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Optimal control problem in a domain with branched structure and homogenization

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We consider a linear parabolic problem in a thick junction domain which is the union of a fixed domain and a collection of periodic branched trees of height of order 1 and small width connected on a part of the boundary. We consider a three-branched structure, but the analysis can be extended to n-branched structures. We use unfolding operator to study the asymptotic behavior of the solution of the problem. In the limit problem, we get a multi-sheeted function in which each sheet is the limit of restriction of the solution to various branches of the domain. Homogenization of an optimal control problem posed on the above setting is also investigated. One of the novelty of the paper is the characterization of the optimal control via the appropriately defined unfolding operators. Finally, we obtain the limit of the optimal control problem. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: optimal control; asymptotic analysis; unfolding operator; domain with branched structures; homogenization

1. Introduction

In this article, we consider a parabolic problem in a thick junction domain $\Omega_\epsilon, \epsilon > 0$, a small parameter, and also the corresponding optimal control problem. Various materials with complex structures including multi-layer thick junctions are widely used in many fields of science. Such structures are usually known as complex structures because of its complexity both in construction and analysis. Other complex structures are perforated domains, composite materials, grid domains, and domains with oscillating boundaries to name a few.

As mentioned earlier, constructions with thick junction (also multi-level) are used in many technologies, like microstrip radiator, nano technologies ([1, 2]), biological systems, fractal-type constructions, etc. Studying PDE problems in such complex structures has paramount importance. We refer to the work in [3–6] and the references therein for the study in multi-branched structures. Although the importance of optimal control may be at the junctions, we consider the controls on the entire oscillating part from which we can also understand the contribution from each branch at each level. One can apply need based controls at the appropriate junctions.

The domain Ω_ϵ under consideration consists of multiple layer thick junctions known as branched structure (Figure 1). Such a domain has a fixed part and lot of thin periodically distributed parts (or handle trees) attached along certain part of the boundary of the domain at different levels. The trees have finite number of branching levels and in this paper, we take three branching levels, but one can consider any finite number of branches. The height of each branch is of $O(1)$, whereas the thickness is of $O(\epsilon)$. We consider the domain in two-dimensional space. Such a domain has already been considered by Mel'nyk ([6]). Indeed the results can be extended to three dimensional problem and higher dimensions as well. Asymptotic analysis for a Robin problem in a thick junction has been investigated in [5]. In fact, our work is motivated from the work of Mel'nyk, where he considers a semi-linear parabolic problem with the source term vanishes on the oscillating interior part. He has studied the problem using the method of asymptotic expansion and derived appropriate error estimates. From the recent work on problems on oscillating boundary domains ([7–9]), in which we have studied optimal control problems with Elliptic PDE using periodic unfolding operators, we learned that the method of unfolding is well suited to study problems on branched structures. In addition to the study of the PDE and its homogenization using unfolding method, our major interest is to study the control problem with controls applied to the oscillating part. Thus, we also need to consider the PDE problem with non-vanishing source on the oscillating part. The approach using unfolding method is very useful to study control problems when the controls are applied in the oscillating interior part.

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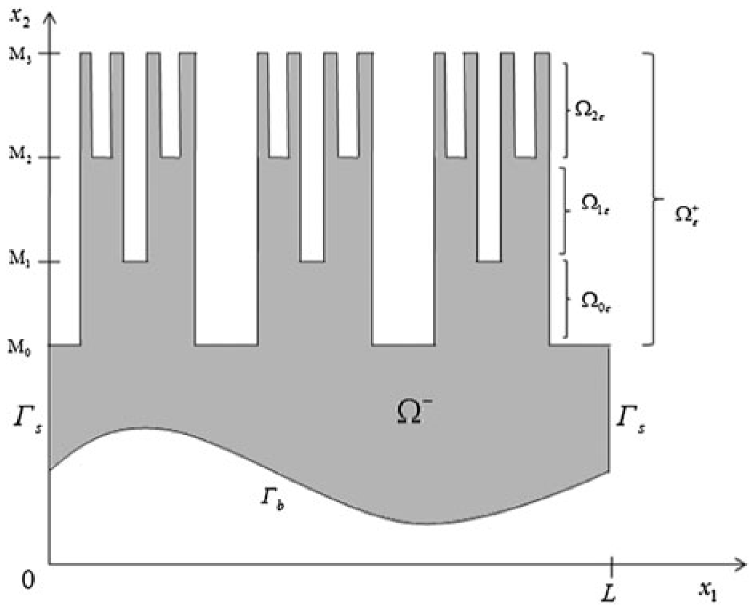

Figure 1. Domain Ω_ϵ.

