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EXACT INTERNAL CONTROLLABILITY FOR THE WAVE EQUATION IN A DOMAIN WITH OSCILLATING BOUNDARY WITH NEUMANN BOUNDARY CONDITION

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Abstract. In this paper, we study the exact controllability of a second order linear evolution equation in a domain with highly oscillating boundary with homogeneous Neumann boundary condition on the oscillating part of boundary. Our aim is to obtain the exact controllability for the homogenized equation. The limit problem with Neumann condition on the oscillating boundary is different and hence we need to study the exact controllability of this new type of problem. In the process of homogenization, we also study the asymptotic analysis of evolution equation in two setups, namely solution by standard weak formulation and solution by transposition method.

1. **Introduction.** In this article, we analyze the exact internal controllability for a boundary-value problem in a domain $\Omega_\varepsilon \subset \mathbf{R}^n$, whose boundary $\partial\Omega_\varepsilon$ contains an oscillating part with respect to ε, as $\varepsilon \to 0$. The oscillating boundary is defined by a set of cylinders with axis Ox_n and $\varepsilon-$periodically distributed on a base Ω^-. Each cylinder has a small cross section of size ε and a fixed height (see Fig.1 for a 3-d example). Boundary-value problems in a domain with highly oscillating boundary are models for problems in biology and in industrial applications: motion of ciliated micro-organisms, flows over rough walls, electromagnetic waves in a region with a rough interface, structures such as bridges on supports, frameworks of houses, etc. Another interesting application is the air flow through compression systems in turbo machines such as jet engine. For example, such a system is modelled by the Viscous-Moore-Greitzer equation derived from Scaled Navier-Stokes equations (see [7], [40], [41]). Here the pitch and size of the rotor - stator pair of blades in the engine provides a small parameter compared to the size of the engine which is oscillatory as well as rotating (moving). The motion of the stator and rotor blades in the compressor produces turbulent flow on a fast time scale. When the engine

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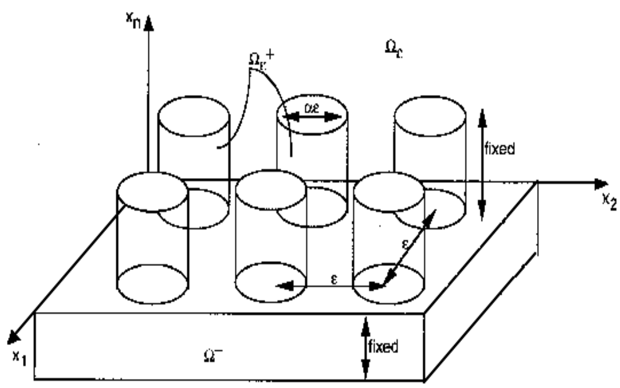

FIGURE 1. Domain Ω_ε

operates close to the optimal parameters, the flow becomes unstable. This model motivates to look into control problems described by Partial Differential Equations (PDEs) of evolution type such as the heat equation or the Navier-Stokes equations.

The computational calculation of the solution of these problems is very complicated, rather impossible due to singularities of the domain. It is much more delicate for control and controllability problems. Therefore, an asymptotic analysis of boundary value problems in such domains gives the possibility to replace the original problem by the corresponding limit problem defined in a "simpler" domain.

In this paper, we plan to study the asymptotic behaviour, as $\varepsilon \to 0$, of an exact controllability for a boundary-value problem described by a hyperbolic equation in a domain Ω_ε with oscillating boundary. A homogeneous Neumann condition is given on the oscillating part of the boundary of the domain. In order to point out the main difficulties, we consider the wave equation. Our approach to the homogenization for the exact controllability problem, for a hyperbolic equation consists in applying the Hilbert Uniqueness Method (HUM) of J.L. Lions (see [32], [33]). The study of the asymptotic behaviour of viscous fluid flow over very rough boundaries was considered in [2], [3], [4].

The limit problem of Neumann boundary-value problems in domains with highly oscillating boundary, that is when the amplitude of the oscillations is constant with respect to ε, are derived in [5], [8], [9], [14], [28], [30]. In [14], R. Brizzi and J.P. Chalot derive the limit problem for the Laplace equation with the homogeneous Neumann boundary condition and with the right-hand side in L^2. For the same problem, a nonoscillating approximation at order $\mathcal{O}\left(\varepsilon^{1-\delta}\right)$, $\delta > 0$, for the H^1-norm is obtained by T.A. Mel'nyk in [37], under an additional assumption on the right-hand side. Optimal control problems and the exact controllability in domains with highly oscillating boundary are considered in [19, 20, 21, 22, 23, 25, 26, 43, 44]. In [43], [44] A.K. Nandakumaran et. al. have also studied the asymptotic analysis of an optimal control problem and error estimates were also obtained in the papers. In [27] L. Faella and C. Perugia have studied the optimal control problem for an evolutionary imperfect transmission problem. The p-Laplacian was studied in [17] by A. Corbo Esposito, P. Donato, A. Gaudiello and C. Picard.

The plan of the paper is as follows. In Section 2, we describe the domain with appropriate spaces required. In fact, the limit problem with Neumann condition is different and hence we need different spaces to study it. In the same section, we introduce the controllability problem and the Hilbert uniqueness method. The important result we show is that the constant appearing in the observability estimate is independent of ε. The main homogenization result is also presented here. The Hilbert uniqueness methods is introduced via a forward and adjoint problem and one of them is with weak data and hence the solution is defined via transposition method. We study the homogenization of these systems for both regular and weak data and this is done in Section 3 . Further, we state the limit controllability problem also in Section 2 and as this is a new problem, we will study the exact controllability of the limit problem in Section 4. The proof of homogenization theorem is then completed in Section 5.

2. Statement of the problem and main results.

2.1. Domain definition and spaces. Let a, $b \in]0,+\infty[$, B and ω be bounded open smooth subsets of $\mathbf{R}^{n-1}$ $(n \geq 2)$ with $\omega \subset\subset]0,1[^{n-1}$, and $\{\varepsilon\}$ be a sequence of positive numbers converging to zero. We introduce the domain $\Omega_\varepsilon \subset \mathbf{R}^n$ with highly oscillating boundary (see Fig.1):

$$\Omega_\varepsilon = (B \times]-a,0[) \cup \left(\bigcup_{\mathbf{k}\in J_\varepsilon} (\varepsilon\omega + \varepsilon\mathbf{k}) \times [0,b[\right),$$

where $J_\varepsilon = \left\{\mathbf{k} \in \mathbf{N}^{n-1} : \varepsilon\omega + \varepsilon\mathbf{k} \subset\subset B\right\}$. Moreover, we also set $\Omega = B \times]-a,b[$, $\Omega^+ = B \times]0,b[$, $\quad \Omega^- = B \times]-a,0[$, $\quad \Sigma = B \times \{0\}$, and $\Omega^+_\varepsilon = \Omega_\varepsilon \cap \Omega^+$. Finally Γ_ε denotes the interface boundary between Ω^+_ε and Ω^-, i. e.

$$\Gamma_\epsilon = \bigcup_{\mathbf{k}\in J_\varepsilon} (\varepsilon\omega + \varepsilon\mathbf{k}) \times \{0\}. \tag{1}$$

We recall that

$$\chi_{\Omega^+_\varepsilon} \rightharpoonup |\omega| \quad \text{weakly } * \ L^\infty\left(\Omega^+\right), \quad \chi_{\Omega_\varepsilon\cap\Sigma} \rightharpoonup |\omega| \quad \text{weakly } * \ L^\infty\left(\Sigma\right), \tag{2}$$

where $|\omega|$ denotes the $(n-1)-$dimensional Lebesgue measure of ω and χ_A the characteristic function of a set A. Define the step function

$$\eta(x) = \begin{cases} |\omega| \ \text{ if } \ x \in \Omega^+, \\ 0 \ \text{ if } \ x \in \Omega^-. \end{cases} \tag{3}$$

In the sequel, $x = (x_1, x_2, .., x_{n-1}, x_n) = (x', x_n)$ will denote a generic point of $\mathbf{R}^n$. Moreover, $\widetilde{v}$ will denote the zero-extension to Ω (resp. $]0,T[\times\Omega$) of a function v defined on A (resp. $]0,T[\times A$) , with $A \subset \Omega$. Furthermore, v^+ (resp. v^-) denote the restriction of v to Ω^+ (resp. Ω^-), if v is defined on Ω; the restriction of v to $]0,T[\times\Omega^+$ (resp. $]0,T[\times\Omega^-$) if v is defined on $]0,T[\times\Omega$.

