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Asymptotic analysis of Neumann periodic optimal boundary control problem

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An optimal boundary control problem in a domain with oscillating boundary has been investigated in this paper. The controls are acting periodically on the oscillating boundary. The controls are applied with suitable scaling parameters. One of the major contribution is the representation of the optimal control using the unfolding operator. We then study the limiting analysis (homogenization) and obtain two limit problems according to the scaling parameters. Another notable observation is that the limit optimal control problem has three controls, namely, a distributed control, a boundary control, and an interface control.

Keywords: optimal control and optimal solution; homogenization; oscillating boundary; boundary control; interior control; adjoint system; unfolding operator; boundary unfolding

1. Introduction

We discuss the homogenization of an optimal control problem associated with the Laplacian in a two-dimensional domain with an oscillating boundary. The domain is a standard one considered by many authors in the literature. See, for example [1–9], and so on. The domain Ω_ϵ consists of bottom and upper parts, respectively, denoted by Ω^- and Ω_ϵ^+ (Figure 1). The region Ω^- is fixed, whereas Ω_ϵ^+ has an oscillatory (rugose) boundary. In fact, the two-dimensional domain Ω_ϵ can be thought of as a cross section of a three-dimensional oscillatory domain (Figure 2). Refer to [10, 11] for the homogenization results in three-dimensional domains. But here, we restrict to the two-dimensional domain, although the results may be extended to three-dimensional domains.

We introduce optimal control problem in Ω_ϵ for the Laplacian operator. The novelty of this article is the consideration of periodic controls acting on the boundary of the oscillating part with appropriate scalings. Another important point is that the periodic controls come from the boundary of a fixed periodic cell (Figure 4), which may be useful in numerics as well. In this article, we characterize the optimal controls via the unfolding operator. This new characterization is also used to study the homogenization of the optimal system and subsequently the limit optimal control problem. We obtain a relation between optimal control and adjoint system using characterization. We remark that different scaling leads to different optimality system.

The motivation of studying a problem defined on oscillatory domain comes from various applications; for example, the need to understand flows in channels with rough boundary and heat transmission in domain with rough interface, to name a few.

In [5–8] and [12, 13], the authors have studied controls problems with control acting away from the oscillating part of the domain. In this paper, we consider controls on the boundary of the oscillating part through Neumann condition which seems to be more complicated. Unlike Dirichlet condition, the limit problem is different in the case of Neumann problem. As remarked earlier, the characterization of the optimal control is given via the unfolding operator. The method of unfolding is introduced and developed in [14–17], and it is well-developed and applied to many problems. Particularly, in [17], the method adapted to oscillatory boundaries. In the past 40 years, several methods have been introduced to study homogenization problems, but we feel that the unfolding method seems to be more amenable in the present situation. We do not find any other way of characterizing optimal controls. In addition to the characterization of optimal controls and difficulties in oscillating domain, we also have to homogenize a coupled optimality system involving optimal state, adjoint state, optimal control, and cost functional.

We briefly describe the layout of the paper. A detailed configuration of the domain is given in Section 2. The minimization problems is described in Section 3 together with the proof of existence and uniqueness of the optimal solution (optimal control and corresponding

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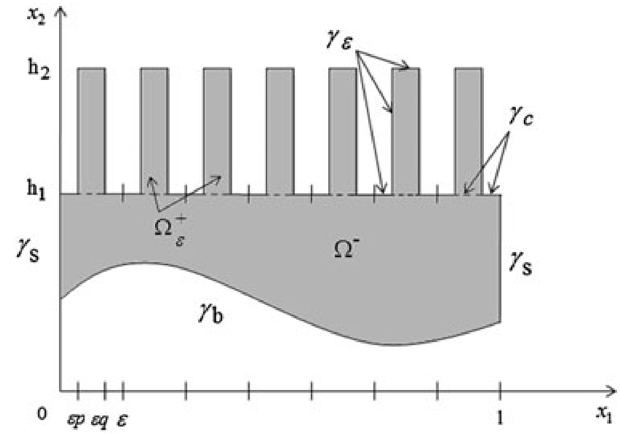

Figure 1. The two-dimensional domain Ω_ϵ.

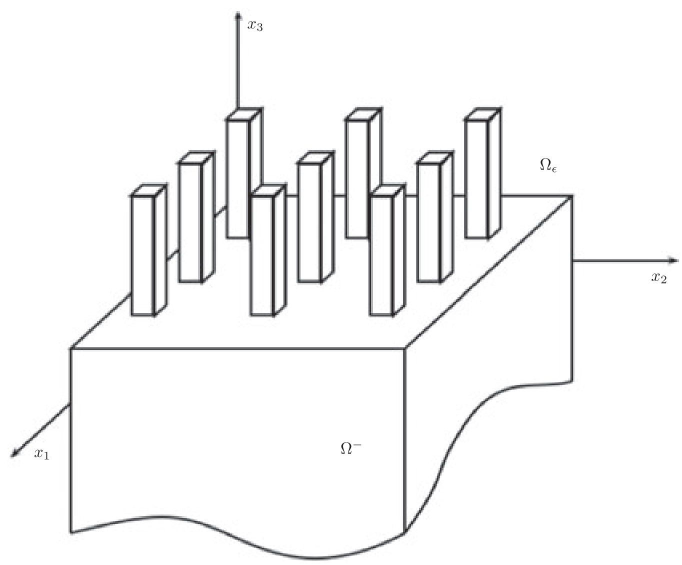

Figure 2. The three-dimensional domain Ω_ϵ.

state) with periodic controls arising from the boundary of a fixed cell. We do use appropriate scaling parameters ϵ^α with $\alpha \geq 1$. Two types of unfolding operators (internal and boundary) are reproduced from [17] (see also [8]) in Section 4. All the results, namely, the optimality system and characterization of optimal controls (Theorem 5.1) and the limit system and two homogenization theorems (Theorem 5.4 for the critical case $\alpha = 1$ and Theorem 5.5 for $\alpha > 1$) are presented in Section 5. In the critical case $\alpha = 1$, the controls on the oscillating boundary splits into three controls in the limit system: a distributed control on the upper part of the domain, a control on the upper boundary, and finally, an interface control between the upper and lower domains. On the other hand $\alpha > 1$, there is no distributed control. The proofs of the theorem can be found in Section 6.

There is also a large amount of literature on the homogenization with oscillating boundaries, which has tremendous applications as well (for example, [1–13], [17–26], and [27]). For some recent work on oscillating boundaries, see [9] and [28–32]. For general literature in homogenization, we refer to [33–36] and the reference therein. Some references regarding the homogenization of the optimal control/ controllability, the reader can refer to [10, 11] and [37–40]. See [41–45] for optimal control problems and derivation of optimality systems.

2. Description of an oscillating domain and notations

The description of the oscillatory domain $\Omega_\epsilon \subset \mathbb{R}^2$ is given in the succeeding paragraphs. For a fixed parameter $\epsilon = \frac{1}{m}$ with $m \in \mathbb{N}$, we consider an oscillating domain Ω_ϵ as given in Figure 1. This can be viewed as the cross section of Figure 2. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth (say, Lipschitz) periodic function with periodic 1 (in fact, one can use any period) and $0 < p < q < 1$. Let ϑ_ϵ be a periodic function defined on [0, 1], with periodic ϵ, defined on $[0, \epsilon]$ by

$$\vartheta_\epsilon(x_1) = \begin{cases} h_2 & \text{if } x_1 \in (\epsilon p, \epsilon q), \\ h_1 & \text{if } x_1 \in [0, \epsilon) \backslash (\epsilon p, \epsilon q), \end{cases}$$

with $h_2 > h_1 > h_0$. Here, h_0 is the maximum value of the smooth function g in $[0, 1]$. We take the domain Ω_ϵ as

$$\Omega_\epsilon = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ 0 < x_1 < 1, \ g(x_1) < x_2 < \vartheta_\epsilon(x_1)\right\}.$$

We decompose the boundary of the domain Ω_ϵ, $\partial\Omega_\epsilon$ into three disjoint parts as $\partial\Omega_\epsilon = \gamma_b \cup \gamma_s \cup \gamma_\epsilon$, where the bottom boundary γ_b and side boundaries γ_s of Ω_ϵ are given by

$$\gamma_b = \{(x_1, x_2) : \ x_2 = g(x_1), \ x_1 \in [0, 1]\},$$

$$\gamma_s = \{(0, x_2) : \ g(0) \le x_2 \le h_1\} \cup \{(1, x_2) : \ g(1) \le x_2 \le h_1\}.$$

The top boundary γ_ϵ is given by $\gamma_\epsilon = \partial\Omega_\epsilon \setminus (\gamma_b \cup \gamma_s)$. Let Ω_ϵ^+ be the top part of the domain Ω_ϵ, which is the union of slabs of height $(h_2 - h_1)$ and width $\epsilon(q - p)$, that is

$$\Omega_\epsilon^+ = \bigcup_{k=0}^{m-1} (k\epsilon + \epsilon p, k\epsilon + \epsilon q) \times (h_1, h_2).$$

Denote Ω^-, the fixed part of the domain Ω_ϵ, which is described by

$$\Omega^- = \{(x_1, x_2) : 0 < x_1 < 1, \ g(x_1) < x_2 < h_1\}.$$

Now note the boundary of Ω^-, namely, $\partial\Omega^- = \gamma_s \cup \gamma_b \cup \gamma_c$, where top boundary of Ω^- is given by

$$\gamma_c = \{(x_1, h_1) : \ 0 \le x_1 \le 1\}.$$

We can also write Ω_ϵ as $\Omega_\epsilon = Int\left(\overline{\Omega_\epsilon^+ \cup \Omega^-}\right)$. We denote the full domain Ω (Figure 3) as $\Omega = \{(x_1, x_2) : 0 < x_1 < 1, \ g(x_1) < x_2 < h_2\}$.

The bottom part of the boundary of Ω is same as Ω_ϵ, which is γ_b. The vertical and top boundaries of Ω denoted by $\gamma_{s'}$ and γ_u, respectively, are given by

$$\gamma_{s'} = \{(0, x_2) : \ g(0) \le x_2 \le h_2\} \cup \{(1, x_2) : \ g(1) \le x_2 \le h_2\}$$

and

$$\gamma_u = \{(x_1, h_2) : \ 0 \le x_1 \le 1\}.$$

Denote Ω^+ as $\Omega^+ = \{(x_1, x_2) : \ 0 < x_1 < 1, h_1 < x_2 < h_2\}$, then we can write $\Omega = Int\left(\overline{\Omega^+ \cup \Omega^-}\right)$. Let γ be the reference boundary (Figure 4), defined as

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4,$$

where

$$\gamma_1 = \{(y_1, h_1) : 0 \le y_1 \le p\} \cup \{(y_1, h_1) : q \le y_1 \le 1\},$$
$$\gamma_2 = \{(y_1, h_2) : p \le y_1 \le q\}, \ \gamma_3 = \{(p, y_2) : h_1 \le y_2 \le h_2\},$$
$$\gamma_4 = \{(q, y_2) : h_1 \le y_2 \le h_2\}.$$

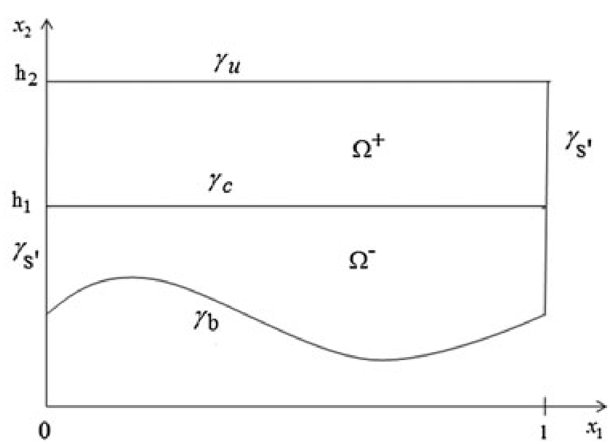

Figure 3. The domain Ω.

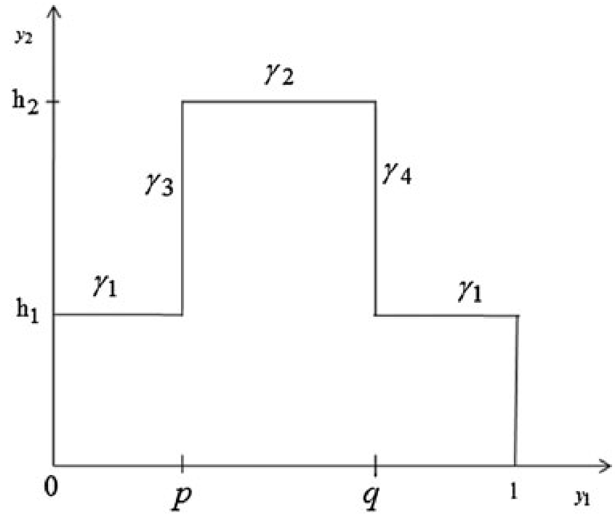

Figure 4. The fixed boundary $\gamma = \bigcup_{i=1}^{4} \gamma_i$.

