a)\frac{\partial v_0^+}{\partial x_2}$. Thus, (6.13) becomes

$$\widetilde{\overline{v}_\epsilon{}^+} \rightharpoonup \frac{b-a}{L}v_0^+ \text{ weakly in } L^2(0,L;H^1(M,M')). \tag{6.14}$$

Since $T^\epsilon(\frac{\partial\overline{v}_\epsilon{}^+}{\partial x_1})$ is bounded in $L^2(\Omega^+\times(a,b))$, we get

$$T^\epsilon\frac{\partial\overline{v}_\epsilon{}^+}{\partial x_1} \rightharpoonup R \text{ weakly in } L^2(\Omega^+\times(a,b)) \tag{6.15}$$

for some $R\in L^2(\Omega^+\times(a,b))$. We now characterize R. Since the sequence $\overline{v}_\epsilon{}^-$ is bounded in $H^1(\Omega^-)$, we get the convergence

$$\overline{v}_\epsilon{}^- \rightharpoonup v_0^- \text{ weakly in } H^1(\Omega^-) \tag{6.16}$$

for some $v_0^- \in H^1(\Omega^-)$. Define v_0 as

$$v_0 = \begin{cases} v_0^+ & \text{if } x \in \Omega^+, \\ v_0^- & \text{if } x \in \Omega^- \end{cases} \tag{6.17}$$

and $v_0 \in V_0(\Omega)$. The proof is similar to the proof in Step 2 in Theorem 5.2.

We now identify R. Consider the test function ϕ^ϵ described as in Step 3 of Theorem 5.2. Now, taking $\phi = \phi^\epsilon$ in (6.6), we get

$$\begin{aligned}\int_{\Omega_\epsilon} \nabla \overline{v}_\epsilon \cdot \nabla \widetilde{\phi^\epsilon} + \int_{\Omega_\epsilon} \overline{v}_\epsilon \widetilde{\phi^\epsilon} &= \int_{\Omega_\epsilon^+} \nabla {\overline{v}_\epsilon}^+ \cdot \nabla \phi^\epsilon + \int_{\Omega_\epsilon^+} {\overline{v}_\epsilon}^+ \phi^\epsilon \\ &= \frac{1}{L} \int_{\Omega^+ \times (a,b)} \left(T^\epsilon \left(\frac{\partial {\overline{v}_\epsilon}^+}{\partial x_1} \right) T^\epsilon \left(\frac{\partial \phi^\epsilon}{\partial x_1} \right) \right. \\ &\quad \left. + T^\epsilon \left(\frac{\partial {\overline{v}_\epsilon}^+}{\partial x_2} \right) T^\epsilon \left(\frac{\partial \phi^\epsilon}{\partial x_2} \right) + T^\epsilon \overline{v}_\epsilon^+ T^\epsilon \phi^\epsilon \right) \\ &\to \frac{1}{L} \int_{\Omega^+ \times (a,b)} R\, \phi(x_1, x_2) \psi'(y_1) \;\; as\ \epsilon \to 0 \end{aligned} \tag{6.18}$$

and

$$\begin{aligned} &\int_{\Omega_\epsilon} \nabla (\overline{u}_\epsilon - u_d) \cdot \nabla \widetilde{\phi^\epsilon} \\ &= \int_{\Omega_\epsilon^+} \nabla ({\overline{u}_\epsilon}^+ - u_d) \cdot \nabla \phi^\epsilon \\ &= \frac{1}{L} \int_{\Omega^+ \times (a,b)} \left(T^\epsilon \frac{\partial ({\overline{u}_\epsilon}^+ - u_d)}{\partial x_1} T^\epsilon \frac{\partial \phi^\epsilon}{\partial x_1} + T^\epsilon \frac{\partial ({\overline{u}_\epsilon}^+ - u_d)}{\partial x_2} T^\epsilon \frac{\partial \phi^\epsilon}{\partial x_2} \right) \\ &\to -\frac{1}{L} \int_{\Omega^+ \times (a,b)} \frac{\partial u_d}{\partial x_1} \cdot \;\; \phi(x_1, x_2) \psi'(y_1) \;\; as\ \epsilon \to 0. \end{aligned} \tag{6.19}$$

Combining (6.18) and (6.19), we arrive at

$$\int_{\Omega^+ \times (a,b)} \left(R + \frac{\partial u_d}{\partial x_1} \right) \phi(x_1, x_2) \psi'(y_1) = \int_{\Omega^+ \times (a,b)} \left(R + \frac{\partial u_d}{\partial x_1} \right) \phi(x_1, x_2) \eta(y_1) = 0.$$

