Numerical Functional Analysis and Optimization

ISSN: 0163-0563 (Print) 1532-2467 (Online) Journal homepage: http://www.tandfonline.com/loi/lnfa20

A FrameWork for the Error Analysis of Discontinuous Finite Element Methods for Elliptic Optimal Control Problems and Applications to C ^{0} IP Methods

Sudipto Chowdhury , Thirupathi Gudi & A. K. Nandakumaran

Numerical Functional Analysis and Optimization, 36:1388–1419, 2015

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630563.2015.1068182

A FRAMEWORK FOR THE ERROR ANALYSIS OF DISCONTINUOUS FINITE ELEMENT METHODS FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS AND APPLICATIONS TO C^0 IP METHODS

Sudipto Chowdhury, Thirupathi Gudi, and A. K. Nandakumaran

Department of Mathematics, Indian Institute of Science, Bangalore, India

□ *In this article, an abstract framework for the error analysis of discontinuous Galerkin methods for control constrained optimal control problems is developed. The analysis establishes the best approximation result from* a priori *analysis point of view and delivers a reliable and efficient* a posteriori *error estimator. The results are applicable to a variety of problems just under the minimal regularity possessed by the well-posedness of the problem. Subsequently, the applications of C^0 interior penalty methods for a boundary control problem as well as a distributed control problem governed by the biharmonic equation subject to simply supported boundary conditions are discussed through the abstract analysis. Numerical experiments illustrate the theoretical findings.*

Keywords Biharmonic; C^0IP method; Discontinuous Galerkin; Error bounds; Finite element; Optimal control; Simply supported plate.

Mathematics Subject Classification 65N30; 65N15.

1. INTRODUCTION

The optimal control problems have been playing a very important role in the modern scientific world. The numerical analysis for this class of problems dates back to the 1970s [19, 28]. There are many landmark results on the finite element analysis of optimal control problems. It is difficult to cite all of the articles here, but the relevant work can be found in the references of some of the articles that we discuss. We refer to the monograph [41] for the theory of optimal control problems and

Received 30 October 2014; Revised 28 June 2015; Accepted 29 June 2015.
Address correspondence to Thirupathi Gudi, Department of Mathematics, Indian Institute of Science, Bangalore 560012, India; E-mail: gudi@math.iisc.ernet.in
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/lnfa.

their numerical algorithmic aspects. Therein, the primal-dual active set algorithm developed in [29] is also discussed in the context of the optimal control problems. Also refer to the books [31, 36] for the theory and numerical analysis of optimal control problems. Apart from this, we refer to [38] for a super-convergence result using a post-processed control for constrained control problems. A variational discretization method is introduced in [30] to derive optimal error estimates by exploiting the relation between the control and the adjoint state. For the numerical approximation of Neumann boundary control problem with graded mesh refinement refer to [1], and for the numerical treatment of the Dirichlet boundary control problems refer to [12, 15, 27, 37, 39] and references there in. Simultaneously, there has also been a lot of interest in the numerical approximation of the state constrained control problems, for examples see [14, 26, 40] and references therein. On the other hand, while the adaptive finite element methods based on *a posteriori* error estimators have grown in popularity, the study of *a posteriori* error analysis for optimal control problems has also gained a lot of interest in the recent years. In particular, the control in control constrained problem can exhibit kinks and hence lacks smoothness. In this context, adaptive finite element methods would be useful to enhance accuracy. An *a posteriori* error analysis of a conforming finite element method for control constrained problems is derived in [32]. Recently, a general framework for *a posteriori* error analysis of conforming finite element methods for optimal control problems with constraints on controls is derived in [35]. The result therein is obtained by the help of appropriate auxiliary linear problems. In the context of higher order problems, recently in [10, 18], mixed finite element methods have been proposed and analyzed for a distributed control problem governed by the biharmonic equation subject to the Dirichlet boundary conditions, while a C^0 interior penalty method is analyzed in [24] for the clamped plate control problem.

There are only a handful of results on the analysis of discontinuous Galerkin (DG) methods for optimal control problems, in particular for higher order problems. In this article, we develop an abstract framework for the error analysis of discontinuous finite element methods applied to control constrained optimal control problems. The outcome of the result is a best approximation result for the numerical method and a reliable and efficient *a posteriori* error estimator. It is important to note that these best approximation results are key estimates in establishing the optimality of adaptive finite element methods, see for examples [11, 33]. Also it is worth noting that the standard error analysis of DG methods require additional regularity which does not exist in several cases, for example in simply supported plates or mixed boundary value problems, e.g., see the discussions in [8, 9, 22, 23]. Therefore, the error analysis of DG methods has to be treated carefully. To this end, we introduce two auxiliary problems; one dealing with a projection in *a priori* analysis and the other is based on a reconstruction in *a posteriori* analysis. Subsequently, Theorem 2.2 and Theorem 2.5 are proved, which play an important role in the analysis. We believe that the results in this article present a framework for the error analysis of discontinuous finite element methods for control problems with limited regularity. Moreover, the *a posteriori* error estimator is useful in adaptive mesh refinement algorithms.

C^0 interior penalty methods became very attractive in the recent past for approximating the solutions of higher order problems [5, 7–9, 17, 23]. This is due to the fact that the conforming and mixed methods are complicated and the nonconforming methods do not come in a natural hierarchy. In this article, we propose and analyze a C^0 interior penalty method for optimal control problems (both distributed and Neumann boundary control) governed by the biharmonic equation subject to simply supported boundary conditions. Note that the analysis of Dirichlet boundary control problems in general is a subtle issue, since the arguments for that particular problem need to be addressed using a very weak formulation or an equivalent one, e.g., see [12, 37]. The analysis in this article differs from the one in [24], in particular an abstract framework for obtaining energy norm estimates, and in *a posteriori* error analysis. Also, we analyze here the boundary control problems. It is shown in [34] that the C^0 interior penalty solution of the biharmonic problem has connection to the divergence-conforming solution of the Stokes problem. Therefore, our results will also be useful in the context of control problems for Stokes equations.

In summary, the results in the article are outlined as follows:

- An abstract framework for the error analysis of discontinuous Galerkin methods for elliptic optimal control problems with control constraints is derived. The results are applicable to the classical nonconforming methods, and the discontinuous Galerkin methods applied to $2k$-th order elliptic control problems, where $k \geq 1$.
- The best approximation error estimates are derived under the minimum regularity on the state and the costate variables. These results have implications in the optimality of adaptive algorithms.
- The analysis also provides a framework for the derivation of a posteriori error estimates.
- Applications of the abstract analysis are discussed for a simply supported plate control problem with control acting through either an interior force or a Neumann boundary condition.
- Numerical experiments are performed to illustrate the theoretical results for both the distributed and the boundary control problems.

The rest of the article is organized as follows. In Section 2, we set up the abstract framework for the error analysis of discontinuous finite element methods and derive therein some abstract error estimates that form the basis for *a priori* and *a posteriori* error analysis. Section 3 introduces two model examples that are under discussion. In Section 4, we develop the discrete setting and discuss the applications to the model problems introduced in Section 3. In Section 5, we present some numerical examples to illustrate the theoretical results. Finally we conclude the article in Section 6.

2. ABSTRACT SETTING AND ANALYSIS

In this section, we develop an abstract framework for the error analysis of discontinuous and nonconforming methods for approximating the solutions of optimal control problems with either boundary control or distributed control. All the vector spaces introduced below are assumed to be real.

Let V be a Sobolev-Hilbert space with the norm $\|\cdot\|_V$ and with the dual denoted by V'. The space V will be an admissible space for state and adjoint state variables. Let W be a Hilbert space such that $V \subset W \subset V'$ (Gelfand triplet) and the inclusion is continuous. The inner product and norm on W is denoted by $(\cdot,\cdot)$ and $\|\cdot\|_W$, respectively. Let Q be a Hilbert space that will be used for seeking the control variable. The norm and the inner-product on Q will be denoted by $\|\cdot\|_Q$ and $\langle\cdot,\cdot\rangle$ respectively. Let $B : V \to Q$ be a linear and continuous operator. Let $Q_{ad} \subset Q$ be a nonempty, closed and convex subset.

Assume that $(u, \Box, q) \in V \times V \times Q_{ad}$ solve the system

$$a(u, v) = f(v) + \langle q, Bv\rangle \quad \forall v \in V, \tag{2.1}$$

$$a(v, \Box) = (u - u_d, v) \quad \forall v \in V, \tag{2.2}$$

$$\langle B\Box + \Box q, p - q\rangle \geq 0 \quad \forall p \in Q_{ad}, \tag{2.3}$$

where $f \in V'$, $u_d \in W$, $\Box > 0$ are given and $a : V \times V \to R$ is a continuous and elliptic bilinear form in the sense that there exist positive constants C and c such that

$$|a(u, v)| \leq C\|u_h\|_V \, \|v\|_V \quad \forall u,\, v \in V,$$

$$a(v, v) \geq c\|v\|_V^2 \quad \forall v \in V\Box$$

Next, we introduce the corresponding discrete setting. Let $V_h \subset W$ be a finite dimensional (finite element) subspace and there is a norm $\|\cdot\|_h$ on $V_h + V$ such that $\|v\|_h = \|v\|_V$ for all $v \in V$. Let $a_h : V_h \times V_h \to R$ be a continuous and elliptic bilinear form, i.e., there exist positive constants $\tilde{C}$ and $\tilde{c}$ such that

$$|a_h(u_h, v_h)| \leq \tilde{C}\|u\|_h \, \|v\|_h \quad \forall u_h, \, v_h \in V_h,$$

$$a_h(v_h, v_h) \geq \tilde{c}\|v_h\|_h^2 \quad \forall v_h \in V_h \square$$

Similarly, assume that $Q_h \subset Q$ is a finite dimensional (finite element) subspace and $Q_{ad}^h \subset Q_{ad}$ is a nonempty, closed and convex subset of Q_h. Further assume that $f \in V_h'$.

Suppose that the discrete variables $(u_h, \square_h, q_h) \in V_h \times V_h \times Q_{ad}^h$ solve the system

$$a_h(u_h, v_h) = f(v_h) + \langle q_h, B_h v_h \rangle \quad \forall v_h \in V_h, \tag{2.4}$$

$$a_h(v_h, \square_h) = (u_h - u_d, v_h) \quad \forall v_h \in V_h, \tag{2.5}$$

$$\langle B_h \square_h + \square q_h, p_h - q_h \rangle \geq 0 \quad \forall p_h \in Q_{ad}^h, \tag{2.6}$$

where $B_h : V_h + V \to Q_h$ is a discrete counterpart of B such that $B_h v = Bv$ for all $v \in V$.

Throughout this section, we assume that the following hold true:
Assumption (P-T): There hold

$$\|v\|_W \leq C\|v\|_h \quad \forall v \in V + V_h, \tag{2.7}$$

$$\|B_h(v - v_h)\|_Q \leq C\|v - v_h\|_h \quad \text{for } v \in V, \; v_h \in V_h \square \tag{2.8}$$

It will be seen in subsequent sections that (2.7) corresponds to a Poincaré type inequality and (2.8) corresponds to a trace inequality on broken Sobolev spaces.