We now introduce the following spaces.

$$\mathcal{V}\left(\Omega_\varepsilon\right) = \left\{ z : z \in H^1\left(\Omega_\varepsilon\right) : \ z_{|_{\partial\Omega^-\setminus\Sigma}} = 0 \right\},$$

$$\mathcal{V}\left(\Omega\right) = \left\{ z : z \in H^1\left(\Omega\right) : \ z_{|_{\partial\Omega^-\setminus\Sigma}} = 0 \right\},$$

$$\mathcal{V}\left(\Omega^{-}\right)=\left\{z: z\in H^{1}\left(\Omega^{-}\right):\ z_{|_{\partial\Omega^{-}\setminus\Sigma}}=0\right\},$$

and

$$\mathcal{W}_{\epsilon}=\left\{v: v\in L^{2}\left(0,T;\ \mathcal{V}\left(\Omega_{\varepsilon}\right)\right),\ v'\in L^{2}\left(0,T;\ L^{2}\left(\Omega_{\varepsilon}\right)\right)\right\}.$$

We observe that the spaces $\mathcal{V}(\Omega_{\varepsilon})$, $\mathcal{V}(\Omega^{-})$ and $\mathcal{V}(\Omega)$ endowed with the norm of gradient in L^2 are Hilbert and $\mathcal{W}_\epsilon$ is a Banach space with respect to the graph norm defined by

$$\|v\|_{\mathcal{W}_{\varepsilon}}=\|v\|_{L^{2}(0,T;\ \mathcal{V}(\Omega_{\varepsilon}))}+\|v'\|_{L^{2}(0,T;\ L^{2}(\Omega_{\varepsilon}))}.$$

Since the limit problem is going to be drastically different in the case of Neumann problem, we also need the following space

$$V\left(\Omega\right)=\left\{z\in L^{2}\left(\Omega\right)\quad:\quad \frac{\partial z}{\partial x_{n}}\in L^{2}\left(\Omega^{+}\right),\ z\in H^{1}\left(\Omega^{-}\right),\ z^{+}=z^{-}\text{ on }\Sigma,\ z_{|_{\partial\Omega^{-}\setminus\Sigma}}=0\right\}. \tag{4}$$

The space $V(\Omega)$ endowed with the norm

$$\|z\|^{2}_{V(\Omega)}=\|z\|^{2}_{L^{2}(\Omega^{+})}+\left\|\frac{\partial z}{\partial x_{n}}\right\|^{2}_{L^{2}(\Omega^{+})}+\|z\|^{2}_{H^{1}(\Omega^{-})}\qquad z\in V\left(\Omega\right)$$

is a Hilbert space and $H^1(\Omega)$ is dense in $V(\Omega)$ with continuous injection (see [17]). The dual of $V(\Omega)$ is denoted by $V(\Omega)'$.

2.2. **Exact controllability problem in Ω_ε.** Now, we formulate our exact controllability problem for a hyperbolic equation in Ω_ε. For a control $\theta_\varepsilon \in L^2(0,T;\ L^2(\Omega_\varepsilon))$, the state u_ε of the system solves the following problem:

$$\begin{cases} u_{\varepsilon}''-\Delta u_{\varepsilon}+u_{\varepsilon}=\theta_{\varepsilon} & \text{in }]0,T[\times\Omega_{\varepsilon},\\ \dfrac{\partial u_{\varepsilon}}{\partial\nu}=0 & \text{in }]0,T[\times\left(\partial\Omega_{\varepsilon}^{+}\setminus\Gamma_{\varepsilon}\right),\\ u_{\varepsilon}=0 & \text{in }]0,T[\times\left(\partial\Omega^{-}\setminus\Sigma\right),\\ u_{\varepsilon}(0)=u_{\varepsilon}^{0},\ u_{\varepsilon}'(0)=u_{\varepsilon}^{1} & \text{in } \Omega_{\varepsilon}, \end{cases} \tag{5}$$

where $(u^0_\varepsilon, u^1_\varepsilon)\in H^1(\Omega_\varepsilon)\times L^2(\Omega_\varepsilon)$, $T>0$ and ν denotes the exterior unit normal to Ω_ε. It is well known (see [32] Lemma 2.1 p.181) that problem (5) admits a unique weak solution $u_\varepsilon = u_\varepsilon(\theta_\varepsilon)$:

$$\begin{cases} u_{\varepsilon}\in\mathcal{W}_{\varepsilon},\\ \displaystyle\int_{0}^{T}\int_{\Omega_{\varepsilon}}u_{\varepsilon}zh''+\nabla_{x}u_{\varepsilon}\nabla zh+u_{\varepsilon}zh\ dxdt\\ \displaystyle=\int_{0}^{T}\int_{\Omega_{\varepsilon}}\theta_{\varepsilon}zh\ dxdt\qquad\forall z\in\mathcal{V}\left(\Omega_{\varepsilon}\right),\ \forall h\in C_{0}^{\infty}\left(]0,T[\right),\\ u_{\varepsilon}(0)=u_{\varepsilon}^{0},\ u_{\varepsilon}'(0)=u_{\varepsilon}^{1}\quad\text{in } \Omega_{\varepsilon}. \end{cases} \tag{6}$$

Remark 1. Let us point out that the solution u_ε of problem (5) has more regularity. In fact, we have $u_\varepsilon\in C([0,T];\ \mathcal{V}(\Omega_\varepsilon))\cap C^1([0,T];\ L^2(\Omega_\varepsilon))$ and $u''_\varepsilon\in L^2(0,T;\ (\mathcal{V}(\Omega_\varepsilon))')$ (see [46], [49]).

Definition 2.1 (Exact Controllability)**.** We say that system (5) is exactly controllable at time T if for every $\left(u_\varepsilon^0, u_\varepsilon^1\right), \left(v_\varepsilon^0, v_\varepsilon^1\right) \in \mathcal{V}\left(\Omega_\varepsilon\right) \times L^2\left(\Omega_\varepsilon\right)$, there exists a control $\theta_\varepsilon \in L^2\left(0, T; L^2\left(\Omega_\varepsilon\right)\right)$ such that the corresponding solution of problem (5) satisfies

$$u_\varepsilon(T) = v_\varepsilon^0, \quad u_\varepsilon'(T) = v_\varepsilon^1.$$

It is well known that for the above linear system, driving the system to any state is equivalent of driving the system to null state and this is known as null controllability. In other words, (5) is *null controllable* if there exists a control $\theta_\varepsilon \in L^2\left(0, T; L^2\left(\Omega_\varepsilon\right)\right)$ such that $u_\varepsilon(T) = u_\varepsilon'(T) = 0$.

A constructive method to determine the control θ_ε such that $u_\varepsilon(T) = 0$ and $u_\varepsilon'(T) = 0$ is the Hilbert Uniqueness Method (HUM) introduced by Lions (see [32], [33]). The idea is to build a control as the solution of a transposed problem associated to some initial conditions. These initial conditions are obtained by calculating at zero time the solution of a backward problem. The source term of the backward problem is the unique solution of the transposed problem. The control obtained by HUM is also a energy minimizing control. We briefly outline the HUM procedure. Let $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) \in L^2\left(\Omega_\varepsilon\right) \times \left(\mathcal{V}\left(\Omega_\varepsilon\right)\right)'$ and consider the problem

$$\begin{cases} \varphi_\varepsilon'' - \Delta\varphi_\varepsilon + \varphi_\varepsilon = 0 & \text{in }]0, T[\times \Omega_\varepsilon, \\ \dfrac{\partial \varphi_\varepsilon}{\partial \nu} = 0 & \text{in }]0, T[\times \left(\partial\Omega_\varepsilon^+ \setminus \Gamma_\varepsilon\right), \\ \varphi_\varepsilon = 0 & \text{in }]0, T[\times \left(\partial\Omega^- \setminus \Sigma\right), \\ \varphi_\varepsilon(0) = \varphi_\varepsilon^0,\ \varphi_\varepsilon'(0) = \varphi_\varepsilon^1 & \text{a.e. in } \Omega_\varepsilon. \end{cases} \tag{7}$$

Since the initial data is in a weak space, one need to apply the so called transposition method (see [36], Example 4 p. 296) to obtain a unique solution $\varphi_\varepsilon \in C\left([0, T];\ L^2\left(\Omega_\varepsilon\right)\right) \cap C^1\left([0, T];\ \left(\mathcal{V}\left(\Omega_\varepsilon\right)\right)'\right)$ to the problem (7). Now, let $\psi_\varepsilon \in C\left([0, T];\ \mathcal{V}\left(\Omega_\varepsilon\right)\right) \cap C^1\left([0, T];\ L^2\left(\Omega_\varepsilon\right)\right)$ be the unique solution of the backward problem

$$\begin{cases} \psi_\varepsilon'' - \Delta\psi_\varepsilon + \psi_\varepsilon = -\varphi_\varepsilon & \text{in }]0, T[\times \Omega_\varepsilon, \\ \dfrac{\partial \psi_\varepsilon}{\partial \nu} = 0 & \text{in }]0, T[\times \left(\partial\Omega_\varepsilon^+ \setminus \Gamma_\varepsilon\right), \\ \psi_\varepsilon = 0 & \text{in }]0, T[\times \left(\partial\Omega^- \setminus \Sigma\right), \\ \psi_\varepsilon(T) = 0,\ \psi_\varepsilon'(T) = 0 & \text{in } \Omega_\varepsilon, \end{cases} \tag{8}$$

where φ_ε is the solution of the problem (7). The weak formulation of problem (8) (see [49]) is given by

$$\begin{cases} \psi_\varepsilon \in \mathcal{W}_\varepsilon, \\ \displaystyle\int_0^T \int_{\Omega_\varepsilon} \psi_\varepsilon z h'' + \nabla_x \psi_\varepsilon \nabla z h + \psi_\varepsilon z h \; dxdt \\ \displaystyle = -\int_0^T \int_{\Omega_\varepsilon} \varphi_\varepsilon z h \; dxdt \qquad \forall z \in \mathcal{V}\left(\Omega_\varepsilon\right),\ \ \forall h \in C_0^\infty\left(]0, T[\right), \\ \psi_\varepsilon(T) = 0, \quad \psi_\varepsilon'(T) = 0 \ \text{ in } \Omega_\varepsilon. \end{cases} \tag{9}$$

Inspired by HUM, we introduce the linear operator

$$\Lambda_\varepsilon : L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))' \longrightarrow L^2(\Omega_\varepsilon) \times \mathcal{V}(\Omega_\varepsilon)$$

by setting for all $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$,

$$\Lambda_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1) = (-\psi_\varepsilon'(0), \psi_\varepsilon(0)), \tag{10}$$

where ψ_ε is the solution of the problem (8). Moreover, it results

$$\begin{aligned}\left\langle \Lambda_\varepsilon(\varphi_\varepsilon^0, \varphi_\varepsilon^1), (\varphi_\varepsilon^0, \varphi_\varepsilon^1)\right\rangle &= \left\langle (-\psi_\varepsilon'(0), \psi_\varepsilon(0)), (\varphi_\varepsilon^0, \varphi_\varepsilon^1)\right\rangle \\ &= \left\langle \varphi_\varepsilon^1, \psi_\varepsilon(0)\right\rangle_{(\mathcal{V}(\Omega_\varepsilon))',\, \mathcal{V}(\Omega_\varepsilon)} - \int_{\Omega_\varepsilon} \varphi_\varepsilon^0 \psi_\varepsilon'(0)\, dx,\end{aligned} \tag{11}$$

for every $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$.