Let $\gamma_1^\epsilon := \gamma_\epsilon \cap \gamma_c$, $\gamma_2^\epsilon = \gamma_\epsilon \cap \gamma_u$, and the common boundary between Ω_ϵ^+ and Ω^- denoted by γ_3^ϵ is defined as

$$\gamma_3^\epsilon = \bigcup_{k=0}^{m-1} (k\epsilon + \epsilon p, k\epsilon + \epsilon q) \times \{h_1\}.$$

Notation: Let $A_1 = \gamma_1,\ A_2 = \gamma_2$, and $A_3 = (p,q) \times \{h_1\}$. Let H_ϵ^1 be the space of H^1- periodic functions, which vanishes on the bottom boundary γ_b. A function defined in Ω_ϵ is called γ_s-periodic, if they take the same value on both side of γ_s. For any function u defined on Ω_ϵ, we denote $\tilde{u}$ extension of u by 0 to the hole domain $\overline{\Omega}$.

3. Description of an optimal control problem

For $\theta \in L^2_{per}(\gamma)$, define $\theta^\epsilon = (\chi_{\gamma_1} + \epsilon^\alpha \chi_{\gamma_3} + \chi_{\gamma_2} + \epsilon^\alpha \chi_{\gamma_4})\,\theta \in L^2_{per}(\gamma)$, where the scaling parameter $\alpha \geq 1$. For any set E, χ_E is the characteristic function of the set E. We define the periodic oscillatory controls $\widehat{\theta^\epsilon} \in L^2(\gamma_\epsilon)$ such that

$$\widehat{\theta^\epsilon}(x_1, x_2) = \theta^\epsilon\left(\frac{x_1}{\epsilon}, x_2\right). \tag{3.1}$$

For $f \in L^2_{per}(\Omega)$ and $\widehat{\theta^\epsilon} \in L^2(\gamma_\epsilon)$ defined earlier, consider the following control problem:

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f \quad \text{in } \Omega_\epsilon, \\ \dfrac{\partial u_\epsilon}{\partial \nu} = \widehat{\theta^\epsilon} \quad \text{on } \gamma_\epsilon, \\ u_\epsilon = 0 \quad \text{on } \gamma_b, \\ u_\epsilon \quad \text{is } \gamma_s\text{-periodic.} \end{cases} \tag{3.2}$$

A variational formulation is given as follows: find u_ϵ in H_ϵ^1 such that

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla \phi + \int_{\Omega_\epsilon} u_\epsilon \phi = \int_{\Omega_\epsilon} f\phi + \int_{\gamma_\epsilon} \widehat{\theta^\epsilon} \phi \tag{3.3}$$

for all $\phi \in H_\epsilon^1$. It is known that (3.2) admits a unique weak solution u_ϵ in H_ϵ^1. The solution operator is linear and continuous from $L^2_{per}(\Omega) \times L^2_{per}(\gamma_\epsilon)$ into H_ϵ^1, that is

$$\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C_\epsilon \left(\|f\|_{L^2(\Omega)} + \|\widehat{\theta^\epsilon}\|_{L^2(\gamma_\epsilon)} \right), \tag{3.4}$$

where, in general, $C_\epsilon > 0$ depends on ϵ. Let us consider an L^2-cost functional functional:

$$J_\epsilon(u_\epsilon, \theta) = \frac{1}{2}\int_{\Omega_\epsilon} |u_\epsilon - u_d|^2 + \frac{\beta}{2}\int_\gamma |\theta|^2,$$

where $\theta \in L^2_{per}(\gamma)$, $u_\epsilon = u_\epsilon(\theta)$ is the solution state of (3.2) corresponding to θ and $\beta > 0$ is a regularization parameter. The desired state is denoted by $u_d \in L^2_{per}(\Omega)$. With this cost functional, we consider the following optimal control problem:

$$\inf\left\{J_\epsilon(u_\epsilon,\theta):\ \theta \in L^2(\gamma), \left(u_\epsilon, \widehat{\theta^\epsilon}\right) \text{ satisfies (3.2)}\right\}. \qquad (P_\epsilon)$$

Now, we show that the optimal control problem (P_ϵ) admits a unique solution.

Theorem 3.1
For each $\epsilon > 0$, the minimization problem (P_ϵ) admits a unique solution.

Proof
Because the functional $J_\epsilon(u_\epsilon,\theta) \geq 0$, there exists the infimum $m_\epsilon := \inf\limits_{\theta\in L^2(\gamma)} J_\epsilon(u_\epsilon,\theta)$. Indeed, $0 \leq m_\epsilon < \infty$ because $m_\epsilon \leq J_\epsilon(u_\epsilon,\theta)$ for any fixed $\theta \in L^2(\gamma)$, in particular, $m_\epsilon \leq J_\epsilon(u_\epsilon, 0)$. Hence, there exists a minimizing sequence $(\theta_{n,\epsilon})_{n\geq 1} \in L^2(\gamma)$ such that $J_\epsilon(u^n_\epsilon, \theta_{n,\epsilon}) \to m_\epsilon$ as $n \to \infty$. Without loss of generality, we can suppose that $J_\epsilon(u^n_\epsilon,\theta_{n,\epsilon}) \leq J_\epsilon(u^0_\epsilon, 0)$ for n large enough. Here, u^n_ϵ, u^0_ϵ are solutions of (3.2) corresponding to the data $\theta_{n,\epsilon}$, $\theta = 0$, respectively. When $\theta = 0$, we have the corresponding $\widehat{\theta^\epsilon} = 0$. Then, it is easy to see that the constant in (3.4) is independent of ϵ, that is, $\|u^0_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C$. This implies $\|\theta_{n,\epsilon}\|_{L^2(\gamma)} \leq C$. So there exists a subsequence still denoted by $(\theta_{n,\epsilon})_n$, which converges weakly to some θ_ϵ in $L^2(\gamma)$, that is, $\theta_{n,\epsilon} \rightharpoonup \theta_\epsilon$ in $L^2(\gamma)$. Using the fact that L^2-norm is weakly lower semi-continuous, we have

$$\int_\gamma |\theta_\epsilon|^2 \leq \liminf_{n\to\infty} \int_\gamma |\theta_{n,\epsilon}|^2. \qquad (3.5)$$

We know from norm estimate (3.4) that $\|u^n_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C_\epsilon$, which implies up to a subsequence $u^n_\epsilon \rightharpoonup u_\epsilon$ in $H^1(\Omega_\epsilon)$ as $n \to \infty$.

Claim: The limit u_ϵ is the weak solution corresponding to f and $\widehat{\theta^\epsilon_\epsilon}$, that is, $u_\epsilon = u_\epsilon(f, \widehat{\theta^\epsilon_\epsilon})$.
We know u^n_ϵ solves the partial differential equation (3.2) for $\widehat{\theta^\epsilon} = \widehat{\theta^\epsilon_{n,\epsilon}}$, and we have the following variational formulation:

$$\int_{\Omega_\epsilon} \nabla u^n_\epsilon \cdot \nabla\phi + \int_{\Omega_\epsilon} u^n_\epsilon \phi = \int_{\Omega_\epsilon} f\phi + \int_{\gamma_\epsilon} \widehat{\theta^\epsilon_{n,\epsilon}}\phi, \qquad (3.6)$$

$\forall \phi \in H^1_\epsilon$. To prove our claim, we need to show the following variational formulation:

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla\phi + \int_{\Omega_\epsilon} u_\epsilon \phi = \int_{\Omega_\epsilon} f\phi + \int_{\gamma_\epsilon} \widehat{\theta^\epsilon_\epsilon}\phi \quad \forall \phi \in H^1_\epsilon. \qquad (3.7)$$

Using the convergence $u^n_\epsilon \rightharpoonup u_\epsilon$ in $H^1(\Omega_\epsilon)$ and Trace theorem, we obtain

$$\lim_{n\to\infty} \int_{\Omega_\epsilon} \nabla u^n_\epsilon \cdot \nabla\phi + \int_{\Omega_\epsilon} u^n_\epsilon\phi = \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla\phi + \int_{\Omega_\epsilon} u_\epsilon\phi \qquad (3.8)$$

It remains to prove that

$$\lim_{n\to\infty} \int_{\gamma_\epsilon} \widehat{\theta^\epsilon_{n,\epsilon}}\phi = \int_{\gamma_\epsilon} \widehat{\theta^\epsilon_\epsilon}\phi \quad \text{for } \phi \in L^2(\gamma_\epsilon). \qquad (3.9)$$

Now, to compute the limit, let

$$\begin{aligned}
\int_{\gamma_\epsilon} \widehat{\theta^\epsilon_{n,\epsilon}}\phi = \sum_{k=0}^{m-1} \Bigg(&\int_{k\epsilon}^{k\epsilon+\epsilon p} \widehat{\theta^\epsilon_{n,\epsilon}}(x_1,h_1)\phi(x_1,h_1)dx_1 + \int_{h_1}^{h_2} \widehat{\theta^\epsilon_{n,\epsilon}}(k\epsilon+\epsilon p, x_2)\phi(k\epsilon+\epsilon p, x_2)dx_2 \\
&+ \int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q} \widehat{\theta^\epsilon_{n,\epsilon}}(x_1,h_2)\phi(x_1,h_2)dx_1 + \int_{h_2}^{h_1} \widehat{\theta^\epsilon_{n,\epsilon}}(k\epsilon+\epsilon q, x_2)\phi(k\epsilon+\epsilon q, x_2)dx_2 \\
&+ \int_{k\epsilon+\epsilon q}^{(k+1)\epsilon} \widehat{\theta^\epsilon_{n,\epsilon}}(x_1,h_1)\phi(x_1,h_1)dx_1 \Bigg) \\
= \quad &\epsilon \int_0^p \theta_{n,\epsilon}(y_1,h_1) \sum_{k=0}^{m-1} \phi(k\epsilon+\epsilon y_1, h_1)dy_1 + \epsilon^\alpha \int_{h_1}^{h_2} \theta_{n,\epsilon}(p,x_2)\phi(k\epsilon+\epsilon p, x_2)dx_2 \\
&+ \epsilon \int_p^q \theta_{n,\epsilon}(y_1,h_2) \sum_{k=0}^{m-1} \phi(k\epsilon+\epsilon y_1, h_2))dy_1 + \epsilon^\alpha \int_{h_2}^{h_1} \theta_{n,\epsilon}(q,x_2)\phi(k\epsilon+\epsilon q, x_2)dx_2 \\
&+ \epsilon \int_q^1 \theta_{n,\epsilon}(y_1,h_1) \sum_{k=0}^{m-1} \phi(k\epsilon+\epsilon y_1, h_1)dy_1.
\end{aligned}$$

Taking limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\int_{\gamma_\epsilon}\widehat{\theta_{n,\epsilon}^{\epsilon}}\phi = \epsilon\int_0^p \theta_\epsilon(y_1,h_1)\sum_{k=0}^{m-1}\phi(k\epsilon+\epsilon y_1,h_1)dy_1 + \epsilon^\alpha\int_{h_1}^{h_2}\theta_\epsilon(p,x_2)\phi(k\epsilon+\epsilon p,x_2)dx_2$$
$$+\epsilon\int_p^q \theta_\epsilon(y_1,h_2)\sum_{k=0}^{m-1}\phi(k\epsilon+\epsilon y_1,h_2)dy_1 + \epsilon^\alpha\int_{h_2}^{h_1}\theta_\epsilon(q,x_2)\phi(k\epsilon+\epsilon q,x_2)dx_2 \quad (3.10)$$
$$+\epsilon\int_q^1 \theta_\epsilon(y_1,h_1)\sum_{k=0}^{m-1}\phi(k\epsilon+\epsilon y_1,h_1)dy_1.$$

On the other hand

$$\int_{\gamma_\epsilon}\widehat{\theta_\epsilon^\epsilon}\phi = \sum_{k=0}^{m-1}\left(\int_{k\epsilon}^{k\epsilon+\epsilon p}\theta_\epsilon^\epsilon\left(\frac{x_1}{\epsilon},h_1\right)\phi(x_1,h_1)dx_1 + \epsilon^\alpha\int_{h_1}^{h_2}\theta_\epsilon^\epsilon(k\epsilon+\epsilon p,x_2)\phi(k\epsilon+\epsilon p,x_2)dx_2\right.$$
$$+\int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q}\theta_\epsilon^\epsilon\left(\frac{x_1}{\epsilon},h_2\right)\phi(x_1,h_2)dx_1 + \epsilon^\alpha\int_{h_2}^{h_1}\theta_\epsilon^\epsilon(k\epsilon+\epsilon q,x_2)\phi(k\epsilon+\epsilon q,x_2)dx_2$$
$$\left.+\int_{k\epsilon+\epsilon q}^{(k+1)\epsilon}\theta_\epsilon^\epsilon\left(\frac{x_1}{\epsilon},h_1\right)\phi(x_1,h_1)dx_1\right).$$

By resealing each term, we will end up with the same expression as in (3.10). Hence, (3.9) proved. Because $u_\epsilon^n \rightharpoonup u_\epsilon$ in $H^1(\Omega_\epsilon)$, by weakly lower semi-continuity of L^2- norm gives

$$\int_{\Omega_\epsilon}|u_\epsilon - u_d|^2 \le \liminf_{n\to\infty}\int_{\Omega_\epsilon}|u_\epsilon^n - u_d|^2. \quad (3.11)$$

Hence, combining (3.5) and (3.11), we obtain $J_\epsilon(u_\epsilon,\theta_\epsilon) \le \liminf_{n\to\infty} J_\epsilon(u_\epsilon^n,\theta_{n,\epsilon}) = m_\epsilon$.
Therefore, $(u_\epsilon,\theta_\epsilon)$ is a solution to problem (P_ϵ). Uniqueness follows from the strict convexity of the L^2-cost functional. □

In the next section, we introduce the unfolding operator and its properties required for our article. Then using these operators, we derive the optimality system and characterize the optimal control using unfolding operators.

