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Variational approach to homogenization of doubly-nonlinear flow in a periodic structure

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ABSTRACT

This work deals with the homogenization of an initial- and boundary-value problem for the doubly-nonlinear system

$$D_t w - \nabla \cdot \vec{z} = g(x, t, x/\varepsilon) \tag{0.1}$$

$$w \in \alpha(u, x/\varepsilon) \tag{0.2}$$

$$\vec{z} \in \vec{\gamma}(\nabla u, x/\varepsilon). \tag{0.3}$$

Here ε is a positive parameter; α and $\vec{\gamma}$ are maximal monotone with respect to the first variable and periodic with respect to the second one. The inclusions (0.2) and (0.3) are here formulated as *null-minimization principles*, via the theory of Fitzpatrick [MR 1009594]. As $\varepsilon \to 0$, a two-scale formulation is derived via Nguetseng's notion of two-scale convergence, and a (single-scale) homogenized problem is then retrieved.

1. Introduction

This paper deals with the homogenization of a class of doubly-nonlinear parabolic equations of the form

$$\begin{cases} D_t w_\varepsilon - \nabla \cdot \vec{z}_\varepsilon = g(x, t, x/\varepsilon) \\ w_\varepsilon \in \alpha(u_\varepsilon, x/\varepsilon) \\ \vec{z}_\varepsilon \in \vec{\gamma}(\nabla u_\varepsilon, x/\varepsilon) \end{cases} \quad \text{in } \Omega \times]0, T[. \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^N$, $T > 0$, and ε is a positive parameter. The (possibly multivalued) prescribed mappings

$$\alpha : \mathbb{R} \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}), \qquad \vec{\gamma} : \mathbb{R}^N \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N) \tag{1.2}$$

are assumed to be maximal monotone with respect to the first variable and periodic with respect to the second one. (By $\mathcal{P}$ we denote the set of the parts.) The known source field g is also periodic with respect to the third argument. We also assume that

$$\begin{aligned} &u_\varepsilon = 0 \quad \text{on } \partial\Omega \times]0, T[, \\ &w_\varepsilon(x, 0) = w^0(x, x/\varepsilon) \quad \text{for } x \in \Omega, \end{aligned} \tag{1.3}$$

for a prescribed periodic function w^0. All periods are assumed to coincide.

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Problems of the form (1.1) arise in several physical contexts: e.g., this may represent the entropy balance in diffusion phenomena; α may be the subdifferential of a dissipation potential. Existence of a solution for an associated boundary- and initial-value problem was proved e.g. by DiBenedetto and Showalter [8] and by Alt and Luckhaus [2].

In the case of single-valued operators, the homogenization of a system similar to (1.1) was already studied by H. Jian [13]. This was also used to model filtration in porous media by A.K. Nandakumaran and M. Rajesh [15,17,16]. More precisely, in [15] a quasi-linear equation of the form

$$\partial_t \alpha(u_\varepsilon, x/\varepsilon) - \nabla \cdot \vec{\gamma}(u_\varepsilon, \nabla u_\varepsilon, x/\varepsilon, t/\varepsilon) = g(x,t) \tag{1.4}$$

was studied with appropriate boundary and initial conditions, thus also accounting for high-frequency oscillations with respect to time. The same equation was also addressed by A. K. Nandakumaran and M. Rajesh [17,16], dealing with a porous medium with Neumann and Dirichlet boundary conditions, respectively. In [15,17] two-scale convergence was used extensively. It should be noticed that the Dirichlet condition on the boundary of the holes may yield different homogenized problems, that depend on the asymptotic relation between the size of the holes and the period ε.

The homogenization of quasi-linear equations has been studied by various authors, see e.g. [3,4,11,20]. The homogenization of doubly-nonlinear equations of the form (1.1) occurring in electromagnetic processes in composites and in Stefan-type problems was performed in [22,24].

Each of the inclusions $(1.1)_2$ and $(1.1)_3$ is equivalent to a variational inequality. On the basis of the Fitzpatrick theory [10], here we convert the system (1.1) to a linear PDE coupled with a *null-minimization* problem, along the lines of [26]. We then study the limit behavior for vanishing ε.

This note is organized as follows. First in Section 2 we briefly outline the Fitzpatrick theory for the variational representation of maximal monotone operators. In Section 3 we describe the homogenization problem to be studied, and reformulate it via the Brezis–Ekeland–Nayroles approach, see Problem 3.2. In Section 4 we prove existence of a solution via time-discretization, a priori estimates and passage to the limit, see Theorem 4.3. We then let ε vanish; in Section 5 we formulate the two-scale Problem 5.1, and in Theorem 5.4 we prove two-scale convergence to a solution of that problem. In Section 6 we then formulate the single-scale Problem 5.2, and in Proposition 6.4 we prove that it is equivalent to the two-scale problem. In Theorem 6.5 we then state the desired homogenization theorem. Finally, in an Appendix we briefly review Nguetseng's theory of two-scale convergence and related properties of integral functionals; these also include a result in preparation on the homogenization of maximal monotone operators.

The novelty of this work stays in the use of a Fitzpatrick-type formulation for homogenization, and in the derivation of a two-scale problem as an intermediate step towards homogenization.

The results of this note may be extended in several directions; for instance explicit dependence on time may be assumed in the nonlinear operator, and time-homogenization may also be considered. The homogenization of several other quasilinear equations may also be studied, including doubly-nonlinear systems of the form

$$w_\varepsilon - \nabla \cdot \vec{z}_\varepsilon = g(x,t,x/\varepsilon) \tag{1.5}$$

$$w_\varepsilon \in \alpha(D_t u_\varepsilon, x/\varepsilon) \tag{1.6}$$

$$\vec{z}_\varepsilon \in \vec{\gamma}(\nabla u_\varepsilon, x/\varepsilon), \tag{1.7}$$

with α and $\vec{\gamma}$ as above. Existence of a solution for an associated boundary- and initial-value problem was proved in [7].

2. Preliminaries

In this section we illustrate the tenets of the Fitzpatrick theory on the variational representation of maximal monotone operators, that is at the basis of the procedures of the present work. We also illustrate an idea of Brezis, Ekeland and Nayroles for the variational formulation of monotone flows. We refer the reader e.g. to [27] for a more detailed review.

2.1. Variational representation of maximal monotone operators

Let us first recall the Fenchel system, which is a basic result of the theory of convex analysis, see e.g. [9,21]. Let V be a separable and reflexive real Banach space with dual V', let $\psi : V \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function, and $\psi^* : V' \to \mathbb{R} \cup \{+\infty\}$ be its conjugate function, namely,

$$\psi^*(v') := \sup_{v \in V} \{\langle v', v\rangle - \psi(v)\} \quad \forall v' \in V'. \tag{2.1}$$

It is known that ψ, ψ^* and the *subdifferential* $\partial\psi$ (see e.g. [9,21]), satisfy the following Fenchel system:

$$\begin{cases} \psi(v) + \psi^*(v') \geq \langle v', v\rangle & \forall (v, v') \in V \times V', \\ \psi(v) + \psi^*(v') = \langle v', v\rangle & \text{if and only if } v' \in \partial\psi(v). \end{cases} \tag{2.2}$$

Let now $\alpha : V \times \mathcal{P}(V')$ be a multivalued mapping. In [10] Fitzpatrick introduced the following convex and lower semicontinuous function:

$$\begin{aligned} f_\alpha(v, v') &:= \langle v', v\rangle + \sup \{\langle v' - v'_0, v_0 - v\rangle : \forall v'_0 \in \alpha(v_0)\} \\ &= \sup \{\langle v', v_0\rangle - \langle v'_0, v_0 - v\rangle : \forall v'_0 \in \alpha(v_0)\} \end{aligned} \tag{2.3}$$

for all $(v, v') \in V \times V'$, and proved that, whenever α is maximal monotone,

$$\begin{cases} f_\alpha(v, v') \ge \langle v', v\rangle & \forall (v, v') \in V \times V', \\ f_\alpha(v, v') = \langle v', v\rangle & \text{if and only if } v' \in \alpha(v). \end{cases} \tag{2.4}$$

This system obviously extends (2.2). Nowadays f_α is called the *Fitzpatrick function* of α.

The inclusion $v' \in \alpha(v)$ is thus equivalent to

$$f_\alpha(v, v') - \langle v', v\rangle = \inf \{f_\alpha(r, r') - \langle r', r\rangle : (r, r') \in V \times V'\} = 0, \tag{2.5}$$

that we label as a *null-minimization problem*.

Next we review the notion of *(variational) representation* of monotone operators.