We need the Q-projection defined by the following: For given $q \in Q$, let $\square_h q \in Q_h$ be the solution of

$$\langle \square_h q - q, p_h \rangle = 0 \quad \forall p_h \in Q_h \square \tag{2.9}$$

Assumption (Q): Assume that $\square_h q \in Q_{ad}^h$ whenever $q \in Q_{ad}$.

We turn to deriving some abstract *a priori* error analysis. To this end, we introduce some projections as follows: Let $P_h u \in V_h$ and $\bar{P}_h \square \in V_h$ solve

$$a_h(P_h u, v_h) = f(v_h) + \langle q, B_h v_h \rangle \quad \forall v_h \in V_h, \tag{2.10}$$

$$a_h(v_h, \bar{P}_h \square) = (u - u_d, v_h) \quad \forall v_h \in V_h, \tag{2.11}$$

respectively.

The following lemma is a key in the error analysis:

Lemma 2.1. *There holds*

$$\langle B_h(\square_h - \bar{P}_h\square), q - q_h\rangle \geq \square\|q - q_h\|_Q^2 + \langle B_h\square_h + \square q_h, q - p_h\rangle$$
$$+ \langle B_h(\square - \bar{P}_h\square), q - q_h\rangle \quad \forall p_h \in Q_{ad}^h\square$$

Proof. Since $Q_{ad}^h \subset Q_{ad}$ and $B_h = B$ on V, we find from (2.3) and (2.6) that

$$\langle B_h\square_h + \square q_h, q - q_h\rangle \geq \langle B_h\square_h + \square q_h, q - p_h\rangle \quad \forall p_h \in Q_{ad}^h\square$$
$$-\langle B_h\square + \square q, q - q_h\rangle \geq 0\square$$

We find by adding the above two inequalities that

$$\langle B_h(\square_h - \square) + \square(q_h - q), q - q_h\rangle \geq \langle B_h\square_h + \square q_h, q - p_h\rangle \quad \forall p_h \in Q_{ad}^h,$$

which implies

$$\langle B_h(\square_h - \bar{P}_h\square), q - q_h\rangle \geq \square\|q - q_h\|_Q^2 + \langle B_h\square_h + \square q_h, q - p_h\rangle$$
$$+ \langle B_h(\square - \bar{P}_h\square), q - q_h\rangle \quad \forall p_h \in Q_{ad}^h\square$$

This completes the proof. □

The following theorem derives an abstract *a priori* error estimate for the control.

Theorem 2.2. *There holds*

$$\|q - q_h\|_Q^2 + \|u - u_h\|_W^2 \leq C \square \|B\square - \square_h(B\square)\|_Q^2 + \|q - \square_h q\|_Q^2 + \|\square - \bar{P}_h\square\|_h^2 \square$$
$$+ C\|u - P_h u\|_W^2\square$$

Proof. From (2.4–2.5) and the definition of P_h, we have

$$a_h(P_h u - u_h, v_h) = \langle q - q_h, B_h v_h\rangle \quad \forall v_h \in V_h, \tag{2.12}$$
$$a_h(v_h, \bar{P}_h\square - \square_h) = (u - u_h, v_h) \quad \forall v_h \in V_h\square \tag{2.13}$$

Take $v_h = P_h u - u_h$ in (2.13), $v_h = \bar{P}_h\square - \square_h$ in (2.12) and subtract the resulting equations to find

$$\langle q - q_h, B_h(\bar{P}_h\square - \square_h)\rangle - (u - u_h, P_h u - u_h) = 0\square$$

This implies

$$\langle q - q_h, B_h(\square_h - \bar{P}_h \square)\rangle + \|P_h u - u_h\|_W^2 = (u - P_h u, u_h - P_h u)\square$$

Using Lemma 2.1 with $p_h = \square_h q$, we find that

$$\begin{aligned}
\square\|q - q_h\|_Q^2 + \|P_h u - u_h\|_W^2 &\le -\langle B_h \square_h + \square q_h, q - p_h\rangle - \langle B_h(\square - \bar{P}_h \square), q - q_h\rangle \\
&\quad + (u - P_h u, u_h - P_h u) \\
&= -\langle B_h \square + \square q, q - p_h\rangle - \langle B_h(\square - \bar{P}_h \square), q - q_h\rangle \\
&\quad - \langle B_h(\square_h - \square) + \square(q_h - q), q - p_h\rangle \\
&\quad + (u - P_h u, u_h - P_h u) \\
&= -\langle B\square - \square_h (B\square) + \square(q - p_h), q - p_h\rangle \\
&\quad - \langle B_h(\square - \bar{P}_h \square), q - q_h\rangle + (u - P_h u, u_h - P_h u) \\
&\quad - \langle B_h(\square_h - \square) + \square(q_h - q), q - p_h\rangle \\
&= -\langle B\square - \square_h (B\square) + \square(q - p_h), q - p_h\rangle \\
&\quad - \langle B_h(\square - \bar{P}_h \square), q - q_h\rangle + (u - P_h u, u_h - P_h u) \\
&\quad - \langle B_h(\square_h - \square) + \square(p_h - q), q - p_h\rangle \\
&\le C \square \|B\square - \square_h(B\square)\|_Q^2 + \|q - p_h\|_Q^2 \\
&\quad + \|B_h(\square - \bar{P}_h \square)\|_Q^2 \square \\
&\quad + C \square \|B_h(\square - \square_h)\|_Q \|q - p_h\|_Q + \|u - P_h u\|_W^2 \square \\
&\quad + \frac{1}{2}\|u_h - P_h u\|_W^2 + \frac{\square}{2}\|q - q_h\|_Q^2 \square
\end{aligned} \tag{2.14}$$

From the error equation (2.13), we have

$$a_h(\bar{P}_h \square - \square_h, \bar{P}_h \square - \square_h) = (u - u_h, \bar{P}_h \square - \square_h) \le C\|u - u_h\|_W \|\bar{P}_h \square - \square_h\|_W \square$$

By assumption (2.7) and the ellipticity of a_h, we find

$$\|\bar{P}_h \square - \square_h\|_h \le C\|u - u_h\|_W \square \tag{2.15}$$

Now using assumption (2.8) and (2.15), we find

$$\begin{aligned}
\|B_h(\square - \square_h)\|_Q &\le C\|\square - \square_h\|_h \le C\|\square - \bar{P}_h \square\|_h + C\|\bar{P}_h \square - \square_h\|_h \\
&\le C\|\square - \bar{P}_h \square\|_h + C\|u - u_h\|_W \\
&\le C \square \|\square - \bar{P}_h \square\|_h + \|u - P_h u\|_W + \|P_h u - u_h\|_W \square \square
\end{aligned}$$

Using this estimate in (2.14), we complete the proof. □

We now derive the error estimates for the state and the adjoint state variables.

Theorem 2.3. *There hold*

$$\|\square - \square_h\|_h \leq C \,\square\, \|\square - \bar{P}_h\square\|_h + \|B\square - \square_h(B\square)\|_Q + \|q - \square_h q\|_Q + \|u - P_h u\|_W \,\square\,,$$
$$\|u - u_h\|_h \leq C \,\square\, \|\square - \bar{P}_h\square\|_h + \|B\square - \square_h(B\square)\|_Q + \|q - \square_h q\|_Q + \|u - P_h u\|_h \,\square\, \square$$

Proof. The estimate in (2.15) together with the estimate in Theorem 2.2 and the triangle inequality imply

$$\begin{aligned}\|\square - \square_h\|_h &\leq \|\square - \bar{P}_h\square\|_h + \|\bar{P}_h\square - \square_h\|_h \\ &\leq \|\square - \bar{P}_h\square\|_h + C\|u - u_h\|_W \\ &\leq C \,\square\, \|\square - \bar{P}_h\square\|_h + \|B\square - \square_h(B\square)\|_Q + \|q - \square_h q\|_Q + \|u - P_h u\|_W \,\square\end{aligned}$$

The error equation (2.12) and the assumption (2.8) imply

$$\begin{aligned}a_h(P_h u - u_h, P_h u - u_h) &= \langle q - q_h, B_h(P_h u - u_h)\rangle \\ &= \langle q - q_h, B_h(P_h u - u)\rangle + \langle q - q_h, B_h(u - u_h)\rangle \\ &\leq C\,(\|u - P_h u\|_h + \|u - u_h\|_h)\,\|q - q_h\|_Q \square\end{aligned}$$

The rest of the proof follows from Theorem 2.2. □

Next, we will develop an abstract setting for *a posteriori* error control. To this end, define the reconstructions $Ru \in V$ and $\bar{R}\square \in V$ by

$$a(Ru, v) = f(v) + \langle q_h, Bv\rangle \quad \forall v \in V, \tag{2.16}$$
$$a(v, \bar{R}\square) = (u_h - u_d, v) \quad \forall v \in V\square \tag{2.17}$$

From the above definitions and (2.1)–(2.2), we have

$$a(u - Ru, v) = \langle q - q_h, Bv\rangle \quad \forall v \in V, \tag{2.18}$$
$$a(v, \square - \bar{R}\square) = (u - u_h, v) \quad \forall v \in V\square \tag{2.19}$$

The following lemma will be useful in the subsequent *a posteriori* error analysis:

Lemma 2.4. *There holds*

$$\langle B(\bar{R}\square - \square), q - q_h\rangle \geq \square\|q - q_h\|_Q^2 + \langle B_h\square_h + \square q_h, q - p_h\rangle$$
$$+ \langle B_h(\bar{R}\square - \square_h), q - q_h\rangle \quad \forall p_h \in Q_{ad}^h\square$$

Proof. Using the assumptions $Q_{ad}^h \subset Q_{ad}$, $B_h = B$ on V, and the inequalities (2.3) and (2.6), we find

$$\langle B_h\square_h + \square q_h, q - q_h\rangle \geq \langle B_h\square_h + \square q_h, q - p_h\rangle \quad \forall p_h \in Q_{ad}^h\square$$
$$-\langle B_h\square + \square q, q - q_h\rangle \geq 0\square$$

Add the above two inequalities and find

$$\langle B_h(\square_h - \square) + \square(q_h - q), q - q_h\rangle \geq \langle B_h\square_h + \square q_h, q - p_h\rangle \quad \forall p_h \in Q_{ad}^h\square$$

This trivially implies

$$\langle B(\bar{R}\square - \square), q - q_h\rangle \geq \square\|q - q_h\|_Q^2 + \langle B_h\square_h + \square q_h, q - p_h\rangle$$
$$+ \langle B_h(\bar{R}\square - \square_h), q - q_h\rangle \quad \forall p_h \in Q_{ad}^h\square$$

Hence the proof. □

The first result that will be useful in *a posteriori* error estimates for the control is the following:

Theorem 2.5. *There holds*

$$\|q - q_h\|_Q + \|u - Ru\|_W \leq C\,\square\,\|Ru - u_h\|_W + \|\bar{R}\square - \square_h\|_h\,\square$$
$$+ C\|B_h\square_h - \square_h(B_h\square_h)\|_Q\square$$

Proof. Taking $v = u - Ru$ in (2.19) and $v = \square - \bar{R}\square$ in (2.18) and then subtracting the resulting equations,

$$\langle q - q_h, B(\square - \bar{R}\square)\rangle - (u - u_h, u - Ru) = 0$$

trivially implies

$$\langle q - q_h, B(\bar{R}\square - \square)\rangle + \|u - Ru\|_W^2 = -(Ru - u_h, u - Ru)\square$$

Using the estimate in Lemma 2.4 in the above equation, we find

$$\square\|q - q_h\|_Q^2 + \|u - Ru\|_W^2 \leq -(Ru - u_h, u - Ru) - \langle B_h\square_h + \square q_h, q - p_h\rangle$$
$$- \langle B_h(\bar{R}\square - \square_h), q - q_h\rangle \quad \forall p_h \in Q_{ad}^h\square$$

Let $p_h = \Box_h q \in Q^h_{ad}$. Then

$$\langle B_h\Box_h + \Box q_h, q - \Box_h q\rangle = \langle (B_h\Box_h - \Box_h(B_h\Box_h)),\ q - \Box_h q\rangle$$
$$= \langle B_h\Box_h - \Box_h(B_h\Box_h),\ q - q_h\rangle\Box$$

The proof then follows from the Cauchy-Schwarz inequality and assumption (2.8). $\Box$

Next, the result that will be useful in the *a posteriori* error analysis of the state and the adjoint states is derived below.