4. Unfolding operators and its properties

We define periodic unfolding operator and some of its properties without proof. The proofs can be found in [17] (see also in [8]). For $x \in \mathbb{R}$, we write $[x]$ as the integer part of x, that is, $[x] = k$, where k is the largest integer such that $k \le x$ and $\{x\} = x - [x]$.

Definition 4.1
(The unfolding operator) Let $\phi^\epsilon : \Omega^+ \times (p,q) \to \Omega_\epsilon^+$ be defined by $(x_1,x_2,x_3) \mapsto \left(\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon x_3, x_2\right)$. The ϵ-unfolding of a function $u : \Omega_\epsilon^+ \to \mathbb{R}$ is the composite function $u \circ \phi^\epsilon : \Omega^+ \times (p,q) \to \mathbb{R}$. The operator that maps every function $u : \Omega_\epsilon^+ \to \mathbb{R}$ to its ϵ-unfolding is called the unfolding operator, which we denote by T^ϵ, that is

$$T^\epsilon : \left\{u : \Omega_\epsilon^+ \to \mathbb{R}\right\} \to \left\{v : \Omega^+ \times (p,q) \to \mathbb{R}\right\}$$

defined by

$$T^\epsilon u(x_1,x_2,x_3) = u \circ \phi^\epsilon(x_1,x_2,x_3) = u\left(\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon x_3, x_2\right).$$

If U is an open subset of $\mathbb{R}^2$ containing Ω_ϵ^+ and u a is real valued function on U, then $T^\epsilon u$ will mean T^ϵ acting on the restriction of u to Ω_ϵ^+. The following prosperities of T^ϵ can be obtain from [17].

Proposition 4.2

(i) T^ϵ is linear.
(ii) Let $u_1,\ u_2$ be two functions from $\Omega_\epsilon^+ \to \mathbb{R}$. Then $T^\epsilon(u_1u_2) = T^\epsilon(u_1)T^\epsilon(u_2)$.
(iii) Let $u \in L^1(\Omega_\epsilon^+)$. Then

$$\int_{\Omega^+\times(p,q)} T^\epsilon u\, dx = \int_{\Omega_\epsilon^+} u\, dx.$$

(iv) Let $u \in L^2(\Omega_\epsilon^+)$. Then $T^\epsilon u \in L^2(\Omega^+ \times (p,q))$ and $\|T^\epsilon u\|_{L^2(\Omega^+\times(p,q))} = \|u\|_{L^2(\Omega_\epsilon^+)}$.

(v) Let $u \in H^1(\Omega_\epsilon^+)$. Then $T^\epsilon u \in L^2(0,1;H^1((h_1,h_2)\times(p,q)))$. Moreover, $\frac{\partial}{\partial x_2}(T^\epsilon u) = T^\epsilon\left(\frac{\partial u}{\partial x_2}\right)$, $\frac{\partial}{\partial x_3}(T^\epsilon u) = \epsilon T^\epsilon\left(\frac{\partial u}{\partial x_1}\right)$. Further, $\|T^\epsilon u\|_{L^2(0,1;H^1((h_1,h_2)\times(p,q)))} \leq C\|u\|_{H^1(\Omega_\epsilon^+)}$.

(vi) Let $u \in L^2(\Omega^+)$. Then $T^\epsilon u \to u$ strongly in $L^2(\Omega^+ \times (p,q))$.

(vii) Let $u_\epsilon \to u$ strongly in $L^2(\Omega^+)$. Then $T^\epsilon u_\epsilon \to u$ strongly in $L^2(\Omega^+ \times (p,q))$.

(viii) Let for every ϵ, $u_\epsilon \in L^2(\Omega_\epsilon^+)$ be such that $T^\epsilon u_\epsilon \rightharpoonup u$ weakly in $L^2(\Omega^+ \times (p,q))$. Then

$$\widetilde{u}_\epsilon \rightharpoonup \int_p^q u(x_1,x_2,x_3)\, dx_3 \text{ weakly in } L^2(\Omega^+).$$

(ix) Let $u_\epsilon \in H^1(\Omega_\epsilon^+)$ for every $\epsilon > 0$ be such that $T^\epsilon u_\epsilon \rightharpoonup u$ weakly in $L^2((0,1)\times(p,q);H^1((h_1,h_2)))$. Then $\widetilde{u}_\epsilon \rightharpoonup \int_p^q u(x_1,x_2,x_3)\, dx_3$ weakly in $L^2((0,1);H^1((h_1,h_2)))$. □

4.1. *Unfolding on the boundary*

For our analysis, we define boundary unfolding on γ_1^ϵ, γ_2^ϵ, and γ_3^ϵ.

Definition 4.3

For $i = 1,2,3$, the ϵ-unfolding of a function $u : \gamma_i^\epsilon \to \mathbb{R}$ is the function $T_i^\epsilon u : (0,1)\times A_i \to \mathbb{R}$ defined by $T_i^\epsilon u(x_1,x_2,x_3) = u\left(\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon x_3, x_2\right)$.

If U is an open subset of $\mathbb{R}^2$ such that $\gamma_i^\epsilon \subset \overline{U}$ and $u : U \to \mathbb{R}$, then $T_i^\epsilon u = T_i^\epsilon\left(u|_{\gamma_i^\epsilon}\right)$, for functions u with a well-defined trace on γ_i^ϵ. Some of the essential properties of boundary unfolding operators are stated in the succeeding texts ([17], [8]).

Proposition 4.4

For i=1, 2, 3,

(i) T_i^ϵ is linear, and for functions u_1, u_2 from $\gamma_i^\epsilon \to \mathbb{R}$, we have $T_i^\epsilon(u_1u_2) = T_i^\epsilon(u_1)T_i^\epsilon(u_2)$.

(ii) If $u \in L^2(\gamma_i^\epsilon)$, then $T_i^\epsilon u \in L^2((0,1)\times A_i)$ and $\|T_i^\epsilon u\|_{L^2((0,1)\times A_i)} = \|u\|_{L^2(\gamma_i^\epsilon)}$.

(iii) If $u_\epsilon \longrightarrow u$ strongly in $H^1((0,1)\times(h_1,h_2))$, then $T_i^\epsilon u_\epsilon \longrightarrow u$ strongly in $L^2((0,1)\times A_i)$.

(iv) If u_ϵ be a sequence in $L^2(\gamma_i^\epsilon)$ such that $T_i^\epsilon u_\epsilon \rightharpoonup u$ weakly in $L^2((0,1)\times A_i)$, then $\widetilde{u_\epsilon} \rightharpoonup \int_{A_i} u\, dx_3$ weakly in $L^2(0,1)$. □

5. Main results

In this section, we present our results, namely, the optimality system and the characterization of the optimal control, the limit system, and the main convergence theorems.

5.1. *Optimality system*

Let $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ be the optimal solution to the problem (P_ϵ). Our aim is to derive a characterization of $\overline{\theta}_\epsilon$ with the help of unfolding operators and adjoint state $\overline{v}_\epsilon \in H_\epsilon^1$. The adjoint state $\overline{v}_\epsilon$ solves

$$\begin{cases} -\Delta\overline{v}_\epsilon + \overline{v}_\epsilon = \overline{u}_\epsilon - u_d & \text{in } \Omega_\epsilon, \\ \dfrac{\partial\overline{v}_\epsilon}{\partial\nu} = 0 & \text{on } \gamma_\epsilon, \\ \overline{v}_\epsilon = 0 & \text{on } \gamma_b, \\ \overline{v}_\epsilon & \text{is } \gamma_s\text{-periodic.} \end{cases} \tag{5.1}$$

We now present one of our major contribution, namely, the characterization of the optimal control via the unfolding operators.

Theorem 5.1

Let $f \in L^2(\Omega)$ and $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ be the optimal solution of (P_ϵ). Let $\overline{v}_\epsilon \in H_\epsilon^1$ solves (5.1), then the optimal control is given by

$$\begin{aligned}\overline{\theta}_\epsilon(y_1,y_2) = -\frac{1}{\beta}\Bigg[\chi_{\gamma_1}\int_0^1 T_1^\epsilon(\overline{v}_\epsilon)(x_1,h_1,y_1)\, dx_1 + \chi_{\gamma_3}\epsilon^{\alpha-1}\int_0^1 T^\epsilon\overline{v}_\epsilon(x_1,y_2,p)dx_1 \\ + \chi_{\gamma_2}\int_0^1 T_2^\epsilon\overline{v}_\epsilon(x_1,h_2,y_1)dx_1 + \chi_{\gamma_4}\epsilon^{\alpha-1}\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,y_2,q)\, dx_1\Bigg],\end{aligned}$$

where $T^\epsilon, T_1^\epsilon, T_2^\epsilon$ be the unfolding operators as in Definition 4.1 and Definition 4.3, respectively. Conversely, assume that a pair $(\hat{u}_\epsilon, \hat{v}_\epsilon) \in H_\epsilon^1 \times H_\epsilon^1$ solves the optimality system

$$\begin{cases} -\Delta \hat{u}_\epsilon + \hat{u}_\epsilon = f \text{ in } \Omega_\epsilon, \\ -\Delta \hat{v}_\epsilon + \hat{v}_\epsilon = \hat{u}_\epsilon - u_d \text{ in } \Omega_\epsilon, \\ \dfrac{\partial \hat{u}_\epsilon}{\partial \nu} = \widehat{\hat{\theta}_\epsilon}^{\,\epsilon}, \ \dfrac{\partial \hat{v}_\epsilon}{\partial \nu} = 0 \text{ on } \gamma_\epsilon, \\ \hat{u}_\epsilon = 0, \ \hat{v}_\epsilon = 0 \text{ on } \gamma_b, \\ \hat{u}_\epsilon, \ \hat{v}_\epsilon \text{ are } \gamma_s\text{-periodic.} \end{cases} \tag{5.2}$$

Define $\hat{\theta}_\epsilon$ as

$$\hat{\theta}_\epsilon(y_1, y_2) = -\frac{1}{\beta}\left[\chi_{\gamma_1}\int_0^1 T_1^\epsilon(\hat{v}_\epsilon)(x_1, h_1, y_1)\, dx_1 + \chi_{\gamma_3}\epsilon^{\alpha-1}\int_0^1 T^\epsilon \hat{v}_\epsilon(x_1, y_2, p) dx_1 \right.$$
$$\left. + \chi_{\gamma_2}\int_0^1 T_2^\epsilon \hat{v}_\epsilon(x_1, h_2, y_1) dx_1 + \chi_{\gamma_4}\epsilon^{\alpha-1}\int_0^1 T^\epsilon(\hat{v}_\epsilon)(x_1, y_2, q)\, dx_1\right].$$

Then, the pair $(\hat{u}_\epsilon, \hat{\theta}_\epsilon)$ is the optimal solution to (P_ϵ). □

5.2. *Homogenized systems*

We now consider the limit optimality systems corresponding to scaling parameters $\alpha > 1$ and $\alpha = 1$. Consider the following Banach space:

$$V_0(\Omega) = \left\{\psi \in L^2(\Omega) : \frac{\partial \psi|_{\Omega^-}}{\partial x_1} \in L^2(\Omega^-), \ \frac{\partial \psi}{\partial x_2} \in L^2(\Omega) \text{ and } \psi|_{\gamma_b} = 0\right\},$$

with respect to the norm defined by

$$\|\psi\|^2_{V_0(\Omega)} = \|\psi\|^2_{L^2(\Omega)} + \left\|\frac{\partial \psi}{\partial x_2}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial \psi|_{\Omega^-}}{\partial x_1}\right\|^2_{L^2(\Omega^-)}.$$