Definition 2.1. We shall say that a lower semicontinuous convex function $f : V \times V' \to \mathbb{R} \cup \{+\infty\}$ (variationally) represents a (necessarily monotone) operator $\alpha : V \to \mathcal{P}(V')$ in the sense of Fitzpatrick, whenever

$$\begin{cases} f(v, v') \ge \langle v', v\rangle & \forall (v, v') \in V \times V', \\ f(v, v') = \langle v', v\rangle & \text{if and only if } v' \in \alpha(v). \end{cases} \tag{2.6}$$

Such a function is called a *representative function*. For instance, because of (2.3)–(2.5), α is represented by the function f_α. If $\alpha = \partial\psi$, then because of (2.2) α is also represented by the *Fenchel function* $g_\alpha(v, v') := \psi(v) + \psi^*(v')$.

2.2. The Brezis–Ekeland–Nayroles variational formulation of flows

Let us assume that we are given a triplet of (real) Banach spaces

$$V \subset H = H' \subset V' \quad \text{with continuous and dense injections.} \tag{2.7}$$

On the basis of the Fenchel system (2.2), under suitable restrictions, for any prescribed lower semicontinuous and convex function $\psi : V \to \mathbb{R} \cup \{+\infty\}$, any $u^* \in L^2(0, T; V')$ and any $u^0 \in H$, Brezis and Ekeland [5] and Nayroles [18] independently reformulated the gradient flow

$$\begin{cases} D_t u + \partial\psi(u) = u^* & \text{in }]0, T[\\ u(0) = u^0 \end{cases} \tag{2.8}$$

as the null-minimization of the functional

$$\Phi_1(v, u^*) = \int_0^T [\psi(v) + \psi(u^* - D_t v)]\, dt + \frac{1}{2}(\|v(T)\|_H^2 - \|u^0\|_H^2) - \langle u^*, v\rangle, \tag{2.9}$$

as v ranges in $H^1(0, T; V') \cap L^2(0, T; V)$ $(\subset C^0([0, T]; H))$ (here by $\langle \cdot, \cdot\rangle$ we denote the duality pairing between $L^2(0, T; V')$ and $L^2(0, T; V)$). More generally, see [24], for any maximal monotone $\alpha : V \to \mathcal{P}(V')$, denoting by f_α a representative function of α, the monotone flow

$$\begin{cases} D_t u + \alpha(u) = u^* & \text{in }]0, T[\\ u(0) = u^0 \end{cases} \tag{2.10}$$

may be represented as the null-minimization of the functional

$$\Phi_2(v, u^*) = \int_0^t f_\alpha(v, u^* - D_t v)\, dt + \frac{1}{2}(\|v(T)\|_H^2 - \|u^0\|_H^2) - \langle u^*, v\rangle. \tag{2.11}$$

3. Weak formulation of the ε-problem

In this section we provide two equivalent formulations of the system (1.1) coupled with appropriate initial- and boundary-conditions in a periodic medium.

As the (possibly multivalued) mapping $\alpha(\cdot, y)$ acts on scalar variables, with no loss of generality we may assume that it coincides with the subdifferential $\partial\varphi(\cdot, y)$ of a convex function, for a.e. $y \in Y$.

Let $Y =]0, 1[^N$ be the unit cell, and assume that (denoting by $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(Y)$ the σ-algebras of Borel- and Lebesgue-measurable functions),

$$\begin{aligned} &\varphi : \mathbb{R} \times Y \to \mathbb{R} \cup \{+\infty\} \quad \text{is measurable w.r.t. } \mathcal{B}(\mathbb{R}) \otimes \mathcal{L}(Y), \\ &\varphi(\cdot, y) \quad \text{is convex and lower semicontinuous for a.e. } y, \end{aligned} \tag{3.1}$$

$$\exists c_1, c_2 > 0 : \forall v \in \mathbb{R}, \quad |\varphi(v, y)| \le c_1|v|^2 + c_2 \quad \text{for a.e. } y \in Y. \tag{3.2}$$

By definition of the convex conjugate function $\varphi^*(\cdot, y)$ (see e.g. [9,21]), it follows that

$$\exists L, M > 0 : \forall v \in \mathbb{R}, \quad |\varphi^*(v, y)| \ge L|v|^2 - M \quad \text{for a.e. } y \in Y. \tag{3.3}$$

Let us assume that

$$\begin{aligned} &\vec{\gamma} : \mathbb{R}^N \times Y \to \mathcal{P}(\mathbb{R}^N) \quad \text{is measurable w.r.t. } \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{L}(Y), \\ &\vec{\gamma}(\cdot, y) \quad \text{is maximal monotone for a.e. } y, \end{aligned} \tag{3.4}$$

and that there exist nonnegative constants k, a, b such that

$$|\vec{z}| \le k(1 + |\vec{\zeta}|) \quad \forall(\vec{\zeta}, \vec{z}) \in \text{graph}(\vec{\gamma}(\cdot, y)), \text{ for a.e. } y, \tag{3.5}$$

$$\vec{z} \cdot \vec{\zeta} \ge a(|\vec{z}|^2 + |\vec{\zeta}|^2) - b \quad \forall(\vec{\zeta}, \vec{z}) \in \text{graph}(\vec{\gamma}(\cdot, y)), \text{ for a.e. } y. \tag{3.6}$$

Let us also assume that Ω is a bounded domain of $\mathbb{R}^N$ of Lipschitz class, and that, setting $\Omega_T := \Omega \times]0, T[$,

$$\begin{aligned} &\vec{h} : \Omega_T \times Y \to \mathbb{R}^N \quad \text{is measurable w.r.t. } \mathcal{B}(\Omega_T) \otimes \mathcal{L}(Y), \\ &\vec{h}(\cdot, \cdot, y) \in L^2(\Omega_T)^N \quad \text{for a.e. } y, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &w^0 : \Omega \times Y \to \mathbb{R} \quad \text{is measurable w.r.t. } \mathcal{B}(\Omega) \otimes \mathcal{L}(Y), \\ &w^0(\cdot, y) \in L^2(\Omega) \quad \text{for a.e. } y. \end{aligned} \tag{3.8}$$

We extend all of these functions Y-periodically to $\mathbb{R}^N$ with respect to the argument y, and set

$$\varphi_\varepsilon(v, x) := \varphi(v, x/\varepsilon) \quad \forall v \in \mathbb{R}, \text{ for a.e. } x \in \mathbb{R}^N, \tag{3.9}$$

$$\vec{\gamma}_\varepsilon(v, x) := \vec{\gamma}(v, x/\varepsilon) \quad \forall v \in \mathbb{R}^N, \text{ for a.e. } x \in \mathbb{R}^N, \tag{3.10}$$

$$\vec{h}_\varepsilon(x, t) := \vec{h}(x, t, x/\varepsilon) \quad \text{for a.e. } (x, t) \in \Omega_T, \tag{3.11}$$

$$w^0_\varepsilon(x) := w^0(x, x/\varepsilon) \quad \text{for a.e. } x \in \Omega. \tag{3.12}$$

We shall deal with the homogenization of the following doubly-nonlinear system:

$$D_t w_\varepsilon - \nabla \cdot \vec{z}_\varepsilon = \nabla \cdot \vec{h}_\varepsilon \quad \text{in } \mathcal{D}'(\Omega), \text{ a.e. in }]0, T[, \tag{3.13}$$

$$w_\varepsilon \in \partial\varphi_\varepsilon(u_\varepsilon, x) \quad \text{a.e. in } \Omega_T, \tag{3.14}$$

$$\vec{z}_\varepsilon \in \vec{\gamma}_\varepsilon(\nabla u_\varepsilon, x) \quad \text{a.e. in } \Omega_T, \tag{3.15}$$

$$u_\varepsilon = 0 \quad \text{a.e. on } \partial\Omega \times]0, T[, \tag{3.16}$$

$$w_\varepsilon(\cdot, 0) = w^0_\varepsilon \quad \text{a.e. in } \Omega. \tag{3.17}$$

Examples of data include the following ones:

$$\varphi_\varepsilon(v, x) = a_0 v^2 + a_1(x/\varepsilon)|v| \quad \forall v \in \mathbb{R}, \text{ for a.e. } x \in \Omega, \tag{3.18}$$

$$\vec{\gamma}_\varepsilon(\vec{v}, x) = [b_0 + b_1(x/\varepsilon)]\partial|\vec{v}| + A(x/\varepsilon) \cdot \vec{v} \quad \forall \vec{v} \in \mathbb{R}^N, \text{ for a.e. } x \in \Omega, \tag{3.19}$$

$$\vec{h}_\varepsilon(x, t) = \vec{h}_0(x, t) + c(x/\varepsilon)\vec{h}_1(x, t) \quad \text{for a.e. } (x, t) \in \Omega_T. \tag{3.20}$$

Thus $\partial\varphi_\varepsilon(v, x) = 2a_0 v + a_1(x/\varepsilon)\text{sign}(v)$, $\partial|\vec{v}| = \vec{v}/|\vec{v}|$ if $\vec{v} \ne \vec{0}$, and $\partial|\vec{0}|$ is the whole ball of $\mathbb{R}^N$ with center the origin and radius 1. Here a_0, b_0 are positive constants and $\vec{h}_0, \vec{h}_1 \in L^2(\Omega_T)^N$. Moreover, $a_1, b_1, c : \mathbb{R}^N \to \mathbb{R}$ and $A : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ are prescribed Y-periodic functions, with $a_1, b_1 \ge 0$ and A positive-definite. A need not be symmetric, so that $\vec{\gamma}_\varepsilon$ may have no potential.