Remark 2. For each $\varepsilon > 0$, the operator Λ_ε is linear, continuous and injective. If Λ_ε is surjective then, we define the control $\theta_\varepsilon \in L^2((0,T) \times \Omega_\varepsilon)$ by $\theta_\varepsilon = -\varphi_\varepsilon$, where φ_ε is the solution of the problem (7) with initial data $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) = \Lambda_\varepsilon^{-1}(-u_\varepsilon^1, u_\varepsilon^0)$. The state is given by $u_\varepsilon = \psi_\varepsilon$, where ψ_ε is the solution of the problem (8). So we obtain the exact controllability in $\mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ at time T for the system (5).

The aim of this paper is to study the asymptotic behaviour, as $\varepsilon \to 0$, of the sequence of the control pairs $\{(u_\varepsilon, \theta_\varepsilon)\}_\varepsilon$, under the following assumptions:

$$\begin{cases} \widetilde{u_\varepsilon^0} \rightharpoonup (|\omega| \chi_{\Omega^+} + \chi_{\Omega^-}) u^0 = \eta u^0 & \text{weakly in } L^2(\Omega), \\ \widetilde{u_\varepsilon^1} \rightharpoonup (|\omega| \chi_{\Omega^+} + \chi_{\Omega^-}) u^1 = \eta u^1 & \text{weakly in } L^2(\Omega), \end{cases} \tag{12}$$

where η is defined as in (3).

2.3. **Limit controllability problem.** We now introduce the limit exact controllability problem: Given control θ, consider the exact controllability for the following problem:

$$\begin{cases} u'' - \dfrac{\partial^2 u}{\partial x_n^2} + u = \theta & \text{in }]0,T[\times \Omega^+, \\ u'' - \Delta u + u = \theta & \text{in }]0,T[\times \Omega^-, \\ u^+ = u^-, \quad |\omega| \dfrac{\partial u^+}{\partial x_n} = \dfrac{\partial u^-}{\partial x_n} & \text{on }]0,T[\times \Sigma, \\ \dfrac{\partial u}{\partial x_n} = 0 & \text{on }]0,T[\times (B \times \{b\}), \\ u = 0 & \text{on }]0,T[\times (\partial\Omega^- \setminus \Sigma), \\ u(0) = u^0, \ u'(0) = u^1 & \text{in } \Omega. \end{cases} \tag{13}$$

The above exact controllability problem is new and we will prove in Section 4, the exact controllability for the system. We now state the main result regarding the homogenization which will be proved in Section 5.

Theorem 2.2. *Assume* (12) *and let $T>0$ be the controllability time. Let u_ε be the solution of the controllability problem* (5) *where θ_ε is the exact control given by HUM. Then, there exists $\theta \in L^2\left(0,T;L^2(\Omega)\right)$ such that*

$$\widetilde{\theta_\varepsilon} \rightharpoonup \left(|\omega|\,\chi_{\Omega^+}+\chi_{\Omega^-}\right)\theta=\eta\theta \quad \text{weakly in } L^2\left(0,T;L^2(\Omega)\right) \tag{14}$$

where θ is the exact control for the homogenized system (13) *and η is defined as in* (3).

Moreover

$$\begin{cases} \widetilde{u_\varepsilon(\theta_\varepsilon)} \rightharpoonup |\omega|\,u(\theta) & \text{weakly in } L^2\left(0,T;L^2(\Omega^+)\right), \\ u_\varepsilon(\theta_\varepsilon) \rightharpoonup u(\theta) & \text{weakly in } L^2\left(0,T;H^1(\Omega^-)\right), \\ \widetilde{u'_\varepsilon(\theta_\varepsilon)} = \left(\widetilde{u_\varepsilon(\theta_\varepsilon)}\right)' \rightharpoonup |\omega|\,(u(\theta))' & \text{weakly in } L^2\left(0,T;L^2(\Omega^+)\right), \\ \widetilde{u'_\varepsilon(\theta_\varepsilon)} = \left(\widetilde{u_\varepsilon(\theta_\varepsilon)}\right)' \rightharpoonup (u^-(\theta))' & \text{weakly in } L^2\left(0,T;L^2(\Omega^-)\right), \end{cases} \tag{15}$$

where $u(\theta)$ is the unique solution of the problem (13).

To consider the weak formulation of problem (13), we recall that any function in $L^2(\Omega^+)$ having weak derivative with respect to x_n in $L^2(\Omega^+)$ admits a trace on Σ. We need the space $V(\Omega)$ introduced earlier.

The weak formulation of problem (13) is the following one:

$$\begin{cases} u \in L^2(0,T;V(\Omega)),\ u' \in L^2\left([0,T];L^2(\Omega)\right), \\ |\omega| \displaystyle\int_0^T\int_{\Omega^+} uzh'' + \frac{\partial u}{\partial x_n}\frac{\partial z}{\partial x_n}h + uzh\,dxdt + \int_0^T\int_{\Omega^-} uzh'' + \nabla_x u\nabla zh + uzh\,dxdt \\ = |\omega| \displaystyle\int_0^T\int_{\Omega^+}\theta zh\,dxdt + \int_0^T\int_{\Omega^-}\theta zh\,dxdt \quad \forall z\in V(\Omega),\ \forall h \in C_0^\infty(]0,T[), \\ u(0)=u^0,\ u'(0)=u^1 \quad \text{in } \Omega. \end{cases} \tag{16}$$

The following lemma provides an explicit formula for the operator Λ_ε.

Lemma 2.3. *Let us fix $\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right) \in L^2(\Omega_\varepsilon)\times\left(\mathcal{V}(\Omega_\varepsilon)\right)'$. Let φ_ε be the corresponding solution of problem* (7). *Then the following identity holds*

$$\left\langle \Lambda_\varepsilon\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right),\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\right\rangle = \int_0^T\int_{\Omega_\varepsilon}|\varphi_\varepsilon|^2\,dxdt. \tag{17}$$

Proof. Multiplying equation in (7) by ψ_ε yields

$$0=\int_0^T\int_{\Omega_\varepsilon}\left(\varphi_\varepsilon''-\Delta\varphi_\varepsilon+\varphi_\varepsilon\right)\psi_\varepsilon dxdt = \int_{\Omega_\varepsilon}\left(\varphi_\varepsilon'(T)\,\psi_\varepsilon(T)-\varphi_\varepsilon(T)\,\psi_\varepsilon'(T)\right)dx$$

$$-\int_{\Omega_\varepsilon}\left(\varphi_\varepsilon'(0)\,\psi_\varepsilon(0)-\varphi_\varepsilon(0)\,\psi_\varepsilon'(0)\right)dx+\int_0^T\int_{\Omega_\varepsilon}\left(\psi_\varepsilon''-\Delta\psi_\varepsilon+\psi_\varepsilon\right)\varphi_\varepsilon dxdt.$$

Moreover, by virtue of (8) and (11), identity (17) follows. ∎

As we have mentioned above, our first aim will be to prove that the operator Λ_ε is an isomorphism from $L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$ to $L^2(\Omega_\varepsilon) \times \mathcal{V}(\Omega_\varepsilon)$ for every ε and obtain the estimates independent of ε. This amounts to show the following observability estimate.

Proposition 1. *Let us fix $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$. Let φ_ε be the corresponding solution of problem (7). Then, there exists a positive constant C, independent of ε such that*

$$\left\|\varphi_\varepsilon^0\right\|^2_{L^2(\Omega_\varepsilon)} + \left\|\varphi_\varepsilon^1\right\|^2_{(\mathcal{V}(\Omega_\varepsilon))'} \leq C\int_0^T \int_{\Omega_\varepsilon} |\varphi_\varepsilon|^2 \, dxdt, \tag{18}$$

$$\left\|\Lambda_\varepsilon^{-1}\right\|_{\mathcal{L}(L^2(\Omega_\varepsilon)\times\mathcal{V}(\Omega_\varepsilon),\, L^2(\Omega_\varepsilon)\times(\mathcal{V}(\Omega_\varepsilon))')} \leq C, \tag{19}$$

for every ε.

2.4. **Proof of Proposition 1.** Let us establish a preliminary result.