For a given $f \in L^2(\Omega)$, $\theta \in L^2(h_1, h_2)$, C_1 and C_2 in $\mathbb{R}$, consider two systems for $j = 0, 1$:

$$\begin{cases} -\dfrac{\partial^2 u^+}{\partial x_2^2} + u^+ = f - j\theta\chi_{\Omega^+} \text{ in } \Omega^+, \\ -\Delta u^- + u^- = f \text{ in } \Omega^-, \\ \dfrac{\partial u^+}{\partial \nu} = C_2 \text{ on } \gamma_u, \\ u^+ = u^-, \ \dfrac{\partial u^-}{\partial x_2} - (q-p)\dfrac{\partial u^+}{\partial x_2} = C_1 \text{ on } \gamma_c, \\ u^- = 0 \text{ on } \gamma_b, \ u \text{ is } \gamma_{s'}\text{-periodic.} \end{cases} \tag{5.3}$$

Write $u = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-}$. The linearity of the solution operator of (5.3) is obvious, and we have the continuity of the solution operator. More precisely

$$\|u\|_{V_0(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + j\|\theta\|_{L^2(h_1,h_2)}\right). \tag{5.4}$$

Existence and uniqueness of $u \in V_0(\Omega)$ follow in a standard way. Now, consider the L^2-cost functionals J_1 and J_2 defined by

$$J_1(u, \theta, C_1, C_2) = \frac{1}{2}\int_\Omega ((q-p)\chi_{\Omega^+} + \chi_{\Omega^-})\,|u - u_d|^2 + \frac{(q-p)^2}{4}\int_{h_1}^{h_2}|\theta|^2$$
$$+ \frac{\beta}{2(1-(q-p))}|C_1|^2 + \frac{\beta}{2}|C_2|^2.$$

and

$$J_2(u, C_1, C_2) = \frac{1}{2}\int_\Omega ((q-p)\chi_{\Omega^+} + \chi_{\Omega^-})\,|u - u_d|^2 + \frac{\beta}{2(1-(q-p))}|C_1|^2 + \frac{\beta}{2}|C_2|^2.$$

Associated with these cost functionals, we introduce the following optimal control problems:

$$\inf\left\{J_1(u,\theta,C_1,C_2):\ \theta\in L^2(h_1,h_2),\ C_1,C_2\in\mathbb{R} \text{ and } (u,\theta,C_1,C_2) \text{ obeys (5.3) for } j=1\right\} \qquad (P_1)$$

and

$$\inf\{J_2(u,C_1,C_2):\ C_1,C_2\ \in\mathbb{R} \text{ and } (u,C_1,C_2) \text{ obeys (5.3) for } j=0\}. \qquad (P_2)$$

We will see later that (P_1) corresponds to the case $\alpha = 1$ and (P_2) corresponds to $\alpha > 1$. Further, the problems (P_1) and (P_2) admit unique solutions. First, we characterize optimal controls $\overline{\theta},\ \overline{C_1},\ \overline{C_2}$ of the problem (P_1) using adjoint state $\overline{v}$, which are given in Theorems 5.2 and 5.3. We omit the proof of these theorems. Let $\overline{v}\in V_0(\Omega)$ solves the adjoint problem

$$\begin{cases} -\dfrac{\partial^2\overline{v}^+}{\partial x_2^2}+\overline{v}^+ = (\overline{u}^+-u_d)\ \text{ in } \Omega^+,\\ -\Delta\overline{v}^- + \overline{v}^- = (\overline{u}^- - u_d)\ \text{ in } \Omega^-,\\ \dfrac{\partial\overline{v}^+}{\partial x_2} = 0\ \text{ on } \gamma_u,\\ \overline{v}^+ = \overline{v}^-,\ \dfrac{\partial\overline{v}^-}{\partial x_2} - (q-p)\dfrac{\partial\overline{v}^+}{\partial x_2} = 0 \text{ on } \gamma_c,\\ \overline{v}^- = 0\ \text{ on }\ \gamma_b,\ \overline{v}\ \text{ is } \gamma_{s'}\text{-periodic.}\end{cases} \tag{5.5}$$

Here, we denote $\overline{v} = \overline{v}^+\chi_{\Omega^+} + \overline{v}^-\chi_{\Omega^-}$.

Theorem 5.2
Let $f\in L^2(\Omega)$ and $(\overline{u},\ \overline{\theta},\ \overline{C_1},\ \overline{C_2})$ be the optimal solution of (P_1). Let $\overline{v}\in V_0(\Omega)$ solves (5.5), then the optimal control is given by

$$\begin{aligned} &\overline{\theta} = \overline{\theta}(x_2) = \frac{2}{(q-p)}\int_0^1 \overline{v}(x_1,x_2)dx_1,\\ &\overline{C_1} = -\frac{1-(q-p)}{\beta}\int_0^1 \overline{v}(y,h_1)dy,\ \ \overline{C_2} = -\frac{(q-p)}{\beta}\int_0^1\overline{v}(y,h_2)dy \end{aligned} \tag{5.6}$$

Conversely, assume that a pair $(\hat{u},\hat{v})\in V_0(\Omega)\times V_0(\Omega)$ solves the optimality system

$$\begin{cases} -\dfrac{\partial^2\hat{u}^+}{\partial x_2^2}+\hat{u}^+ = f-\hat{\theta},\ \ -\dfrac{\partial^2\hat{v}^+}{\partial x_2^2}+\hat{v}^+ = (\hat{u}^+-u_d)\ \ in\ \ \Omega^+,\\ -\Delta\hat{u}^- + \hat{u}^- = f,\ \ -\Delta\hat{v}^- + \hat{v}^- = (\hat{u}^- - u_d)\ \ in\ \Omega^-,\\ \dfrac{\partial\hat{u}^+}{\partial x_2} = \hat{C}_2,\ \ \dfrac{\partial\hat{v}^+}{\partial x_2} = 0\ \ on\ \ \gamma_u,\\ \hat{u}^+ = \hat{u}^-,\ \ \dfrac{\partial\hat{u}^-}{\partial x_2} - (q-p)\dfrac{\partial\hat{u}^+}{\partial x_2} = \hat{C}_1\ \ on\ \gamma_c,\\ \hat{v}^+ = \hat{v}^-,\ \ (q-p)\dfrac{\partial\hat{v}^+}{\partial x_2} = \dfrac{\partial\hat{v}^-}{\partial x_2}\ \ \ on\ \gamma_c,\\ \hat{u}^- = 0,\ \ \hat{v}^- = 0\ \ on\ \ \gamma_b,\ \ \hat{u},\ \hat{v}\ are\ \gamma_{s'} - periodic,\\ \hat{\theta} = \dfrac{2}{(q-p)}\displaystyle\int_0^1 \hat{v}(x_1,x_2)dx_1,\\ \hat{C}_1 = -\dfrac{1-(q-p)}{\beta}\displaystyle\int_0^1\hat{v}(y,h_1)dy,\ \ \hat{C}_2 = -\dfrac{(q-p)}{\beta}\displaystyle\int_0^1\hat{v}(y,h_2)dy. \end{cases} \tag{5.7}$$

Then, the pair $(\hat{u},\hat{\theta},\hat{C}_1,\hat{C}_2)$ is the optimal solution to (P_1). □

Similarly, we have the following theorem corresponding to (P_2).

Theorem 5.3
Let $f\in L^2(\Omega)$ and $(\overline{u},\ \overline{C_1},\ \overline{C_2})$ be the optimal solution of (P_2). Let $\overline{v}\in V_0(\Omega)$ solves (5.5), then the optimal control is given by

$$\overline{C_1} = -\frac{1-(q-p)}{\beta}\int_0^1 \overline{v}(y,h_1)dy,\ \ \overline{C_2} = -\frac{(q-p)}{\beta}\int_0^1\overline{v}(y,h_2)dy$$

Conversely, assume that a pair $(\hat{u},\hat{v}) \in V_0(\Omega)\times V_0(\Omega)$ solves the optimality system

$$\begin{cases} -\dfrac{\partial^2 \hat{u}^+}{\partial x_2^2} + \hat{u}^+ = f, \ -\dfrac{\partial^2 \hat{v}^+}{\partial x_2^2} + \hat{v}^+ = (\hat{u}^+ - u_d) \ \ in \ \ \Omega^+, \\ -\Delta \hat{u}^- + \hat{u}^- = f, \ -\Delta \hat{v}^- + \hat{v}^- = (\hat{u}^- - u_d) \ \ in \ \Omega^-, \\ \dfrac{\partial \hat{u}^+}{\partial x_2} = \hat{C}_2, \ \dfrac{\partial \hat{v}^+}{\partial x_2} = 0 \ \ on \ \ \gamma_u, \\ \hat{u}^+ = \hat{u}^-, \ \dfrac{\partial \hat{u}^-}{\partial x_2} - (q-p)\dfrac{\partial \hat{u}^+}{\partial x_2} = \hat{C}_1 \ \ on \ \gamma_c, \\ \hat{v}^+ = \hat{v}^-, \ (q-p)\dfrac{\partial \hat{v}^+}{\partial x_2} = \dfrac{\partial \hat{v}^-}{\partial x_2} \ \ on \ \gamma_c, \\ \hat{u}^- = 0, \ \hat{v}^- = 0 \ \ on \ \ \gamma_b, \ \ \hat{u}, \ \hat{v} \ are \ \gamma_{s'} - periodic, \\ \hat{C}_1 = -\dfrac{1-(q-p)}{\beta}\displaystyle\int_0^1 \hat{v}(y,h_1)dy, \ \ \hat{C}_2 = -\dfrac{(q-p)}{\beta}\displaystyle\int_0^1 \hat{v}(y,h_2)dy. \end{cases} \tag{5.8}$$

Then, the pair $(\hat{u},\hat{C}_1,\hat{C}_2)$ is the optimal solution to (P_2). □

5.3. *Convergence theorems*

We now state the main homogenization theorems. We have the following theorem for $\alpha = 1$.

Theorem 5.4

(Critical case $\alpha = 1$) Let $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ and $(\overline{u}, \overline{\theta}, \overline{C_1}, \overline{C_2})$ be the optimal solution of (P_ϵ) with $\alpha = 1$ and of (P_1), respectively, and $\overline{v}_\epsilon, \overline{v}$ be the corresponding adjoint systems given, respectively, by (5.1) and (5.5). Then

$$\begin{aligned} &\widetilde{\overline{u}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup (q-p)\overline{u}|_{\Omega^+} \text{ weakly in } L^2(0,1;H^1(h_1,h_2)), \\ &\widetilde{\overline{v}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup (q-p)\overline{v}|_{\Omega^+} \text{ weakly in } L^2(0,1;H^1(h_1,h_2)), \\ &\overline{u}_\epsilon|_{\Omega^-} \rightharpoonup \overline{u}|_{\Omega^-} \text{ weakly in } H^1(\Omega^-), \\ &\overline{v}_\epsilon|_{\Omega^-} \rightharpoonup \overline{v}|_{\Omega^-} \text{ weakly in } H^1(\Omega^-), \\ &\left\langle \widehat{\overline{\theta}_\epsilon^\epsilon}, \phi \right\rangle \to \langle \Phi, \phi \rangle \end{aligned}$$

for all $\phi \in H^1(\Omega^+)$, $\Phi = \Phi(\overline{\theta}, \overline{C_1}, \overline{C_2})$ and

$$\langle \Phi, \phi \rangle = \int_0^1 \overline{C_1}\phi(x_1,h_1)dx_1 + \int_0^1 \overline{C_2}\phi(x_1,h_2)dx_1 + \int_{\Omega^+} \overline{\theta}\phi(x_1,x_2)dx_1dx_2$$

and

$$\overline{\theta} = \frac{2}{(q-p)}\int_0^1 \overline{v}(x_1,x_2)dx_1, \ \overline{C_1} = -\frac{1-(q-p)}{\beta}\int_0^1 \overline{v}(y,h_1)dy, \ \overline{C_2} = -\frac{(q-p)}{\beta}\int_0^1 \overline{v}(y,h_2)dy.$$

□

Similarly, we have the following theorem for $\alpha > 1$.