Theorem 2.6. *There holds*

$$\|u - u_h\|_h + \|\Box - \Box_h\|_h \le C \Box \|Ru - u_h\|_h + \|\bar{R}\Box - \Box_h\|_h \Box$$
$$+ C\|B_h\Box_h - \Box_h(B_h\Box_h)\|_Q\Box$$

Proof. By the triangle inequality,

$$\|u - u_h\|_h \le \|u - Ru\|_h + \|Ru - u_h\|_h\Box$$

Taking $v = u - Ru$ in (2.18) and since $\|\cdot\|_h = \|\cdot\|_V$ on V, we find by using the continuity of the operator B that

$$\|u - Ru\|_V \le C\|q - q_h\|_Q\Box$$

The bound for $u - u_h$ then follows by using Theorem 2.5. Similarly by the triangle inequality

$$\|\Box - \Box_h\|_h \le \|\Box - \bar{R}\Box\|_h + \|\bar{R}\Box - \Box_h\|_h\Box$$

Taking $v = \Box - \bar{R}\Box$ in (2.19) and again since $\|\cdot\|_h = \|\cdot\|_V$ on V, we find by using the continuous imbedding of V in W that

$$\|\Box - \bar{R}\Box\|_V \le C\|u - u_h\|_W\Box$$

The rest of the proof follows from assumption (2.7) and the estimate for $\|u - u_h\|_h$. $\Box$

3. MODEL EXAMPLES

In this section, we present two model problems arising from the optimal control of simply supported plate problem. One deals with the distributed control problem and the other with the boundary control problem. In subsequent sections, we discuss the applications of the abstract error analysis developed in Section 2 to these two model problems approximated by the C^0 interior penalty method. However, the abstract analysis that we have developed in Section 2 is not only limited to these problems. Indeed, it can be applied to second and sixth order problems and to the classical nonconforming and discontinuous Galerkin methods. In what follows, we introduce the common data used by the two model problems.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary Γ. Assume that there is some $m \geq 1$ such that the boundary Γ is the union of some line segments Γ_i's $(1 \leq i \leq m)$ whose interior in the induced topology are pair-wise disjoint. Let the admissible space $V := H^2(\Omega) \cap H_0^1(\Omega)$. Denote the $L_2(\Omega)$ and $L_2(\Gamma)$ inner-products by $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$, respectively. Let $f \in H^{-1}(\Omega)$, $u_d \in L_2(\Omega)$ and a real number $\beta > 0$ be given. Define the bilinear form $a : V \times V \to R$ by

$$a(w, v) = (D^2 w, D^2 v), \tag{3.1}$$

where $D^2 w = [w_{x_i x_j}]_{1 \leq i,j, \leq 2}$ is the standard Hessian of w.

Remark 3.1. We may assume that the load function $f \in V^*$, the dual of V. In that case, the numerical method will have to be modified. The analysis in such cases can be handled as in [2, 25].

Model Problem 1. Define the quadratic functional $J : V \times L_2(\Gamma) \to R$ by

$$J(w, p) = \frac{1}{2} \|w - u_d\|^2_{L_2(\Omega)} + \frac{\beta}{2} \|p\|^2_{L_2(\Gamma)}, \quad w \in V, \; p \in L_2(\Gamma). \tag{3.2}$$

For given $\underline{q}, \overline{q} \in \mathbb{R} \cup \{\pm\infty\}$ with $\underline{q} < \overline{q}$, define the admissible set of controls by

$$Q_b = \{p \in L_2(\Gamma) : \underline{q} \leq p(x) \leq \overline{q} \text{ for a.e. } x \in \Gamma\}.$$

Consider the optimal control problem of finding $u \in V$ and $q \in Q_b$ such that

$$J(u, q) = \min_{w \in V, p \in Q_b} J(w, p), \tag{3.3}$$

subject to the condition that $w \in V$ satisfies

$$a(w, v) = f(v) + \langle p, \partial v / \partial n \rangle \quad \forall v \in V. \tag{3.4}$$

Note that the optimal solution $(u, q) \in V \times Q_b$, whenever it exists, satisfies

$$a(u, v) = f(v) + \langle q, \partial v/\partial n\rangle \quad \forall v \in V. \tag{3.5}$$

In order to establish the existence of a solution to (3.3), note that the model problem (3.4) has a unique solution $w \in V$ for given $p \in L_2(\Gamma)$. Define this correspondence as $Sp = w$. From the stability estimates of the solution w, it is easy to check that $S : L_2(\Gamma) \to L_2(\Omega)$ defines a continuous affine operator. Using the operator S, the minimization problem (3.3) can be written in the reduced form of finding $q \in L_2(\Gamma)$ such that

$$j(q) = \min_{p\in Q_b} j(p), \tag{3.6}$$

where

$$j(p) = \frac{1}{2}\|Sp - u_d\|^2_{L_2(\Omega)} + \frac{\beta}{2}\|p\|^2_{L_2(\Gamma)}. \tag{3.7}$$

Using the theory of elliptic optimal control problems [41], the following proposition on the existence and uniqueness of the solution can be proved and the optimality condition can be derived.

Proposition 3.2. *The control problem (3.6) has a unique solution q and correspondingly there exists a unique state $u = Sq$ of (3.5). Furthermore, by introducing the adjoint state $\phi \in V$ such that*

$$a(v, \phi) = (u - u_d, v) \quad \forall v \in V, \tag{3.8}$$

the optimality condition that $j'(q)(p - q) \geq 0$, $\forall p \in Q_b$, can be expressed as

$$\langle \partial\phi/\partial n + \beta q, p - q\rangle \geq 0 \quad \forall p \in Q_b. \tag{3.9}$$

The strong formulation of the optimality conditions satisfied by (u, ϕ, q) is given by the following system of equations:

$$\begin{aligned}
\Delta^2 u &= f \quad \text{in} \quad \Omega,\\
\Delta^2 \phi &= u - u_d \quad \text{in} \quad \Omega,\\
u = 0,\ \frac{\partial^2 u}{\partial n^2} &= q \quad \text{on} \quad \partial\Omega,\\
\phi = 0,\ \frac{\partial^2 \phi}{\partial n^2} &= 0 \quad \text{on} \quad \partial\Omega,\\
q &= \Pi_{[\underline{q},\bar{q}]}\left(-\frac{1}{\beta}\frac{\partial\phi}{\partial n}\right) \quad \text{on} \quad \partial\Omega,
\end{aligned}$$

where $\Pi_{[a,b]}g(x)$ is defined by $\Pi_{[a,b]}g(x) = \min\{b, \max\{a, g(x)\}\}$.

Model Problem 2. In this example, we consider the model of a distributed control problem. For this, define the quadratic functional $J : V \times L_2(\Omega) \to R$ by

$$J(w, p) = \frac{1}{2}\|w - u_d\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\|p\|^2_{L_2(\Omega)}, \quad w \in V, \; p \in L_2(\Omega). \tag{3.10}$$

Let $\underline{q}, \; \overline{q} \in \mathbb{R}$ with $\underline{q} < \overline{q}$ be given. Define $Q_d = \{p \in L_2(\Omega) : \underline{q} \leq p(x) \leq \overline{q}$ for a.e. $x \in \Omega\}$. The distributed control problem consists of finding $u \in V$ and $q \in Q_d$ such that

$$J(u, q) = \min_{w \in V, p \in Q_d} J(w, p), \tag{3.11}$$

where $w \in V$ satisfies

$$a(w, v) = f(v) + (p, v) \quad \forall v \in V. \tag{3.12}$$

It is clear that whenever it exists the optimal solution $(u, q) \in V \times Q_d$ satisfies

$$a(u, v) = f(v) + (q, v) \quad \forall v \in V. \tag{3.13}$$

Note that the model problem (3.12) has a unique solution $w \in H^2(\Omega)$ for given $p \in L_2(\Omega)$. Setting this correspondence as $Sp = w$ and using the stability estimates of w, it is obvious that $S : L_2(\Omega) \to L_2(\Omega)$ defines a continuous affine operator. Then, the minimization problem (3.11) is reduced to find $q \in L_2(\Omega)$ such that

$$j(q) = \min_{p \in Q_d} j(p), \tag{3.14}$$

where

$$j(p) = \frac{1}{2}\|Sp - u_d\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\|p\|^2_{L_2(\Omega)}. \tag{3.15}$$

Again, the theory of elliptic optimal control problems [41] implies that the problem (3.14) has a unique solution q. The corresponding solution of (3.13) is denoted by u. Moreover, as in the earlier case there exists an adjoint state $z \in V$ such that

$$a(v, z) = (u - u_d, v) \quad \forall v \in V, \tag{3.16}$$

and

$$(z + \alpha q, p - q) \geq 0 \quad \forall p \in Q_d. \tag{3.17}$$

Similar to the boundary control problem, it can be checked that the strong formulation of the optimality system satisfied by the solution (u, ϕ, q) is given by

$$
\begin{aligned}
\Delta^2 u &= f + q \quad \text{in } \Omega,\\
\Delta^2 \phi &= u - u_d \quad \text{in } \Omega,\\
u = 0,\ \frac{\partial^2 u}{\partial n^2} &= 0 \quad \text{on } \partial\Omega,\\
\phi = 0,\ \frac{\partial^2 \phi}{\partial n^2} &= 0 \quad \text{on } \partial\Omega,\\
q &= \Pi_{[\underline{q}, \overline{q}]}\left(-\frac{1}{\alpha}\phi\right) \quad \text{in } \Omega,
\end{aligned}
$$

where we recall that $\Pi_{[a,b]}g(x)$ is defined by $\Pi_{[a,b]}g(x) = \min\{b, \max\{a, g(x)\}\}$.