Lemma 2.4. *Let us fix $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$. Then the solution is defined via the standard weak formulation and let φ_ε be the corresponding solution of (7). Then, there exists a positive constant C, independent of ε such that*

$$E(0) \leq C\int_0^T \int_{\Omega_\varepsilon} |\varphi_\varepsilon'|^2 \, dxdt \tag{20}$$

for every ε, where $E(0) = \dfrac{1}{2}\displaystyle\int_{\Omega_\varepsilon} \left|\nabla\varphi_\varepsilon^0\right|^2 + \left|\varphi_\varepsilon^0\right|^2 + \left|\varphi_\varepsilon^1\right|^2 dx.$

Proof. First note that since $(\varphi_\varepsilon^0, \varphi_\varepsilon^1) \in H^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$, we can define the solution φ_ε of (7) by the usual weak formulation.

Then, the energy $E(t) = \dfrac{1}{2}\displaystyle\int_{\Omega_\varepsilon} |\nabla_x\varphi_\varepsilon(t)|^2 + |\varphi_\varepsilon(t)|^2 + |\varphi_\varepsilon'(t)|^2 \, dx$ is conserved (see [32], Lemma 1.2 p. 183), that is

$$E(t) = E(0), \text{ for every } t \in [0,T]. \tag{21}$$

Let $\rho(t)$ be the function defined by

$$\rho(t) = t^2 (T-t)^2,$$

for every $t \in [0,T]$. By choosing $\eta_\varepsilon(x,t) = \rho(t)\varphi_\varepsilon(x,t)$ as a test function in (7) and integrating by parts, we obtain

$$\int_0^T \int_{\Omega_\varepsilon} \rho(t) \left|\varphi_\varepsilon'\right|^2 dxdt + \int_0^T \int_{\Omega_\varepsilon} \rho'(t)\varphi_\varepsilon\varphi_\varepsilon' dxdt = \int_0^T \int_{\Omega_\varepsilon} \rho(t) \left(|\nabla_x\varphi_\varepsilon|^2 + |\varphi_\varepsilon(t)|^2\right) dxdt. \tag{22}$$

Then, by making use of the Young's inequality, we get

$$\begin{aligned}\int_0^T \int_{\Omega_\varepsilon} \rho'(t)\varphi_\varepsilon\varphi_\varepsilon' dxdt &\leq \gamma\int_0^T \int_{\Omega_\varepsilon} \rho(t)|\varphi_\varepsilon(t)|^2 \, dxdt + C(\gamma)\int_0^T \int_{\Omega_\varepsilon} |\varphi_\varepsilon'|^2 \, dxdt \\ &\leq \gamma\int_0^T \int_{\Omega_\varepsilon} \rho(t)\left(|\nabla_x\varphi_\varepsilon|^2 + |\varphi_\varepsilon(t)|^2\right) dxdt \\ &\quad + C(\gamma)\int_0^T \int_{\Omega_\varepsilon} |\varphi_\varepsilon'|^2 \, dxdt\end{aligned} \tag{23}$$

where $\gamma > 0$ and $C(\gamma) = \frac{1}{4\gamma}\left\|\frac{(\rho')^2}{\rho}\right\|_{L^\infty(0,T)}$. From (22) and (23), it follows that

$$(1-\gamma)\int_0^T\int_{\Omega_\varepsilon}\rho(t)\left(|\nabla_x\varphi_\varepsilon|^2+|\varphi_\varepsilon(t)|^2\right)dxdt\leq\int_0^T\int_{\Omega_\varepsilon}\rho(t)\,|\varphi_\varepsilon'|^2dxdt$$

$$+C(\gamma)\int_0^T\int_{\Omega_\varepsilon}|\varphi_\varepsilon'|^2dxdt$$

for every $0 < \gamma < 1$. Thus, there exists a positive constant C, independent of ε such that

$$\int_0^T\int_{\Omega_\varepsilon}\rho(t)\left(|\nabla_x\varphi_\varepsilon|^2+|\varphi_\varepsilon(t)|^2\right)dxdt\leq C\int_0^T\int_{\Omega_\varepsilon}|\varphi_\varepsilon'|^2\,dxdt. \tag{24}$$

Multiplying the equation in (21) by $\rho(t)$ and integrating from 0 to T, we obtain

$$E(0)\int_0^T\rho(t)\,dt=\frac{1}{2}\left(\int_0^T\rho(t)\int_{\Omega_\varepsilon}\left(|\nabla_x\varphi_\varepsilon(t)|^2+|\varphi_\varepsilon(t)|^2+|\varphi_\varepsilon'(t)|^2\right)dx\right)dt. \tag{25}$$

By virtue of (24) and (25), estimate (20) follows. □

Now, let us fix $(\varphi_\varepsilon^0,\varphi_\varepsilon^1)\in L^2(\Omega_\varepsilon)\times(\mathcal{V}(\Omega_\varepsilon))'$. Let $\pi_\varepsilon\in H^1(\Omega_\varepsilon)$ be the unique solution of the problem

$$\begin{cases} -\Delta\pi_\varepsilon+\pi_\varepsilon=\varphi_\varepsilon^1 & \text{in } \Omega_\varepsilon,\\ \pi_\varepsilon=0 & \text{in } \partial\Omega^-\setminus\Sigma,\\ \dfrac{\partial\pi_\varepsilon}{\partial\nu}=0 & \text{on } \partial\Omega_\varepsilon^+\setminus\Gamma_\varepsilon. \end{cases}$$

There exists a positive constant C, independent of ε such that

$$\|\pi_\varepsilon\|_{\mathcal{V}(\Omega_\varepsilon)}\leq C\left\|\varphi_\varepsilon^1\right\|_{(\mathcal{V}(\Omega_\varepsilon))'}. \tag{26}$$

Let φ_ε be the transposition solution of (7) corresponding to the initial date $(\varphi_\varepsilon^0,\varphi_\varepsilon^1)\in L^2(\Omega_\varepsilon)\times(\mathcal{V}(\Omega_\varepsilon))'$. Then, the function

$$w_\varepsilon(x,t)=\int_0^t\varphi_\varepsilon(x,s)\,ds+\pi_\varepsilon(x)$$

satisfies the problem

$$\begin{cases} w_\varepsilon''-\Delta w_\varepsilon+w_\varepsilon=0 & \text{in }]0,T[\times\Omega_\varepsilon,\\ \dfrac{\partial w_\varepsilon}{\partial\nu}=0 & \text{on }]0,T[\times\partial\Omega_\varepsilon^+\setminus\Gamma_\varepsilon,\\ w_\varepsilon=0 & \text{in }]0,T[\times(\partial\Omega^-\setminus\Sigma),\\ w_\varepsilon(0)=\pi_\varepsilon,\ w_\varepsilon'(0)=\varphi_\varepsilon^0 & \text{in } \Omega_\varepsilon. \end{cases} \tag{27}$$

Observe that the solution w_ε is defined by usual weak formulation. Hence by applying Lemma 2.4, there exists a positive constant C, independent of ε such that

$$\int_{\Omega_\varepsilon}\left(|\nabla\pi_\varepsilon|^2+|\pi_\varepsilon|^2+|\varphi_\varepsilon^0|^2\right)dx\leq C\int_0^T\int_{\Omega_\varepsilon}|w_\varepsilon'|^2\,dxdt. \tag{28}$$

The inequality (18) is then a direct consequence of (26) and (28).

Now, we prove (19). By making use of Young's inequality, (18) and (17), it follows that

$$\begin{aligned} &\left\|\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\right\|^2_{L^2(\ \Omega_\varepsilon)\times(\mathcal{V}(\Omega_\varepsilon))'} \\ &\le 2\left(\left\|\varphi_\varepsilon^0\right\|^2_{L^2(\ \Omega_\varepsilon)}+\left\|\varphi_\varepsilon^1\ \right\|^2_{(\mathcal{V}(\Omega_\varepsilon))'}\right) \\ &\le C\int_0^T\int_{\Omega_\varepsilon}\left|\varphi_\varepsilon\right|^2 dxdt \\ &= C\left\langle\Lambda_\varepsilon\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right),\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\right\rangle \\ &\le C\left\|\Lambda_\varepsilon\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\right\|_{L^2(\Omega_\varepsilon)\times\mathcal{V}(\Omega_\varepsilon)}\left\|\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\right\|_{L^2(\Omega_\varepsilon)\times(\mathcal{V}(\Omega_\varepsilon))'}. \end{aligned} \tag{29}$$

From (29) and taking into account that Λ_ε is a isomorphism, we obtain

$$\begin{aligned} &\left\|\Lambda_\varepsilon^{-1}\right\|_{\mathcal{L}(L^2(\Omega_\varepsilon)\times\mathcal{V}(\Omega_\varepsilon),\ L^2(\Omega_\varepsilon)\times(\mathcal{V}(\Omega_\varepsilon))')} \\ &= \sup\left\{\frac{\left\|\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\right\|_{L^2\times(\mathcal{V}(\Omega_\varepsilon))'}}{\left\|\Lambda_\varepsilon\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\right\|_{L^2(\Omega_\varepsilon)\times\mathcal{V}(\Omega_\varepsilon)}}:\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\in L^2\left(\Omega_\varepsilon\right)\times\left(\mathcal{V}\left(\Omega_\varepsilon\right)\right)'\right\}\le C, \end{aligned}$$

from which estimate (19) follows. □

2.5. **A priori norm-estimates.** In this subsection, we deduce some a priori norm-estimates for the initial conditions $\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)$ of problem (7), for the control θ_ε and for the corresponding solution u_ε of problem (5). We have the following proposition which is a consequence of (19) of Proposition 1.