Theorem 5.5

(Case $\alpha > 1$) Let $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ and $(\overline{u}, \overline{C_1}, \overline{C_2})$ be the optimal solution of (P_ϵ) with $\alpha > 1$ and (P_2), respectively, and $\overline{v}_\epsilon, \overline{v}$ be the corresponding adjoint systems given, respectively, by (5.1) and (5.5). Then

$$\begin{aligned} &\widetilde{\overline{u}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup (q-p)\overline{u}|_{\Omega^+} \text{ weakly in } L^2(0,1;H^1(h_1,h_2)), \\ &\widetilde{\overline{v}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup (q-p)\overline{v}|_{\Omega^+} \text{ weakly in } L^2(0,1;H^1(h_1,h_2)), \\ &\overline{u}_\epsilon|_{\Omega^-} \rightharpoonup \overline{u}|_{\Omega^-} \text{ weakly in } H^1(\Omega^-), \\ &\overline{v}_\epsilon|_{\Omega^-} \rightharpoonup \overline{v}|_{\Omega^-} \text{ weakly in } H^1(\Omega^-), \\ &\widehat{\overline{\theta}_\epsilon^\epsilon} \rightharpoonup \Phi = \Phi(\overline{C_1}, \overline{C_2}) \text{ weakly in } \left(H^1(\Omega^+)\right)^*, \end{aligned}$$

where

$$\langle \Phi, \phi \rangle = \int_0^1 \overline{C_1}\phi(x_1,h_1)dx_1 + \int_0^1 \overline{C_2}\phi(x_1,h_2)dx_1dx_2$$

and

$$\overline{C_1} = -\frac{1-(q-p)}{\beta}\int_0^1 \overline{v}(y,h_1)dy, \;\; \overline{C_2} = -\frac{(q-p)}{\beta}\int_0^1 \overline{v}(y,h_2)dy. \qquad \square$$

5.4. A priori estimates

Assume that $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ is the optimal solution of (P_ϵ). Let $u_\epsilon(0)$ be the solution of the problem (3.2) corresponding to $\theta = 0$, then we obtain

$$\|u_\epsilon(0)\|_{H^1(\Omega_\epsilon)} \le C, \tag{5.9}$$

where $C > 0$ is independent of ϵ. Using optimality of the solution $(\overline{u}_\epsilon, \overline{\theta}^\epsilon_\epsilon)$, we obtain

$$\int_{\Omega_\epsilon} |\overline{u}_\epsilon - u_d|^2 + \frac{\beta}{2}\int_\gamma |\overline{\theta}_\epsilon|^2 \le \int_{\Omega_\epsilon} |u_\epsilon(0) - u_d|^2 \le C. \tag{5.10}$$

Thus, we have

$$\|\overline{\theta}_\epsilon\|_{L^2(\gamma)} \le C \text{ and } \|\overline{u}_\epsilon\|_{L^2(\Omega_\epsilon)} \le C. \tag{5.11}$$

From the weak formulation of the adjoint problem (5.1), we have

$$\|\overline{v}_\epsilon\|_{H^1(\Omega_\epsilon)} \le C \tag{5.12}$$

where C is independent of ϵ.

Lemma 5.6
For any u_ϵ in $H^1(\Omega_\epsilon)$, there exists constant $C > 0$ independent of ϵ such that

$$\|u_\epsilon\|_{L^2(\gamma_\epsilon\cap\gamma_c)} \le C\|u_\epsilon\|_{H^1(\Omega_\epsilon)},$$
$$\|u_\epsilon\|_{L^2(\gamma_\epsilon\cap\gamma_u)} \le C\|u_\epsilon\|_{H^1(\Omega_\epsilon)}.$$

Proof
By Trace theorem, there exists a positive constant $C > 0$ independent of ϵ such that

$$\|u_\epsilon\|^2_{L^2(\gamma_\epsilon\cap\gamma_c)} = \int_{\gamma_\epsilon\cap\gamma_c} |u_\epsilon|^2 \le \int_{\gamma_c} |u_\epsilon|^2 \le C\|u_\epsilon\|^2_{H^1(\Omega-)} \le C\|u_\epsilon\|^2_{H^1(\Omega_\epsilon)}.$$

Again, by Trace theorem and Hölders inequality, we obtain

$$\begin{aligned}
\|u_\epsilon\|^2_{L^2(\gamma_\epsilon\cap\gamma_u)} = \int_{\gamma_\epsilon\cap\gamma_u} u_\epsilon^2 &= \sum_{k=0}^{m-1}\int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q} (u_\epsilon(x_1,h_2))^2 dx_1 \\
&= \sum_{k=0}^{m-1}\int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q} \left(\int_{g(x_1)}^{h_2} \frac{\partial u_\epsilon}{\partial x_2}(x_1,x_2)dx_2 + u_\epsilon(x_1,g(x_1))\right)^2 dx_1 \\
&\le C\sum_{k=0}^{m-1}\int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q} \left(\left(\int_{g(x_1)}^{h_2} \frac{\partial u_\epsilon}{\partial x_2}(x_1,x_2)dx_2\right)^2 + u_\epsilon^2(x_1,g(x_1))\right) dx_1 \\
&\le C\left(\sum_{k=0}^{m-1}\int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q}\int_{g(x_1)}^{h_2} \left|\frac{\partial u_\epsilon}{\partial x_2}\right|^2 + \|u_\epsilon\|^2_{L^2(\gamma_b)}\right) \\
&\le C\left(\left\|\frac{\partial u_\epsilon}{\partial x_2}\right\|^2_{L^2(\Omega_\epsilon)} + \|u_\epsilon\|^2_{H^1(\Omega-)}\right).
\end{aligned}$$

Thus, we have

$$\|u_\epsilon\|_{L^2(\gamma_\epsilon\cap\gamma_u)} \le C\|u_\epsilon\|_{H^1(\Omega_\epsilon)}. \qquad \square$$

Proposition 5.7
Let $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ be the optimal solution of (P_ϵ). For $\alpha \ge 1$, there exist a positive constant $C > 0$ independent of ϵ such that $\|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)} \le C$.

Proof
Taking $\phi = \overline{u}_\epsilon$ in the variational formulation (3.3), we obtain

$$\|\overline{u}_\epsilon\|^2_{H^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} f\overline{u}_\epsilon + \int_{\gamma_\epsilon} \widehat{\overline{\theta}^\epsilon_\epsilon}\,\overline{u}_\epsilon. \tag{5.13}$$

Using Cauchy–Schwarz inequality, we obtain

$$\int_{\Omega_\epsilon} f\overline{u}_\epsilon \le \|f\|_{L^2(\Omega_\epsilon)}\|\overline{u}_\epsilon\|_{L^2(\Omega_\epsilon)} \le \|f\|_{L^2(\Omega)}\|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)}. \tag{5.14}$$

We now estimate the second term of the right-hand side of (5.13):

$$\begin{aligned}\int_{\gamma_\epsilon} \widehat{\overline{\theta}^\epsilon_\epsilon}\,\overline{u}_\epsilon &\le \int_{\gamma_\epsilon} |\widehat{\overline{\theta}^\epsilon_\epsilon}|\,|\overline{u}_\epsilon| \\ &= \sum_{k=0}^{m-1}\Bigg\{\int_{k\epsilon}^{k\epsilon+\epsilon p} |\widehat{\overline{\theta}^\epsilon_\epsilon}(x_1,h_1)||\overline{u}_\epsilon(x_1,h_1)|dx_1 \\ &\quad + \int_{h_1}^{h_2} |\widehat{\overline{\theta}^\epsilon_\epsilon}(k\epsilon+\epsilon p,x_2)||\overline{u}_\epsilon(k\epsilon+\epsilon p,x_2)|dx_2 \\ &\quad + \int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q} |\widehat{\overline{\theta}^\epsilon_\epsilon}(x_1,h_1)||\overline{u}_\epsilon(x_1,h_1)|dx_1 \\ &\quad + \int_{h_1}^{h_2} |\widehat{\overline{\theta}^\epsilon_\epsilon}(k\epsilon+\epsilon q,x_2)||\overline{u}_\epsilon(k\epsilon+\epsilon q,x_2)|dx_2 \\ &\quad + \int_{k\epsilon+\epsilon q}^{k\epsilon+\epsilon} |\widehat{\overline{\theta}^\epsilon_\epsilon}(x_1,h_2)||\overline{u}_\epsilon(x_1,h_2)|dx_1\Bigg\}\end{aligned}$$

Therefore, by Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\int_{\gamma_\epsilon} \widehat{\overline{\theta}^\epsilon_\epsilon}\,\overline{u}_\epsilon \le \sum_{k=0}^{m-1}\Bigg\{&\epsilon^{1/2}\left(\int_0^p |\overline{\theta}_\epsilon\,(y_1,h_1)\,|^2dy_1\right)^{1/2}\left(\int_{k\epsilon}^{k\epsilon+\epsilon p}|\overline{u}_\epsilon(x_1,h_1)|^2dx_1\right)^{1/2} \\ &+\epsilon^{\alpha}\left(\int_{h_1}^{h_2}|\overline{\theta}_\epsilon(p,y_2)|^2dy_2\right)^{1/2}\left(\int_{h_1}^{h_2}|\overline{u}_\epsilon(k\epsilon+\epsilon p,x_2)|^2dx_2\right)^{1/2} \\ &+\epsilon^{1/2}\left(\int_p^q|\overline{\theta}_\epsilon(y_1,h_2)|^2dy_1\right)^{1/2}\left(\int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q}|\overline{u}_\epsilon(x_1,h_2)|^2dx_1\right)^{1/2} \\ &+\epsilon^{\alpha}\left(\int_{h_1}^{h_2}|\overline{\theta}_\epsilon(q,y_2)|^2dy_2\right)^{1/2}\left(\int_{h_1}^{h_2}|\overline{u}_\epsilon(k\epsilon+\epsilon q,x_2)|^2dx_2\right)^{1/2} \\ &+\epsilon^{1/2}\left(\int_q^1|\overline{\theta}_\epsilon\,(y_1,h_1)\,|^2dy_1\right)^{1/2}\left(\int_{k\epsilon+\epsilon q}^{k\epsilon+\epsilon}|\overline{u}_\epsilon(x_1,h_1)|dx_1\right)^{1/2}\Bigg\} \\ \le\ & \epsilon^{1/2}\|\overline{\theta}_\epsilon\|_{L^2(\gamma)}\sum_{k=0}^{m-1}\Bigg\{\left(\int_{k\epsilon}^{k\epsilon+\epsilon p}|\overline{u}_\epsilon(x_1,h_1)|^2dx_1\right)^{1/2}+\epsilon^{\alpha-1/2}\left(\int_{h_1}^{h_2}|\overline{u}_\epsilon(k\epsilon+\epsilon p,x_2)|^2dx_2\right)^{1/2} \\ &+\left(\int_{k\epsilon+\epsilon p}^{k\epsilon+\epsilon q}|\overline{u}_\epsilon(x_1,h_2)|^2dx_1\right)^{1/2}+\epsilon^{\alpha-1/2}\left(\int_{h_1}^{h_2}|\overline{u}_\epsilon(k\epsilon+\epsilon q,x_2)|^2dx_2\right)^{1/2} \\ &+\left(\int_{k\epsilon+\epsilon q}^{k\epsilon+\epsilon}|\overline{u}_\epsilon(x_1,h_1)|dx_1\right)^{1/2}\Bigg\} \\ \le\ & \epsilon^{1/2}\|\overline{\theta}_\epsilon\|_{L^2(\gamma)}\left(\epsilon^{-1/2}\|\overline{u}_\epsilon\|_{L^2(\gamma_\epsilon\cap\gamma_u)}+\epsilon^{-1/2}\|\overline{u}_\epsilon\|_{L^2(\gamma_\epsilon\cap\gamma_c)}\right) \\ &+\epsilon^{\alpha-1/2}\|\overline{\theta}_\epsilon\|_{L^2(\gamma)}\sum_{k=0}^{m-1}\Bigg\{\left(\int_{k\epsilon}^{k\epsilon+\epsilon}\int_{h_1}^{h_2}|T^\epsilon\overline{u}_\epsilon(x_1,x_2,p)|^2dx_1dx_2\right)^{1/2} \\ &+\left(\int_{k\epsilon}^{k\epsilon+\epsilon}\int_{h_1}^{h_2}|T^\epsilon\overline{u}_\epsilon(x_1,x_2,q)|^2dx_1dx_2\right)^{1/2}\Bigg\} \\ \le&\|\overline{\theta}_\epsilon\|_{L^2(\gamma)}\left(\|\overline{u}_\epsilon\|_{L^2(\gamma_\epsilon\cap\gamma_u)}+\|\overline{u}_\epsilon\|_{L^2(\gamma_\epsilon\cap\gamma_c)}\right) \\ &+\epsilon^{\alpha-1}\|\overline{\theta}_\epsilon\|_{L^2(\gamma)}\left(\|T^\epsilon\overline{u}_\epsilon|_{x_3=p}\|_{L^2(\Omega^+)}+\|T^\epsilon\overline{u}_\epsilon|_{x_3=q}\|_{L^2(\Omega^+)}\right).\end{aligned}$$

By Trace theorem and Proposition 4.2(v)

$$\int_{\gamma_\epsilon} \widehat{\overline{\theta}_\epsilon^{\epsilon}}\, \overline{u}_\epsilon \leq \quad \|\overline{\theta}_\epsilon\|_{L^2(\gamma)} \left(\|\overline{u}_\epsilon\|_{L^2(\gamma_\epsilon \cap \gamma_u)} + \|\overline{u}_\epsilon\|_{L^2(\gamma_\epsilon \cap \gamma_c)} + \epsilon^{\alpha-1} \|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon^+)} \right). \tag{5.15}$$

Therefore, combining (5.11), (5.13), (5.14), (5.15), Lemma 5.6 and because $\alpha \geq 1$, we obtain

$$\|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C.$$

□

6. Proof of theorems

In this section, we give the proof of all the theorem presented in Section 5.