Remark 3.3. The quadratic functionals J in (3.2) or (3.11) may consists of $\|w - u_d\|_{H^k(\Omega)}$ for $k = 1$ or 2. The analysis in the forthcoming section can easily be extended to these cases as well.

4. DISCRETE PROBLEMS

4.1. Notations

Denote the norm and semi-norm on $H^k(D)$ $(k \geq 0)$ for any open domain $D \subset \mathbb{R}^s$ $(s \geq 1)$ by $\|v\|_{k,D}$ and $|v|_{k,D}$. Note that the semi-norm $|\cdot|_{2,\Omega}$ defines a norm on $V = H^2(\Omega) \cap H_0^1(\Omega)$ which is equivalent to $\|\cdot\|_{2,\Omega}$. Let $\mathcal{T}_h$ be a regular simplicial subdivision of Ω. Denote the set of all interior edges/faces of $\mathcal{T}_h$ by $\mathcal{E}_h^i$, the set of boundary edges/faces by $\mathcal{E}_h^b$, and define $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$. Let h_T=diam(T) and $h = \max\{h_T : T \in \mathcal{T}_h\}$. The diameter of any edge/face $e \in \mathcal{E}_h$ will be denoted by h_e. We define the Sobolev space $H^s(\Omega, \mathcal{T}_h)$ associated with the subdivision $\mathcal{T}_h$ as follows:

$$H^s(\Omega, \mathcal{T}_h) = \{v \in L_2(\Omega) : v|_T \in H^s(T)\ \forall\ T \in \mathcal{T}_h\}.$$

The discontinuous finite element space is

$$V_h = \{v \in H_0^1(\Omega) : v|_T \in \mathbb{P}_2(T)\ \forall\ T \in \mathcal{T}_h\}, \tag{4.1}$$

where $\mathbb{P}_2(D)$ is the space of polynomials of degree less than or equal to 2 restricted to the set D. It is clear that $V_h \subset H_0^1(\Omega) \cap H^s(\Omega, \mathcal{T}_h)$ for any positive integer s.

For any $e \in \mathcal{E}_h^i$, there are two elements T_+ and T_- such that $e = \partial T_+ \cap \partial T_-$. Let n_- be the unit normal of e pointing from T_- to T_+ and set $n_+ = -n_-$. For any $v \in H^2(\Omega, \mathcal{T}_h)$, we define the jump of the normal derivative of v on e by

$$[\![\nabla v]\!] = \nabla v_+\big|_e \cdot n_+ + \nabla v_-\big|_e \cdot n_-$$

where $v_\pm = v\big|_{T_\pm}$. For any $v \in H^3(\Omega, \mathcal{T}_h)$, we define the mean and jump of the second order normal derivative of v across e by

$$\{\!\{\partial^2 v/\partial n^2\}\!\} = \frac{1}{2}\left(\partial^2 v_+/\partial n^2 + \partial^2 v_-/\partial n^2\right),$$

and

$$[\![\partial^2 v/\partial n^2]\!] = \left(\partial^2 v_+/\partial n^2 - \partial^2 v_-/\partial n^2\right),$$

respectively, where n is either n_+ or n_- (the sign of n will not change the above quantities).

For notational convenience, we also define jump and average on the boundary edges. For any $e \in \mathcal{E}_h^b$, there is an element $T \in \mathcal{T}_h$ such that $e = \partial T \cap \partial\Omega$. Let n_e be the unit normal of e that points outside T. For any $v \in H^2(T)$, we set on e

$$[\![\nabla v]\!] = \nabla v \cdot n_e,$$

and for any $v \in H^3(T)$, we set

$$\left\{\!\!\left\{\partial^2 v/\partial n^2\right\}\!\!\right\} = \partial^2 v/\partial n^2.$$

We require the following trace inequality [20]:

Lemma 4.1. *There holds for $v \in H^2(\Omega) \cap H_0^1(\Omega)$ that*

$$\|\partial v/\partial n\|_{L_2(\Gamma_i)} \leq C\|v\|_{2,\Omega} \quad \forall 1 \leq i \leq m.$$

We also use the following inverse inequality on V_h [6, 13]:

Lemma 4.2. *For $v_h \in V_h$, there holds*

$$\|v_h\|_{L_2(e)} \leq C h_e^{-1/2} \|v_h\|_{L_2(T)} \quad \forall\, T \in \mathcal{T}_h,$$

where e is an edge of T.

4.1.1. Enriching Map

Let $V_c \subset H^2(\Omega) \cap H_0^1(\Omega)$ be the Hsieh-Clough-Toucher C^1 finite element space associated with the triangulation $\mathcal{T}_h$ (see [6, 9, 13]). In the error analysis of discontinuous Galerkin methods, we use an enriching map $E_h : V_h \to V_c$ that plays an important role. As it is done in [9], we define $E_h : V_h \to V_c$ as follows: Let N be any degree of freedom of V_c, i.e., N is either the evaluation of a shape function, its first order derivatives at any vertex, or the evaluation of the normal derivative of shape function at the midpoint of any edge in $\mathcal{T}_h$. Then, for any $v \in V_h$,

$$N(E_h v_h) = \frac{1}{|\mathcal{T}_N|} \sum_{T\in\mathcal{T}_N} N(v_T),$$

where $\mathcal{T}_N$ is the set of triangles sharing the degree of freedom N and $|\mathcal{T}_N|$ denotes the cardinality of $\mathcal{T}_N$.

The following lemma states the approximation properties satisfied by the map E_h [9], also see [7]:

Lemma 4.3. *Let $v \in V_h$. It holds that*

$$\sum_{T\in\mathcal{T}_h}\left(h_T^{-4}\|E_h v - v\|_{0,T}^2 + h_T^{-2}\|\nabla(E_h v - v)\|_{0,T}^2\right) \le C\left(\sum_{e\in\mathcal{E}_h^i}\frac{1}{h_e}\left\|\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right\|_{0,e}^2\right) \quad \forall\, v \in V_h,$$

and

$$\sum_{T\in\mathcal{T}_h}|E_h v - v|_{H^2(T)}^2 \le C\left(\sum_{e\in\mathcal{E}_h^i}\frac{1}{h_e}\left\|\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right\|_{0,e}^2\right) \quad \forall\, v \in V_h.$$

Following [9], the bilinear form for the numerical method is defined by

$$\begin{aligned} a_h(w,v) = {} & \sum_{T\in\mathcal{T}_h}\int_T D^2 w : D^2 v\, dx - \sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\partial^2 w/\partial n^2\right\}\!\!\right\}\left[\!\left[\partial v/\partial n\right]\!\right] ds \\ & - \sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\partial^2 v/\partial n^2\right\}\!\!\right\}\left[\!\left[\partial w/\partial n\right]\!\right] ds + \sum_{e\in\mathcal{E}_h^i}\int_e \frac{\sigma}{h_e}\left[\!\left[\partial w/\partial n\right]\!\right]\left[\!\left[\partial v/\partial n\right]\!\right] ds, \quad (4.2) \end{aligned}$$

where $\sigma > 0$ is a real number. Define the following norm for $v \in H^s(\Omega, \mathcal{T}_h)$ for $s \geq 2$:

$$\|v\|_h^2 = \left(\sum_{T\in\mathcal{T}_h} \|D^2 v\|^2_{L_2(T)} + \sum_{e\in\mathcal{E}_h^i} \int_e \frac{\sigma}{h_e} [\![\partial v/\partial n]\!]^2 \, ds \right).$$

We refer to [5, 9] for a proof of the following lemma.

Lemma 4.4. *It holds that*

$$a_h(w, v) \leq \tilde{C}\|w\|_h\|v\|_h \quad \forall w, v \in V_h.$$

For sufficiently large σ, it holds that

$$\tilde{c}\|v\|_h^2 \leq a_h(v, v) \quad \forall v \in V_h.$$

4.2. Discrete Boundary Control Problem

The model we study in this case is the Model Problem 1 described in Section 3. In this model, the space $V = H^2(\Omega) \cap H_0^1(\Omega)$ and V_h is the one defined in (4.1). The space $Q = L_2(\Gamma)$ and $W = L_2(\Omega)$. Set $Q_{ad} = Q_b$, where Q_b is defined in Section 3. The discrete control space Q_h is defined by $Q_h = \{p_h \in L_2(\Gamma) : p_h|_e \in P_0(e), \forall e \in \mathcal{E}_h^b\}$ and define the admissible set $Q_{ad}^h = \{p_h \in Q_h : \underline{q} \leq p_h \leq \overline{q}\}$. It is clear that $Q_{ad}^h \subset Q_{ad}$ and $\Pi_h q \in Q_{ad}^h$ whenever $q \in Q_{ad}$. The operator $B : V \to Q$ is nothing but the piece-wise (Γ_i-wise) normal derivative on Γ and $B_h : V_h \to Q_h$ is defined by the piecewise (edge-wise) normal derivative, i.e., $B_h v|_e = (\partial v_T/\partial n)|_e$, where $v_T = v|_T$ and T are the triangle having the edge e on its boundary. We now verify assumptions (2.7) and (2.8). The inequality (2.7) follows from the results on Poincaré type inequalities in [4]. The estimate in (2.8) follows from the well-known trace inequality on $H^2(\Omega)$ and the properties of enriching function E_h as follows: Let $v \in V$ and $v_h \in V_h$. Then

$$\begin{aligned}
\sum_{e\in\mathcal{E}_h^b} \|\partial(v - v_h)/\partial n\|^2_{0,e} &\leq 2 \sum_{e\in\mathcal{E}_h^b} \left(\|\partial(v - E_h v_h)/\partial n\|^2_{0,e} + \|\partial(v_h - E_h v_h)/\partial n\|^2_{0,e} \right) \\
&= 2 \sum_{1\leq i\leq m} \|\partial(v - E_h v_h)/\partial n\|^2_{0,\Gamma_i} \\
&\quad + 2 \sum_{e\in\mathcal{E}_h^b} \|\partial(v_h - E_h v_h)/\partial n\|^2_{0,e}.
\end{aligned}$$

Since $(v - E_h v_h) \in H^2(\Omega) \cap H_0^1(\Omega)$, the trace inequality in Lemma 4.1 implies that

$$\sum_{e \in \Box_h^b} \|\partial(v - v_h)/\partial n\|_{0,e}^2 \leq C\|v - E_h v_h\|_{2,\Omega}^2 + 2 \sum_{e \in \Box_h^b} \|\partial(v_h - E_h v_h)/\partial n\|_{0,e}^2,$$

then the triangle inequality yields

$$\sum_{e \in \Box_h^b} \|\partial(v - v_h)/\partial n\|_{0,e}^2 \leq C\left(\|v - v_h\|_h^2 + \sum_{e \in \Box_h^b} \|\partial(v_h - E_h v_h)/\partial n\|_{0,e}^2 \right).$$

The trace-inverse inequality in Lemma 4.2 on discrete spaces and Lemma 4.3 complete the proof of (2.8).