By applying Lemma A.6 with $D = \Omega$ and either $B = L^2(\Omega)$ or $B = L^2(\Omega)^N$, we see that there exist measurable selections $w_\varepsilon \in \partial\varphi_\varepsilon(u_\varepsilon, \cdot)$ and $\vec{z}_\varepsilon \in \vec{\gamma}_\varepsilon(\nabla u_\varepsilon, \cdot)$. (The same will apply repeatedly in this work.)

Next we introduce the Hilbert triplet

$$V = H_0^1(\Omega) \subset H = L^2(\Omega) = H' \subset V' = H^{-1}(\Omega) \tag{3.21}$$

(with continuous and dense injections), and reformulate the system (3.13)–(3.17) in weak form as follows, for any $\varepsilon > 0$.

Problem 3.1. Find $(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon) \in L^2(0,T;V) \times L^2(\Omega_T) \times L^2(\Omega_T)^N$ such that

$$\iint_{\Omega_T} [(w_\varepsilon^0 - w_\varepsilon) D_t v + (\vec{z}_\varepsilon + \vec{h}_\varepsilon) \cdot \nabla v]\, dxdt = 0 \quad \forall v \in H^1(0,T;V),\ v(\cdot,T) = 0, \tag{3.22}$$

$$w_\varepsilon \in \partial\varphi_\varepsilon(u_\varepsilon, x) \quad \text{a.e. in } \Omega_T, \tag{3.23}$$

$$\vec{z}_\varepsilon \in \vec{\gamma}_\varepsilon(\nabla u_\varepsilon, x) \quad \text{a.e. in } \Omega_T. \tag{3.24}$$

Eq. (3.22) yields

$$D_t w_\varepsilon - \nabla \cdot \vec{z}_\varepsilon = \nabla \cdot \vec{h}_\varepsilon \quad \text{in } V', \text{ a.e. in }]0,T[. \tag{3.25}$$

By comparing the terms of this equation we get $D_t w_\varepsilon \in L^2(0,T;V')$, whence

$$w_\varepsilon \in H^1(0,T;V') \subset C^0([0,T];V') \quad \text{(by an obvious identification).} \tag{3.26}$$

Eq. (3.22) then also entails (3.17). Conversely, (3.17) and (3.25) yield (3.22).

Next we reformulate (3.23) and (3.24) via the Fitzpatrick theorem (2.4). First we denote by $f_{\vec{\gamma}_\varepsilon}(\cdot,\cdot,x)$ a representative function of $\vec{\gamma}_\varepsilon(\cdot,x)$ for a.e. x. For any $(u, w, \vec{z})$ that satisfies (3.22), we set

$$\Phi_\varepsilon(u, w, \vec{z}) := \iint_{\Omega_T} [\varphi_\varepsilon(u,x) + \varphi_\varepsilon^*(w,x) - wu + f_{\vec{\gamma}_\varepsilon}(\nabla u, \vec{z}, x) - \nabla u \cdot \vec{z}]\, dxdt, \tag{3.27}$$

where φ_ε is defined as in (3.9). We then define the infinite-dimensional manifold

$$X_\varepsilon = \{(w_\varepsilon, \vec{z}_\varepsilon) \in L^2(\Omega_T) \times L^2(\Omega_T)^N \text{ that fulfill (3.22)}\}. \tag{3.28}$$

For any $\varepsilon > 0$ we shall consider the following problem, in which a PDE is coupled with a null-minimization problem.

Problem 3.2. Find $u_\varepsilon \in L^2(0,T;V)$ and $(w_\varepsilon, \vec{z}_\varepsilon) \in X_\varepsilon$ such that

$$\Phi_\varepsilon(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon) = \inf_{L^2(0,T;V)\times X_\varepsilon} \Phi_\varepsilon = 0. \tag{3.29}$$

Theorem 3.3. *For any $\varepsilon > 0$, Problems 3.1 and 3.2 are mutually equivalent.*

Proof. By the Fenchel and Fitzpatrick systems (2.2) and (2.4), the functional Φ_ε is nonnegative. The null-minimization (3.29) is thus equivalent to the inequality

$$\Phi_\varepsilon(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon) \le 0, \tag{3.30}$$

or also to the system of the two inequalities

$$\iint_{\Omega_T} [\varphi_\varepsilon(u_\varepsilon,x) + \varphi_\varepsilon^*(w_\varepsilon,x) - w_\varepsilon u_\varepsilon]\, dxdt \le 0, \tag{3.31}$$

$$\iint_{\Omega_T} [f_{\vec{\gamma}_\varepsilon}(\nabla u_\varepsilon, \vec{z}_\varepsilon, x) - \nabla u_\varepsilon \cdot \vec{z}_\varepsilon]\, dxdt \le 0. \tag{3.32}$$

The integrand of either functional is pointwise nonnegative, so that by (2.2) and (2.4) these inequalities are respectively equivalent to (3.23) and (3.24). □

4. Existence of a solution of the ε-problems

In this section we prove existence of a solution of Problem 3.1. Although this result has already been proved e.g. in [8], here we present an argument based on the equivalence with Problem 3.2. We do so for the sake of completeness, and also because this argument provides the uniform estimates that we shall use in the homogenization procedure in the next section.

4.1. Approximation by time-discretization

Let us fix any $\varepsilon > 0$, any $m \in \mathbb{N}$, set $k = T/m$ and

$$\vec{h}^n_{\varepsilon m} = \frac{1}{k}\int_{(n-1)k}^{nk} \vec{h}_\varepsilon(\cdot, t)\, dt \ (\in L^2(\Omega)^N) \quad \text{for } n = 1, \dots, m. \tag{4.1}$$

For any $\varepsilon > 0$ and any m, let us then consider the following time-discretized problem.

Problem 4.1. Find $(u^n_{\varepsilon m}, w^n_{\varepsilon m}, \vec{z}^n_{\varepsilon m}) \in V \times H \times H^N$ $(n = 1, \dots, m)$, such that, setting $w^0_{\varepsilon m} = w^0_\varepsilon$, for $n = 1, \dots, m$

$$w^n_{\varepsilon m} - k\nabla \cdot \vec{z}^n_{\varepsilon m} = w^{n-1}_{\varepsilon m} + k\nabla \cdot \vec{h}^n_{\varepsilon m} \quad \text{in } V', \tag{4.2}$$

$$w^n_{\varepsilon m} \in \partial\varphi_\varepsilon(u^n_{\varepsilon m}, x) \quad \text{a.e. in } \Omega, \tag{4.3}$$

$$\vec{z}^n_{\varepsilon m} \in \vec{\gamma}_\varepsilon(\nabla u^n_{\varepsilon m}, x) \quad \text{a.e. in } \Omega. \tag{4.4}$$

Defining $\Lambda_{\varepsilon,m}(v) = \partial\varphi_\varepsilon(v, x) - k\nabla \cdot \vec{\gamma}_\varepsilon(\nabla v, x) \in \mathcal{P}(V')$ for any $v \in V$, the system (4.2)–(4.4) is equivalent to

$$\Lambda_{\varepsilon,m}(u^n_{\varepsilon m}, x) \ni w^{n-1}_{\varepsilon m} + k\nabla \cdot \vec{h}^n_{\varepsilon m} \quad \text{in } V', \ n = 1, \dots, m. \tag{4.5}$$

By the assumptions (3.1)–(3.8), for any ε, m the operator $\Lambda_{\varepsilon,m} : V \to \mathcal{P}(V')$ is maximal monotone and coercive. The inclusion (4.5) has then at least one solution, and this solves Problem 4.1.

Let us now define time-interpolate functions as follows. For any family $\{v^n_m\}_{n=0,\dots,m} \subset \mathbb{R}$, let us denote by v_m the piecewise-linear time-interpolate of $v^0_m := v^0, v^1_m, \dots, v^m_m$ a.e. in Ω. Let us denote by $\bar{v}_m$ the corresponding piecewise-constant interpolate function, that is, $\bar{v}_m(t) := v^n_m$ if $(n-1)k < t \le nk$ for $n = 1, \dots, m$.