Proof of Theorem 5.1:
We know that (P_ϵ) admits a unique solution by Theorem 3.2 , say $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$, where $\overline{\theta}_\epsilon$ is the optimal control and $\overline{u}_\epsilon$ is the optimal state. For $\theta \in L^2(\gamma)$, let $F(\theta) = J_\epsilon(u_\epsilon(f, \widehat{\theta^\epsilon}), \theta)$.
Because $\overline{\theta}_\epsilon$ is optimal, for any $\mu > 0$, we have

$$\frac{1}{\mu}\left(F(\overline{\theta}_\epsilon + \mu\theta) - F(\overline{\theta}_\epsilon)\right) \geq 0.$$

Now calculate

$$\begin{aligned} F(\overline{\theta}_\epsilon + \mu\theta) - F(\overline{\theta}_\epsilon) &= \frac{1}{2}\int_{\Omega_\epsilon} |u_{\epsilon,\mu} - u_d|^2 + \frac{\beta}{2}\int_\gamma \left|\overline{\theta}_\epsilon + \mu\theta\right|^2 - \frac{1}{2}\int_{\Omega_\epsilon} |\overline{u}_\epsilon - u_d|^2 - \frac{\beta}{2}\int_\gamma \left|\overline{\theta}_\epsilon\right|^2 \\ &= \frac{1}{2}\int_{\Omega_\epsilon} (u_{\epsilon,\mu} - \overline{u}_\epsilon)(u_{\epsilon,\mu} + \overline{u}_\epsilon - 2u_d) + \frac{\beta}{2}\int_\gamma (2\mu\overline{\theta}_\epsilon\theta + \mu^2\theta^2) \end{aligned}$$

where $u_{\epsilon,\mu} = u_\epsilon\left(f, \widehat{\overline{\theta}_\epsilon^{\epsilon}} + \mu\widehat{\theta^\epsilon}\right)$ is the solution of (3.2) with non-homogeneous boundary term $\widehat{\overline{\theta}_\epsilon^{\epsilon}} + \mu\widehat{\theta^\epsilon}$. Note that $w_{\epsilon,\mu} = u_{\epsilon,\mu} - \overline{u}_\epsilon$ is the solution to the equation

$$\begin{cases} -\Delta w + w = 0 \ \text{ in } \Omega_\epsilon, \\ \dfrac{\partial w}{\partial \nu} = \mu\widehat{\theta^\epsilon} \ \text{ on } \gamma_\epsilon, \\ w = 0 \ \text{ on } \gamma_b, \\ w \ \text{ is } \gamma_s\text{-periodic.} \end{cases}$$

Using the continuity of solution operator, we obtain

$$\|w_{\epsilon,\mu}\|_{H^1(\Omega_\epsilon)} \leq C_\epsilon |\mu| \|\widehat{\theta^\epsilon}\|_{L^2(\gamma_\epsilon)}.$$

Thus, $w_{\epsilon,\mu} \to 0$ strongly in $H^1(\Omega_\epsilon)$ as $\mu \to 0$, and hence the sequence $(u_{\epsilon,\mu})_{\mu>0}$ converges to $\overline{u}_\epsilon$ strongly in $H^1(\Omega_\epsilon)$. Set $w_{\widehat{\theta^\epsilon},\epsilon} = \frac{1}{\mu} w_{\epsilon,\mu}$. Notice $w_{\widehat{\theta^\epsilon},\epsilon} \in H^1(\Omega_\epsilon)$ satisfies equation

$$\begin{cases} -\Delta w + w = 0 \ \text{ in } \Omega_\epsilon, \\ \dfrac{\partial w}{\partial \nu} = \widehat{\theta^\epsilon} \ \text{ on } \gamma_\epsilon, \\ w = 0 \ \text{ on } \gamma_b, \\ w \ \text{ is } \gamma_s\text{-periodic.} \end{cases} \tag{6.16}$$

Thus, $w_{\widehat{\theta^\epsilon},\epsilon}$ is independent of μ, and hence

$$0 \leq \lim_{\mu\to 0} \frac{1}{\mu}\left(F(\overline{\theta}_\epsilon + \mu\theta) - F(\overline{\theta}_\epsilon)\right) = \int_{\Omega_\epsilon} (\overline{u}_\epsilon - u_d) w_{\widehat{\theta^\epsilon},\epsilon} + \beta\int_\gamma \overline{\theta}_\epsilon\theta.$$

Hence, $F'(\overline{\theta}_\epsilon)\theta \geq 0$, $\forall\ \theta \in L^2(\gamma)$, which in turn implies that $F'(\overline{\theta}_\epsilon)\theta = 0$, $\ \forall\ \theta \in L^2(\gamma)$. Thus for the optimal solution, we obtain

$$\int_{\Omega_\epsilon^+} (\overline{u}_\epsilon - u_d) w_{\widehat{\theta^\epsilon},\epsilon} = -\beta\int_\gamma \overline{\theta}_\epsilon\theta. \tag{6.17}$$

We now derive the characterization of $\overline{\theta}_\epsilon$. Because $\overline{v}_\epsilon$ satisfies the system (5.1) and $w_{\widehat{\theta^\epsilon},\epsilon}$ satisfies (6.16), we have

$$\int_{\gamma_\epsilon} \overline{v}_\epsilon \widehat{\theta^\epsilon} = \int_{\Omega_\epsilon} (\overline{u}_\epsilon - u_d) w_{\widehat{\theta^\epsilon},\epsilon} = -\beta \int_\gamma \overline{\theta}_\epsilon \theta. \tag{6.18}$$

We know

$$\int_{\gamma_\epsilon} \overline{v}_\epsilon \widehat{\theta^\epsilon} = \int_{\gamma_1^\epsilon} \overline{v}_\epsilon \widehat{\theta^\epsilon} + \int_{\gamma_2^\epsilon} \overline{v}_\epsilon \widehat{\theta^\epsilon} + \int_{\gamma_\epsilon \setminus (\gamma_1^\epsilon \cup \gamma_2^\epsilon)} \overline{v}_\epsilon \widehat{\theta^\epsilon}. \tag{6.19}$$

Using the unfolding operator

$$\begin{aligned}
\int_{\gamma_1^\epsilon} \overline{v}_\epsilon \widehat{\theta^\epsilon} &= \int_{(0,1)\times A_1} T_1^\epsilon(\overline{v}_\epsilon)(x_1,h_1,x_3) T_1^\epsilon(\widehat{\theta^\epsilon})(x_1,h_1,x_3)\, dx_1 dx_3 \\
&= \int_{(0,1)\times A_1} T_1^\epsilon(\overline{v}_\epsilon)(x_1,h_1,x_3) \widehat{\theta^\epsilon}\left(\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon x_3, h_1\right) dx_1 dx_3 \\
&= \int_{(0,1)\times A_1} T_1^\epsilon(\overline{v}_\epsilon)(x_1,h_1,x_3) \theta(x_3,h_1)\, dx_1 dx_3 \\
&= \int_{A_1} \left\{ \int_0^1 T_1^\epsilon(\overline{v}_\epsilon)(x_1,h_1,x_3)\, dx_1 \right\} \theta(x_3,h_1) dx_3.
\end{aligned}$$

Similarly

$$\int_{\gamma_2^\epsilon} \overline{v}_\epsilon \widehat{\theta^\epsilon} = \int_{A_2} \left\{ \int_0^1 T_2^\epsilon(\overline{v}_\epsilon)(x_1,h_2,x_3)\, dx_1 \right\} \theta(x_3,h_2) dx_3,$$

and

$$\begin{aligned}
\int_{\gamma_\epsilon \setminus (\gamma_1^\epsilon \cup \gamma_2^\epsilon)} \overline{v}_\epsilon \widehat{\theta^\epsilon} &= \sum_{k=0}^{m-1} \left\{ \int_{h_1}^{h_2} \overline{v}_\epsilon(k\epsilon + \epsilon p, x_2) \widehat{\theta^\epsilon}(k\epsilon + \epsilon p, x_2) dx_2 \right. \\
&\quad \left. + \int_{h_1}^{h_2} \overline{v}_\epsilon(k\epsilon + \epsilon q, x_2) \widehat{\theta^\epsilon}(k\epsilon + \epsilon q, x_2) dx_2 \right\} \\
&= \frac{\epsilon^\alpha}{\epsilon} \sum_{k=0}^{m-1} \left\{ \int_{h_1}^{h_2} \left(\int_{k\epsilon}^{(k+1)\epsilon} \overline{v}_\epsilon\left(\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon p, x_2\right) dx_1 \right) \theta(p,x_2) dx_2 \right. \\
&\quad \left. + \int_{h_1}^{h_2} \left(\int_{k\epsilon}^{(k+1)\epsilon} \overline{v}_\epsilon\left(\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon q, x_2\right) dx_1 \right) \theta(q,x_2) dx_2 \right\} \\
&= \epsilon^{\alpha-1} \left\{ \int_{h_1}^{h_2} \left(\int_0^1 T^\epsilon \overline{v}_\epsilon(x_1,x_2,p) dx_1 \right) \theta(p,x_2) dx_2 \right. \\
&\quad \left. + \int_{h_1}^{h_2} \left(\int_0^1 T^\epsilon \overline{v}_\epsilon(x_1,x_2,q) dx_1 \right) \theta(q,x_2) dx_2 \right\}.
\end{aligned}$$

Now using (6.18) and (6.19), because θ is arbitrary, we arrive at the characterization of the optimal control $\overline{\theta}_\epsilon$ as in the theorem.

To prove the converse, suppose that $(\hat{u}_\epsilon, \hat{v}_\epsilon) \in H^1_\epsilon \times H^1_\epsilon$ and $\hat{\theta}_\epsilon$ obeys the optimality system (5.2). For $\theta \in L^2(\gamma)$, we have

$$F(\hat{\theta}_\epsilon + \theta) - F(\hat{\theta}_\epsilon) = \frac{1}{2}\int_{\Omega_\epsilon} |u_{\epsilon,1} - \hat{u}_\epsilon|^2 + \frac{\beta}{2}\int_\gamma |\theta_\epsilon|^2 + \int_{\Omega_\epsilon} (u_{\epsilon,1} - \hat{u}_\epsilon)(\hat{u}_\epsilon - u_d) + \beta \int_\gamma \hat{\theta}_\epsilon \theta.$$

where $u_{\epsilon,1} = u_\epsilon\left(f, \widehat{\hat{\theta}_\epsilon}^{\,\epsilon} + \widehat{\theta^\epsilon}\right)$. Observe that

$$\begin{aligned}
\int_{\Omega_\epsilon} (u_{\epsilon,1} - \hat{u}_\epsilon)(\hat{u}_\epsilon - u_d) &= \int_{\Omega_\epsilon} \nabla(u_{\epsilon,1} - \hat{u}_\epsilon) \cdot \nabla \hat{v}_\epsilon + \int_{\Omega_\epsilon} (u_{\epsilon,1} - \hat{u}_\epsilon)\hat{v}_\epsilon + \int_{\partial\Omega_\epsilon} \frac{\partial \hat{v}_\epsilon}{\partial \nu}(u_{\epsilon,1} - \hat{u}_\epsilon) \\
&= \int_{\gamma_\epsilon} \hat{v}_\epsilon \widehat{\theta^\epsilon} = -\beta \int_\gamma \hat{\theta}_\epsilon \theta.
\end{aligned}$$

Hence, $F(\hat{\theta}_\epsilon + \theta) - F(\hat{\theta}_\epsilon) \geq 0$. Thus $(\hat{u}_\epsilon, \hat{\theta}_\epsilon)$ is the optimal solution to (P_ϵ). □

Proof of Theorem 5.4:
We know from Proposition 5.7 that we have

$$\|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)} \le C \tag{6.20}$$

where C is constant independent of ϵ. Let us denote $\overline{u}_\epsilon{}^+$ as the restriction to $\overline{u}_\epsilon$ in Ω_ϵ^+ and $\overline{u}_\epsilon{}^-$ the restriction of $\overline{u}_\epsilon$ to Ω^-.

