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HOMOGENIZATION OF A HYPERBOLIC EQUATION WITH HIGHLY CONTRASTING DIFFUSIVITY COEFFICIENTS

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Abstract. We study a hyperbolic problem in the framework of periodic homogenization assuming a high contrast between the diffusivity coefficients of the two components M_ε and B_ε of the heterogeneous medium. There are three regimes depending on the ratio between the size of the period and the amplitude α_ε of the diffusivity in B_ε. For the critical regime $\alpha_\varepsilon \simeq \varepsilon$, the limit problem is a strongly coupled system involving both the macroscopic and the microscopic variables. We also include the results in the non critical case.

1. Introduction

Multi-scale problems have been studied extensively in the last four decades using classical homogenization(see [2, 8, 23, 25] and the references therein). But a direct approach of homogenization fails in many interesting problems and homogenization of periodic composites with highly contrasting diffusive properties falls in this category. To be more precise, we consider a bounded domain $\Omega \subset \mathbb{R}^n$ which is a periodic composite of the form $\Omega = M_\varepsilon \bigcup B_\varepsilon$, $\varepsilon > 0$ a small parameter, is the period of the composite distribution. Here, M_ε is the material with $O(1)$ conductivity and B_ε is the material with $O(\varepsilon^2)$ conductivity. This constitutes a composite with highly contrasting conductivities. Studying homogenization of problems in such domains requires refined methods. Observe that the problem looses its uniform ellipticity with respect to the small parameter ε as $\varepsilon \to 0$ due to the high contrast between the diffusivity of the two media under consideration. Other references in

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this direction may be found in [3], [5], [6], [18], [19] [21], [24] and references therein. On the other hand, homogenization of hyperbolic equation in composites with imperfect inclusions can be found in [10], [11].

Taking terminologies from elasticity, the part B_ε is referred to as *soft component* as it allows large deformations compared to the part M_ε which in turn is known as *stiff component (matrix part)*. The literature is mainly available for second order elliptic problems with high contrast in coefficients. In this article, we plan to study such a high contrast problem for a hyperbolic equation. In the sequel, we see that the hyperbolic problem poses new challenges for the future. In fact, it turns out that the only reasonable way to formulate the homogenized problem, at least in the critical case, is the coupled limit system involving the microscopic variable y together with the macroscopic one x in contrast with the stationary case, (see for instance [1] and [20]). Indeed, the main part of the article is when the soft material B_ε has conductivity $O(\varepsilon^2)$. But, it is also possible to consider the two other regimes $\alpha = 0$ and $\alpha = +\infty$, that is, B_ε has conductivity $O(\alpha_\varepsilon^2)$, where $\alpha := \lim_\varepsilon \frac{\alpha_\varepsilon}{\varepsilon}$. We discuss these two regimes in Section 5.

1.1. **Notations.** Let $\Omega \subset \mathbb{R}^n$ be the reference domain of the periodic composite. Let $Y = (-\frac{1}{2}, \frac{1}{2})^n$ be the basic cell and $B = B(0, r) \subset Y$ be the ball of radius r with $0 < r < \frac{1}{2}$ which eventually gives the soft part of the composite and the stiff part will be obtained from $M = Y \setminus \bar{B}$. Define the ε periodic cells $Y_\varepsilon^k = \varepsilon Y + \varepsilon k$ for all $k \in \mathbb{Z}^n$. The set of inclusions B_ε and matrix M_ε are defined as

$$B_\varepsilon = \bigcup_{k \in I_\varepsilon} B_\varepsilon^k, \quad M_\varepsilon = \Omega \setminus \bar{B}_\varepsilon = \bigcup_{k \in I_\varepsilon} M_\varepsilon^k,$$

where $k \in I_\varepsilon = \{k \in \mathbb{Z}^n : Y_\varepsilon^k \subset \Omega\}$, $B_\varepsilon^k = \varepsilon B(0, r) + \varepsilon k$, $M_\varepsilon^k = Y_\varepsilon^k \setminus \bar{B}_\varepsilon^k$. We denote $C_\#(Y)$, the space of continuous functions defined on Y which are $Y-$ periodic. We also use standard Sobolev spaces like $H^1(\Omega)$, $H_0^1(\Omega)$ etc. and the spaces required for studying time dependent problems like $L^p(0, T, L^q(\Omega))$, more generally $L^p(0, T, X)$ with appropriate norms. Let $A(x, y) = [a_{ij}(x, y)]$ be a smooth, periodic in y and elliptic matrix, that is, A satisfies: $\exists\, m > 0,$ for almost all $x \in \Omega$, $y \in Y$ and for all $\xi \in \mathbb{R}^n$

$$\begin{cases} y \to a_{ij}(x, y) \text{ is } Y- \text{ periodic} \\ a_{ij} \in L^\infty(\Omega; C_\#(Y)), \quad A(x, y)\xi \cdot \xi \geq m|\xi|^2. \end{cases} \tag{1.1}$$

For any set E, we denote χ_E, the characteristic function of E, while C denotes any positive constant the value of which may change from a line to another.

2. Problem Description and Apriori Estimates

In this article, we consider the following hyperbolic equation with high contrast diffusivities, namely,

$$(P_\varepsilon) \qquad \begin{cases} L_\varepsilon u_\varepsilon := u''_\varepsilon - \operatorname{div}\left(a_\varepsilon(x) A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon\right) + u_\varepsilon = f_\varepsilon \text{ in } \Omega_T, \\ u_\varepsilon = 0 \text{ on } (0,T) \times \Gamma, \quad u_\varepsilon(0) = u^0_\varepsilon,\ u'_\varepsilon(0) = u^1_\varepsilon, \end{cases}$$

where $\Omega_T = (0,T) \times \Omega$ and $\Gamma = \partial\Omega$. We assume that the coefficient a_ε takes the form

$$a_\varepsilon(x) = \varepsilon^2 \chi_{B_\varepsilon} + \chi_{M_\varepsilon}, \tag{2.1}$$

and A is uniformly elliptic as in (1.1) which gives the high contrasting properties in B_ε and M_ε. Notice that as in the elliptic case (see [20]), one can consider other situations assuming the coefficient a_ε given by

$$a_\varepsilon(x) = \alpha_\varepsilon^2 \chi_{B_\varepsilon} + \chi_{M_\varepsilon}, \tag{2.2}$$