The system (4.2)–(4.4) then also reads

$$D_t w_{\varepsilon m} - \nabla \cdot \bar{\vec{z}}_{\varepsilon m} = \nabla \cdot \bar{\vec{h}}_{\varepsilon m} \quad \text{in } V', \text{a.e. in }]0, T[, \tag{4.6}$$

$$\bar{w}_\varepsilon \in \partial\varphi_\varepsilon(\bar{u}_{\varepsilon m}, x) \quad \text{a.e. in } \Omega_T, \tag{4.7}$$

$$\bar{\vec{z}}_{\varepsilon m} \in \vec{\gamma}_\varepsilon(\nabla \bar{u}_{\varepsilon m}, x) \quad \text{a.e. in } \Omega_T, \tag{4.8}$$

$$w_{\varepsilon m}(\cdot, 0) = w^0_\varepsilon \quad \text{in } V', \tag{4.9}$$

which is equivalent to the approximate weak equation

$$\iint_{\Omega_T} [(w^0_\varepsilon - w_{\varepsilon m})D_t v + (\bar{\vec{z}}_{\varepsilon m} + \bar{\vec{h}}_{\varepsilon m}) \cdot \nabla v]\, dxdt = 0 \quad \forall v \in H^1(0, T; V), v(\cdot, T) = 0. \tag{4.10}$$

By mimicking the procedure of Theorem 3.3, it is promptly checked that (4.7) and (4.8) may be replaced by the two inequalities

$$\iint_{\Omega_T} [\varphi_\varepsilon(\bar{u}_{\varepsilon m}, x) + \varphi^*_\varepsilon(\bar{w}_{\varepsilon m}, x) - \bar{w}_{\varepsilon m}\bar{u}_{\varepsilon m}]\, dxdt \le 0, \tag{4.11}$$

$$\iint_{\Omega_T} [f_{\vec{\gamma}_\varepsilon}(\nabla \bar{u}_{\varepsilon m}, \bar{\vec{z}}_{\varepsilon m}, x) - \nabla \bar{u}_{\varepsilon m} \cdot \bar{\vec{z}}_{\varepsilon m}]\, dxdt \le 0. \tag{4.12}$$

Defining Φ_ε as in (3.27) and the space

$$X_{\varepsilon m} = \{(w_m, \bar{\vec{z}}_m) \in L^2(\Omega_T) \times L^2(\Omega_T)^N \text{ that fulfill (4.10)}\}, \tag{4.13}$$

we conclude that Problem 4.1 is equivalent to the following null-minimization problem:

Problem 4.2. Find $u_{\varepsilon m} \in L^2(0, T; V)$ and $(w_{\varepsilon m}, \vec{z}_{\varepsilon m}) \in X_{\varepsilon m}$ such that

$$\Phi_\varepsilon(u_{\varepsilon m}, w_{\varepsilon m}, \vec{z}_{\varepsilon m}) = \inf_{L^2(0,T;V)\times X_{\varepsilon m}} \Phi_\varepsilon = 0. \tag{4.14}$$

4.2. A priori estimates

By the Fenchel inequality (2.2), the inequality (4.11) is tantamount to (4.7). By (4.10) and (4.7)

$$\begin{aligned} -\iint_{\Omega_T} \nabla \bar{u}_{\varepsilon m} \cdot \bar{\vec{z}}_{\varepsilon m}\, dxdt &= \int_0^T \langle D_t w_{\varepsilon m}, \bar{u}_{\varepsilon m}\rangle_{V',V}\, dt + \iint_{\Omega_T} \nabla \bar{u}_{\varepsilon m} \cdot \bar{\vec{h}}_{\varepsilon m}\, dxdt \\ &= \iint_{\Omega_T} D_t \varphi_\varepsilon^*(w_{\varepsilon m}(x,t), x)\, dxdt + \iint_{\Omega_T} \nabla \bar{u}_{\varepsilon m} \cdot \bar{\vec{h}}_{\varepsilon m}\, dxdt \\ &= \int_\Omega [\varphi_\varepsilon^*(w_{\varepsilon m}(x,T), x) - \varphi_\varepsilon^*(w_\varepsilon^0, x)]\, dx + \iint_{\Omega_T} \nabla \bar{u}_{\varepsilon m} \cdot \bar{\vec{h}}_{\varepsilon m}\, dxdt. \end{aligned} \tag{4.15}$$

By (3.3)

$$\int_\Omega g^*\left(\bar{w}_{\varepsilon m}(x,T), x/\varepsilon\right)\, dx \ge L \int_\Omega |\bar{w}_{\varepsilon m}(\cdot,T)|^2 - M|\Omega|. \tag{4.16}$$

On the other hand, as the function $f_{\vec{\gamma}_\varepsilon}$ *represents* the operator $\vec{\gamma}_\varepsilon$ (in the sense of the theory of Fitzpatrick), (3.6) yields

$$\begin{aligned} \iint_{\Omega_T} f_{\vec{\gamma}_\varepsilon}(\nabla \bar{u}_\varepsilon, \bar{\vec{z}}_\varepsilon, x)\, dxdt &\ge \int_0^T (\nabla \bar{u}_{\varepsilon m}, \bar{\vec{z}}_{\varepsilon m})\, dt \\ &\ge a\left(\|\nabla \bar{u}_{\varepsilon m}\|^2_{L^2(\Omega)^N} + \|\bar{\vec{z}}_{\varepsilon m}\|^2_{L^2(\Omega)^N}\right) - b|\Omega|. \end{aligned} \tag{4.17}$$

By (4.12), then

$$\begin{aligned} & a\left(\|\nabla \bar{u}_{\varepsilon m}\|^2_{L^2(\Omega)} + \|\bar{\vec{z}}_{\varepsilon m}\|^2_{L^2(\Omega)^N}\right) - b|\Omega| + L\int_\Omega |\bar{w}_{\varepsilon m}(\cdot,T)|^2 - M|\Omega| \\ &\le \int_\Omega \varphi_\varepsilon^*(w_\varepsilon^0)\, dx - \iint_{\Omega_T} \bar{\vec{h}}_{\varepsilon m} \cdot \nabla \bar{u}_{\varepsilon m}\, dxdt \\ &\le \int_\Omega \varphi_\varepsilon^*(w_\varepsilon^0)\, dx + \|\bar{\vec{h}}_{\varepsilon m}\|_{L^2(\Omega_T)^N} \|\bar{u}_{\varepsilon m}\|_{L^2(0,T;V)}. \end{aligned} \tag{4.18}$$

As in these inequalities one may replace T by any $t \in]0, T]$, we get the uniform estimates

$$\|\bar{u}_{\varepsilon m}\|_{L^2(0,T;V)} \le C_1, \qquad \|\bar{\vec{z}}_{\varepsilon m}\|_{L^2(\Omega_T)^N} \le C_2, \tag{4.19}$$

where by $C_1, C_2, \ldots$ we denote constants independent of ε. By the above computation, we also infer that

$$\|\bar{w}_{\varepsilon m}\|_{L^\infty(0,T;H)} \le C_3, \tag{4.20}$$

and by comparing the terms of (4.6) we conclude that $w_{\varepsilon m} \in H^1(0,T;V')$ and

$$\|w_{\varepsilon m}\|_{H^1(0,T;V')} \le C_4. \tag{4.21}$$

Analogous estimates to (4.19) and (4.20) hold for the piecewise interpolate functions, that is, $w_{\varepsilon m}, u_{\varepsilon m}, \vec{z}_{\varepsilon m}$. On the other hand, obviously (4.21) does not apply to $\bar{w}_{\varepsilon m}$.

4.3. Passage to the limit

On the basis of the above a priori estimates, there exist $u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon$ such that, up to extracting subsequences,[1]

$$u_{\varepsilon m} \rightharpoonup u_\varepsilon \quad \text{in } L^2(0,T;V), \tag{4.22}$$

$$w_{\varepsilon m} \overset{*}{\rightharpoonup} w_\varepsilon \quad \text{in } L^\infty(0,T;H) \cap H^1(0,T;V'), \tag{4.23}$$

$$\vec{z}_{\varepsilon m} \rightharpoonup \vec{z}_\varepsilon \quad \text{in } L^2(\Omega)^N. \tag{4.24}$$

Moreover,

$$\vec{h}_{\varepsilon m} \rightharpoonup \vec{h}_\varepsilon \quad \text{in } L^2(\Omega_T)^N, \tag{4.25}$$

$$w^0_{\varepsilon m} \rightharpoonup w^0_\varepsilon \quad \text{in } L^2(\Omega). \tag{4.26}$$

[1] With standard notation, we shall denote the (single-scale) strong and weak convergence by $\to$ and $\rightharpoonup$, respectively.