The abstract error estimates in Theorem 2.2 and Theorem 2.3 are valid to the model problem under discussion.

Using the error analysis in [9] and [8, 25], we deduce the following error estimates for the projections $P_h u$ and $\bar{P}_h \Box$:

$$\|u - P_h u\|_h \leq C\left(\inf_{v_h \in V_h} \|u - v_h\|_h + h\|f\|_{-1,\Omega} + h^{1/2} \inf_{p_h \in Q_h} \|q - p_h\|_{0,\Omega} \right),$$

$$\|\Box - \bar{P}_h \Box\|_h \leq C\left(\inf_{v_h \in V_h} \|\Box - v_h\|_h + h^2\|u - u_d\|_{0,\Omega} \right).$$

Using these estimates, Theorem 2.2, and Theorem 2.3, we obtain the following error estimate:

$$\begin{aligned}
&\|q - q_h\|_{0,\Omega} + \|u - u_h\|_h + \|\Box - \Box_h\|_h \\
&\quad \leq C\Big(\inf_{v_h \in V_h} \|\Box - v_h\|_h + \inf_{v_h \in V_h} \|u - v_h\|_h + h^2\|u - u_d\|_{0,\Omega} + h\|f\|_{-1,\Omega} \\
&\qquad + \|\partial\Box/\partial n - \Box_h(\partial\Box/\partial n)\|_{0,\Omega} + \|q - \Box_h q\|_{0,\Omega} \\
&\qquad + h^{1/2} \inf_{p_h \in Q_h} \|q - p_h\|_{0,\Omega} \Big).
\end{aligned}$$

Now we can apply the elliptic regularity to derive concrete error estimates. Note that by the well-posedness of the problem, $u, \Box \in H^2(\Omega)$ and $q \in L_2(\Omega)$. Then the optimality condition (3.9) implies that

$$q = \Box_{[\underline{q}, \bar{q}]}\left(-\frac{1}{\Box}\frac{\partial \Box}{\partial n} \right).$$

The elliptic regularity on polygonal domains [3, 21] implies that $\psi \in H^{2+s}(\Omega)$ for some $s \in (0,1]$, which depends on the interior angles of the domain Ω. Then

$$q|_{\Gamma_i} \in H^{1/2+s}(\Gamma_i) \quad \text{for all } 1 \leq i \leq m,$$

since $(\partial\psi/\partial n)|_{\Gamma_i} \in H^{1/2+s}(\Gamma_i)$ for $1 \leq i \leq m$. Using this and since $\partial^2 u/\partial n^2 = q$ on $\partial\Omega$, we also deduce that $u \in H^{2+s}(\Omega)$.

Thus we have proved the following theorem:

Theorem 4.5. *Let $s \in (0,1]$ be the elliptic regularity index. Then there holds*

$$\begin{aligned}
&\|q - q_h\|_{0,\Gamma} + \|u - u_h\|_h + \|\psi - \psi_h\|_h \\
&\quad \leq Ch^s \left(\|u\|_{2+s,\Omega} + \|\psi\|_{2+s,\Omega} + h\|u - u_d\|_{0,\Omega} \right) \\
&\quad\quad + Ch^s \left(h^{1-s}\|f\|_{-1,\Omega} + h^{1/2} \sum_{i=1}^{m} \|q\|_{s,\Gamma_i} \right).
\end{aligned}$$

Define the estimators

$$\begin{aligned}
\eta_u^2 = &\sum_{T\in\mathcal{T}_h} h_T^2\|f\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h^i} \int_e \left(h_e [\![\partial^2 u_h/\partial n^2]\!]^2 + h_e^{-1} [\![\nabla u_h]\!]^2 \right) ds \\
&+ \sum_{e\in\mathcal{E}_h^b} \int_e h_e \left(\partial^2 u_h/\partial n^2 - q_h \right)^2 ds,
\end{aligned}$$

and

$$\begin{aligned}
\eta_\psi^2 = &\sum_{T\in\mathcal{T}_h} h_T^2\|u_h - u_d\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h^i} \int_e \left(h_e [\![\partial^2 \psi_h/\partial n^2]\!]^2 + h_e^{-1} [\![\nabla \psi_h]\!]^2 \right) ds \\
&+ \sum_{e\in\mathcal{E}_h^b} \int_e h_e \left(\partial^2 \psi_h/\partial n^2 \right)^2 ds.
\end{aligned}$$

Again, the error analysis in [9] conclude the following error estimates:

$$\|Ru - u_h\|_h \leq C\eta_u,$$

$$\|\bar{R}\psi - \psi_h\|_h \leq C\eta_\psi.$$

The following theorem is the consequence of the above two estimates, Theorem 2.5, and Theorem 2.6:

Theorem 4.6. *There holds*

$$\|q-q_h\|_{0,\square} + \|u-u_h\|_h + \|\square-\square_h\|_h \le C\left(\square_u + \square_\square\right)$$
$$+C\|\square\square_h/\square n - \square_h(\square\square_h/\square n)\|_{0,\square}.$$

Define the oscillations of a given function $g \in L_2(T)$ by

$$osc(g,T) = \min_{\bar{g}\in P_0(T)} \|g-\bar{g}\|_{0,T}.$$

The local efficiency of the error estimators is presented in the following theorem:

Theorem 4.7. *Let $\square_e$ be the set of two triangles sharing the edge $e \in \square_h^i$. Then there hold*

$$h_T\|f\|_{0,T} \le C\left(|u-u_h|_{2,T} + +osc(f,T)\right),$$
$$h_T\|u_h-u_d\|_{0,T} \le C\left(|u-u_h|_{0,T} + |\square-\square_h|_{2,T} + osc(u_d,T)\right),$$
$$h_e^{1/2}\|[\![\square^2 u_h/\square n^2]\!]\|_{0,e} \le C\left(\sum_{T\in\square_e}\left(|u-u_h|_{2,T} + osc(f,T)\right)\right),$$
$$h_e^{1/2}\|[\![\square^2 \square_h/\square n^2]\!]\|_{0,e} \le C\left(\sum_{T\in\square_e}\left(|u-u_h|_{0,T} + |\square-\square_h|_{2,T} + osc(u_d,T)\right)\right),$$
$$\|\square\square_h/\square n - \square_h(\square\square_h/\square n)\|_{0,e} \le C\left(\|\square(\square_h-\square)/\square n\|_{0,e} + \|\square\square/\square n - \square_h(\square\square/\square n)\|_{0,e}\right),$$

for all $e \in \square_h^i$ and $T \in \square_h$. Further for any boundary edge $e \in \square_h^b$, there hold

$$h_e^{1/2}\|\square^2 u_h/\square n^2 - q_h\|_{0,e} \le C\left(|u-u_h|_{2,T} + \|q-q_h\|_{0,e} + osc(f,T)\right),$$
$$h_e^{1/2}\|\square^2 \square_h/\square n^2\|_{0,e} \le C\left(|u-u_h|_{0,T} + |\square-\square_h|_{2,T} + osc(u_d,T)\right),$$

where $T \in \square_h$ is the triangle sharing the edge e.

Proof. Firstly, note that the triangle inequality and the stability of the projection $\square_h$ imply

$$\|\square\square_h/\square n - \square_h(\square\square_h/\square n)\|_{0,e}$$
$$\le \|\square\square_h/\square n - \square\square/\square n\|_{0,e} + \|\square\square/\square n - \square_h(\square\square/\square n)\|_{0,e}$$
$$+ \|\square_h(\square\square/\square n) - \square_h(\square\square_h/\square n)\|_{0,e}$$
$$\le 2\|\square\square_h/\square n - \square\square/\square n\|_{0,e} + \|\square\square/\square n - \square_h(\square\square/\square n)\|_{0,e}.$$

This proves the efficiency of the term involving the projection $\square_h$. The proof of the remaining estimates of the theorem follow by the standard bubble function techniques. $\square$

4.3. Discrete Distributed Control Problem

The model problem in this subsection is Model Problem 2 introduced in Section 3. Set $V = H^2(\square) \cap H_0^1(\square)$, $W = L_2(\square)$ and $Q = L_2(\square)$. The set $Q_{ad} = Q_d$, where Q_d is defined in Section 3. The discrete set V_h is the same as in (4.1). Define the discrete space $Q_h = \square p_h \in L_2(\square) : p_h|_T \in P_0(T), \forall T \in \square_h \square$ and the admissible set $Q_{ad}^h = \square p_h \in Q_h : \underline{q} \leq p_h \leq \overline{q} \square$. It is easy to check that $Q_{ad}^h \subset Q_{ad}$ and $\square_h q \in Q_{ad}^h$ for $q \in Q_{ad}$. The operator $B : V \to Q$ and $B_h : V_h \to Q_h$ are inclusion (identity) maps. Assumptions (2.7) and (2.8) are the Poincaré type inequalities derived in [4].

The error analysis in [9] and [8, 25] implies the following error estimates for the projections $P_h u$ and $\bar{P}_h \square$:

$$\|u - P_h u\|_h \leq C \Big(\inf_{v_h \in V_h} \|u - v_h\|_h + h\|f\|_{-1,\square} + h^2 \inf_{p_h \in Q_h} \|q - p_h\|_{0,\square} \Big),$$

$$\|\square - \bar{P}_h \square\|_h \leq C \Big(\inf_{v_h \in V_h} \|\square - v_h\|_h + h^2 \|u - u_d\|_{0,\square} \Big)$$

Using Theoremz 2.2, Theorem 2.3, and above estimates, we find

$$\begin{aligned} &\|q - q_h\|_{0,\square} + \|u - u_h\|_h + \|\square - \square_h\|_h \\ &\quad \leq C \Big(\inf_{v_h \in V_h} \|\square - v_h\|_h + \inf_{v_h \in V_h} \|u - v_h\|_h + h^2 \|u - u_d\|_{0,\square} + h\|f\|_{-1,\square} \\ &\qquad + \|\square - \square_h \square\|_{0,\square} + \|q - \square_h q\|_{0,\square} + h^2 \inf_{p_h \in Q_h} \|q - p_h\|_{0,\square} \Big) \end{aligned}$$

We invoke the elliptic regularity now to derive the concrete error estimates. Note that

$$q = \square_{[\underline{q}, \overline{q}]} \Big(-\frac{1}{\square} \square \Big)$$

By the elliptic regularity [21], there is some $s \in (0, 1]$ which depends on the interior angles of the domain $\square$ such that $u, \square \in H^{2+s}(\square)$ and hence $q \in W^{1,\infty}(\square)$.

Thus we deduce the following theorem as in the case of a boundary control problem.