where α_ε is a positive sequence which goes to zero when $\varepsilon \to 0$. Then, we should have to consider the three cases $\alpha = 0$; $\alpha = +\infty$ and $0 < \alpha < +\infty$ where $\alpha := \lim_\varepsilon \frac{\alpha_\varepsilon}{\varepsilon}$. For the sake of brevity, we only write the complete proof of the results in the critical case $0 < \alpha < +\infty$ assuming without loss of generality that $\alpha_\varepsilon = \varepsilon$ for every ε while we will only state and sketch the proofs for the two other cases. Our aim is to study the limiting behavior of the solution of the above system and obtain the homogenized equation. We make the following assumptions on the data

$$(A1) \qquad \int_\Omega |u^0_\varepsilon|^2 + \int_{B_\varepsilon} \varepsilon^2 |\nabla u^0_\varepsilon|^2 + \int_{M_\varepsilon} |\nabla u^0_\varepsilon|^2 + \int_\Omega |u^1_\varepsilon|^2 < C$$

$$(A2) \qquad \|f_\varepsilon\|_{L^2(\Omega_T)} \le C,$$

where C is a positive constant independent of ε. The assumption $(A1)$ on the initial values is natural and it is based on energy estimates which we are going to see shortly while deriving the apriori estimates. However, in order to make more precise the homogenized problem, we assume in addition that the initial data $(u^0_\varepsilon, u^1_\varepsilon)$ are given in the space $H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ and that f_ε is given in $W^{1,1}(0,T; L^2(\Omega))$; it is then well known that under this assumption, for each $\varepsilon > 0$, the problem (P_ε) admits a unique solution u_ε with the regularity

$$(R1) \qquad u_\varepsilon \in C^2([0,T]; L^2(\Omega) \cap C^1([0,T]; H^1_0(\Omega) \cap C([0,T]; H^2(\Omega)).$$

Of course, due to the degenerating character of the problem, uniform estimates with respect to ε in the above spaces are out of reach. To get the limit problem, we only need the natural assumption $(A1)$ on $(u^0_\varepsilon, u^1_\varepsilon)$ and $(A2)$ on f_ε.

The following (weak) formulation of (P_ε) will be helpful when we will seek for the initial data of the limits:

$$\int_{\Omega_T} u_\varepsilon \phi^{''} dxdt + \int_\Omega (u^0_\varepsilon \phi^{'}(0,x) - u^1_\varepsilon \phi(0,x))\ dx \qquad (2.3)$$
$$+ \int_{\Omega_T} a_\varepsilon(x) A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \nabla \phi dxdt + \int_{\Omega_T} u_\varepsilon \phi\ dxdt = \int_{\Omega_T} f_\varepsilon \phi\ dxdt,$$

$\forall \phi \in C^\infty([0,T]; H^1_0(\Omega),\ \phi(T,x) = 0$. The solution u_ε is uniformly bounded only in $L^2(\Omega_T)$ and we represent the L^2 weak limit u in a convenient way. Even though, the solution is not uniformly bounded in $L^\infty(0,T; H^1_0(\Omega))$, we do get uniform bound of the matrix (stiff) part. We use this property to study the limiting analysis. In the remaining part of this section, we give apriori estimates. We also recall the preliminaries required for our analysis. Our main tool is two-scale convergence together with an extension lemma. In Section 3, we prove the required convergence and identify the limits. Section 4 is devoted to the passage to the limit and study the homogenization theorem (Theorem 4.1). Decomposition of the two-scale system is also presented here. In section 5, we state and sketch the proofs in the cases $\alpha = 0$ and $\alpha = +\infty$.

2.1. **Apriori Estimates.** We define the energy of the system as

$$E_\varepsilon(t) = \frac{1}{2}\left\{ \int_\Omega |u'_\varepsilon(t)|^2 + \int_\Omega |u_\varepsilon(t)|^2 + \int_\Omega \varepsilon^2 \chi_{B_\varepsilon} A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \right. \qquad (2.4)$$
$$\left. + \int_\Omega \chi_{M_\varepsilon} A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \right\} = E^1_\varepsilon(t) + E^2_\varepsilon(t) + E^3_\varepsilon(t) + E^4_\varepsilon(t),$$

where $E^i_\varepsilon(t),\ i = 1,2,3,4$ are the respective terms in order. At this stage, we remark that the assumption $(A1)$ is nothing but the boundedness of the initial energy $E_\varepsilon(0)$. As a first step, we have the following energy estimates.

Proposition 2.1. *There exists a constant $C > 0$ independent of ε such that $E_\varepsilon(t) \le C$.*

The above proposition will give us the following estimates

Proposition 2.2. *There exists a constant $C > 0$ independent of ε such that*

$$\|u_\varepsilon\|_{L^\infty_t L^2_x} \le C,\ \|u'_\varepsilon\|_{L^\infty_t L^2_x} \le C, \qquad (2.5)$$
$$\|\varepsilon \nabla u_\varepsilon\|_{L^\infty_t L^2_x(B_\varepsilon)} \le C,\ \|\nabla u_\varepsilon\|_{L^\infty_t L^2_x(M_\varepsilon)} \le C.$$

Note that the short notation $L_t^\infty L_x^2$ represent the space with respect to the time domain $[0,T]$ and spatial domain Ω.

For example, $L_t^\infty L_x^2 = L^\infty(0,T;L^2(\Omega))$. When the domain is not the full domain Ω, we use $L_t^\infty L_x^2(B_\varepsilon) = L^\infty(0,T;L^2(B_\varepsilon))$. We use these notations when there is no confusion.

Remark 2.3. Thus, we have the correct L^2 estimates for the solution and its time-derivative. Also gradient estimate in the stiff part. The difficulty in this problem is that the gradient estimate in the soft inclusions is of order ε^{-1} and hence, in general H^1 estimate is of order ε^{-1}. This also motivates the assumption (A1) on the initial data.

Since the gradient estimate in the stiff part is bounded, we use the extension operators (see [7]) to extend the solution from the stiff part to the soft part in a continuous and bounded way. This is required for our analysis. Our second main tool is the two-scale convergence introduced by G. Nguetseng [16] and developed by G. Allaire [1]. For more details see [17]. Also see A. K. Nandakumaran et. al [13], [14], [15], [12] for the application of two scale convergence. Since there is no oscillations with respect to $t \in (0,T)$, we will use an adapted definition of two-scale convergence in which the variable t plays a role of parameter as presented in the work by G.W. Clark and L.A. Packer [9], also used in the work by M. Sfaxi and A. Sili [21]. We recall the extension operators and two-scale convergence in the preliminaries.