Proposition 2. *Let $\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\in L^2\left(\Omega_\varepsilon\right)\times\left(\mathcal{V}\left(\ \Omega_\varepsilon\right)\right)'$ be the initial conditions of problem (7). Then, there exists a constant C, independent of ε, such that*

$$\left\|\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\right\|_{L^2(\ \Omega_\varepsilon)\times(\mathcal{V}(\Omega_\varepsilon))'}\le C, \tag{30}$$

for every ε.

Proposition 3. *Let $\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)\in L^2\left(\Omega_\varepsilon\right)\times\mathcal{V}\left(\Omega_\varepsilon\right)'$ be the initial conditions of problem (7). Let φ_ε and ψ_ε be, respectively, the unique solutions of problems (7) and (8). Then, there exists a constant C, independent of ε, such that*

$$\left\|\varphi_\varepsilon\ \right\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\le C, \tag{31}$$

$$\left\|\varphi_\varepsilon'\ \right\|_{L^2(0,T;(\mathcal{V}(\ \Omega_\varepsilon))')}\le C, \tag{32}$$

$$\left\|\psi_\varepsilon\ \right\|_{L^\infty(0,T;\mathcal{V}(\Omega_\varepsilon))}\ \le C, \tag{33}$$

$$\left\|\psi_\varepsilon'\ \right\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}\le C, \tag{34}$$

for every ε.

Proof. The proof follows by Proposition 2, Remark 9.11 p.290 of [36] and Lemma 2.1 p. 181 of [32]. □

3. **Homogenization of wave equation in domain with oscillating boundary.** In this section, we prove two homogenization results, namely one for the wave equation with regular initial data to obtain the limit equation corresponding to the solution ψ_ε, where the solution is defined via the standard weak formulation. This is done in the next subsection. Secondly, we also study the homogenization of the wave with weak data whose solution is defined by the method of transposition. This is necessary to obtain the homogenized equation corresponding to φ_ε.

3.1. **Homogenization with regular data.** Let us consider the problem

$$\begin{cases} y_\varepsilon'' - \Delta y_\varepsilon + y_\varepsilon = f_\varepsilon \qquad \text{in }]0,T[\times\Omega_\varepsilon, \\ \dfrac{\partial y_\varepsilon}{\partial \nu} = 0 \qquad \text{in }]0,T[\times\left(\partial\Omega_\varepsilon^+ \setminus \Gamma_\varepsilon\right), \\ y_\varepsilon = 0 \qquad \text{in }]0,T[\times\left(\partial\Omega^- \setminus \Sigma\right), \\ y_\varepsilon(0) = y_\varepsilon^0,\ y_\varepsilon'(0) = y_\varepsilon^1 \qquad \text{in } \Omega_\varepsilon, \end{cases} \tag{35}$$

where $f_\varepsilon \in L^2\left(0,T;L^2(\Omega_\varepsilon)\right)$ and $(y_\varepsilon^0, y_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$. It is well known (see [35]) that problem (35) admits a unique weak solution y_ε:

$$\begin{cases} y_\varepsilon \in \mathcal{W}_\varepsilon, \\ \displaystyle\int_0^T \int_{\Omega_\varepsilon} y_\varepsilon z h'' + \nabla_x y_\varepsilon \nabla z h + y_\varepsilon z h \; dxdt \\ \displaystyle= \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon z h \; dxdt \qquad \forall z \in \mathcal{V}(\Omega_\varepsilon),\ \ \forall h \in C_0^\infty(]0,T[), \\ y_\varepsilon(0) = y_\varepsilon^0 \qquad \text{in } \Omega_\varepsilon, \\ y_\varepsilon'(0) = y_\varepsilon^1 \qquad \text{in } \Omega_\varepsilon. \end{cases} \tag{36}$$

Now, we recall the following result (See [35] Chapter 3, Remark 8.2, Theorem 8.2 and Lemma 8.3).

Lemma 3.1. *The solution y_ε of problem (36) satisfies the following estimate:*

$$\begin{aligned} \|y_\varepsilon\|_{L^\infty(0,T;\mathcal{V}(\Omega_\varepsilon))} + \|y_\varepsilon'\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq & D\Big(\|y_\varepsilon^0\|_{\mathcal{V}(\Omega_\varepsilon)} + \|y_\varepsilon^1\|_{L^2(\Omega_\varepsilon)} \\ & + \|f_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\Big), \end{aligned} \tag{37}$$

where D is a positive constant depending on T. Moreover it holds that $y_\varepsilon \in C\left(0,T;\ \mathcal{V}(\Omega_\varepsilon)\right) \cap C^1\left(0,T;\ L^2(\Omega_\varepsilon)\right)$.

As far as the weak formulation of problems (35) is concerned, we prefer to use the following form which is equivalent to the usual one (see [25], Proposition 3.4):

$$\begin{cases} y_\varepsilon \in \mathcal{W}_\varepsilon, \\ i)\ \displaystyle\int_0^T \langle y_\varepsilon''(t,\cdot), \psi(t,\cdot)\rangle_{((H^1(\Omega_\varepsilon))', H^1(\Omega_\varepsilon))} \, dt + \int_0^T \int_{\Omega_\varepsilon} \nabla_x y_\varepsilon \nabla_x \psi \; dxdt \\ \displaystyle= \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon \psi \; dxdt \qquad \forall \psi \in L^2\left(0,T;\mathcal{V}(\Omega_\varepsilon)\right), \\ ii)\ y_\varepsilon(0) = y_\varepsilon^0,\ y_\varepsilon'(0) = y_\varepsilon^1 \qquad \text{in } \Omega_\varepsilon. \end{cases} \tag{38}$$

The aim of this section is to study the asymptotic behaviour, as $\varepsilon \to 0$, of the sequence of solutions $(y_\varepsilon)_\varepsilon$, under the following assumptions:

$$\begin{cases} \widetilde{y_\varepsilon^0} \rightharpoonup (|\omega| \chi_{\Omega^+} + \chi_{\Omega^-}) \, y^0 = \eta y^0 & \text{weakly in } L^2(\Omega), \\ \widetilde{y_\varepsilon^1} \rightharpoonup (|\omega| \chi_{\Omega^+} + \chi_{\Omega^-}) \, y^1 = \eta y^1 & \text{weakly in } L^2(\Omega), \\ \widetilde{f_\varepsilon} \rightharpoonup (|\omega| \chi_{\Omega^+} + \chi_{\Omega^-}) \, f = \eta f & \text{weakly in } L^2\left(0,T; L^2(\Omega)\right) \end{cases} \tag{39}$$

where η is defined as in (3).

The above convergence together with Lemma 3.1 gives the following proposition.

Proposition 4. *Assume* (39)*. Let y_ε be the solution of problem* (35)*. Then, there exists a constant C, independent of ε, such that*

$$\|y_\varepsilon\|_{L^\infty(0,T;\ \mathcal{V}(\Omega_\varepsilon))} \leq C, \tag{40}$$

$$\|y_\varepsilon'\|_{L^\infty(0,T;\ L^2(\Omega_\varepsilon))} \leq C, \tag{41}$$

for every ε.

Now, we give the homogenization of the wave equation (35).

Theorem 3.2. *Assume* (39)*. Let y_ε be the solution of the problem* (35)*. Then, we have*

$$\begin{cases} \widetilde{y_\varepsilon} \rightharpoonup |\omega| \, y & \textit{weakly } * \textit{ in } L^\infty\left(0,T; L^2(\Omega^+)\right), \\ y_\varepsilon \rightharpoonup y & \textit{weakly in } * \textit{ in } L^\infty\left(0,T; H^1(\Omega^-)\right), \\ \widetilde{y_\varepsilon'} = (\widetilde{y_\varepsilon})' \rightharpoonup |\omega| \, y' & \textit{weakly in } * \textit{ in } L^\infty\left(0,T; L^2(\Omega^+)\right), \\ \widetilde{y_\varepsilon'} = (\widetilde{y_\varepsilon})' \rightharpoonup (y^-)' & \textit{weakly in } * \textit{ in } L^\infty\left(0,T; L^2(\Omega^+)\right), \end{cases} \tag{42}$$

where $y \in L^2\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; H^1(\Omega^-)\right)$ is the unique solution of the following problem

$$\begin{cases} y'' - \dfrac{\partial^2 y}{\partial x_n^2} + y = f & \textit{in }]0,T[\times \Omega^+, \\ y'' - \Delta y + y = f & \textit{in }]0,T[\times \Omega^-, \\ y^+ = y^-, \quad |\omega| \dfrac{\partial y^+}{\partial x_n} = \dfrac{\partial y^-}{\partial x_n} & \textit{on }]0,T[\times \Sigma, \\ \dfrac{\partial y}{\partial x_n} = 0 & \textit{on }]0,T[\times (B \times \{b\}), \\ y = 0 & \textit{on }]0,T[\times (\partial\Omega^- \setminus \Sigma), \\ y(0) = y^0, \, y'(0) = y^1 & \textit{in } \Omega. \end{cases}$$

Proof. See [25] . ∎

3.2. **Homogenization for the transposition solution.** In the previous section, we have obtained the homogenized equation with regular initial data in $(\mathcal{V}(\Omega_\varepsilon)) \times L^2(\Omega_\varepsilon)$. Now to study the homogenization of the controllability problem with the initial data $\left(u_\varepsilon^0, u_\varepsilon^1\right) \in (\mathcal{V}(\Omega_\varepsilon)) \times L^2(\Omega_\varepsilon)$, we have the initial data $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) = \Lambda_\varepsilon^{-1}\left(-u_\varepsilon^1, u_\varepsilon^0\right) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$ for the problem (7) which eventually gives the optimal control. Hence we need to study the homogenization of (7) with the weak data and we do it in this chapter.