Theorem 4.8. *Let $s \in (0,1]$ be the elliptic regularity index. Then there holds*

$$\begin{aligned}\|q-q_h\|_{0,\Omega} + \|u-u_h\|_h + \|\varphi-\varphi_h\|_h &\le Ch^s\left(\|u\|_{2+s,\Omega} + \|\varphi\|_{2+s,\Omega} + \|u-u_d\|_{0,\Omega}\right)\\ &\quad + Ch\left(\|f\|_{-1,\Omega} + h^2\|q\|_{1,\Omega}\right).\end{aligned}$$

Define the estimators

$$\begin{aligned}\eta_u^2 &= \sum_{T\in\mathcal{T}_h} h_T^2\|f+q_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h^i}\int_e \left(h_e\left[\!\left[\partial^2 u_h/\partial n^2\right]\!\right]^2 + h_e^{-1}[\![\nabla u_h]\!]^2\right) ds\\ &\quad + \sum_{e\in\mathcal{E}_h^b}\int_e h_e\left(\partial^2 u_h/\partial n^2\right)^2 ds,\end{aligned}$$

and

$$\begin{aligned}\eta_\varphi^2 &= \sum_{T\in\mathcal{T}_h} h_T^2\|u_h-u_d\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h^i}\int_e \left(h_e\left[\!\left[\partial^2 \varphi_h/\partial n^2\right]\!\right]^2 + h_e^{-1}[\![\nabla \varphi_h]\!]^2\right) ds\\ &\quad + \sum_{e\in\mathcal{E}_h^b}\int_e h_e\left(\partial^2 \varphi_h/\partial n^2\right)^2 ds.\end{aligned}$$

As in the case of a boundary control problem, the following theorem on *a posteriori* error estimates is a consequence of the results in [9], Theorem 2.5, and Theorem 2.6:

Theorem 4.9. *There holds*

$$\|q-q_h\|_{0,\Omega} + \|u-u_h\|_h + \|\varphi-\varphi_h\|_h \le C\left(\eta_u + \eta_\varphi + \|\varphi_h - \Pi_h(\varphi_h)\|_{0,\Omega}\right).$$

The following theorem on the local efficiency can be deduced by the standard bubble function techniques:

Theorem 4.10. *Let $\mathcal{T}_e$ be the set of two triangles sharing the edge $e \in \mathcal{E}_h^i$. Then there hold*

$$\begin{aligned}h_T\|f+q_h\|_{0,T} &\le C\left(|u-u_h|_{2,T} + \|q-q_h\|_{0,T} + osc(f,T)\right),\\ h_T\|u_h-u_d\|_{0,T} &\le C\left(|u-u_h|_{0,T} + |\varphi-\varphi_h|_{2,T} + osc(u_d,T)\right),\\ h_e^{1/2}\left\|\left[\!\left[\partial^2 u_h/\partial n^2\right]\!\right]\right\|_{0,e} &\le C\sum_{T\in\mathcal{T}_e}\left(|u-u_h|_{2,T} + \|q-q_h\|_{0,T} + osc(f,T)\right),\\ h_e^{1/2}\left\|\left[\!\left[\partial^2 \varphi_h/\partial n^2\right]\!\right]\right\|_{0,e} &\le C\sum_{T\in\mathcal{T}_e}\left(|u-u_h|_{0,T} + |\varphi-\varphi_h|_{2,T} + osc(u_d,T)\right),\end{aligned}$$

$$\|\square_h - \square_h \square_h\|_{0,T} \leq C \left(\|\square_h - \square\|_{0,T} + \|\square - \square_h \square\|_{0,T} \right),$$

for all $e \in \square_h^i$ *and* $T \in \square_h$. *Further for any boundary edge* $e \in \square_h^b$, *there hold*

$$h_e^{1/2} \|\square^2 u_h / \square n^2\|_{0,e} \leq C \left(|u - u_h|_{2,T} + \|q - q_h\|_{0,T} + osc(f, T) \right),$$

$$h_e^{1/2} \|\square^2 \square_h / \square n^2\|_{0,e} \leq C \left(|u - u_h|_{0,T} + |\square - \square_h|_{2,T} + osc(u_d, T) \right),$$

where $T \in \square_h$ *is the triangle sharing the edge* e.

5. NUMERICAL EXPERIMENTS

In this section, we present some numerical experiments to illustrate the theoretical results derived in the article. In all of the examples below, we choose the penalty parameter $\square = 10$. The discrete solution is computed by using the primal-dual active set algorithm in [41]. For each of the examples below, we test the order of convergence obtained on a sequence of uniformly refined meshes with mesh parameter h as is shown in Table 1 and Table 2. Further, we test the performance of the *a posteriori* error estimator on a sequence of adaptively refined meshes using the following adaptive refinement algorithm:

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

We compute the discrete solutions, and then we compute the error estimator and mark the elements using the Dörlfer marking technique [16] with bulk parameter $\square = 0\square3$. We refine the marked elements using the newest vertex bisection algorithm and obtain a new mesh.

Example 1. In this example, we consider a model of distributed optimal control problem with homogeneous simply supported plate boundary conditions. The computational domain is chosen to be $\square = (0,1)^2$. We set the parameters $\underline{q} = -750$, $\overline{q} = -50$ and $\square = 10^{-3}$. The data of the model problem is constructed in such a way that the exact solution is known. This is done by choosing the state variable u and the adjoint variable $\square$ as

$$u(x,y) = \square(x,y) = \sin^3(\square x)\sin^3(\square y),$$

TABLE 1 Errors and orders of convergence for Example 1

h	$\|u - u_h\|_h$	order	$\|\square - \square_h\|_h$	order	$\|q - q_h\|_{0,\square}$	order
1/8	6.5598	–	6.5644	–	58.6117	–
1/16	3.3721	0.9600	3.3808	0.9573	29.7993	0.9759
1/32	1.6701	1.0137	1.6719	1.0158	14.6533	1.0241
1/64	0.8286	1.0112	0.8289	1.0123	7.3130	1.0027
1/128	0.4133	1.0037	0.4133	1.0040	3.6581	0.9994

TABLE 2 Errors and orders of convergence for Example 2

h	$\|u-u_h\|_h$	order	$\|\Box-\Box_h\|_h$	order	$\|q-q_h\|_{0,\Box}$	order
1/8	37.6642	–	0.0867	–	6.1533e+002	–
1/16	7.8031	2.2711	0.0363	1.2563	3.4702e+002	0.8264
1/32	2.0838	1.9049	0.0178	1.0256	1.8139e+002	0.9359
1/64	0.9318	1.1611	0.0090	0.9830	9.2646e+001	0.9693
1/128	0.2908	1.6799	0.0045	1.0008	4.6435e+001	0.9965

and the control q as $q(x) = \Box_{[-750,-50]}\left(-\frac{1}{\Box}\Box(x)\right)$. The source term f and the observation u_d are then computed by using

$$f = \Box^2 u - q, \quad u_d = u - \Box^2 \Box\Box$$

The exact errors and orders of convergence have been computed on a sequence of uniformly refined meshes and shown in Table 1. The results clearly predict the linear rate of convergence derived in Theorem 4.8.

Now, we test the performance of the *a posteriori* error estimator in Theorem 4.9 for the above distributed control problem. Note that the state and the adjoint state are smooth but not the control. Figure 1 shows the behavior of the estimator and the errors $\|u-u_h\|_h$, $\|\Box-\Box_h\|_h$ and $\|q-q_h\|_{0,\Box}$ with the increasing number of degrees of freedom N

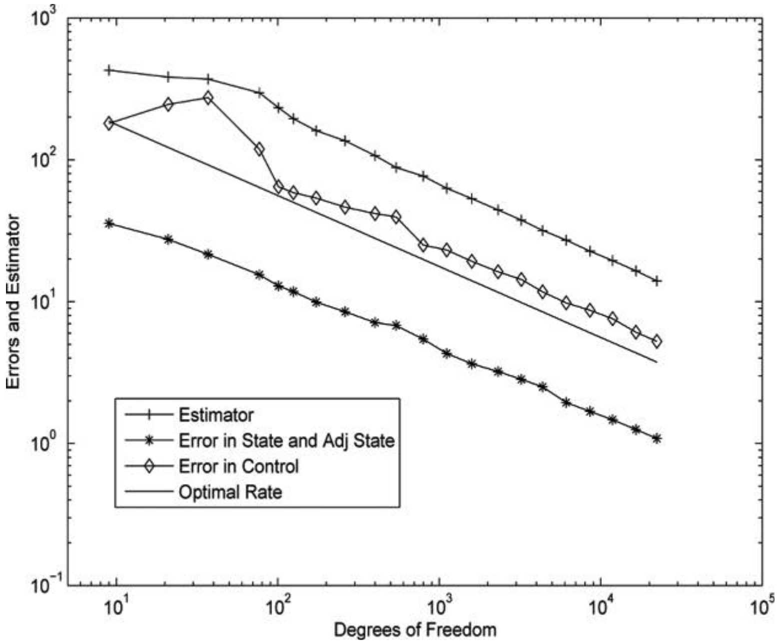

FIGURE 1 Errors and estimator for Example 1.

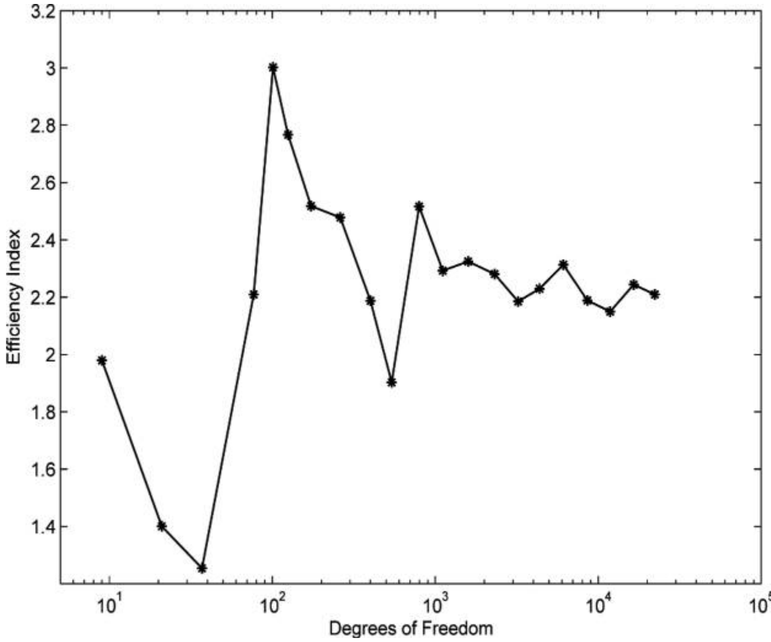

FIGURE 2 Efficiency Index for Example 1.