Proof of Propositions 2.1 and 2.2. The method is standard for hyperbolic equations. Multiply the equation (P_ε) by u_ε' and integrate by parts to get

$$\int_0^t \int_\Omega \frac{1}{2}\frac{d}{dt}|u_\varepsilon'(t)|^2 \quad + \quad \frac{1}{2}\frac{d}{dt}\left[\left(\varepsilon^2\chi_{B_\varepsilon} + \chi_{M_\varepsilon}\right) A\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon \cdot \nabla u_\varepsilon\right] + \int_0^t \int_\Omega \frac{1}{2}\frac{d}{dt}|u_\varepsilon(t)|^2 = \int_0^t \int_\Omega f_\varepsilon u_\varepsilon'.$$

The left hand side of the expression is $E_\varepsilon(t) - E_\varepsilon(0)$. Now, using the assumption $(A1)$ and $(A2)$, we get

$$E_\varepsilon(t) \le C_1 + C_2 \left\|u_\varepsilon'\right\|_{L_t^\infty L_x^2}.$$

Using the first term in $E_\varepsilon(t)$, we have

$$\left\|u_\varepsilon'(t)\right\|_{L^2(\Omega)}^2 \le C(1 + \left\|u_\varepsilon'\right\|_{L_t^\infty L_x^2}),$$

for some constant $C > 0$ which implies the estimate in the Proposition 2.1. Rewriting, we get the Proposition 2.2. □

2.2. **Preliminaries.** In the weak convergence, the weak limit averages out all the oscillations present and the two-scale convergence is introduced to capture the oscillations through the limit and hence the limit is defined on the fast and slow variables. See the references given earlier. In the sequel we adopt the definition of two-scale convergence given in [9] or in [21] which is more convenient for our evolution problem without oscillations in time.

Definition 2.1 (Two-scale convergence)**.** *A sequence of functions* $\{v_\varepsilon\}$ *in* $L^2(\Omega_T)$ *is said to* two-scale converge *to a limit* $v \in L^2(\Omega_T \times Y)$ (*denoted as* $v_\varepsilon \xrightarrow{2s} v$) *if*

$$\int_{\Omega_T} v_\varepsilon \phi\left(t, x, \frac{x}{\varepsilon}\right) dx\, dt \to \int_{\Omega_T}\int_Y v(t,x,y)\phi(t,x,y)\, dy\, dx\, dt$$

$\forall \phi \in L^2\big(\Omega_T; C_\#(Y)\big)$. *Further, if* v_0 *is the weak limit of* $\{v_\varepsilon\}$ *in* $L^2(\Omega_T)$, *then*

$$v_0(t,x) = \int_Y v(t,x,y)dy.$$

We have the following compactness theorem.

Theorem 2.4 (Compactness)**.** *For any bounded sequence* v_ε *in* $L^2(\Omega_T)$, *there exist a subsequence and* $v \in L^2(\Omega_T \times Y)$ *such that,* v_ε *two-scale converges to* v *along the subsequence. Also, if* v_ε *is bounded in* $L^2(0,T;H^1(\Omega))$, *then* v *is independent of* y *and is in* $L^2(0,T;H^1(\Omega))$, *and there exists a* $v_1 \in L^2(\Omega_T; H^1_\#(Y))$ *such that, up to a subsequence,* ∇v_ε *two-scale converges to* $\nabla v + \nabla_y v_1$.

Note that the proof of Theorem 2.4 is exactly the compactness theorem given in [1] replacing the open domain Ω by the cylinder Ω_T. There are many interesting properties of two-scale convergence and the reader can see the references cited above.

We also need the following extension lemma which is available for instance in Cioranescu-Donato [7].

Lemma 2.5 (Extension Operator)**.** *There exists a linear continuous operator* $P^\varepsilon : L^\infty(0,T;H^k(M_\varepsilon)) \to L^\infty(0,T;H^k(\Omega))$, *that is,*

$$P^\varepsilon \in \mathcal{L}\left(L^\infty(0,T;H^k(M_\varepsilon));\, L^\infty(0,T;H^k(\Omega))\right),$$

where $k = 0,1$ *such that, for some constant* C *independent of* ε*: for any* $\phi \in L^\infty(0,T;H^k(M_\varepsilon))$;

$$P^\varepsilon\phi = \phi \text{ in } \mathrm{M}_\varepsilon \times (0,\mathrm{T}), \quad \mathrm{P}^\varepsilon\phi' = (\mathrm{P}^\varepsilon\phi)' \text{ in } \Omega \times (0,\mathrm{T}),$$
$$\|P^\varepsilon\phi\|_{L^\infty(0,T;L^2(\Omega))} \le C\,\|\phi\|_{L^\infty(0,T;L^2(M_\varepsilon))},$$

$$\|P^\varepsilon \phi'\|_{L^\infty(0,T;L^2(\Omega))} \le C\,\|\phi'\|_{L^\infty(0,T;L^2(M_\varepsilon))},$$
$$\|\nabla(P^\varepsilon \phi)\|_{L^\infty(0,T;L^2(\Omega))} \le C\,\|\nabla\phi\|_{L^\infty(0,T;L^2(M_\varepsilon))}.$$

3. Convergence

Using the assumption (A2) and two-scale convergence, we get $f_\varepsilon \rightharpoonup f$ in $L^2(\Omega_T)$ *weak* and $f_\varepsilon \overset{2s}{\rightharpoonup} f_0$, where $f \in L^2(\Omega_T)$ and $f_0 = f_0(t,x,y) \in L^2(\Omega_T \times Y)$. Hence,

$$f(t,x) = \int_Y f_0(t,x,y)dy. \tag{3.1}$$

We denote $u_\varepsilon^{0M} = u_\varepsilon^0|_{M_\varepsilon}$ and $u_\varepsilon^M = u_\varepsilon|_{M_\varepsilon}$, respectively, the restrictions of the initial data u_ε^0 and the solution u_ε to M_ε. We extend u_ε^{0M} and u_ε^M to all of Ω_T using the extension Lemma 2.5. Denote the extensions as $\tilde{u}_\varepsilon^0 = P_\varepsilon u_\varepsilon^{0M}$ and $\tilde{u}_\varepsilon = P_\varepsilon u_\varepsilon^M$. Then, using assumption (A1), apriori estimates and extension lemma, it follows that $\|\tilde{u}_\varepsilon^0\|_{H^1(\Omega)} \le C$ and

$$\|\tilde{u}_\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} \le C \text{ and } \|\tilde{u}_\varepsilon'\|_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{3.2}$$