By passing to the limit in (4.10), we get Eq. (3.22); namely, $(w_\varepsilon, \vec{z}_\varepsilon) \in X_\varepsilon$. Let us next derive (4.14) by passing to the inferior limit in (3.29). By the sequential weak lower semicontinuity of φ_ε and by (4.15), we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} - \iint_{\Omega_T} \nabla \bar{u}_{\varepsilon m} \cdot \vec{\bar{z}}_{\varepsilon m}\, dxdt &\geq \int_\Omega [\varphi_\varepsilon^*(w_\varepsilon(x,T),x) - \varphi_\varepsilon^*(w_\varepsilon^0, x)]\, dx + \iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec{h}_\varepsilon\, dxdt \\ &= \int_0^T \langle D_t w_\varepsilon, u_\varepsilon\rangle_{V',V}\, dt + \iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec{h}_\varepsilon\, dxdt \\ &\overset{(3.22)}{=} - \iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec{z}_\varepsilon\, dxdt. \end{aligned} \tag{4.27}$$

By the sequential weak lower semicontinuity of φ_ε, φ_ε^* and f_ε, we then infer that

$$\iint_{\Omega_T} [\varphi_\varepsilon(u_\varepsilon, x) + \varphi_\varepsilon^*(w_\varepsilon, x) - w_\varepsilon u_\varepsilon]\, dxdt \leq 0, \tag{4.28}$$

$$\iint_{\Omega_T} [f_{\vec{\gamma}_\varepsilon}(\nabla u_\varepsilon, \vec{z}_\varepsilon, x) - \nabla u \cdot \vec{z}_\varepsilon]\, dxdt \leq 0, \tag{4.29}$$

namely

$$\Phi_\varepsilon(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon) \leq 0; \tag{4.30}$$

that is, $(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon)$ solves Problem 3.2. We have thus proved the following assertion.

Theorem 4.3. *Let the conditions (3.1)–(3.8) be fulfilled for any fixed $\varepsilon > 0$, and that*

$$\vec{h}_\varepsilon \in L^2(\Omega)^N, \qquad w_\varepsilon^0 \in L^2(\Omega) \quad \text{and are uniformly bounded w.r.t. } \varepsilon \text{ in these spaces}. \tag{4.31}$$

The solutions $(u_{\varepsilon m}, w_{\varepsilon m}, \vec{z}_{\varepsilon m})$ of Problem 4.1 then satisfy the uniform estimates (4.19)–(4.21). Therefore there exists $(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon)$ such that, up to extracting subsequences, (4.22)–(4.24) hold.

The triplet $(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon)$ is then a solution of Problem 3.2 (equivalently, of Problem 3.1). Finally, the following uniform estimates hold:

$$\|u_\varepsilon\|_{L^2(0,T;V)}, \|\vec{z}_\varepsilon\|_{L^2(\Omega_T)^N}, \|w_\varepsilon\|_{L^\infty(0,T;H)\cap H^1(0,T;V')} \leq Constant. \tag{4.32}$$

5. Two-scale formulation

In this section we introduce two mutually equivalent two-scale formulations, that we then derive by passing to the limit as $\varepsilon \to 0$ in Problem 3.1 (or 3.2).

We shall denote by $H^1_\#(Y)$ the subspace of the functions of $H^1(Y)$ that have equal traces on opposite faces of Y; these coincide with the restrictions of the Y-periodic functions of $H^1(\mathbb{R}^N)$.

We introduce two equivalent two-scale formulation, in which the constitutive relations are respectively expressed either as inclusions or as null-minimization principles.

Problem 5.1. Find

$$\begin{aligned} &u \in L^2(0,T;V), \qquad u_1 \in L^2(\Omega_T; H^1_\sharp(Y)),\\ &w \in L^2(\Omega_T \times Y) \cap H^1(0,T; L^2(Y;V')), \qquad \vec{z} \in L^2(\Omega_T \times Y)^N, \end{aligned} \tag{5.1}$$

such that

$$\iiint_{\Omega_T\times Y} \big[(w_0 - w)D_t v + (\vec{z} + \vec{h})\cdot(\nabla v + \nabla_y v_1)\big]\, dxdtdy = 0$$

$$\forall v \in H^1(0,T;V),\ v|_{t=T} = 0,\ \forall v_1 \in L^2(\Omega_T; H^1_\sharp(Y)), \tag{5.2}$$

$$w \in \partial\varphi(u,y) \quad \text{a.e. in } \Omega_T \times Y, \tag{5.3}$$

$$\vec{z} \in \vec{\gamma}(\nabla u + \nabla_y u_1, y) \quad \text{a.e. in } \Omega_T \times Y. \tag{5.4}$$

Let us next define the spaces

$$S = L^2(0,T;V) \times L^2(\Omega_T; H^1_\sharp(Y)), \qquad X_0 = \{(w,\vec{z}) \text{ as in (5.1) that fulfill (5.2)}\}, \tag{5.5}$$

and the functional

$$\Phi_0(u,u_1,w,\vec{z}) := \iiint_{\Omega_T\times Y} [\varphi(u,y) + \varphi^*(w,y) - wu + f_{\vec{\gamma}}(\nabla u + \nabla_y u_1, \vec{z}, y) - (\nabla u + \nabla_y u_1)\cdot \vec{z}\,]\, dxdtdy \quad \forall (u,u_1) \in S, \forall (w,\vec{z}) \in X_0. \tag{5.6}$$

We are now able to introduce our second two-scale formulation.

Problem 5.2. Find $(u,u_1) \in S$ and $(w,\vec{z}) \in X_0$ such that

$$\Phi_0(u,u_1,w,\vec{z}) = \inf_{S\times X_0} \Phi_0 = 0. \tag{5.7}$$

Proposition 5.3. *The two-scale Problems 5.1 and 5.2 are mutually equivalent.*

Proof. This argument mimics that of Theorem 3.3. The null-minimization of Φ_0 is equivalent to the system of the two inequalities

$$\iiint_{\Omega_T\times Y} [\varphi(u,y) + \varphi^*(w,y) - wu]\, dxdtdy \le 0, \tag{5.8}$$

$$\iiint_{\Omega_T\times Y} [f_{\vec{\gamma}}(\nabla u + \nabla_y u_1, \vec{z}, y) - (\nabla u + \nabla_y u_1)\cdot \vec{z}\,]\, dxdtdy \le 0, \tag{5.9}$$

which are respectively equivalent to (5.3) and (5.4). □

Theorem 5.4. *Let the assumptions (3.1)–(3.8), (4.31) be fulfilled. For any $\varepsilon > 0$, let $(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon)$ be a solution of Problem 3.1 or equivalently of Problem 3.2 (this exists by Theorem 4.3). Then there exist $u, w, \vec{z}$ as in (5.1) such that, as $\varepsilon \to 0$ along a suitable sequence,*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^2(0,T;V), \tag{5.10}$$

$$\nabla u_\varepsilon \underset{2}{\rightharpoonup} \nabla u + \nabla_y u_1 \quad \text{in } L^2(\Omega_T \times Y)^N, \tag{5.11}$$

$$w_\varepsilon \underset{2}{\rightharpoonup} w \quad \text{in } L^2(\Omega_T \times Y), \tag{5.12}$$

$$\vec{z}_\varepsilon \underset{2}{\rightharpoonup} \vec{z} \quad \text{in } L^2(\Omega_T \times Y)^N. \tag{5.13}$$

Moreover, $(u,u_1,w,\vec{z})$ is then a solution of Problem 5.1, or equivalently of Problem 5.2.

Proof. (i) By Theorem 4.3 the family of solutions $\{(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon)\}$ fulfills the uniform estimates (4.32). By Theorems A.2 and A.5 in the Appendix, then there exist $u, w, \vec{z}$ as in (5.1) that fulfill (5.10)–(5.13) as $\varepsilon \to 0$ along a suitable sequence. By (3.11) and (3.12)

$$\vec{h}_\varepsilon \underset{2}{\to} \vec{h} \quad \text{in } L^2(\Omega_T \times Y)^N, \tag{5.14}$$

$$w^0_\varepsilon \underset{2}{\to} w^0 \quad \text{in } L^2(\Omega \times Y). \tag{5.15}$$

By passing to the limit in (3.22) we then get Eq. (5.2).

(ii) Next we prove (5.3). The null-minimization (4.14) is tantamount to

$$\iint_{\Omega_T} [\varphi_\varepsilon(u_\varepsilon, x) + \varphi^*_\varepsilon(w_\varepsilon, x) - w_\varepsilon u_\varepsilon]\, dxdt = 0, \tag{5.16}$$

$$\iint_{\Omega_T} [f_{\vec{\gamma}_\varepsilon}(\nabla u_\varepsilon, \vec{z}_\varepsilon, x) - \nabla u_\varepsilon \cdot \vec{z}_\varepsilon]\, dxdt = 0. \tag{5.17}$$

By (5.10) and (5.12), recalling that $\{w_\varepsilon\}$ is also uniformly bounded in $H^1(0,T;V')$, we have

$$\iint_{\Omega_T} w_\varepsilon u_\varepsilon \, dxdt \to \iiint_{\Omega_T\times Y} wu \, dxdtdy. \tag{5.18}$$

By (5.16) and (A.12), we then infer that

$$\iiint_{\Omega_T \times Y} [\varphi(u, y) + \varphi^*(w, y) - wu]\, dxdtdy \le 0, \tag{5.19}$$

and this is equivalent to (5.3).