Since ϕ and η are arbitrary, we get

$$R = -\frac{\partial u_d}{\partial x_1}. \tag{6.20}$$

To end the proof of the main theorem, we need to find the equation satisfied by the adjoint limit v_0. Taking $\psi \in \{\phi \in C^\infty(\overline{\Omega}) |\ \phi|_{\Gamma_b} = 0\}$ in the left- and right-hand side of (6.6), we get

$$\begin{aligned}
&\int_{\Omega_\epsilon} \nabla\overline{v}_\epsilon \cdot \nabla\psi + \overline{v}_\epsilon\psi \\
&\quad = \int_{\Omega_\epsilon^+} \left(\nabla\overline{v}_\epsilon{}^+ \cdot \nabla\psi + \overline{v}_\epsilon{}^+\psi\right) + \int_{\Omega^-} \left(\nabla\overline{v}_\epsilon{}^- \cdot \nabla\psi + \overline{v}_\epsilon{}^-\psi\right) \\
&\quad = \frac{1}{L}\int_{\Omega^+\times(a,b)} \left(T^\epsilon\left(\frac{\partial\overline{v}_\epsilon{}^+}{\partial x_1}\right) T^\epsilon\left(\frac{\partial\psi}{\partial x_1}\right) + T^\epsilon\left(\frac{\partial\overline{v}_\epsilon{}^+}{\partial x_2}\right) T^\epsilon\left(\frac{\partial\psi}{\partial x_2}\right)\right) \\
&\qquad + \frac{1}{L}\int_{\Omega^+\times(a,b)} T^\epsilon(\overline{v}_\epsilon{}^+)T^\epsilon(\psi) + \int_{\Omega^-} \left(\nabla\overline{v}_\epsilon{}^- \cdot \nabla\psi + \overline{v}_\epsilon{}^-\psi\right) \\
&\quad \to -\frac{1}{L}\int_{\Omega^+\times(a,b)} \frac{\partial u_d}{\partial x_1}\frac{\partial\psi}{\partial x_1} + \frac{1}{L}\int_{\Omega^+\times(a,b)} \left(\frac{\partial v_0^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + v_0^+\psi\right) \\
&\qquad + \int_{\Omega^-} \left(\nabla v_0{}^- \cdot \nabla\psi + v_0{}^-\psi\right) \quad as\ \epsilon \to 0
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega_\epsilon} \nabla(\overline{u}_\epsilon - u_d)\cdot\nabla\psi \to -\frac{1}{L}\int_{\Omega^+\times(a,b)} \frac{\partial u_d}{\partial x_1}\frac{\partial\psi}{\partial x_1} &+ \frac{1}{L}\int_{\Omega^+\times(a,b)} \frac{\partial(u_0^+ - u_d)}{\partial x_2}\frac{\partial\psi}{\partial x_2} \\
&+ \int_{\Omega^-} \nabla(u_0{}^- - u_d)\cdot\nabla\psi \ \ as\ \epsilon \to 0.
\end{aligned}$$

Thus, it follows that (since v_0 and u_0 are independent of y_1)

$$\left\{\begin{aligned}
&\frac{b-a}{L}\int_{\Omega^+} \left(\frac{\partial v_0^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + v_0{}^+\psi\right) + \int_{\Omega^-} \left(\nabla v_0{}^- \cdot \nabla\psi + v_0{}^-\psi\right) \\
&\quad = \frac{b-a}{L}\int_{\Omega^+} \frac{\partial(u_0^+ - u_d)}{\partial x_2}\frac{\partial\psi}{\partial x_2} + \int_{\Omega^-} \nabla(u_0{}^- - u_d)\cdot\nabla\psi,
\end{aligned}\right. \tag{6.21}$$

which in fact holds true for all $\psi \in V_0(\Omega)$ by density. Hence, $v_0 \in V_0(\Omega)$ satisfies

$$\left\{\begin{aligned}
&-\frac{\partial^2 v_0^+}{\partial x_2^2} + v_0^+ = -\frac{\partial^2(u_0{}^+ - u_d)}{\partial x_2^2} \ \text{ in } \ \Omega^+, \\
&-\Delta v_0^- + v_0^- = -\Delta(u_0{}^- - u_d) \ \text{ in } \Omega^-, \\
&\frac{\partial v_0^+}{\partial x_2} = 0 \ \text{ on } \ \Gamma_u, \\
&v_0{}^+ = v_0{}^-, \quad \frac{b-a}{L}\frac{\partial v_0{}^+}{\partial x_2} = \frac{\partial v_0{}^-}{\partial x_2} \ \text{ on } \Gamma, \\
&v_0^- = 0 \ \text{ on } \ \Gamma_b, \ \ v_0^- \text{ is } \Gamma_{s'} - \text{periodic}.
\end{aligned}\right. \tag{6.22}$$

We also have the convergence $\overline{\theta}_\epsilon \rightharpoonup \theta_0$ in $L^2(\Lambda^+)$, $T^\epsilon(\overline{v}_\epsilon|_{\Omega_\epsilon^+}) \rightharpoonup v_0|_{\Omega^+}$. Thus, we get

$$\theta_0 = -\frac{1}{\beta}\left[\frac{1}{L}\int_0^L v_0 dx_1\right]. \tag{6.23}$$

Therefore, we get the optimality system corresponding to the minimization problem (P_2). According to Theorem 6.1, its optimal solution is given by (u_0, θ_0). Thus, by uniqueness we have

$$\overline{u} = u_0, \ \ \overline{v} = v_0 \ \text{ and } \overline{\theta} = \theta_0.$$

Hence, the main theorem is proved. □

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