Thus, we can deduce that

$$\tilde{u}_\varepsilon \rightharpoonup u \text{ weak* in } L_t^\infty H_0^1(\Omega), \quad \tilde{u}_\varepsilon' \rightharpoonup u' \text{ weak* in } L_t^\infty L_x^2(\Omega). \tag{3.3}$$

This implies that, extracting a subsequence, (see [22], Corollary 4, p. 85)

$$\tilde{u}_\varepsilon \to u \text{ strongly in } \mathrm{C}([0,\mathrm{T}],\mathrm{L}^2(\Omega)). \tag{3.4}$$

In particular, we get from (3.4) that

$$\tilde{u}_\varepsilon(0,x) \to u(0,x) \text{ strongly in } \mathrm{L}^2(\Omega). \tag{3.5}$$

On the other hand, from equation (P_ε), we deduce the estimate

$$\|u_\varepsilon''\|_{L^\infty(0,T;H^{-1}(\Omega))} \le C,$$

which implies

$$\|\tilde{u}_\varepsilon''\|_{L^\infty(0,T;H^{-1}(\Omega))} \le C;$$

taking into account the second estimate in (3.2), we can therefore use [22], Corollary 4, p. 85, to get that

$$\tilde{u}_\varepsilon' \to u' \text{ strongly in } \mathrm{C}([0,\mathrm{T}],\mathrm{H}^{-1}(\Omega)). \tag{3.6}$$

As a consequence of (3.6), we infer

$$\tilde{u}_\varepsilon'(0,x) \to u'(0,x) \text{ strongly in } \mathrm{H}^{-1}(\Omega). \tag{3.7}$$

Now, we will look into the two-scale convergence of u_ε. Since u_ε, $\varepsilon\nabla u_\varepsilon$ are bounded in $L^2(\Omega_T)$, we apply two-scale convergence to get

$$v_0 = v_0(t,x,,y) \in L^2(\Omega_T; H_\sharp^1(Y)), \quad v_0' \in L^2(\Omega_T \times Y),$$

such that

$$u_\varepsilon \xrightarrow{2s} v_0, \quad \varepsilon \nabla u_\varepsilon \xrightarrow{2s} \nabla_y v_0, \;\; u'_\varepsilon \xrightarrow{2s} v'_0. \tag{3.8}$$

Note that the third convergence in (3.8) is a consequence of the boundedness of u'_ε in $L^\infty_t(L^2(\Omega))$ together with the first convergence in (3.8).

Remark 3.1. The following proposition shows that, as it can be expected, the function v_0 depends on the variable y only inside the part B of the cell Y since in B_ε only $\varepsilon \nabla u_\varepsilon$ is bounded in L^2 while in the region M_ε, the sequence ∇u_ε is bounded in L^2.

Proposition 3.2. *The limit v_0 given by* (3.8) *satisfies*

$$\nabla_y v_0(t,x,y) = 0 \text{ in } \Omega \times M. \tag{3.9}$$

In addition, the function v_0 is such that

$$v_0 \in L^2(\Omega_T; H^1_\sharp(Y)) \cap L^\infty_t(L^2(\Omega; H^1(M)) \text{ and} \tag{3.10}$$
$$v'_0 \in L^\infty_t(L^2(\Omega \times M)) \cap L^2(\Omega_T \times Y).$$

Proof. Observe that $\chi_{M_\varepsilon} \nabla u_\varepsilon$ is bounded in $L^2(\Omega_T)$. Choose test functions of the type $\phi = \phi(t,x,y) \in (\mathcal{D}(\Omega_T \times M))^n$, then

$$\int_{\Omega_T} \chi_{M_\varepsilon} \nabla u_\varepsilon \cdot \phi\left(t,x,\frac{x}{\varepsilon}\right) = \sum_i \int_{\Omega_T} \chi_{M^i_\varepsilon} \nabla u_\varepsilon \cdot \phi\left(t,x,\frac{x}{\varepsilon}\right)$$
$$= -\sum_i \int_{\Omega_T} \chi_{M^i_\varepsilon} u_\varepsilon \left(\mathrm{div}_x \phi\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon^{-1} \mathrm{div}_y \phi\left(t,x,\frac{x}{\varepsilon}\right)\right) dx\, dt$$
$$= -\int_{\Omega_T} \chi_{M_\varepsilon} u_\varepsilon \left(\mathrm{div}_x \phi\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon^{-1} \mathrm{div}_y \phi\left(t,x,\frac{x}{\varepsilon}\right)\right) dx\, dt.$$

Multiplying by ε and passing to the limit, we get

$$\int_{\Omega_T} \int_Y \chi_M(y) v_0(t,x,y) \mathrm{div}_y \phi(x,y) dx dy dt = 0.$$

Since ϕ is arbitrary, we have (3.9). One can precise (3.10) by passing to the limit in the equality $\tilde{u}_\varepsilon \chi_{M_\varepsilon} = u_\varepsilon \chi_{M_\varepsilon}$. Indeed, thanks to convergence (3.4) and (3.8) together with the two-scale convergence of χ_{M_ε} to $\chi_M(y)$, we get at the limit $u \chi_M(y) = v_0 \chi_M(y)$, which means that $v_0(t,x,y) = u(t,x)$ in $\Omega_T \times M$. The L^∞ regularity (3.10) is then a consequence of the L^∞ regularity (3.3) of u. Thus, the proposition. □

Define now the function

$$v(t,x,y) := v_0(t,x,y) - u(t,x) \text{ for } (t,x,y) \in \Omega_T \times Y. \tag{3.11}$$

From the previous considerations, we have $v \in L^2(\Omega_T; H_0^1(B))$, $v' \in L^2(\Omega_T \times Y)$; $u \in L_t^\infty H_0^1(\Omega)$, $u' \in L_t^\infty(L^2(\Omega))$ while the first convergence in (3.8) takes the form $u_\varepsilon \xrightarrow{2s} u(t,x) + v(t,x,y)$.

Our aim now is to look for a two scale limit system satisfied by the pair $(u(t,x), v(t,x,y))$.