(ii) We are left with the proof of (5.4). By (A.12)

$$\liminf_{\varepsilon\to 0} \iint_{\Omega_T} f_{\vec\gamma_\varepsilon}(\nabla u_\varepsilon, \vec z_\varepsilon, x)\, dxdtdy \ge \iiint_{\Omega_T\times Y} f_{\vec\gamma}(\nabla u + \nabla_y u_1, \vec z, y)\, dxdtdy. \tag{5.20}$$

On the other hand, using (3.13) and (3.23) and mimicking (4.27), we have

$$\begin{aligned} -\iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec z_\varepsilon\, dxdt &= \int_0^T \langle D_t w_\varepsilon, u_\varepsilon\rangle_{V',V}\, dt + \iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec h_\varepsilon\, dxdt \\ &= \iint_{\Omega_T} D_t \varphi_\varepsilon^*(w_\varepsilon(x,t), x)\, dxdt + \iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec h_\varepsilon\, dxdt \\ &= \int_\Omega [\varphi_\varepsilon^*(w_\varepsilon(x,T), x) - \varphi_\varepsilon^*(w_\varepsilon^0, x)]\, dx + \iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec h_\varepsilon\, dxdt. \end{aligned} \tag{5.21}$$

By (5.11), (5.12) and (A.12),

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \left\{ \int_\Omega [\varphi_\varepsilon^*(w_\varepsilon(x,T), x) - \varphi_\varepsilon^*(w_\varepsilon^0, x)]\, dx + \iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec h_\varepsilon\, dxdt \right\} \\ &\quad \ge \iint_{\Omega\times Y} [\varphi^*(w(x,T)) - \varphi^*(w^0)]\, dxdy + \iiint_{\Omega_T\times Y} (\nabla u + \nabla_y u_1)\cdot \vec h\, dxdtdy. \end{aligned} \tag{5.22}$$

Here also we may drop the term in $\nabla_y u_1$. Moreover, by (5.2) and (5.3), recalling that ∇u is independent of y,

$$\begin{aligned} &-\iiint_{\Omega_T\times Y} (\nabla u + \nabla_y u_1)\cdot(\vec z + \vec h)\, dxdtdy \\ &\quad = \int_Y dy \int_0^T \langle D_t w, u\rangle_{V',V}\, dt = \iiint_{\Omega_T\times Y} D_t \varphi^*(w(x,y,t))\, dxdtdy \\ &\quad = \iint_{\Omega\times Y} [\varphi^*(w(x,y,T)) - \varphi^*(w^0(x,y))]\, dxdy. \end{aligned} \tag{5.23}$$

By (5.21), using (5.22) and (5.23), we have

$$\liminf_{\varepsilon\to 0} -\iint_{\Omega_T} \nabla u_\varepsilon \cdot \vec z_\varepsilon\, dxdt \ge -\iint_{\Omega_T} \nabla u \cdot \vec z\, dxdt. \tag{5.24}$$

By passing to the inferior limit in (5.17) and using (A.12), we then get

$$\iiint_{\Omega_T\times Y} [f_{\vec\gamma}(\nabla u + \nabla_y u_1, \vec z, y) - (\nabla u + \nabla_y u_1)\cdot \vec z\,]\, dxdtdy \le 0, \tag{5.25}$$

which is tantamount to (5.4). □

6. Single-scale formulation (homogenization)

In this section we derive a single-scale formulation (i.e., a homogenized problem) from the two equivalent two-scale Problems 5.1 and 5.2, and prove a homogenization theorem. Along the lines of the previous sections, we introduce two equivalent formulations, in which the constitutive relations are respectively expressed either as inclusions or as null-minimization principles.

Let the convex function φ_0 and the maximal monotone map $\vec\gamma_0$ be respectively defined as in (A.16) and (A.17). Here is our first single-scale formulation.

Problem 6.1. Find

$$u \in L^2(0,T;V), \qquad w \in L^2(\Omega_T) \cap H^1(0,T;V'), \qquad \vec z \in L^2(\Omega_T)^N, \tag{6.1}$$

such that

$$\iint_{\Omega_T} \big[(w_0 - w)D_t v + (\vec{z} + \vec{h}) \cdot \nabla v\big]\, dxdt = 0 \quad \forall v \in H^1(0,T;V),\ v|_{t=T} = 0, \tag{6.2}$$

$$w \in \partial\varphi_0(u) \quad \text{a.e. in } \Omega_T, \tag{6.3}$$

$$\vec{z} \in \vec{\gamma}_0(\nabla u) \quad \text{a.e. in } \Omega_T. \tag{6.4}$$

We already know that the weak equation (6.2) is equivalent to the PDE

$$D_t w - \nabla\cdot\vec{z} = \nabla\cdot\vec{h} \quad \text{in } V', \text{a.e. in }]0,T[, \tag{6.5}$$

coupled with the initial condition

$$w(\cdot,0) = w^0 \quad \text{a.e. in } \Omega. \tag{6.6}$$

Let us next define the space

$$\widetilde{X}_0 = \{(w,\vec{z}) \text{ as in (6.1) that fulfill (6.2)}\}, \tag{6.7}$$

the mutually orthogonal spaces

$$\begin{gathered} W = \{\nabla\phi : \phi \in W^{1,p}_\sharp(Y)\}, \\ Z = \left\{\vec{v} \in L^{p'}(Y) : \int_Y w(y)\,dy = 0, \nabla\cdot\vec{v} = 0\right\}, \end{gathered} \tag{6.8}$$

and the functionals

$$F_0(\vec{\xi},\vec{\eta}) = \inf_{\vec{v}\in W,\vec{w}\in Z} \int_Y f_{\vec{\gamma}}(\vec{\xi}+\vec{v}(y),\vec{\eta}+\vec{w}(y),y)\,dy \quad \forall\vec{\xi},\vec{\eta}\in\mathbb{R}^N, \tag{6.9}$$

$$\widetilde{\Phi}_0(u,w,\vec{z}) := \iint_{\Omega_T} [\varphi_0(u) + \varphi_0^*(w) - wu + F_0(\nabla u,\vec{z}) - \nabla u\cdot\vec{z}\,]\,dxdt \quad \forall u \in L^2(0,T;V),\ \forall(w,\vec{z})\in\widetilde{X}_0. \tag{6.10}$$

We are now able to introduce another single-scale formulation.

Problem 6.2. Find $u \in L^2(0,T;V)$ and $(w,\vec{z}) \in \widetilde{X}_0$ such that

$$\widetilde{\Phi}_0(u,w,\vec{z}) = \inf_{L^2(0,T;V)\times\widetilde{X}_0} \widetilde{\Phi}_0 = 0. \tag{6.11}$$

Proposition 6.3. *The single-scale Problems 6.1 and 6.2 are mutually equivalent.*

Proof. This argument mimics that of Theorem 3.3. The null-minimization of $\widetilde{\Phi}_0$ is equivalent to the system of the two inequalities

$$\iint_{\Omega_T} [\varphi_0(u) + \varphi_0^*(w) - wu]\,dxdt \le 0, \tag{6.12}$$

$$\iint_{\Omega_T} [F_0(\nabla u,\vec{z}) - \nabla u\cdot\vec{z}\,]\,dxdt \le 0, \tag{6.13}$$

which are respectively equivalent to (6.3) and (6.4). □

We shall use the two-scale decomposition

$$\begin{aligned} \widehat{u}(x) &:= \int_Y u(x,y)\,dy \\ \widetilde{u}(x,y) &:= u(x,y) - \widehat{u}(x) \end{aligned} \qquad \text{for a.e. } (x,y)\in\Omega\times Y. \tag{6.14}$$

Proposition 6.4. *If $(u,u_1,w,\vec{z})$ is a solution of Problem 5.1 or equivalently of Problem 5.2 (such a solution exists by Theorem 5.4), then $(u,\widehat{w},\widehat{\vec{z}})$ is a solution of Problem 6.1 or equivalently of Problem 6.2.*

Proof. Selecting either $v = 0$ or $v_1 = 0$ in Eq. (5.2), we respectively get

$$\iint_{\Omega_T} \big[(\widehat{w_0} - \widehat{w})D_t v + (\widehat{\vec{z}} + \widehat{\vec{h}}) \cdot \nabla v\big]\, dxdt = 0 \quad \forall v \in H^1(0, T; V),\ v|_{t=T} = 0, \tag{6.15}$$

$$\iiint_{\Omega_T \times Y} (\widetilde{\vec{z}} + \widetilde{\vec{h}}) \cdot \nabla_y v_1 \, dxdtdy = 0 \quad \forall v_1 \in L^2(\Omega_T; H^1_\sharp(Y)). \tag{6.16}$$

These integral equations respectively correspond to the following coarse- and fine-scale PDEs:

$$D_t \widehat{w} - \nabla \cdot \widehat{\vec{z}} = \nabla \cdot \widehat{\vec{h}} \quad \text{in } V', \text{ a.e. in }]0, T[, \tag{6.17}$$

$$-\nabla_y \cdot \widetilde{\vec{z}} = \nabla_y \cdot \widetilde{\vec{h}} \quad \text{in } H^1_\sharp(Y)', \text{ a.e. in } \Omega_T. \tag{6.18}$$

By Propositions A.8 and A.9, the single-scale constitutive relations (6.3) and (6.4) follow from (5.3) and (5.4). □

Theorem 6.5. *Let the assumption (3.1)–(3.8), (4.31) be fulfilled. For any $\varepsilon > 0$, let $(u_\varepsilon, w_\varepsilon, \vec{z}_\varepsilon)$ be a solution of Problem 3.1 or equivalently of Problem 3.2 (this exists by Theorem 4.3). Then there exist $u, w, \vec{z}$ as in (6.1) such that, as $\varepsilon \to 0$ along a suitable sequence,*

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^2(0, T; V), \tag{6.19}$$

$$w_\varepsilon \rightharpoonup w \quad \text{in } L^\infty(0, T; H) \cap H^1(0, T; V'), \tag{6.20}$$

$$\vec{z}_\varepsilon \rightharpoonup \vec{z} \quad \text{in } L^2(\Omega_T)^N. \tag{6.21}$$

This entails that $(u, w, \vec{z})$ is a solution of the homogenized Problem 6.1, or equivalently of Problem 6.2.