From estimates (31) and (32), we deduce that there exist $\left(\varphi^0, \varphi^1\right) \in L^2(\Omega_\varepsilon) \times (\mathcal{V}(\Omega_\varepsilon))'$, and two subsequences of $\left(\varphi_\varepsilon^0\right)$ and of $\left(\varphi_\varepsilon^1\right)$, still denoted by $\left(\varphi_\varepsilon^0\right)$ and $\left(\varphi_\varepsilon^1\right)$ respectively such that

$$\begin{cases} \widetilde{\varphi_\varepsilon^0} \rightharpoonup (|\omega|\,\chi_{\Omega^+} + \chi_{\Omega^-})\,\varphi^0 = \eta\phi^0 & \text{weakly in } L^2(\Omega), \\ \widetilde{\varphi_\varepsilon^1} \rightharpoonup (|\omega|\,\chi_{\Omega^+} + \chi_{\Omega^-})\,\varphi^1 = \eta\phi^1 & \text{weakly in } (\mathcal{V}(\Omega))' \end{cases} \tag{43}$$

where η is defined as in (3).

Proposition 5. *Let φ_ε be the unique solution of problem (7) corresponding to the initial data given above. Then, there exists a subsequence of $\{\varphi_\varepsilon\}$, still denoted by $\{\varphi_\varepsilon\}$ such that as $\varepsilon \to 0$*

$$\begin{cases} \widetilde{\varphi}_\varepsilon \rightharpoonup (|\omega|\,\chi_{\Omega^+} + \chi_{\Omega^-})\,\varphi & \textit{weakly in } L^2\left(0,T; L^2(\Omega)\right), \\ \widetilde{\varphi_\varepsilon'} = (\widetilde{\varphi}_\varepsilon)' \rightharpoonup (|\omega|\,\chi_{\Omega^+} + \chi_{\Omega^-})\,\varphi' & \textit{weakly in } L^2\left(0,T;\ (\mathcal{V}(\Omega))'\right), \end{cases} \tag{44}$$

where φ is solution of the problem

$$\begin{cases} \varphi'' - \dfrac{\partial^2\varphi}{\partial x_n^2} + \varphi = 0 & \textit{in }]0,T[\times\Omega^+, \\ \varphi'' - \Delta\varphi + \varphi = 0 & \textit{in }]0,T[\times\Omega^-, \\ \varphi^+ = \varphi^-, \quad |\omega|\,\dfrac{\partial\varphi^+}{\partial x_n} = \dfrac{\partial\varphi^-}{\partial x_n} & \textit{on }]0,T[\times\Sigma, \\ \dfrac{\partial\varphi}{\partial x_n} = 0 & \textit{on }]0,T[\times(B\times\{b\}), \\ \varphi = 0 & \textit{on }]0,T[\times(\partial\Omega^-\setminus\Sigma), \\ \varphi(0) = \varphi^0,\ \varphi'(0) = \varphi^1 & \textit{in } \Omega. \end{cases} \tag{45}$$

Proof. Estimates (31) and (32) provide the existence of a subsequence of $\{\varphi_\varepsilon\}$, still denoted by $\{\varphi_\varepsilon\}$, and a function $\varphi \in L^2\left(0,T; L^2(\Omega)\right)$ with $\varphi' \in L^2\left(0,T;\ (\mathcal{V}(\Omega))'\right)$ such that

$$\begin{cases} \widetilde{\varphi}_\varepsilon \rightharpoonup (|\omega|\,\chi_{\Omega^+} + \chi_{\Omega^-})\,\varphi = \eta\varphi & \text{weakly in } L^2\left(0,T; L^2(\Omega)\right), \\ \widetilde{\varphi_\varepsilon'} = (\widetilde{\varphi}_\varepsilon)' \rightharpoonup (|\omega|\,\chi_{\Omega^+} + \chi_{\Omega^-})\,\varphi' = \eta\varphi' & \text{weakly in } L^2\left(0,T;\ (\mathcal{V}(\Omega))'\right). \end{cases} \tag{46}$$

Let us prove that φ is solution of the system (45). Let $\xi_\varepsilon \in \mathcal{V}(\Omega_\varepsilon)$ be the unique solution of the system

$$\begin{cases} -\Delta\xi_\varepsilon + \xi_\varepsilon = -\varphi_\varepsilon^1 \quad \text{in } \Omega_\varepsilon, \\ \dfrac{\partial \xi_\varepsilon}{\partial \nu} = 0 \qquad \text{on } \partial\Omega_\varepsilon^+ \setminus \Gamma_\varepsilon, \\ \xi_\varepsilon = 0 \qquad \text{in } \partial\Omega^- \setminus \Sigma. \end{cases} \tag{47}$$

Let us consider the function

$$\sigma_\varepsilon(x,t) = \int_0^t \varphi_\varepsilon(x,s)\, ds + \xi_\varepsilon(x). \tag{48}$$

We do observe that this transformation leads to a problem for which initial data are more regular than the initial data $(\varphi_\varepsilon^0, \varphi_\varepsilon^1)$ of the problem (7). Indeed, σ_ε is solution of the problem

$$\begin{cases} \sigma_\varepsilon'' - \Delta\sigma_\varepsilon + \sigma_\varepsilon = 0 \quad \text{in }]0,T[\times \Omega_\varepsilon, \\ \dfrac{\partial \sigma_\varepsilon}{\partial \nu} = 0 \quad \text{on }]0,T[\times (\partial\Omega_\varepsilon^+ \setminus \Gamma_\varepsilon), \\ \sigma_\varepsilon = 0 \qquad \text{in}]0,T[\times (\partial\Omega^- \setminus \Sigma), \\ \sigma_\varepsilon(0) = \xi_\varepsilon, \, \sigma_\varepsilon'(0) = \varphi_\varepsilon^0 \quad \text{in } \Omega_\varepsilon \end{cases} \tag{49}$$

with $(\xi_\varepsilon\ , \varphi_\varepsilon^0) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$. Moreover, by (47) and (30), there exists a positive constant C, independent of ε such that

$$\|\xi_\varepsilon\|_{\mathcal{V}(\ \Omega_\varepsilon)} \le C.$$

Consequently, (see [17] Proposition 2.2) there exists a subsequence of $\{\xi_\varepsilon\}$, still denoted by $\{\xi_\varepsilon\}$, and a function ξ in $\mathcal{V}(\Omega)$ such that

$$\begin{cases} \widetilde{\xi_\varepsilon} \rightharpoonup |\omega|\, \xi & \text{weakly in } L^2(\Omega^+), \\ \xi_\varepsilon \rightharpoonup \xi & \text{weakly in } \mathcal{V}(\Omega^-), \\ \dfrac{\partial \widetilde{\xi_\varepsilon}}{\partial x_n} = \widetilde{\dfrac{\partial \xi_\varepsilon}{\partial x_n}} \rightharpoonup |\omega| \dfrac{\partial \xi}{\partial x_n} & \text{weakly in } L^2(\Omega^+), \\ \widetilde{\dfrac{\partial \xi_\varepsilon}{\partial x_i}} \rightharpoonup 0 & \text{weakly in } L^2(\Omega^+),\ for\ i \in \{1, ..., n-1\}. \end{cases} \tag{50}$$

Then, by (50), (43) and [28] , we have

$$\begin{cases} -\dfrac{\partial^2 \xi^+}{\partial x_n^2} + \xi^+ = -\varphi^1 \qquad \text{in } \Omega^+, \\ \\ -\Delta\xi^- + \xi^- = -\varphi^1 \qquad \text{in } \Omega^-, \\ \\ \xi^+ = \xi^-, \quad |\omega| \dfrac{\partial \xi^+}{\partial x_n} = \dfrac{\partial \xi^-}{\partial x_n} \qquad \text{on } \Sigma, \\ \\ \dfrac{\partial \xi}{\partial x_n} = 0 \qquad \text{on } B \times \{b\}, \\ \\ \xi = 0 \quad \text{on } \partial\Omega^- \setminus \Sigma. \end{cases}$$

Now, applying Theorem 3.2 to problem (49), it results

$$\begin{cases} \widetilde{\sigma}_\varepsilon \rightharpoonup |\omega|\, \sigma \quad \text{weakly in } L^2\left(0,T; L^2(\Omega^+)\right), \\ \\ \sigma_\varepsilon \rightharpoonup \sigma \quad \text{weakly in } L^2(0,T; \mathcal{V}(\Omega^-)), \end{cases} \tag{51}$$

and σ is the solution of the homogenized system:

$$\begin{cases} \sigma'' - \dfrac{\partial^2 \sigma}{\partial x_n^2} + \sigma = 0 & \text{in }]0,T[\times \Omega^+, \\ \sigma'' - \Delta\sigma + \sigma = 0 & \text{in }]0,T[\times \Omega^-, \\ \sigma^+ = \sigma^-, \quad |\omega| \dfrac{\partial \sigma^+}{\partial x_n} = \dfrac{\partial \sigma^-}{\partial x_n} & \text{on }]0,T[\times \Sigma, \\ \dfrac{\partial \sigma}{\partial x_n} = 0 & \text{on }]0,T[\times (B \times \{b\}), \\ \sigma = 0 & \text{on }]0,T[\times (\partial\Omega^- \setminus \Sigma), \\ \sigma(0) = \xi, \, \sigma'(0) = \varphi^0 & \text{in } \Omega. \end{cases}$$