3.1. **Identification of the limits of $\chi_{M_\varepsilon} \nabla u_\varepsilon$ and $\varepsilon \chi_{B_\varepsilon} \nabla u_\varepsilon$.** From the second convergence in (3.8), we get with the help of (3.11)

$$\varepsilon \chi_{B_\varepsilon} \nabla u_\varepsilon \xrightarrow{2s} \chi_B(y) \nabla_y v. \tag{3.12}$$

Let $K(t,x,y)$ be the two-scale limit of $\chi_{M_\varepsilon} \nabla u_\varepsilon$. Take any smooth vector function $\phi = \phi(t,x,y)$ such that $\operatorname{supp} \phi(t,x,\cdot) \subset M$ and $\operatorname{div}_y \phi(t,x,\cdot) = 0$. Then

$$\int_{\Omega_T} \chi_{M_\varepsilon} \nabla u_\varepsilon \cdot \phi\left(t, x, \frac{x}{\varepsilon}\right) = - \int_{\Omega_T} \chi_{M_\varepsilon}\ \tilde{u}_\varepsilon \operatorname{div}_x \phi(t,x,y),$$

since $\operatorname{div}_y \phi(t,x,\cdot) = 0$. Indeed, the term on the left hand side converges to

$$\int_{\Omega_T} \int_Y \chi_M(y) K(t,x,y) \phi(t,x,y).$$

On the other hand, the term on the right hand side converges to

$$- \int_{\Omega_T} \int_Y \chi_M(y) u(t,x) \operatorname{div}_x \phi(t,x,y) = \int_{\Omega_T} \int_Y \chi_M(y) \nabla_x u(t,x) \cdot \phi(t,x,y).$$

Thus, we have,

$$\int_{\Omega_T} \chi_M(y) \left(K(t,x,y) - \nabla_x u(t,x)\right) \cdot \phi(t,x,y) = 0,$$

for all ϕ as above. Hence, by defining the space $H^1_\#(M)$ (recall that $M = Y \setminus \overline{B}$) as $H^1_\#(M) := \{u \in H^1(M)),\, u \,\text{is}\, Y-\text{periodic}\}$, one can conclude that there exists $u_1 \in L^2(\Omega_T; H^1_\#(M))$ such that

$$K(t,x,y) - \nabla_x u(t,x) = \nabla_y u_1(t,x,y) \ \text{ in } \Omega_T \times M. \tag{3.13}$$

Hence, $K(t,x,y)$ is given by the equality

$$K(t,x,y) = \nabla_x u(t,x) + \nabla_y u_1(t,x,y) \ \text{ in } \Omega_T \times M. \tag{3.14}$$

In order to state the homogenized system, we first need to introduce the following matrix A^*: Introduce first, for *a.e.*, $x \in \Omega$, $w_i(x,\cdot)$, the solution of the cell problem in M as

$$\begin{cases} -\operatorname{div}_y \left(A(x,y)(\nabla_y w_i(x,\cdot) + e_i)\right) = 0 \,\text{in}\, M \\ w_i(x,\cdot) \text{ is } Y-\text{periodic}, \quad w_i(x,y) = 0 \,\text{on}\, \partial B, \end{cases} \tag{3.15}$$

where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$. Then the entries of the matrix A^* are given by

$$A^*_{i,j}(x) = \int_M A(x,y)(\nabla_y w_j(x,\cdot) + e_j) \cdot (\nabla_y w_i(x,\cdot) + e_i) dy. \tag{3.16}$$

4. Homogenization in the critical case

Having obtained the required limits, we now state and prove the main theorem of this article. Let $u^0 = u^0(x,y)$ and $u^1 = u^1(x,y)$ be the two-scale limits of the initial values u^0_ε and u^1_ε, respectively, that is,

$$u^0_\varepsilon \xrightarrow{2s} u^0, \qquad u^1_\varepsilon \xrightarrow{2s} u^1. \tag{4.1}$$

Theorem 4.1 (Homogenization)**.** *Let the given data f_ε, u^0_ε, u^1_ε satisfy the assumptions* (A1) *and* (A2), *a_ε be given as in* (2.1). *Let u_ε be the unique solution to the problem (P_ε). Then, $u_\varepsilon \xrightarrow{2s} u(t,x) + v(t,x,y)$, where the pair*

$$(u,v) \in L^\infty(0,T;H^1_0(\Omega)) \times L^2(\Omega_T; H^1_0(B))$$

is the unique solution of the coupled system

$$\begin{cases} u \in L^\infty(0,T;H^1_0(\Omega)),\ u' \in L^\infty(0,T;L^2(\Omega)), \\ u'' + u + \displaystyle\int_B \big(v'' + v\big)\, dy - div_x A^*(x)\nabla u = \int_Y f_0 dy \quad in\ \ \Omega_T, \\ u(0,x) = \displaystyle⨍_M u^0(x,y)\, dy \quad in\ \Omega, \\ u'(0,x) = \displaystyle⨍_M u^1(x,y)\, dy \quad in\ \Omega, \\ v \in L^2(\Omega_T; H^1_0(B)),\ v' \in L^2(\Omega_T \times Y), \\ v'' + v + u'' + u - div_y A(x,y)\nabla_y v = f_0\ in\ \ \Omega_T \times B, \\ v(0,x,y) = u^0(x,y) - \displaystyle⨍_M u^0(x,y)\, dy \quad in\ \Omega \times B, \\ v'(0,x,y) = u^1(x,y) - \displaystyle⨍_M u^1(x,y)\, dy \quad in\ \Omega \times B. \end{cases} \tag{4.2}$$

Note that by virtue of (3.5) and (3.7), the initial conditions $u(0,x)$ and $u'(0,x)$ are well defined; for $v(0,x,y)$, one can remark that since $v \in L^2(\Omega_T; H^1_0(B))$, $v' \in L^2(\Omega_T \times Y)$, we have $v \in C([0,T]; L^2(\Omega \times Y))$ (see [22], Lemma 4), on the other hand, the second equation in (4.2) shows that

$v'' \in L^2(0,T;H^{-1}(\Omega\times B))$ so that $v' \in C([0,T];H^{-1}(\Omega\times B))$. Hence, the initial conditions arising in (4.2) are well defined.