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Appendix

Here we briefly review the notion of two-scale convergence, and some related properties of integral functionals.

A.1. Two-scale convergence

This notion was introduced by Nguetseng [19], and was further developed by Allaire and others, see e.g. [1]; see also the survey [14].

Let us denote by $Y =]0, 1[^N$ the fundamental periodicity-cell, and by $\varepsilon > 0$ a small parameter which we shall let eventually vanish. Let us fix any $p \in]1, +\infty[$ and define the conjugate index $p' := p/(p-1)$. Let us denote by $C_\sharp(Y)$ ($W^{1,p}_\sharp(Y)$, resp.) the space of continuous ($W^{1,p}_\sharp$, resp.) functions $\mathbb{R}^N \to \mathbb{R}$ that are Y-periodic and have equal traces on opposite faces of Y. By the index $*$ we shall denote subspaces of functions with vanishing average: e.g., $L^1_*(Y) = \{w \in L^1(Y) : \int_Y w(y)\, dy = 0\}$.

Definition A.1 (*Weak Two-Scale Convergence*). We shall say that a sequence $\{u_\varepsilon\}$ of functions in $L^p(\Omega)$ *weakly two-scale converges* to a limit function $u \in L^p(\Omega \times Y)$, and write $u_n \underset{2}{\rightharpoonup} u$, whenever

$$\int_\Omega u_\varepsilon(x)\phi(x, x/\varepsilon)\, dx \to \iint_{\Omega \times Y} u(x, y)\phi(x, y)\, dxdy \quad \forall \phi \in L^{p'}(\Omega; C_\sharp(Y)). \tag{A.1}$$

For instance, $x\sin(2\pi x/\varepsilon) \underset{2}{\rightharpoonup} x\sin(2\pi y)$ in $L^p(]0,1[\times Y)$ for any $p\in]1,+\infty[$. Notice that the weak two-scale limit is unique, if it exists.

This definition is trivially extended to time-dependent functions. For any $p, r \in]1,+\infty[$, we shall say that a family $\{u_\varepsilon\}$ of functions in $L^r(0,T,L^p(\Omega))$ *weakly two-scale converges* to a limit $u\in L^r(0,T,L^p(\Omega\times Y))$ whenever

$$\iint_{\Omega_T} u_\varepsilon(x,t)\phi(x,x/\varepsilon,t)\,dxdt \to \iiint_{\Omega_T\times Y} u(x,y,t)\phi(x,y,t)\,dxdydt \quad \forall\phi\in L^{r'}(0,T,L^{p'}(\Omega;C_\sharp(Y))). \tag{A.2}$$

The results that follow also trivially take over to time-dependent functions.

Theorem A.2. *If $\{u_\epsilon\}$ is a bounded sequence in $L^p(\Omega)$ ($p\in]1,+\infty[$), then there exists $u\in L^p(\Omega\times Y)$ such that, as $\varepsilon\to 0$ along a suitable subsequence, $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\Omega\times Y)$.*

For any measurable function $u:\Omega\times Y\to\mathbb{R}$ such that $u(x,\cdot)\in L^1(Y)$ for a.e. $x\in\Omega$, we define the average component $\widehat{u}$ and the fluctuating component $\widetilde{u}$ as follows:

$$\begin{aligned} \widehat{u}(x) &:= \int_Y u(x,y)\,dy \\ \widetilde{u}(x,y) &:= u(x,y)-\widehat{u}(x) \end{aligned} \qquad \text{for a.e. } (x,y)\in\Omega\times Y. \tag{A.3}$$

Thus $\widetilde{u}(x,\cdot)\in L^1_*(Y)$ for a.e. $x\in\Omega$.

Proposition A.3. *If $\{u_\varepsilon\}$ is a sequence in $L^p(\Omega)$ ($p\in]1,+\infty[$) that two-scale converges to $u\in L^p(\Omega\times Y)$, then*

$$\liminf_{\varepsilon\to 0}\|u_\varepsilon\|_{L^p(\Omega)} \ge \|u\|_{L^p(\Omega\times Y)} \ge \|\widehat{u}\|_{L^p(\Omega)}. \tag{A.4}$$

Definition A.4 (*Strong Two-Scale Convergence*)**.** We shall say that a sequence $\{u_\varepsilon\}$ of functions in $L^p(\Omega)$ ($p\in]1,+\infty[$) strongly two-scale converges to $u=u(x,y)$ in $L^p(\Omega\times Y)$, and write $u_\varepsilon \underset{2}{\to} u$, whenever

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega\times Y), \quad \text{and} \quad \|u_\varepsilon\|_{L^p(\Omega)} \to \|u\|_{L^p(\Omega\times Y)}.$$

For instance the sequence $x\sin(2\pi x/\varepsilon)$ strongly two-scale converges to $x\sin(2\pi y)$, whereas $x\sin(2\pi x/\varepsilon)+x\sin(2\pi x/\varepsilon^2)$ just weakly two-scale converges to $x\sin(2\pi y)$.

The next result is one of the major tools for the application of two-scale convergence to the homogenization of PDEs.

Theorem A.5. *Let $\{u_\varepsilon\}$ be a bounded sequence in $W^{1,p}(\Omega)$ ($p\in]1,+\infty[$). Then there exist $(u,u_1)\in W^{1,p}(\Omega)\times L^p(\Omega, W^{1,p}_\sharp(Y))$ such that, as $\varepsilon\to 0$ along a suitable subsequence,*

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } W^{1,p}(\Omega), \\ u_\varepsilon \underset{2}{\to} u & \text{in } L^p(\Omega), \\ \nabla u_\varepsilon \underset{2}{\rightharpoonup} \nabla u + \nabla_y u_1 & \text{in } L^p(\Omega)^N. \end{cases} \tag{A.5}$$

A.2. On the measurability of multivalued mappings

Let us assume that D is a domain of $\mathbb{R}^N$ (e.g., referring to previous notation, $D=\Omega$ or $D=Y$). Let us denote by $\mathcal{B}(D)$ and $\mathcal{L}(D)$ the σ-algebras of Borel- and Lebesgue-measurable functions $D\to\mathbb{R}$, respectively, and by $\mathcal{B}(B)\otimes\mathcal{L}(D)$ the σ-algebra generated by the Cartesian product. Let B be a separable and reflexive real Banach space with dual B'. We remind the reader that a multivalued mapping $g:D\to\mathcal{P}(B')$ is called *measurable* if

$$g^{-1}(R) := \left\{x\in D : g(x)\cap R\neq\emptyset\right\} \quad \text{is measurable, for any open set } R\subset B'. \tag{A.6}$$

By a classical theorem of Pettis, see e.g. [28], it is equivalent to refer to measurability with respect to the weak or to the strong topology of the separable space B'.

Moreover g is called *closed-valued* if $g(x)$ is closed for a.e. $x\in D$. It is known that if g is closed-valued and measurable, then it has a measurable selection, see e.g. Section III.2 of [6] or Section 8.1 of [12]. This means that there exists a measurable single-valued mapping $f:D\to B'$ such that $f(x)\in g(x)$ for a.e. $x\in D$.

Lemma A.6. *Let us assume that*

$$\begin{aligned} &\alpha : B \times D \to \mathcal{P}(B') \quad \text{is measurable w.r.t. } \mathcal{B}(B) \otimes \mathcal{L}(D), \\ &\alpha(\xi, x) \text{ is closed for any } \xi \text{ and a.e. } x. \end{aligned} \tag{A.7}$$

For any $\mathcal{L}(D)$-measurable mapping $v : D \to B$, the multivalued mapping $\beta : D \to \mathcal{P}(B') : x \mapsto \alpha(v(x), x)$ is then closed-valued and measurable.