Moreover, by regularity results for hyperbolic equation we have

$$\sigma \in C\left(0,T;\ \mathcal{V}\left(\Omega^-\right)\right) \cap C^1\left(0,T;\ L^2\left(\Omega^-\right)\right) \cap C^2\left(0,T;\ \left(\mathcal{V}\left(\Omega^-\right)\right)'\right).$$

Now, we observe that

$$\begin{cases} \sigma''(0) = \dfrac{\partial^2 \sigma(0)}{\partial x_n^2} - \sigma(0) = \dfrac{\partial^2 \xi^+}{\partial x_n^2} - \xi^+ = \varphi^1 & \text{in } \Omega^+, \\ \sigma''(0) = \Delta\sigma(0) - \sigma(0) = \Delta\xi - \xi = \varphi^1 & \text{in } \Omega^- \end{cases}$$

and so it results that

$$\sigma''(0) = \varphi^1 \text{ in } \Omega.$$

Then the function $\sigma' = W$ satisfies the problem

$$\begin{cases} W'' - \dfrac{\partial^2 W}{\partial x_n^2} + W = 0 & \text{in }]0,T[\times \Omega^+, \\ W'' - \Delta W + W = 0 & \text{in }]0,T[\times \Omega^-, \\ W^+ = W^-, \quad |\omega| \dfrac{\partial W^+}{\partial x_n} = \dfrac{\partial W^-}{\partial x_n} & \text{on }]0,T[\times \Sigma, \\ \dfrac{\partial W}{\partial x_n} = 0 & \text{on }]0,T[\times (B \times \{b\}), \\ W = 0 & \text{on }]0,T[\times (\partial\Omega^- \setminus \Sigma), \\ W(0) = \varphi^0, \, W'(0) = \varphi^1 & \text{in } \Omega. \end{cases}$$

Here W is defined in the sense of transposition. By (48) it results

$$\widetilde{\sigma'_\varepsilon} = \widetilde{\varphi_\varepsilon}.$$

Moreover, by definition of distributional derivative, one has

$$\int_0^T \int_{\Omega_\varepsilon^+} \sigma''_\varepsilon zh \; dxdt = \int_0^T \int_{\Omega^+} \widetilde{\sigma_\varepsilon} zh'' dxdt \tag{52}$$

for every $h \in \mathcal{D}((0,T))$. Passing to the limit in (52) as $\varepsilon \to 0$, using (51) the right hand side converges to

$$|\omega| \int_0^T \int_{\Omega^+} \sigma z h'' dx dt = -|\omega| \int_0^T \int_{\Omega^+} \sigma' z h' dx dt. \tag{53}$$

Concerning the left hand side of (52), we have

$$\int_0^T \int_{\Omega_\varepsilon^+} \sigma_\varepsilon'' z h \, dx dt = -\int_0^T \int_{\Omega_\varepsilon^+} \widetilde{\sigma_\varepsilon'} z h' \, dx dt = -\int_0^T \int_{\Omega_\varepsilon^+} \widetilde{\varphi_\varepsilon} z h' \, dx dt. \tag{54}$$

Finally, passing to the limit in (52) as $\varepsilon \to 0$, by (53), (54) and (44), we have

$$W = \varphi \quad \text{in } \Omega^+.$$

In same way, we have that

$$W = \varphi \quad \text{in } \Omega^-.$$

Since the problem (45) admits a unique solution, the convergence (44) holds true for the whole sequence. The proof is complete. □

4. **Exact controllability of the limit system (13).** We prove the following controllability result for the limit problem (13) using Hilbert uniqueness method (HUM). In the process, we need to prove an observability estimate corresponding to the system (45).

Theorem 4.1. *The system (13) is exactly controllable in the space $V(\Omega) \times L^2(\Omega)$. That is, for given $(u^0, u^1) \in V(\Omega) \times L^2(\Omega)$, there exists a control $\theta \in L^2(0,T; L^2(\Omega))$ such that the solution of the problem (13) satisfies $u(T) = 0 = u'(T)$.*

We briefly describe HUM to get the right observability estimate to be proved. Let $(\varphi^0, \varphi^1) \in L^2(\Omega) \times V(\Omega)'$ and φ be the transposition solution to the system (45). Now, let $\psi \in C(0,T;\ V(\Omega)) \cap C^1(0,T;\ L^2(\Omega))$ be the solution in the weak formulation to the adjoint system

$$\begin{cases} \psi'' - \dfrac{\partial^2 \psi}{\partial x_n^2} + \psi = -\varphi & \text{in }]0,T[\times \Omega^+, \\ \psi'' - \Delta\psi + \psi = -\varphi & \text{in }]0,T[\times \Omega^-, \\ \psi^+ = \psi^-, \quad |\omega| \dfrac{\partial \psi^+}{\partial x_n} = \dfrac{\partial \psi^-}{\partial x_n} & \text{on }]0,T[\times \Sigma, \\ \dfrac{\partial \psi}{\partial x_n} = 0 & \text{on }]0,T[\times (B \times \{b\}), \\ \psi = 0 & \text{on }]0,T[\times (\partial\Omega^- \setminus \Sigma), \\ \psi(T) = 0 = \psi'(T) & \text{in } \Omega. \end{cases} \tag{55}$$

At this stage, define an operator Λ as

$$\Lambda : L^2(\Omega) \times (\mathcal{V}(\Omega))' \longrightarrow L^2(\Omega) \times \mathcal{V}(\Omega)$$

by setting, for all $(\varphi^0, \varphi^1) \in L^2(\Omega) \times (V(\Omega))'$,

$$\Lambda(\varphi^0, \varphi^1) = (-\eta\psi'(0), \eta\psi(0))$$

where η is defined as in (3).

Then, we have

$$\left\langle \Lambda\left(\varphi^0,\varphi^1\right),\,\left(\varphi^0,\varphi^1\right)\right\rangle = \left\langle \varphi^1,\,\eta\psi(0)\right\rangle_{V(\Omega)',V(\Omega)} - \int_\Omega \eta\varphi^0\psi'(0). \tag{56}$$

Multiply the first and second equations in (45), respectively, by $|\omega|\psi^+$ and ψ^- and similarly multiply the first and second equations in (55), respectively, by $|\omega|\varphi^+$ and φ^-, subtracting and using (56) easily we obtain

$$\left\langle \Lambda\left(\varphi^0,\varphi^1\right),\,\left(\varphi^0,\varphi^1\right)\right\rangle = \int_\Omega \eta\varphi^2. \tag{57}$$

Proof of Theorem 4.1. If the operator Λ is an isomorphism, then the proof of the theorem is complete. To see this, let $(u^0,u^1)\in V(\Omega)\times L^2(\Omega)$. Since Λ is surjective, the equation

$$\Lambda\left(\varphi^0,\varphi^1\right) = \left(-\eta u^1,\eta u^0\right)$$

has a solution for $\left(\varphi^0,\varphi^1\right)\in L^2(\Omega)\times(V(\Omega))'$. For φ given by (45) with $\left(\varphi^0,\varphi^1\right)$ as initial conditions, we consider the corresponding solution ψ of problem (55). By definition of Λ, we get

$$\Lambda\left(\varphi^0,\varphi^1\right) = \left(-\eta\psi'(0),\eta\psi(0)\right).$$

Under the assumption that Λ is an isomorphism, we see that $\psi(0)=u^0$ and $\psi'(0)=u^1$. Thus the controllability problem is solved by taking the control $-\varphi$ and the controlled solution as $u=\psi$.

Thus remains to show that Λ is an isomorphism which follows from the following *observability estimate*

Lemma 4.2. *There exists a constant $C>0$ such that*

$$\left\|\varphi^0\right\|^2_{L^2(\Omega)} + \left\|\varphi^1\right\|^2_{(V(\Omega))'} \le C\int_0^T\int_\Omega \eta|\varphi|^2 \tag{58}$$

for all $\left(\varphi^0,\varphi^1\right)\in L^2(\Omega)\times(V(\Omega))'$, where η is defined as in (3) and φ is the corresponding solution of the system (45).

First, we prove the following proposition with regular data in (45).

Lemma 4.3. *Let $\left(\varphi^0,\varphi^1\right)\in V(\Omega)\times L^2(\Omega)$. Then the solution φ of (45) is defined via the classical weak formulation and satisfies the estimate*

$$\left\|\varphi^0\right\|^2_{(V(\Omega))} + \left\|\varphi^1\right\|^2_{L^2(\Omega)} \le C\int_0^T\int_\Omega \eta|\varphi'|^2, \tag{59}$$

where C is a positive constant.