Proof. Consider the test functions of the form

$$\phi_\varepsilon(t,x) = \bar u(t,x) + \varepsilon \bar u_1\left(t,x,\tfrac{x}{\varepsilon}\right) + \bar v\left(t,x,\tfrac{x}{\varepsilon}\right),$$

where $\bar u \in \mathcal{D}(\Omega_T)$, $\bar u_1 \in \mathcal{D}(\Omega_T\times M)$ and $\bar v \in \mathcal{D}(\Omega_T\times B)$ are arbitrary. Multiplying the equation in (P_ε) by ϕ_ε and integrating by parts, we may write $I_1^\varepsilon + I_2^\varepsilon + I_{3,1}^\varepsilon + I_{3,2}^\varepsilon = I^\varepsilon$. Here, I^ε, I_i^ε are given by

$$I^\varepsilon = \int_{\Omega_T} f_\varepsilon(t,x)\phi_\varepsilon(t,x), \quad I_1^\varepsilon = \langle u_\varepsilon'', \bar u + \varepsilon\bar u_1 + \bar v\rangle, \quad I_2^\varepsilon = \langle u_\varepsilon, \bar u + \varepsilon \bar u_1 + \bar v\rangle$$

$$I_{3,1}^\varepsilon = \int_{\Omega_T} \varepsilon^2 \chi_{B_\varepsilon} A\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\left(\nabla\bar u + \nabla \bar v + \varepsilon^{-1}\nabla_y \bar v\right),$$

$$I_{3,2}^\varepsilon = \int_{\Omega_T} \chi_{M_\varepsilon} A\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\left(\nabla\bar u + \varepsilon\nabla \bar u_1 + \nabla_y \bar u_1\right).$$

4.1. **Passage to the limit term by term.** The term I_1^ε

$$I_1^\varepsilon = \int_{\Omega_T} u_\varepsilon \bar u'' + \int_{\Omega_T} \varepsilon u_\varepsilon \bar u_1''\left(t,x,\frac{x}{\varepsilon}\right) + \int_{\Omega_T} u_\varepsilon \bar v''\left(t,x,\frac{x}{\varepsilon}\right),$$

which converges to

$$\int_{\Omega_T}\int_Y (u+v)\,\bar u'' + 0 + \int_{\Omega_T}\int_Y (u+v)\,\bar v''. \tag{4.3}$$

Thus,

$$I_1^\varepsilon \to I_1^0 := \int_{\Omega_T}\int_Y (u+v)\,(\bar u + \bar v)''. \tag{4.4}$$

Now,

$$I_2^\varepsilon \to I_2^0 := \int_{\Omega_T}\int_Y (u+v)\,(\bar u+\bar v), \quad I^\varepsilon \to I^{0,1} := \int_{\Omega_T}\int_Y f_0(\bar u + \bar v),$$

$$I_{3,1}^\varepsilon \to I_{3,1}^0 := 0 + 0 + \int_{\Omega_T}\int_Y \chi_B(y)A(x,y)\nabla_y v\cdot\nabla_y \bar v.$$

Similarly,

$$I_{3,2}^\varepsilon \to I_{3,2}^0 := \int_{\Omega_T}\int_Y \chi_M(y)A(x,y)(\nabla_x u + \nabla_y u_1)\cdot(\nabla_x\bar u + \nabla_y \bar u_1).$$

Thus, we have the system

$$I_1^0 + I_2^0 + I_{3,1}^0 + I_{3,2}^0 = I^{0,1}, \tag{4.5}$$

for all test functions $\bar{u}$, $\bar{u}_1$, $\bar{v}$ as taken earlier. We represent the limit $I^0_{3,2}$ using the solutions w_i of the cell problem. If we take $\bar{u} = 0$ and $\bar{v} = 0$ in the above equation, (4.5) becomes

$$\int_{\Omega_T}\int_M A(x,y)(\nabla_x u + \nabla_y u_1)\cdot\nabla_y \bar{u}_1 = 0.$$

At this stage, choose $\bar{u}_1$ of the form, $\bar{u}_1 = \bar{u}_1(x,y)s(t)$ with $\bar{u}_1 \in \mathcal{D}(\Omega \times M)$ and $s \in \mathcal{D}(0,T)$, then the above equation becomes

$$\begin{aligned}&\int_{\Omega_T}\int_M A(x,y)\nabla_y u_1(t,x,y)\cdot\nabla_y \bar{u}_1(x,y)s(t)\\ &= -\int_{\Omega_T}\int_M A(x,y)\nabla_x u\cdot\nabla_y\bar{u}_1(x,y)s(t).\end{aligned}$$

Assuming u is known, for *a.e.*, $x \in \Omega$, $t \in (0,T)$, the above equation is the elliptic weak formulation for $u_1(t,x,\cdot)$ in M. Hence, using the test function w_i introduced by (3.15), we may represent u_1 as

$$u_1(t,x,y) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(t,x)w_i(x,y). \tag{4.6}$$

Thus, we have a representation for $\bar{u}_1$ and now consider (4.5) with $\bar{u}_1 = 0$. In this case, using (4.6), $I^0_{3,2}$ becomes

$$I^0_{3,2} = \int_{\Omega_T}\int_M A(x,y)\Big(\nabla_x u + \sum_{i=1}^n \frac{\partial u}{\partial x_i}\nabla_y w_i\Big)\cdot\nabla\bar{u} = \int_{\Omega_T} A^*(x)\nabla u\cdot\nabla\bar{u},$$

where A^* is given by (3.16). By representing u_1 in terms of u, we essentially eliminated the test function $\bar{u}_1$ in the weak formulation (4.5). Thus, for all test functions $\bar{u} \in \mathcal{D}(\Omega_T)$ and $\bar{v} \in \mathcal{D}(\Omega_T \times B)$, we get the two-scale limit system

$$\begin{aligned}&\int_{\Omega_T}\int_Y (u+v)\,(\bar{u}+\bar{v})'' + \int_{\Omega_T}\int_Y (u+v)\,(\bar{u}+\bar{v})\\ &+\int_{\Omega_T}\int_B A(x,y)\nabla_y v\cdot\nabla_y\bar{v} + \int_{\Omega_T} A^*(x)\nabla u\cdot\nabla\bar{u} = \int_{\Omega_T}\int_Y f_0(\bar{u}+\bar{v}).\end{aligned} \tag{4.7}$$

The first Euler equation arising in (4.2) is then easily obtained from (4.7) by choosing $\bar{v} = 0$ while we obtain the second Euler equation of (4.2) with the choice $\bar{u} = 0$ in (4.7). It remains to identify the initial conditions on u and v.