The sequence $T^\epsilon \overline{u}_\epsilon{}^+$ is bounded in the space $L^2(0,1;H^1((h_1,h_2)\times(p,q)))$. It follows from Proposition 4.2(v) and (6.20) that there exists u_0^+ in $L^2(0,1;H^1((h_1,h_2)\times(p,q)))$ such that up to a subsequence

$$T^\epsilon \overline{u}_\epsilon{}^+ \rightharpoonup u_0^+ \text{ weakly in } L^2(0,1;H^1((h_1,h_2)\times(p,q))). \tag{6.21}$$

From Proposition 4.2(v) and (6.21), it follows that

$$T^\epsilon \overline{u}_\epsilon{}^+ \rightharpoonup u_0^+ \text{ weakly in } L^2(\Omega^+\times(p,q)), \tag{6.22}$$

$$T^\epsilon\left(\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_2}\right) \rightharpoonup \frac{\partial u_0^+}{\partial x_2} \text{ weakly in } L^2(\Omega^+\times(p,q)), \tag{6.23}$$

$$\epsilon T^\epsilon\left(\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}\right) \rightharpoonup \frac{\partial u_0^+}{\partial x_3} \text{ weakly in } L^2(\Omega^+\times(p,q)). \tag{6.24}$$

Again from Proposition 4.2(iv), we have

$$\left\| T^\epsilon \frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}\right\|_{L^2(\Omega^+\times(p,q))} = \left\|\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}\right\|_{L^2(\Omega_\epsilon^+)} \le \|\overline{u}_\epsilon\|_{H^1(\Omega_\epsilon)}$$

which implies the boundedness of the sequence $T^\epsilon\left(\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1}\right)$ in the space $L^2(\Omega^+\times(p,q))$ from (6.20). Hence, there exist an element $P\in L^2(\Omega^+\times(p,q))$ such that

$$T^\epsilon \frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1} \rightharpoonup P \text{ weakly in } L^2(\Omega^+\times(p,q)). \tag{6.25}$$

Thus, from (6.24), it follows that $\frac{\partial u_0^+}{\partial x_3} = 0$, and hence u_0^+ is independent of x_3. Further, using Proposition 4.2(ix)

$$\widetilde{\overline{u}_\epsilon}{}^+ \rightharpoonup \int_p^q u_0^+ dx_3 = (q-p)u_0^+ \text{ weakly in } L^2(0,1;H^1(h_1,h_2)). \tag{6.26}$$

Because $\overline{u}_\epsilon{}^-$ is bounded in $H^1(\Omega^-)$ by (6.20), up to a subsequence (still denoted by ϵ), we obtain

$$\overline{u}_\epsilon{}^- \rightharpoonup u_0^- \text{ weakly in } H^1(\Omega^-). \tag{6.27}$$

Define u_0 as

$$u_0(x) = \begin{cases} u_0^+ & \text{if } x\in\Omega^+, \\ u_0^- & \text{if } x\in\Omega^-. \end{cases} \tag{6.28}$$

It can be proved that $u_0\in V_0(\Omega)$; see the proof of Theorem 5.3 in [8].

Claim: The limit $P=0$. Let $\phi\in\mathcal{D}(\Omega^+)$ and $\eta\in C^\infty[0,1)$ be arbitrary and let $\psi=\eta'$. Now choose the test function

$$\phi^\epsilon(x) = \epsilon\phi(x)\psi\left(\left\{\frac{x_1}{\epsilon}\right\}\right).$$

Note that ϕ^ϵ is continuous in each strip of Ω_ϵ^+, which are disjoint and hence continuous on Ω_ϵ^+. From definition of ϵ-unfolding of ϕ^ϵ and by Proposition 4.2, we obtain

$$
\begin{aligned}
T^\epsilon \phi^\epsilon &= \epsilon\phi\left(\epsilon\left[\frac{x_1}{\epsilon}\right]+\epsilon x_3, x_2\right)\psi(x_3),\\
T^\epsilon\left(\frac{\partial\phi^\epsilon}{\partial x_1}\right) &= \frac{1}{\epsilon}\frac{\partial}{\partial x_3}(T^\epsilon\phi^\epsilon),\\
&= \epsilon\frac{\partial\phi}{\partial x_1}\left(\epsilon\left[\frac{x_1}{\epsilon}\right]+\epsilon x_3, x_2\right)\psi(y_1)+\phi\left(\epsilon\left[\frac{x_1}{\epsilon}\right]+\epsilon x_3, x_2\right)\psi'(x_3),\\
T^\epsilon\left(\frac{\partial\phi^\epsilon}{\partial x_2}\right) &= \epsilon\frac{\partial\phi}{\partial x_2}\left(\epsilon\left[\frac{x_1}{\epsilon}\right]+\epsilon x_3, x_2\right)\psi(x_3).
\end{aligned}
$$

On convergence, as $\epsilon \to 0$, we obtain

$$T^\epsilon\phi^\epsilon \to 0 \text{ in } L^2(\Omega^+\times(p,q)) \tag{6.29}$$

$$T^\epsilon\frac{\partial\phi^\epsilon}{\partial x_1} \to \phi(x_1,x_2)\psi'(x_3) \text{ in } L^2(\Omega^+\times(p,q)) \tag{6.30}$$

$$T^\epsilon\frac{\partial\phi^\epsilon}{\partial x_2} \to 0 \text{ in } L^2(\Omega^+\times(p,q)). \tag{6.31}$$

From the variational formulation (3.3) for $\widehat{\theta^\epsilon} = \widehat{\overline{\theta_\epsilon}^\epsilon}$, we obtain

$$\lim_{\epsilon\to 0}\left(\int_{\Omega_\epsilon}\nabla\overline{u}_\epsilon\cdot\nabla\widetilde{\phi^\epsilon}+\int_{\Omega_\epsilon}\overline{u}_\epsilon\widetilde{\phi^\epsilon}\right) = \lim_{\epsilon\to 0}\left(\int_{\Omega_\epsilon}f\widetilde{\phi^\epsilon}+\int_{\gamma_\epsilon}\widehat{\overline{\theta_\epsilon}^\epsilon}\widetilde{\phi^\epsilon}\right). \tag{6.32}$$

Here, $\widetilde{\phi^\epsilon}$ of ϕ^ϵ to Ω^- by 0. Now notice

$$
\begin{aligned}
\int_{\Omega_\epsilon}\nabla\overline{u}_\epsilon\cdot\nabla\widetilde{\phi^\epsilon}+\int_{\Omega_\epsilon}\overline{u}_\epsilon\widetilde{\phi^\epsilon} &= \int_{\Omega_\epsilon^+}\nabla\overline{u}_\epsilon{}^+\cdot\nabla\phi^\epsilon+\int_{\Omega_\epsilon^+}\overline{u}_\epsilon{}^+\phi^\epsilon\\
&= \int_{\Omega^+\times(p,q)}T^\epsilon\frac{\partial\overline{u}_\epsilon{}^+}{\partial x_1}T^\epsilon\frac{\partial\phi^\epsilon}{\partial x_1}+T^\epsilon\frac{\partial\overline{u}_\epsilon{}^+}{\partial x_2}T^\epsilon\frac{\partial\phi^\epsilon}{\partial x_2}\\
&\quad+\int_{\Omega^+\times(p,q)}T^\epsilon\overline{u}_\epsilon{}^+T^\epsilon\phi^\epsilon\\
&\to\int_{\Omega^+\times(p,q)}P\phi(x_1,x_2)\psi'(x_3)
\end{aligned}
\tag{6.33}
$$

as $\epsilon \to 0$, and

$$
\begin{aligned}
&\int_{\Omega_\epsilon}f\widetilde{\phi^\epsilon}+\int_{\gamma_\epsilon}\widehat{\overline{\theta_\epsilon}^\epsilon}\widetilde{\phi^\epsilon} = \int_{\Omega_\epsilon^+}f\phi^\epsilon+\int_{\gamma_\epsilon\setminus(\gamma_c\cup\gamma_u)}\overline{\theta_\epsilon}^\epsilon\phi^\epsilon\int_{\Omega^+\times(p,q)}T^\epsilon f\,T^\epsilon\phi^\epsilon\\
&+\epsilon^{\alpha+1}\sum_{k=0}^{m-1}\left\{\int_{h_1}^{h_2}\overline{\theta}_\epsilon(p,y_2)\phi(k\epsilon+\epsilon p,y_2)\psi(p)+\int_{h_1}^{h_2}\overline{\theta}_\epsilon(q,y_2)\phi(k\epsilon+\epsilon q,y_2)\psi(q)\right\}\\
&\to 0,\ \text{as } \epsilon\to 0.
\end{aligned}
\tag{6.34}
$$

Combing (6.33)and (6.34), from (6.32) we obtain

$$\int_{\Omega^+\times(p,q)}P\phi(x_1,x_2)\eta(x_3) = 0$$

Because ϕ and η are arbitrary, we obtain $P = 0$ a.e. $(x_1,x_2)\in\Omega^+, x_3\in(p,q)$ and hence the claim.

Similarly, we find the following convergence for the adjoint state $\overline{v}_\epsilon$ described in (5.1).

$$T^\epsilon\left(\overline{v}_\epsilon|_{\Omega_\epsilon^+}\right) \rightharpoonup v_0|_{\Omega^+} \text{ weakly in } L^2(0,1;H^1((h_1,h_2)\times(p,q))), \tag{6.35}$$

$$\widetilde{\overline{v}_\epsilon|_{\Omega_\epsilon^+}} \rightharpoonup (q-p)v_0|_{\Omega^+} \text{ weakly in } L^2(0,1;H^1(h_1,h_2)), \tag{6.36}$$

$$\overline{v}_\epsilon|_{\Omega^-} \rightharpoonup v_0|_{\Omega^-} \text{ weakly in } H^1(\Omega^-), \tag{6.37}$$

where $v_0\in V_0(\Omega)$ satisfies (5.5) for $\overline{u} = u_0$.

Now choose a test function $\phi \in C^\infty(\overline{\Omega})$ such that $\phi|_{\gamma_b} = 0$ in the variational formulation (3.3) for $\widehat{\theta^\epsilon} = \widehat{\overline{\theta}_\epsilon}^\epsilon$. As $\epsilon \to 0$, the left-hand side of (3.3) becomes

$$\begin{aligned}\int_{\Omega_\epsilon} \nabla \overline{u}_\epsilon \cdot \nabla \phi + \overline{u}_\epsilon \phi &= \int_{\Omega^+ \times (p,q)} \left(T^\epsilon \left(\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_1} \right) T^\epsilon \left(\frac{\partial \phi}{\partial x_1} \right) + T^\epsilon \left(\frac{\partial \overline{u}_\epsilon{}^+}{\partial x_2} \right) T^\epsilon \left(\frac{\partial \phi}{\partial x_2} \right) \right) \\ &\quad + \int_{\Omega^+ \times (p,q)} T^\epsilon \overline{u}_\epsilon{}^+ T^\epsilon \phi + \int_{\Omega^-} \nabla \overline{u}_\epsilon{}^- \cdot \nabla \phi + \int_{\Omega^-} \overline{u}_\epsilon{}^- \phi \\ &\to \int_{\Omega^+ \times (p,q)} \left(\frac{\partial u_0^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + u_0^+ \phi \right) + \int_{\Omega^-} \nabla u_0{}^- \cdot \nabla \phi + u_0^- \phi.\end{aligned} \tag{6.38}$$

The right-hand side of (3.3) becomes

$$\int_{\Omega_\epsilon} f\phi + \int_{\gamma_\epsilon} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi = \int_{\Omega_\epsilon^+} f\phi + \int_{\Omega^-} f\phi + \int_{\gamma_\epsilon} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi. \tag{6.39}$$

Using Proposition 4.2(*vi*), we obtain

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon^+} f\phi = \int_{\Omega^+ \times (p,q)} T^\epsilon f \, T^\epsilon \phi = (q-p) \int_{\Omega^+} f\phi. \tag{6.40}$$

Further

$$\int_{\gamma_\epsilon} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi = \int_{\gamma_1^\epsilon} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi + \int_{\gamma_2^\epsilon} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi + \int_{\gamma_\epsilon \setminus (\gamma_1^\epsilon \cup \gamma_2^\epsilon)} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi. \tag{6.41}$$

Now using Proposition 4.4(*iii*) and the characterization of the optimal control $\overline{\theta}_\epsilon$, we obtain

$$\begin{aligned}\int_{\gamma_1^\epsilon} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi &= \int_{(0,1)\times A_1} T_1^\epsilon \left(\widehat{\overline{\theta}}_\epsilon^\epsilon \right) (x_1, h_1, x_3) \, T_1^\epsilon(\phi)(x_1, h_1, x_3) \, dx_1 dx_3 \\ &= \int_{(0,1)\times A_1} \widehat{\overline{\theta}}_\epsilon^\epsilon \left(\epsilon \left[\frac{x_1}{\epsilon} \right] + \epsilon x_3, h_1 \right) T_1^\epsilon(\phi)(x_1, h_1, x_3) \, dx_1 dx_3 \\ &= \int_{(0,1)\times A_1} \overline{\theta}_\epsilon(x_3, h_1) \, T_1^\epsilon(\phi)(x_1, h_1, x_3) \, dx_1 dx_3 \\ &= -\frac{1}{\beta} \int_{(0,1)\times A_1} \left(\int_0^1 T_1^\epsilon(\overline{v}_\epsilon)(y, h_1, x_3) dy \right) T_1^\epsilon(\phi)(x_1, h_1, x_3) dx_1 dx_3.\end{aligned}$$