Proof. Define the energy $E(t)$ as

$$E(t)=\frac{1}{2}\left[\|\eta\varphi'(t)\|^2_{L^2(\Omega)}+\|\eta\varphi(t)\|^2_{L^2(\Omega)}+\|\nabla\varphi(t)\|^2_{L^2(\Omega^-)}+\left\|\eta\frac{\partial\varphi}{\partial x_n}(t)\right\|^2_{L^2(\Omega^+)}\right].$$

One can multiply the equations in (45) by $\eta\varphi'$ and it is easy to see that the energy is conserved, that is

$$E(t)=E(0)=\frac{1}{2}\left[\left\|\eta\varphi^1(t)\right\|^2_{L^2(\Omega)}+\left\|\eta\varphi^0\right\|^2_{L^2(\Omega)}+\left\|\nabla\varphi^0\right\|^2_{L^2(\Omega^-)}+\left\|\eta\frac{\partial\varphi^0}{\partial x_n}(t)\right\|^2_{L^2(\Omega^+)}\right].$$

Now consider the test function $h(x,t) = \rho(t)\eta(x)\varphi(x,t)$, multiply the equation (45) by h, integrate by parts to get

$$\begin{aligned}
&\int_0^T \int_\Omega \rho(t)\eta(x)|\varphi'|^2 + \int_0^T \int_\Omega \rho'(t)\eta(x)\varphi\varphi' \\
&= \int_0^T \int_\Omega \rho(t)\eta(x)\left|\varphi\right|^2 + \int_0^T \int_{\Omega+} \rho(t)|\omega| \left|\frac{\partial\varphi}{\partial x_n}\right|^2 + \int_0^T \int_{\Omega-} \rho(t)\left|\nabla\varphi\right|^2 \qquad (60)\\
&= \int_0^T E_\rho(t)dt,
\end{aligned}$$

where

$$E_\rho(t) = \int_\Omega \rho(t)\eta(x)\left|\varphi\right|^2 + \int_{\Omega+} \rho(t)|\omega| \left|\frac{\partial\varphi}{\partial x_n}\right|^2 + \int_{\Omega-} \rho(t)\left|\nabla\varphi\right|^2.$$

Now,

$$\begin{aligned}
\int_\Omega \rho'(t)\eta(x)\varphi\varphi' &= \int_{\Omega+} \rho'(t)|\omega|\varphi\varphi' + \int_{\Omega-} \rho'(t)\varphi\varphi' \\
&\leq \left[\gamma \int_{\Omega+} \rho(t)|\omega||\varphi|^2 + C(\gamma)\int_{\Omega+} |\omega||\varphi'|^2\right] \\
&\quad + \left[\gamma \int_{\Omega-} \rho(t)|\varphi|^2 + C(\gamma)\int_{\Omega-} |\varphi'|^2\right] \\
&\leq \gamma E_\rho(t) + C(\gamma)\int_\Omega \eta|\varphi'|^2,
\end{aligned}$$

where $\gamma > 0$ is a positive real number and $C(\gamma) = \dfrac{1}{4\gamma}\left\|\dfrac{(\rho')^2}{\rho}\right\|_{L^\infty(0,T)}$. Thus for $0 < \gamma < 1$, from (60), it follows that

$$(1-\gamma)\int_0^T E_\rho(t)dt \leq \int_0^T \int_\Omega \rho(t)\eta(x)|\varphi'|^2 + C(\gamma)\int_0^T \int_\Omega \eta|\varphi'|^2.$$

Since ρ is bounded above and γ is fixed, there exists a constant $T > 0$ such that

$$\int_0^T E_\rho(t)dt \leq C\int_0^T \int_\Omega \eta|\varphi'|^2.$$

Thus using the conservation of the energy, we have

$$E(0)\int_0^T \rho(t)dt = \int_0^T \rho(t)E(t)dt = \int_0^T E_\rho(t)dt + \int_\Omega \eta|\varphi'|^2.$$

It follows, then that

$$E(0) \leq \int_0^T \int_\Omega \eta|\varphi'|^2.$$

The proof of lemma is complete from the definition of η. □

Proof of Lemma 4.2. Now, let $(\varphi^0, \varphi^1) \in L^2(\Omega) \times (V(\Omega))'$ and φ be the corresponding solution of the system (45). Introduce $\pi \in V(\Omega)$ which solves the following problem

$$\begin{cases} -\dfrac{\partial^2 \pi^+}{\partial x_n^2} + \pi^+ = -\varphi^1 & \text{in } \Omega^+, \\ -\Delta \pi^- + \pi^- = -\varphi^1 & \text{in } \Omega^-, \\ \pi^+ = \pi, \quad |\omega| \dfrac{\partial \pi^+}{\partial x_n} = \dfrac{\partial \pi^-}{\partial x_n} & \text{on } \Sigma, \\ \dfrac{\partial \pi^+}{\partial x_n} = 0 & \text{on } B \times \{b\}, \\ \pi^- = 0 & \text{on } \partial\Omega^- \setminus \Sigma. \end{cases} \tag{61}$$

Define $w(x,t) = \int_0^t \varphi(x,s)ds + \pi(x)$ which satisfies a similar system as in (45) with the regular data $w(0) = \pi$ and $w'(0) = \varphi^0$, that is $(w(0), w'(0) \in V(\Omega) \times L^2(\Omega)$ with same system (45). Thus, we can apply Lemma 4.3 to get the estimate

$$\|\pi\|_{V(\Omega)}^2 + \left\|\varphi^0\right\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\Omega \eta |w'|^2.$$

Since π is the unique solution of (61) corresponding to the data φ^1 , we get the required estimate (58) since $w' = \varphi$. This completes the proof of Lemma 4.2 and hence the exact controllability for the limit system.

Now we can complete the proof of Theorem 2.2. □

5. **Proof of Theorem 2.2.** The proof essentially follows by the convergence analysis in Section 3 and required estimates in Section 2. Let us consider $(u_\varepsilon^0, u_\varepsilon^1) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ satisfying (12). Let $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right)$ be the unique solution of equation:

$$\Lambda_\varepsilon \left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) = \left(-u_\varepsilon^1, u_\varepsilon^0\right). \tag{62}$$

Then the initial condition $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right)$ is uniformly bounded due to the results in Section 2. Let us pose

$$\theta_\varepsilon = -\varphi_\varepsilon, \tag{63}$$

where φ_ε is the unique solution of problem (7) with initial conditions $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right)$. By (10), it results

$$\left(-\psi_\varepsilon'(0), \psi_\varepsilon(0)\right) = \left(-u_\varepsilon^1, u_\varepsilon^0\right),$$

where ψ_ε is the unique solution of problem (8). By uniqueness theorem of the solution of problem (5), we obtain

$$u_\varepsilon = \psi_\varepsilon. \tag{64}$$

By final condition of problem (8), (64) and from the continuity of solution, we have

$$u_\varepsilon(T) = 0, \quad u_\varepsilon'(T) = 0.$$

And so, θ_ε is the exact control for system (5). Moreover, by propositions (3), (63) and (64), we have that, up to subsequences,

$$\begin{cases} \widetilde{\theta_\varepsilon} \rightharpoonup (|\omega|\,\chi_{\Omega^+} + \chi_{\Omega^-})\,\theta = \eta\theta & \text{weakly } * \text{ in } L^2\left(0,T;L^2\left(\Omega\right)\right), \\ \widetilde{u_\varepsilon} \rightharpoonup |\omega|\,u & \text{weakly } * \text{in } L^\infty\left(0,T;L^2\left(\Omega^+\right)\right), \\ u_\varepsilon \rightharpoonup u & \text{weakly } * \text{ in } L^\infty\left(0,T;H^1\left(\Omega^-\right)\right), \\ \widetilde{u'_\varepsilon} = (\widetilde{u_\varepsilon})' \rightharpoonup |\omega|\,u' & \text{weakly } * \text{ in } L^\infty\left(0,T;L^2\left(\Omega^+\right)\right), \\ \widetilde{u'_\varepsilon} = (\widetilde{u_\varepsilon})' \rightharpoonup (u^-)' & \text{weakly } * \text{ in } L^\infty\left(0,T;L^2\left(\Omega^+\right)\right), \end{cases} \tag{65}$$

with

$$\theta = \begin{cases} -|\omega|\,\varphi & \text{in } \;]0,T[\times\Omega^+, \\ -\varphi & \text{in } \;]0,T[\times\Omega^-. \end{cases} \tag{66}$$

By applying Theorem 3.2 to problem (5) with $f_\varepsilon = \theta_\varepsilon$ and from convergence (65), we obtain

$$\begin{cases} \widetilde{u_\varepsilon} \rightharpoonup |\omega|\,u & \text{weakly } * \text{in } L^\infty\left(0,T;L^2\left(\Omega^+\right)\right), \\ u_\varepsilon \rightharpoonup u & \text{weakly in } * \text{ in } L^\infty\left(0,T;H^1\left(\Omega^-\right)\right), \\ \widetilde{u'_\varepsilon} = (\widetilde{u_\varepsilon})' \rightharpoonup |\omega|\,u' & \text{weakly } * \text{ in } L^\infty\left(0,T;L^2\left(\Omega^+\right)\right), \\ \widetilde{u'_\varepsilon} = (\widetilde{u_\varepsilon})' \rightharpoonup (u^-)' & \text{weakly in } * \text{ in } L^\infty\left(0,T;L^2\left(\Omega^+\right)\right), \end{cases}$$

where u is the unique solution of the problem (13) , with θ given by (66). Finally, from (64) and (44) by applying Theorem 3.2 to problem (8) with $f_\varepsilon = -\varphi_\varepsilon$, we obtain the limit problem for $\psi = u$ as in (55).

In the last section, we have proved that the operator Λ is an isomorphism by HUM. Let $\left(u^0, u^1\right)$ be the initial condition for the problem (13) . From the convergence of $u_\varepsilon = \psi_\varepsilon$, it follows that $u^0 = \psi(0)$ and $u^1 = \psi'(0)$. Thus

$$\Lambda\left(\varphi^0, \varphi^1\right) = \left(-u^1, u^0\right).$$

Hence the limit problem is indeed the exact controllability problem. In other words, $\theta = -\varphi$ is the exact limit control. By uniqueness of the solution of the limit problem we obtain that the whole sequences $\left(\widetilde{\theta_\varepsilon}\right)$ and $(\widetilde{u_\varepsilon})$ converge. The proof of Theorem 2.2 is complete .

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