The sequence $u_\varepsilon^0(x)\chi_{M_\varepsilon}$ two-scale converges to $u^0(x,y)\chi_M(y)$ and therefore weakly converges in $L^2(\Omega)$ to

$$\int_Y u^0(x,y)\chi_M(y)\ dy = \int_M u^0(x,y)\ dy.$$

On the other hand, the sequence χ_{M_ε} weakly converges in $L^2(\Omega)$ to $|M|$ (the measure of M). We can therefore pass to the limit in the equality

$$\tilde{u}_\varepsilon(0,x)\chi_{M_\varepsilon} = u_\varepsilon^0(x)\chi_{M_\varepsilon},$$

with the help of the convergence (3.5) to get the first initial condition on u. In a similar way, using the equality

$$\tilde{u}'_\varepsilon(0,x)\chi_{M_\varepsilon} = u_\varepsilon^1(x)\chi_{M_\varepsilon},$$

we can pass to the limit with the help of convergence (3.7) to get

$$u'(0,x)|M| = \int_Y u^1(x,y)\chi_M(y)\ dy = \int_M u^1(x,y)\ \ dy,$$

which is nothing but the second initial condition on u.

To identify the initial condition on v, the lack of compactness in B_ε leads us to proceed differently. Taking $\phi(t,x) = \bar{v}(t,x,\frac{x}{\varepsilon})$ in (2.3), with

$$\bar{v} \in C^\infty(0,T;\mathcal{D}(\Omega\times B)), \quad v(T) = 0,$$

and passing to the limit, we get

$$\begin{aligned} &\int_{\Omega_T}\int_Y (u+v)\,\bar{v}'' + \int_\Omega\int_Y (u^0(x,y)\bar{v}'(0,x,y) - u^1(x,y)\bar{v}(0,x,y)) \\ &+ \int_{\Omega_T}\int_Y (u+v)\,\bar{v} + \int_{\Omega_T}\int_B A(x,y)\nabla_y v\cdot\nabla_y\bar{v} = \int_{\Omega_T}\int_Y f_0\bar{v}. \end{aligned} \tag{4.8}$$

On the other hand, multiplying the second equation of (4.2) by the same test function $\bar{v}$, we obtain after an integration by parts

$$\begin{aligned} &\int_{\Omega_T}\int_Y (u+v)\bar{v}'' + \int_\Omega\int_Y (u(0)\bar{v}'(0,x,y) + v(0)\bar{v}'(0,x,y)) \\ &- \int_{\Omega\times Y} (u'(0)\bar{v}(0,x,y) + v'(0)\bar{v}(0,x,y)) + \int_{\Omega_T}\int_Y (u+v)\,\bar{v} \\ &+ \int_{\Omega_T}\int_B A(x,y)\nabla_y v\cdot\nabla_y\bar{v} = \int_{\Omega_T}\int_Y f_0\bar{v}. \end{aligned} \tag{4.9}$$

From (4.8) and (4.9), we deduce (recall that supp $\bar{v}\subset B$)

$$\int_\Omega\int_B (u^0(x,y)\bar{v}'(0,x,y) - u^1(x,y)\bar{v}(0,x,y)) \tag{4.10}$$

$$= \int_\Omega \int_B \left(u(0)\bar{v}'(0,x,y) + v(0)\bar{v}'(0,x,y)\right) - u'(0)\bar{v}(0,x,y) - v'(0)\bar{v}(0,x,y)\right).$$

We then derive the initial conditions on v by the use of the expressions of $u(0,x)$ and $u'(0,x)$ found above.

4.2. **Existence and Uniqueness.** The equation (4.7) defines a hyperbolic system with appropriate elliptic part. Denote $X = H_0^1(\Omega)$, $Z = L^2(\Omega; H_0^1(B))$, then $(u,v) \in L^\infty(0,T;X) \times L^2(0,T;Z)$.

Elliptic Bilinear Form: Define $\mathcal{A} : X \times Z \to \mathbb{R}$ by

$$\mathcal{A}(U_1,U_2) = \int_\Omega u_1 u_2 \quad + \int_{\Omega\times B} v_1 v_2 + \int_\Omega A^*(x)\nabla u_1 \cdot \nabla u_2 + \int_{\Omega\times B} A(x,y)\nabla_y v_1 \cdot \nabla_y v_2,$$

where $U_i = (u_i, v_i) \in X \times Z$ for $i = 1,2$. Define the norm on $X \times Z$ as

$$\|U\|_{X\times Z}^2 := \|u\|_{H_0^1(\Omega)}^2 + \|v\|_{L^2(\Omega\times B)}^2 + \|\nabla_y v\|_{L^2(\Omega\times B)}^2,$$

which is equivalent to $\|\nabla_x u\|_{L^2(\Omega)}^2 + \|\nabla_y v\|_{L^2(\Omega\times B)}^2$. Clearly, $\mathcal{A}$ is continuous. Further,

$$\mathcal{A}(U,U) = \int_\Omega |u|^2 + \int_{\Omega\times B} |v|^2 + \int_\Omega A^*(x)\nabla u \cdot \nabla u + \int_{\Omega\times B} A(x,y)\nabla_y v \cdot \nabla_y v$$
$$\geq C\left[\int_\Omega |\nabla u|^2 + \int_{\Omega\times B} |\nabla_y v|^2\right] \geq C\|U\|_{X\times Z}^2.$$

Thus, we have the hyperbolic Euler system in the standard form

$$U'' + \mathcal{A}U = F, \qquad U(0), U'(0), \text{ are given}$$

so that the existence and the uniqueness of the solution of (4.7) follow.

Remark 4.2. In contrast with the elliptic case (see [20]), we do not know how to separate u and v so that we have a complete homogenized equation involving the macro variable alone, namely an equation for u by eliminating v using cell problems. This is due to the term u'' in the second equation involving v. Hence, for a given u and unless to use the expression of the unitary group associated to the operator

$$-\text{div}_y A(x,.)\nabla_y + Id,$$

(v solves a so-called Klein Gordon equation for given u and given f), we cannot give a simple expression of v in terms of u even in the particular case $f_0 = f_0(x)$. It is not clear from the numerical point of view that the use of the above unitary group would lead to a numerical solution without a high cost in the calculations. Hence, it appears that for the critical hyperbolic case, the only reasonable limit model is system (4.7) which involves both the macroscopic variable and the microscopic one.