Also, we have the convergence $\overline{v}_\epsilon^- \rightharpoonup v_0^-$ in $H^1(\Omega^-)$, by Trace theorem $\overline{v}_\epsilon^- \to v_0^-$ in $L^2(\gamma_c)$, and by Proposition 4.4(*iv*), (*v*), we conclude that

$$\begin{aligned}\lim_{\epsilon \to 0} \int_{\gamma_1^\epsilon} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi &= -\frac{1}{\beta} \int_{(0,1)\times A_1} \left(\int_0^1 v_0^-(y, h_1) dy \right) \phi(x_1, h_1) dx_1 dx_3 \\ &= -\frac{1-(q-p)}{\beta} \int_0^1 \left(\int_0^1 v_0^-(y, h_1) dy \right) \phi(x_1, h_1) dx_1.\end{aligned}$$

Similarly

$$\begin{aligned}\int_{\gamma_2^\epsilon} \widehat{\overline{\theta}}_\epsilon^\epsilon \phi &= \int_{(0,1)\times A_2} T_2^\epsilon \left(\widehat{\overline{\theta}}_\epsilon^\epsilon \right) (x_1, h_2, x_3) \, T_2^\epsilon(\phi)(x_1, h_2, x_3) \, dx_1 dx_3 \\ &= \int_{(0,1)\times A_2} \widehat{\overline{\theta}}_\epsilon^\epsilon \left(\epsilon \left[\frac{x_1}{\epsilon} \right] + \epsilon x_3, h_2 \right) T_2^\epsilon(\phi)(x_1, h_2, x_3) \, dx_1 dx_3 \\ &= \int_{(0,1)\times A_2} \overline{\theta}_\epsilon(x_3, h_2) \, T_2^\epsilon(\phi)(x_1, h_2, x_3) \, dx_1 dx_3 \\ &= -\frac{1}{\beta} \int_{(0,1)\times A_2} \left(\int_0^1 T_2^\epsilon(\overline{v}_\epsilon)(y, h_2, x_3) dy \right) T_2^\epsilon(\phi)(x_1, h_2, x_3) dx_1 dx_3.\end{aligned}$$

Because $T^\epsilon(\overline{v}_\epsilon) \rightharpoonup v_0^+$ in $L^2(0,1;H^1((h_1,h_2)\times(p,q)))$ and $T_2^\epsilon(\overline{v}_\epsilon) = T^\epsilon(\overline{v}_\epsilon)|_{\gamma_u}$, we obtain

$$\lim_{\epsilon\to 0}\int_{\gamma_2^\epsilon}\widehat{\overline{\theta}_\epsilon^\epsilon}\phi = -\frac{1}{\beta}\int_{(0,1)\times A_2}\left(\int_0^1 v_0^+(y,h_2)dy\right)\phi(x_1,h_2)dx_1dx_3$$
$$= -\frac{(q-p)}{\beta}\int_0^1\left(\int_0^1 v_0^+(y,h_2)dy\right)\phi(x_1,h_2)dx_1.$$

and

$$\begin{aligned}\int_{\gamma_\epsilon\setminus(\gamma_1^\epsilon\cup\gamma_2^\epsilon)}\widehat{\overline{\theta}_\epsilon^\epsilon}\phi &= \sum_{k=0}^{m-1}\left\{\int_{h_1}^{h_2}\widehat{\overline{\theta}_\epsilon^\epsilon}(k\epsilon+\epsilon p,x_2)\phi(k\epsilon+\epsilon p,x_2)dx_2\right.\\ &\quad\left.+\int_{h_1}^{h_2}\widehat{\overline{\theta}_\epsilon^\epsilon}(k\epsilon+\epsilon q,x_2)\phi(k\epsilon+\epsilon q,x_2)dx_2\right\}\\ &= \sum_{k=0}^{m-1}\epsilon^\alpha\left\{\int_{h_1}^{h_2}\overline{\theta}_\epsilon(p,x_2)\phi(k\epsilon+\epsilon p,x_2)dx_2+\int_{h_1}^{h_2}\overline{\theta}_\epsilon(q,x_2)\phi(k\epsilon+\epsilon q,x_2)dx_2\right\}.\end{aligned}$$

Using the characterization of optimal control in terms of unfolding operator, we obtain

$$\begin{aligned}\int_{\gamma_\epsilon\setminus(\gamma_1^\epsilon\cup\gamma_2^\epsilon)}\widehat{\overline{\theta}_\epsilon^\epsilon}\phi &= \sum_{k=0}^{m-1}\epsilon^{2\alpha-1}\left\{\int_{h_1}^{h_2}\left(\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,x_2,p)dx_1\right)\phi(k\epsilon 1+\epsilon p,x_2)dx_2\right.\\ &\quad\left.+\int_{h_1}^{h_2}\left(\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,x_2,q)dx_1\right)\phi(k\epsilon+\epsilon q,x_2)dx_2\right\}\\ &= \frac{\epsilon^{2\alpha-1}}{\epsilon}\sum_{k=0}^{m-1}\int_{h_1}^{h_2}\left(\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,x_2,p)dx_1\right)\left(\int_{k\epsilon}^{(k+1)\epsilon}T^\epsilon(\phi)(x_1,x_2,p)dx_1\right)dx_2\\ &\quad+\frac{\epsilon^{2\alpha-1}}{\epsilon}\sum_{k=0}^{m-1}\int_{h_1}^{h_2}\left(\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,x_2,q)dx_1\right)\left(\int_{k\epsilon}^{(k+1)\epsilon}T^\epsilon(\phi)(x_1,x_2,q)dx_1\right)dx_2\\ &= \epsilon^{2\alpha-2}\sum_{k=0}^{m-1}\int_{h_1}^{h_2}\left(\int_{k\epsilon}^{(k+1)\epsilon}\left(\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,x_2,p)dx_1\right)T^\epsilon(\phi)(x_1,x_2,p)dx_1\right)dx_2\\ &\quad+\epsilon^{2\alpha-2}\sum_{k=0}^{m-1}\int_{h_1}^{h_2}\left(\int_{k\epsilon}^{(k+1)\epsilon}\left(\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,x_2,q)dx_1\right)T^\epsilon(\phi)(x_1,x_2,q)dx_1\right)dx_2\\ &= \epsilon^{2\alpha-2}\left\{\int_{\Omega+}\left(\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,x_2,p)dx_1\right)T^\epsilon(\phi)(x_1,x_2,p)dx_1dx_2\right.\\ &\quad\left.+\int_{\Omega+}\left(\int_0^1 T^\epsilon(\overline{v}_\epsilon)(x_1,x_2,q)dx_1\right)T^\epsilon(\phi)(x_1,x_2,q)dx_1dx_2\right\}.\end{aligned} \tag{6.42}$$

Now, we consider the case $\alpha = 1$ in (6.42), and passing to the limit $\epsilon\to 0$, we obtain

$$\begin{aligned}\int_{\gamma_\epsilon\setminus(\gamma_1^\epsilon\cup\gamma_2^\epsilon)}\widehat{\theta_\epsilon^\epsilon}\phi \to &\left\{\int_{\Omega+}\left(\int_0^1 v_0(x_1,x_2)dx_1\right)\phi(x_1,x_2)dx_1dx_2\right.\\ &\left.+\int_{\Omega+}\left(\int_0^1 v_0(x_1,x_2)dx_1\right)\phi(x_1,x_2)dx_1dx_2\right\}.\end{aligned} \tag{6.43}$$

Therefore, we obtain the following limit equation:

$$\left\{\begin{array}{l}(q-p)\displaystyle\int_{\Omega+}\left(\frac{\partial u_0^+}{\partial x_2}\frac{\partial\phi}{\partial x_2}+u_0^+\phi\right)+\int_{\Omega-}\left(\nabla u_0^-\cdot\nabla\phi+u_0^-\phi\right) = (q-p)\int_{\Omega+}f\phi+\int_{\Omega-}f\phi\\ -\displaystyle\int_0^1\left(\frac{1-(q-p)}{\beta}\int_0^1 v_0^-(y,h_1)dy\right)\phi(x_1,h_1)dx_1-\int_0^1\left(\frac{(q-p)}{\beta}\int_0^1 v_0^+(y,h_2)dy\right)\phi(x_1,h_2)dx_1\\ +(q-p)\displaystyle\int_{\Omega+}\left(\frac{2}{(q-p)}\int_0^1 v_0(x_1,x_2)dx_1\right)\phi(x_1,x_2)dx_1dx_2\end{array}\right.$$

for all $\phi \in C^\infty(\overline{\Omega})$ with $\phi|_{\gamma_b} = 0$, and hence true for all ψ in $V_0(\Omega)$ by density. Therefore, u_0 satisfies the differential equation (5.3) for $j = 1$ with $\theta = \theta^0,\ C_1 = C_1^0,\ C_2 = C_2^0$, where

$$\theta^0 = \frac{2}{(q-p)} \int_0^1 v_0(x_1, x_2)dx_1,$$
$$C_1^0 = -\frac{1-(q-p)}{\beta} \int_0^1 v_0^-(y, h_1)dy,\ C_2^0 = -\frac{(q-p)}{\beta} \int_0^1 v_0^+(y, h_2)dy.$$

Therefore, we obtain the optimality system corresponding to the minimization problem (P_1). According to Theorem 5.2, the optimal solution is given by $(u_0, \theta^0, C_1^0, C_2^0)$. Thus, by uniqueness, we have

$$\overline{u} = u_0,\ \ \overline{v} = v_0 \ \text{ and } \overline{\theta} = \theta^0,\ \overline{C_1} = C_1^0,\ \overline{C_2} = C_2^0.$$

This completes the proof of Theorem 5.4. □

Proof of the Theorem 5.5 :
The proof is similar till the equation (6.42). Now, take $\alpha > 1$ in (6.42) and pass to the limit $\epsilon \to 0$, to obtain

$$\int_{\gamma_\epsilon \setminus (\gamma_1^\epsilon \cup \gamma_2^\epsilon)} \widehat{\overline{\theta}_\epsilon^\epsilon} \phi \to 0. \tag{6.44}$$

Thus, we have the following limit problem:

$$\begin{cases} (q-p)\displaystyle\int_{\Omega+} \left(\frac{\partial u_0^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + u_0^+ \phi \right) + \int_{\Omega-} \left(\nabla u_0^- \cdot \nabla \phi + u_0^- \phi \right) = (q-p) \int_{\Omega+} f\phi + \int_{\Omega-} f\phi \\ -\displaystyle\int_0^1 \left(\frac{1-(q-p)}{\beta} \int_0^1 v_0^-(y, h_1)dy \right) \phi(x_1, h_1)dx_1 - \int_0^1 \left(\frac{(q-p)}{\beta} \int_0^1 v_0^+(y, h_2)dy \right) \phi(x_1, h_2)dx_1 \end{cases}$$

which is true for all ψ in $V_0(\Omega)$. Hence u_0 satisfies (5.3) for $j = 0$ with $C_1 = C_1^0,\ C_2 = C_2^0$, where

$$C_1^0 = -\frac{1-(q-p)}{\beta} \int_0^1 v_0^-(y, h_1)dy,\ C_2^0 = -\frac{(q-p)}{\beta} \int_0^1 v_0^+(y, h_2)dy.$$

Using Theorem 5.3, the optimal solution is given by (u_0, C_1^0, C_2^0). Thus, by uniqueness, we have

$$\overline{u} = u_0,\ \ \overline{v} = v_0,\ \ \overline{C_1} = C_1^0,\ \overline{C_2} = C_2^0.$$

Hence the proof. □

7. Conclusion and remarks

In this article, we have characterized optimal control in terms of the unfolding operator. This characterization is then used to study the limiting behavior of the optimality system and adjoint state. Finally, we have shown that the limit is indeed the optimal solution to the appropriate limit problems. There are two types of limiting problems corresponding to the cases $\alpha > 1$ and $\alpha = 1$ (critical case). Here, we would like to remark that there are three controls appearing in the limiting problem, namely, an interior control, boundary control at the upper part of Ω, and interface control between Ω^+ and Ω^-. In fact, one can see that these three controls are, respectively, the contribution from three different parts of the controls on γ_ϵ, namely, $\gamma_\epsilon \cap \Omega^+,\ \gamma_\epsilon \cap \gamma_u$, and $\gamma_\epsilon \cap \gamma_c$.

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