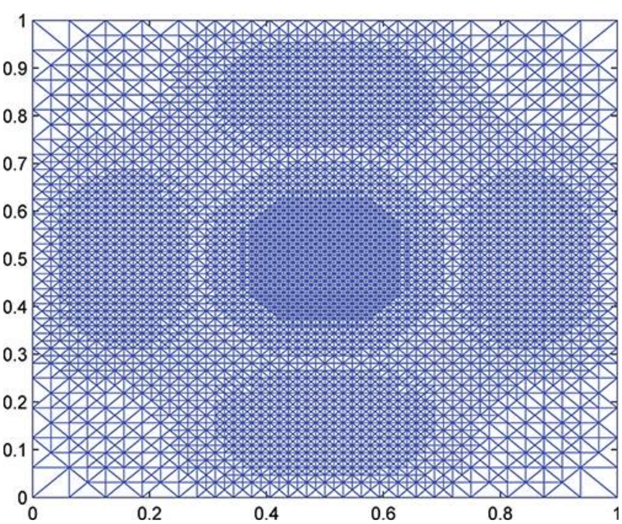

FIGURE 3 Adaptive mesh refinement for Example 1.

(number of unknowns for state variable). We observe that the estimator is reliable. The errors in state, adjoint state, and control converge at the optimal rate of $1/\sqrt{N}$. The efficiency of the estimator is depicted through the efficiency indices $\text{estimator}/(\|u - u_h\|_h + \|\Box - \Box_h\|_h + \|q - q_h\|_{0,\Box})$ in Figure 2. Finally, Figure 3 shows the adaptive mesh refinement.

Example 2. In this example, we consider a boundary control problem defined on a square domain $\Box = (0,1) \times (0,1)$. In order to form the problem with known solution, we slightly modify the model problem to the following:

$$\Box^2 u = f \quad \text{in} \quad \Box,$$
$$u = 0, \quad \frac{\Box^2 u}{\Box n^2} = q + q_d \quad \text{on} \quad \Box\Box,$$

where q_d is a given function. We set the state variable to be $u(x, y) = \sin(\Box x)\sin(\Box y)$, the adjoint variable to be $\Box(x, y) = (x - 2x^3 + x^4)(y - 2y^3 + y^4)$, and the control to be

$$q = -\Box_{[500,2000]} \Box \frac{1}{\Box} \frac{\Box\Box}{\Box n} \Box,$$

where $\Box = 10^{-4}$. The data of the problem is then computed by

$$f = \Box^2 u, \quad u_d = u - \Box^2 \Box, \quad \text{and} \quad q_d = \frac{\Box^2 u}{\Box n^2} - q\Box$$

In the first experiment, we test the order of convergence on a sequence of uniform meshes. The linear rate of convergence can be observed in Table 2 as it is derived in Theorem 4.5.

In the second experiment, we test the performance of the error estimator in Theorem 4.5 using the adaptive algorithm. Figure 4 illustrates the reliability of the error estimator and the optimal rate of convergence of the adaptive refinement algorithm. Efficiency of the error estimator is shown in Figure 5. The local mesh refinement on the boundary is realized near the kinks of the control, see Figure 6.

Example 3. In this example, we test the performance of the *a posteriori* error estimator for a distributed control problem in the presence of re-entrant corners. The domain $\Box$ is set to be L–shaped as is shown in Figure 8. We set the source term $f = 1$ and the observation $u_d = 1$. The parameters $\underline{q}$, $\overline{q}$ and $\Box$ are taken as in Example 1. In this case, since we do not have exact solutions at hand, we test the optimal convergence of the error estimator and its performance in capturing the re-entrant corner. The numerical experiment shows that the error estimator converges optimally (see Figure 7) and refines the mesh locally at the re-entrant corner (see Figure 8) as is expected.

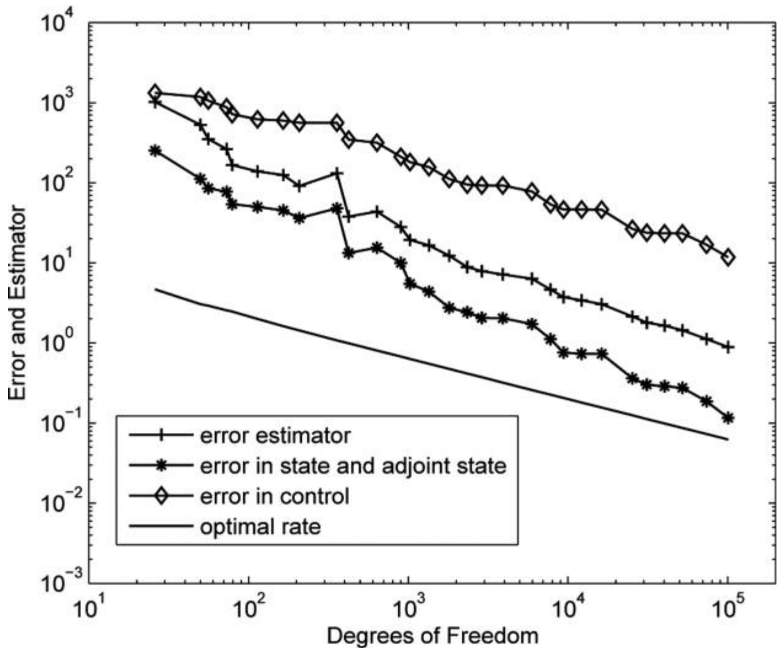

FIGURE 4 Errors and estimator for Example 2.

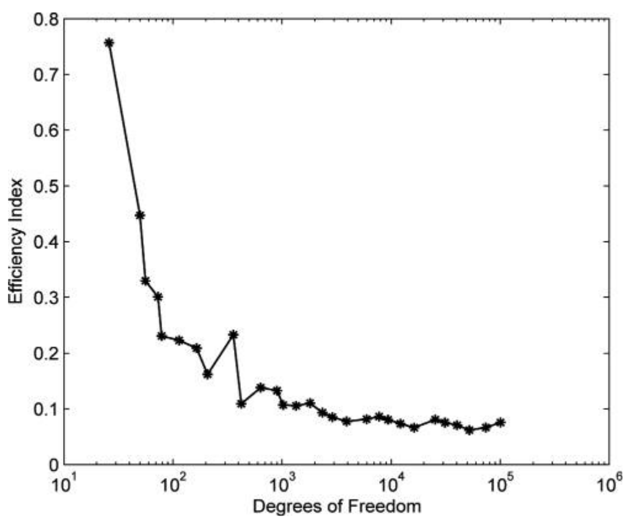

FIGURE 5 Efficiency index for Example 2.

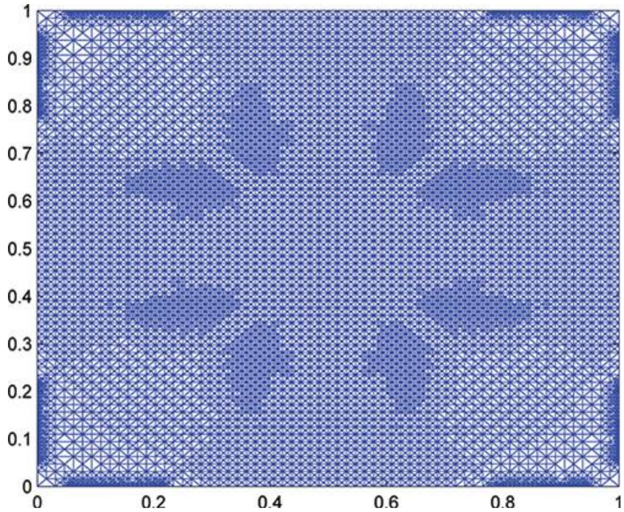

FIGURE 6 Adaptive mesh refinement for Example 2.

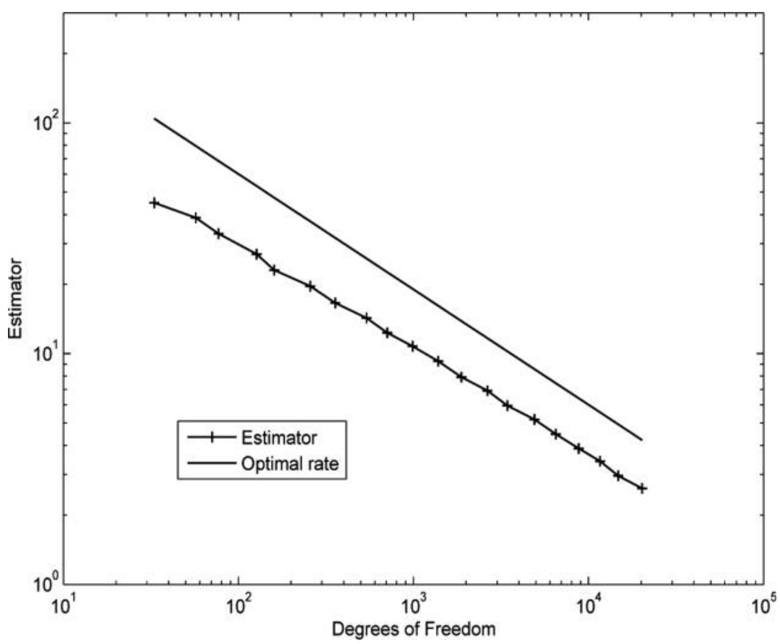

FIGURE 7 Errors and estimator for Example 3.

Example 4. Similar to Example 3, we test the performance of the *a posteriori* error estimator for a boundary control problem on the $L-$shape shown in Figure 10. We consider the source term $f = 1$ and the observation

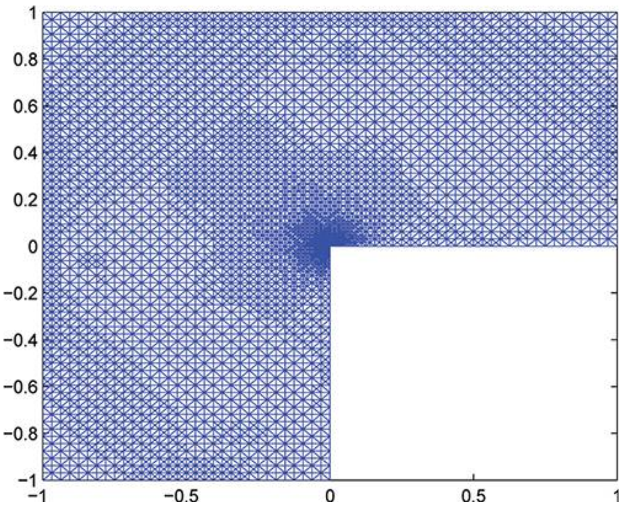

FIGURE 8 Mesh refinement for Example 3.