Proof. Let us set $\gamma_v(x) = (v(x), x)$ for any $x \in D$, so that $\beta = \alpha \circ \gamma_v$ in D. The mapping $\gamma_x : D \to B \times D$ is clearly measurable. Because of (A.7) $_2$ the set $\beta(x)$ is closed for a.e. x. For any open set $R \subset B'$, by (A.7)$_1$$\alpha^{-1}(R) \in \mathcal{B}(B) \otimes \mathcal{L}(D)$. By the $\mathcal{L}(D)$-measurability of v, we conclude that $\beta^{-1}(R) = \gamma_v^{-1}(\alpha^{-1}(R)) = \{x \in D : (v(x), x) \in \alpha^{-1}(R)\} \in \mathcal{L}(D)$. □

A.3. Two-scale limit of integral functionals

Proposition A.7 *([23]).*

(i) *If $\phi : \mathbb{R}^N \times Y \to \mathbb{R} \cup \{+\infty\}$ is $\mathcal{B}(\mathbb{R}^N) \bigotimes \mathcal{L}(Y)$-measurable, then for any measurable function $v : \Omega \to \mathbb{R}^N$, the mappings $x \mapsto \phi(v(x), x/\varepsilon)$ and $(x, y) \mapsto \phi(v(x, y), y)$ are measurable.*

(ii) *Let ϕ be also convex with respect to the first variable for a.e. y, and assume that there exist $C \in \mathbb{R}^N$ and $h \in L^1(Y)$ such that*

$$\phi(\vec{v}, y) \geq C \cdot \vec{v} + h(y) \quad \forall \vec{v} \in \mathbb{R}^N, \ \text{for a.e. } y \in Y. \tag{A.8}$$

Let us define the functionals $\Psi_\varepsilon : L^p(\Omega)^N \to \mathbb{R} \cup \{+\infty\}$ and $\Psi : L^p(\Omega \times Y)^N \to \mathbb{R} \cup \{+\infty\}$ by

$$\Psi_\varepsilon(\vec{v}) := \int_\Omega \phi(\vec{v}(x), x/\varepsilon)\, dx \quad \forall \vec{v} \in L^p(\Omega)^N, \tag{A.9}$$

$$\Psi(\vec{v}) := \iint_{\Omega \times Y} \phi(\vec{v}(x, y), y)\, dxdy \quad \forall \vec{v} \in L^p(\Omega \times Y)^N. \tag{A.10}$$

These functionals are well-defined, convex and lower semicontinuous, respectively in $L^p(\Omega)^N$ and $L^p(\Omega \times Y)^N$.

(iii) *Under the above assumptions, for any sequence $\{\vec{v}_\varepsilon\}$ in $L^p(\Omega)^N$,*

$$\vec{u}_\varepsilon \underset{2}{\to} \vec{u} \quad \text{in } L^p(\Omega \times Y)^N \Rightarrow \Psi_\varepsilon(\vec{v}_\varepsilon) \to \Psi(\vec{v}), \tag{A.11}$$

$$\vec{u}_\varepsilon \underset{2}{\rightharpoonup} \vec{u} \quad \text{in } L^p(\Omega \times Y)^N \Rightarrow \liminf_{\varepsilon \mapsto 0} \Psi_\varepsilon(\vec{v}_\varepsilon) \geq \Psi(\vec{v}). \tag{A.12}$$

It is known that the convex conjugate functionals Ψ_ε^* and Ψ^* then coincide with the integral functionals of the convex conjugate of the respective integrands.

A.4. Scale-integration of cyclically maximal monotone operators

Let us first set

$$W = \{\nabla \phi : \phi \in W^{1,p}_\sharp(Y)\}, \qquad Z = \{\vec{v} \in L^{p'}_*(Y) : \nabla \cdot \vec{v} = 0\}, \tag{A.13}$$

and notice the following orthogonality relation:

$$\int_Y \vec{u}(y) \cdot \vec{w}(y)\, dy = 0 \quad \forall \vec{u} \in W, \ \forall \vec{w} \in Z. \tag{A.14}$$

Let us assume that

$$\begin{aligned} &\varphi : \mathbb{R}^N \times Y \to \mathbb{R}^N \quad \text{is measurable w.r.t. } \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{L}(Y), \\ &\exists p \in]1, +\infty[, \exists c_1, \ldots, c_4 > 0 : c_1|\vec{\xi}|^p - c_2 \leq \varphi(\vec{\xi}, y) \leq c_3|\vec{\xi}|^p + c_4 \quad \forall \vec{\xi} \in \mathbb{R}^N, \ \text{for a.e. } y \in Y, \end{aligned} \tag{A.15}$$

and set

$$\varphi_0(\vec{\xi}) = \inf_{\vec{v} \in W} \int_Y \varphi(\vec{\xi} + \vec{v}(y), y)\, dy \quad \forall \vec{\xi} \in \mathbb{R}^N, \tag{A.16}$$

$$\psi_0(\vec{\eta}) = \inf_{\vec{v} \in Z} \int_Y \varphi^*(\vec{\eta} + \vec{v}(y), y)\, dy \quad \forall \vec{\eta} \in \mathbb{R}^N. \tag{A.17}$$

Proposition A.8 ([25]). *The function ψ_0 is the convex conjugate of φ_0.*
If $\vec{u} \in L^p(Y)^N$ and $\vec{w} \in L^{p'}(Y)^N$ are such that (using the notation (6.14))

$$\widetilde{\vec{u}} \in W, \qquad \widetilde{\vec{w}} \in Z, \tag{A.18}$$

$$\vec{w}(y) \in \partial\varphi(\vec{u}(y), y) \quad \text{for a.e. } y \in Y, \tag{A.19}$$

then

$$\widehat{\vec{w}} \in \partial\varphi_0(\widehat{\vec{u}}), \qquad \widehat{\vec{u}} \in \partial\psi_0(\widehat{\vec{w}}), \tag{A.20}$$

$$\varphi_0(\widehat{\vec{u}}) = \int_Y \varphi(\vec{u}(y), y)\, dy, \qquad \psi_0(\widetilde{\vec{w}}) = \int_Y \varphi^*(\vec{w}(y), y)\, dy. \tag{A.21}$$

(The reader will notice that the vectors $\widehat{\vec{w}}$ and $\widehat{\vec{u}}$ are constants.)

Next we see that this result takes over to noncyclically maximal monotone operators.

A.5. Scale-integration of maximal monotone operators

Let us assume that

$$\vec{\gamma} : \mathbb{R}^N \times Y \to \mathcal{P}(\mathbb{R}^N) \quad \text{is measurable w.r.t. } \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{L}(Y), \text{ is maximal monotone w.r.t. the first argument for a.e. } y \in Y, \text{ and is represented by a function } f_{\vec{\gamma}}(\cdot, y) \text{ for a.e. } y \in Y. \tag{A.22}$$

Defining W and Z as in (A.13), let us then set

$$F_0(\vec{\xi}, \vec{\eta}) = \inf_{\vec{v} \in W, \vec{w} \in Z} \int_Y f_{\vec{\gamma}}(\vec{\xi} + \vec{v}(y), \vec{\eta} + \vec{w}(y), y)\, dy \quad \forall \vec{\xi}, \vec{\eta} \in \mathbb{R}^N. \tag{A.23}$$

As $f_{\vec{\gamma}}$ is a representative map, for any $\vec{\xi}, \vec{\eta}, \vec{v}, \vec{w}$ as above we have

$$\int_Y f_{\vec{\gamma}}(\vec{\xi} + \vec{v}(y), \vec{\eta} + \vec{w}(y), y)\, dy \geq \int_Y [\vec{\xi} + \vec{v}(y)] \cdot [\vec{\eta} + \vec{w}(y)]\, dy \stackrel{(A.14)}{=} \vec{\xi} \cdot \vec{\eta}. \tag{A.24}$$

By taking the infimum with respect to $\vec{v} \in W$ and $\vec{w} \in Z$ we thus get

$$F_0(\vec{\xi}, \vec{\eta}) \geq \vec{\xi} \cdot \vec{\eta} \quad \forall \vec{\xi}, \vec{\eta} \in \mathbb{R}^N \tag{A.25}$$

so that F_0 is also a representative function.

Proposition A.9. *The function F_0 represents a maximal monotone map $\vec{\gamma}_0 : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$. If $\vec{u} \in L^p(Y)^N$ and $\vec{w} \in L^{p'}(Y)^N$ are such that*

$$\widetilde{\vec{u}} \in W, \qquad \widetilde{\vec{w}} \in Z, \tag{A.26}$$

$$\vec{w} \in \vec{\gamma}(\vec{u}, y) \quad \text{for a.e. } y \in Y, \tag{A.27}$$

then

$$\widehat{\vec{w}} \in \vec{\gamma}_0(\widehat{\vec{u}}). \tag{A.28}$$

(Here also with constant vectors $\widehat{\vec{w}}$ and $\widehat{\vec{u}}$.)

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