Remark 4.3. Note also that we did not study in this paper the effect of the oscillations of the initial data on the behavior of the solution u_ε or on the behavior of the associated sequence of energies; this question was addressed in the non-periodic setting but for equicoercive and equibounded operators in [4], while for a periodic fibered medium case and for degenerate parabolic operators the problem was considered in [21].

5. HOMOGENIZATION FOR THE TWO OTHER REGIMES

We are interested in this section in the case where a_ε is given by (2.2) when $\alpha := \lim\limits_{\varepsilon} \dfrac{\alpha_\varepsilon}{\varepsilon} = 0$ or $\alpha = +\infty$. We only sketch the proofs.

5.1. **The case $\alpha = 0$.** In this case, instead of the sequence $\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon}$ of the above critical case, we have to consider the sequence $\alpha_\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon}$ which is easily seen to be bounded in $L^\infty_t(L^2(\Omega))$. Note that all the results obtained above for the stiff part M_ε remain valid by the use of the extension theorem.

Let us prove that since α_ε is too small compared to the size ε of the period,

$$\alpha_\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon} \xrightarrow{2s} 0.$$

The sequence $\frac{\alpha_\varepsilon}{\varepsilon} u_\varepsilon$ converges strongly (and then two-scale converges) to zero in $L^2(\Omega_T)$ since u_ε is bounded in $L^2(\Omega_T)$. On the other hand, the sequence

$$\varepsilon \nabla (\frac{\alpha_\varepsilon}{\varepsilon} u_\varepsilon) = \alpha_\varepsilon \nabla u_\varepsilon,$$

is bounded in $L^2(\Omega_T)$; hence from a classical result of two-scale convergence, there exists a function $k(t,x,y) \in L^2(\Omega_T; H^1_\#(Y))$ such that $\dfrac{\alpha_\varepsilon}{\varepsilon} u_\varepsilon \xrightarrow{2s} k$ and $\varepsilon \nabla (\dfrac{\alpha_\varepsilon}{\varepsilon} u_\varepsilon) \xrightarrow{2s} \nabla_y k$. Since $k = 0$, we have the desired result. The next step is to take a test function $\phi_\varepsilon(t,x) = \bar{u}(t,x) + \varepsilon \bar{u}_1\left(t, x, \frac{x}{\varepsilon}\right) + \bar{v}\left(t, x, \frac{x}{\varepsilon}\right)$ as above and then to pass to the limit; we obtain the variational equation

$$\int_{\Omega_T \times Y} [(u+v)(\bar{u}+\bar{v})'' + (u+v)(\bar{u}+\bar{v})] + \int_{\Omega_T} A^*(x) \nabla u \cdot \nabla \bar{u} = \int_{\Omega_T \times Y} f_0(\bar{u}+\bar{v}). \tag{5.1}$$

Choosing $\bar{u} = 0$, we get

$$(u+v)'' + u + v = f_0 \quad \text{in } \Omega_T \times B. \tag{5.2}$$

This equation provides the expression of

$$\int_B (v'' + v)\ dy,$$

in terms of f_0 and of u, u''. Turning back to the equation satisfied by u (obtained from (5.1) with the choice $\bar{v} = 0$), we get the final homogenized equation

$$(1 - |B|)(u'' + u) - \mathrm{div}_x A^*(x)\nabla u = \int_M f_0\ dy, \quad \text{in } \Omega_T, \tag{5.3}$$

with the same initial conditions as in the section above. The contribution of the soft part B_ε in the homogenized equation is seen through the measure of B in the final macroscopic equation.

5.2. **The case $\alpha = +\infty$.** This case corresponds to the best compactness compared to the two previous cases. Due to the a priori estimates, the sequence $\alpha_\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon}$ is still bounded in $L^2(\Omega_T)$ so that

$$\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon} = \frac{\varepsilon}{\alpha_\varepsilon}(\alpha_\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon}),$$

strongly converges to zero in $L^2(\Omega_T)$. Hence, the two-scale limit v_0 defined in Section 3 devoted to the critical case is now constant with respect to y over the whole of Y since

$$\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon} \xrightarrow{2s} \nabla_y v_0 \chi_B(y) = 0 = \nabla_y v_0 \chi_M(y).$$

Consequently, we obtain $v_0(t, x, y) = u(t, x)$ in $\Omega_T \times Y$ so that the function $v := v_0(t, x, y) - u(t, x)$ defined in Section 3 is now equal to zero and the following two-scale convergence holds $u_\varepsilon \xrightarrow{2s} u(t, x)$. To get the variational limit equation we need to identify the two-scale limit of the sequence $\alpha_\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon}$. To that aim (see [20]), we first prove that the sequence

$$w_\varepsilon := \sum_{i \in I_\varepsilon} \frac{\alpha_\varepsilon}{\varepsilon}(u_\varepsilon - \fint_{B_\varepsilon^i} u_\varepsilon\ dx)\chi_{B_\varepsilon^i},$$

is bounded in $L^2(\Omega_T)$ and it two-scale converges to some $w \in L^2(\Omega_T \times Y)$. We then prove that actually $w \in L^2(\Omega_T; H^1(B))$ and that

$$\alpha_\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon} \xrightarrow{2s} \nabla_y w.$$

Taking a test function ϕ in the form

$$\phi = \bar{u}(t, x) + \varepsilon \bar{u}_1(t, x, \tfrac{x}{\varepsilon}) + \frac{\varepsilon}{\alpha_\varepsilon} \bar{w}(t, x, \tfrac{x}{\varepsilon}),$$

with $\bar{w}$ in the space $\mathcal{D}(\Omega_T \times B)$ which is dense in $L^2(\Omega_T; H^1(B))$, we can pass to the limit to get the variational equation

$$\int_{\Omega_T} u(\bar{u}'' + \bar{u}) + \int_{\Omega_T} A^*(x)\nabla u \cdot \nabla \bar{u} + \int_{\Omega_T \times B} A(x,y))\nabla_y w \cdot \nabla_y \bar{w} = \int_{\Omega_T \times Y} f_0 \bar{u}. \tag{5.4}$$

Taking $\bar{w} = w$, $\bar{u} = 0$ in equation (5.4), we get $w = 0$; hence this regime coincides with the classical one for which the sequence of gradient of the solutions u_ε is bounded in the whole of Ω_T: the macroscopic equation for this regime is the classical one:

$$u'' + u - \mathrm{div}_x A^*(x)\nabla u = \int_Y f_0 \, dy, \quad \text{in } \Omega_T, \tag{5.5}$$

with the same initial conditions as in the critical case.

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