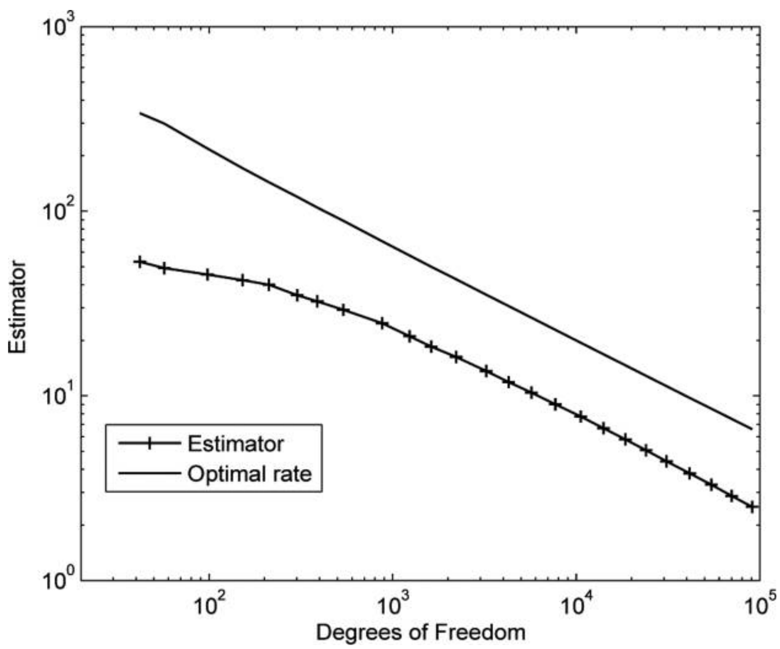

FIGURE 9 Errors and estimator for Example 4.

$u_d = 1$. We test the optimal convergence of the error estimator and its performance in capturing the re-entrant corner and possibly the singularities in the control on the boundary. We consider the parameters

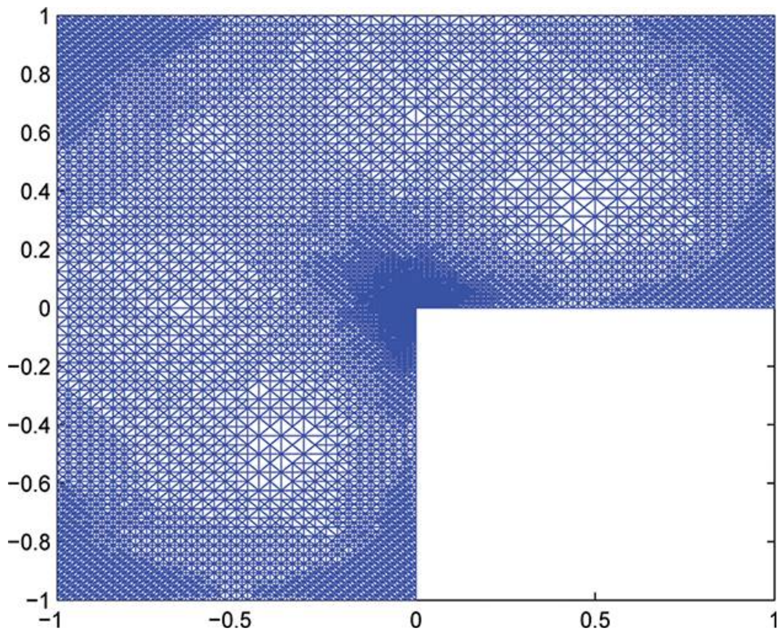

FIGURE 10 Mesh refinement for Example 4.

$\underline{q} = -200$, $\overline{q} = 200$ and □$= 10^{-3}$. The experiment shows the optimal convergence of the error estimator (see Figure 9) and the local mesh refinement at the re-entrant corner (see Figure 10) and on the boundary.

6. CONCLUSIONS

We have developed a framework for the error analysis of discontinuous finite element methods for linear elliptic optimal control problems with control constraints. The abstract analysis provides the best approximation results, which will be useful in convergence of adaptive methods and delivers a reliable and efficient *a posteriori* error estimator. The results are applicable to a variety of discontinuous Galerkin methods (including classical nonconforming methods) applied to elliptic optimal control problems (distributed and Neumann) with constraints on control. Applications to C^0 interior penalty methods for optimal control problems governed by the biharmonic equation with simply supported boundary conditions are established. Numerical experiments illustrate the theoretical findings. The results in the article will not directly cover the analysis of nonlinear elliptic optimal control problems, however they will be useful to analyze the nonlinear problems.

ACKNOWLEDGEMENTS

The authors would like to thank the referees for their comments and suggestions which have helped the authors to improve the presentation. The first author would like to acknowledge the support from NBHM-India, the second author would like to acknowledge the support from DST fast track project and all the authors would like to thank the UGC Center for Advanced Study.

REFERENCES

1. T. Apel, P. Johannes, and A. Rösch (2012). Finite element error estimates for Neumann boundary control problems on graded meshes. *Comput. Optim. Appl.* 52:3–28.
2. S. Badia, R. Codina, T. Gudi, and J. Guzman (2014). Error analysis of discontinuous Galerkin methods for Stokes problem under minimal regularity. *IMA J. Numer. Anal.* 34:800–819.
3. H. Blum and R. Rannacher (1980). On the boundary value problem of the biharmonic operator on domains with angular corners. *Math. Methods Appl. Sci.* 2:556–581.
4. S. C. Brenner, K. Wang, and J. Zhao (2004). Poincaré-Friedrichs inequalities for piecewise H^2 functions. *Numer. Funct. Anal. Optim.* 25: 463–478.
5. S. C. Brenner and L. -Y. Sung (2005). C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. *J. Sci. Comput.* 22/23:83–118.
6. S. C. Brenner and L. R. Scott (2008). *The Mathematical Theory of Finite Element Methods* (Third Ed.). New York: Springer-Verlag.
7. S. C. Brenner, T. Gudi and L. -Y. Sung (2010). An a posteriori error estimator for a quadratic C^0 interior penalty method for the biharmonic problem. *IMA J. Numer. Anal.* 30:777–798.
8. S. C. Brenner, S. Gu, T. Gudi and L. -Y. Sung (2012). A quadratic C^0 interior penalty method for linear fourth order boundary value problems with boundary conditions of the Cahn-Hilliard type. *SIAM J. Numer. Anal.* 50:2088–2110.
9. S. C. Brenner and M. Neilan (2011). A C^0 interior penalty method for a fourth order elliptic singular perturbation problem. *SIAM J. Numer. Anal.* 49:869–892.
10. W. Cao and D. Yang (2009). Ciarlet-Raviart mixed finite element approximation for an optimal control problem governed by the first bi-harmonic equation. *Journal of Computational and Applied Math.*, 233:372–388.
11. C. Carstensen, D. Gallistl, and M. Schedensack (2013). Discrete reliability of Crouzeix-Raviart FEMs. *SIAM J. Numer. Anal.* 51:2935–2955.
12. E. Casas and J. P. Raymond (2006). Error estimates for the numerical approximation of dirichlet boundary control for semilinear elliptic equations. *SIAM J. Control Optim.* 45:1586–1611.
13. P. G. Ciarlet. (1978). *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam.
14. K. Deckelnick and M. Hinze (2007). Convergence of a finite element approximation to a state-constrained elliptic control problem. *SIAM J. Numer. Anal.* 45:1937–1953.
15. K. Deckelnick, A. Günther, and M. Hinze (2009). Finite element approximation of dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains. *SIAM J. Numer. Anal.* 48:2798–2819.
16. W. Dörlfer (1996). A convergent adaptive algorithm for Poisson's equation. *SIAM J. Numer. Anal.* 33:1106–1124.
17. G. Engel, K. Garikipati, T. J. R. Hughes, M. G. Larson, L. Mazzei, and R. L. Taylor (2002). Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. *Comput. Methods Appl. Mech. Engrg.* 191:3669–3750.
18. S. Frei, R. Rannacher, and W. Wollner (2013). A priori error estimates for the finite element discretization of optimal distributed control problems governed by the biharmonic operator. *Calcolo* 50:165–193.
19. T. Geveci (1979). On the approximation of the solution of an optimal control problems governed by an elliptic equation. *Math. Models Numer. Anal.* 13:313–328.

20. P. Grisvard (1985). *Elliptic Problems in Nonsmooth Domains*. Boston: Pitman.
21. P. Grisvard (1992). *Singularities In Boundary Value Problems*. Springer, Masson, Paris.
22. T. Gudi (2010). A new error analysis for discontinuous finite element methods for linear elliptic problems. *Math. Comp.* 79:2169–2189.
23. T. Gudi and M. Neilan (2011). An interior penalty method for a sixth order elliptic problem. *IMA J. Numer. Anal.* 31:1734–1753.
24. T. Gudi, N. Nataraj, and K. Porwal (2014). An interior penalty method for distributed optimal control problems governed by the biharmonic operator. *Comput. Math. Appl.* 68:2205–2221.
25. T. Gudi, H. S. Gupta, and N. Nataraj (2013). Analysis of an interior penalty method for fourth order problems on polygonal domains. *J. Sci. Comput.* 54:177–199.
26. A. Günther and M. Hinze (2011). Elliptic control problems with gradient constraints-variational discrete versus piecewise constant controls. *Comput. Optim. Appl.* 40:549–566.
27. M.D. Gunzburger, L.S. Hou, and T. Swobodny (1991). Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls. *Math. Model. Numer. Anal.* 25:711–748.
28. R. Falk (1973). Approximation of a class of optimal control problems with order of convergence estimates. *J. Math. Anal. Appl.* 44:28–47.
29. M. Hintermüller, K. Ito, and K. Kunish (2003). The primal-dual active set strategy as a semismooth Newton method. *SIAM J. Optim.* 13:865–888.
30. M. Hinze (2005). A variational discretization concept in control constrained optimization: The linear-quadratic case. *Comput. Optim. Appl.* 30:45–61.
31. M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich (2009). *Optimization with PDE Constraints*. New York: Springer.
32. M. Hintermüller, R. H. W. Hoppe, Y. Iliash, and M. Kiewag (2008). An a posteriori error analysis of adaptive finite element methods for distributed elliptic control problems with control constraints. *ESAIM: Control, Optimisation and Calculus of Variations* 14:540–560.
33. J. Hu, Z.C. Shi, and J. Xu (2012). Convergence and optimality of the adaptive Morley element method. *Numer. Math.* 121:731–752.
34. G. Kanshat and N. Sharma (2014). Divergence-conforming Galerkin methods and C^0 interior penalty method. *SIAM J. Numer. Anal.* 52:1822–1842.
35. K. Kohls, A. Rösch, and K. Siebert (2014). A posteriori error analysis of optimal control problems with control constraints. *SIAM J. Control & Optimization* 52:1832–1861.
36. W. Liu and N. Yan (2008). *Adaptive Finite Element Methods: Optimal Control Governed by PDEs*, Beijing Science Press, Beijing.
37. S. May, R. Rannacher, and B. Vexler (2013). Error analysis for a finite element approximation of elliptic dirichlet boundary control problems. *SIAM J. Control Optim.* 51:2585–2611.
38. C. Meyer and A. Rösch (2004). Superconvergence properties of optimal control problems. *SIAM J. Control & Optimization* 43:970–985.
39. G. Of, T. X. Phan, and O. Steinbach (2015). An energy space finite element approach for elliptic Dirichlet boundary control problems. *Numer. Math.* 129:723–748.
40. C. Ortner and W. Wollner (2011). A priori error estimates for optimal control problems with pointwise constraints on the gradient of the state. *Numer. Math.* 118:587–600.
41. F. Tröltzsch (2005). *Optimale Steuerung Partieller Differentialgleichungen*. (1st ed.) Vieweg, Cambridge University Press, Wiesbaden.