Homogenization of a Laplace equation in a highly oscillating boundary domain is studied in [10]. Using boundary layer corrector, the authors obtain the approximation of the homogenized solution of order $\epsilon^{3/2}$ in H^1 norm in a subset of the oscillating domain. In [11, 12], the authors have studied homogenization of PDEs posed on the oscillating boundary domains using Tartar's Oscillating test functions method. In [13], Mel'nyk and Vashchuk have studied the homogenization of Poisson equation on a thick two level junction with varying boundary conditions. In [14, 15], he has derived H^1 norm estimates for the homogenized solutions of elliptic and parabolic type PDEs. In [16–19], asymptotic analysis of optimal control problems posed on various PDEs were investigated using oscillating test functions method. In [3, 4], the authors have used the Buttazzo–Dal Maso abstract scheme for variational convergence of constrained minimization problems to study the asymptotic analysis of optimal control problem in thick multi-level junctions. There are more articles which deals with the homogenization of PDEs posed on an oscillating boundary domain. For example, homogenization of a p-Laplacian operator is discussed in [20] using Γ-convergence; Laplace equation with inhomogeneous Neumann boundary condition is studied in [21] and in [22]; the authors have used extension operators for the homogenization of Ginzburg–Landau equation.

In the first part of this paper, we consider a linear parabolic problem and study the limiting analysis (homogenization) using the method of unfolding. We introduce appropriate unfolding operators for each periodic tree at all levels and also suitable boundary unfolding operators. The unfolding operator was first introduced by D. Cioranescu, A. Damlamian, and G. Griso in 2002 ([23]). The method of unfolding is a well-developed method in homogenization, and it is widely used by various authors ([24–26]) in the study of homogenization problems. The authors D. Blanchard, A. Gaudiello, and G. Griso have first used the unfolding method for homogenization of oscillating boundaries in [27, 28] for elastic rods with a 3D plane as well as thin plate. In [29], strongly contrasting diffusivity problem in highly oscillating boundaries has been studied. For general homogenization, see the books [30–32] and control, we refer to [33–37]. In the second part of this article, we consider an associated optimal control problem with quadratic cost functional in the multi-level thick junction domain. One of our main aim is to characterize the optimal control via the unfolding operator. In fact, we can characterize the optimal control in each branch separately by introducing corresponding unfolding operators. This, we consider as a novel approach. Then, we study the homogenization and obtain the limit (homogenized) equation. It is to be noted that we consider the controls on the oscillating branched part of the domain which are periodically distributed. Interestingly, the limit is given by a multi-sheeted function through which we can see the contribution of different branches at different levels. The asymptotic expansion method may be perhaps too tedious to study such problems, at least we do not see how to characterize the controls and hence a homogenization.

A brief layout of the article as follows. In Section 2, we describe the domain Ω_ϵ, the problem description in Section 3, and the unfolding operators with properties in Section 4. The convergence analysis of the uncontrolled problem is studied in Section 5. The solution belongs to a multi-sheeted function space introduced by Mel'nyk in [6]. We use the same function space. In Section 6, we study the corresponding optimal control problem. The characterization of the optimal control, optimality system, limit analysis, etc., are carried out through various subsections.

2. Multi-branched oscillating domain

Let $L > 0$ and for a small parameter $\epsilon = \frac{L}{N}, N \in \mathbb{Z}^+$, we consider an oscillating domain Ω_ϵ as given in the Figure 1. We now describe the domain Ω_ϵ and its boundaries. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth and periodic function with period L. Again, let $0 < b_1 < b_2 < \cdots < b_8 < L$

and η be a function defined on $[0, L]$ as

$$\eta(x_1) = \begin{cases} M_0 & \text{if } x_1 \in]0, L[\setminus]b_1, b_8[, \\ M_1 & \text{if } x_1 \in [b_4, b_5], \\ M_2 & \text{if } x_1 \in [b_2, b_3] \cup [b_6, b_7], \\ M_3 & \text{if } x_1 \in \bigcup_{i=1}^{4}]b_{2i-1}, b_{2i}[. \end{cases}$$

with $M_3 > M_2 > M_1 > M_0 > \mathfrak{m}$, where $\mathfrak{m} > \max_{x \in [0,L]} |g(x)|$. Extend η to the whole real line periodically with period L.

Now, define η_ϵ on the cell $[0, L]$ by $\eta_\epsilon(x_1) = \eta(\frac{x_1}{\epsilon})$, which is an ϵL-periodic function. We define the domain Ω_ϵ as, $\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < L, g(x_1) < x_2 < \eta_\epsilon(x_1)\}$. Note that η_ϵ describes the periodic and oscillatory part of the domain with multiple sheets.

Let the top part Ω_ϵ^+ and the bottom (fixed) part Ω^- of the domain Ω_ϵ are, respectively defined as $\Omega_\epsilon^+ = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < L, M_0 < x_2 < \eta_\epsilon(x_1)\}$ and $\Omega^- = \{(x_1, x_2) : 0 < x_1 < L, g(x_1) < x_2 < M_0\}$.

Let us define the reference intervals $Y_{i,m}$ for $i = 0, 1, 2$ and $m = 1, \cdots, 2^i$ as follows: $Y_{0,1} =]b_1, b_8[$, $Y_{1,1} =]b_1, b_4[$, $Y_{1,2} =]b_5, b_8[$ and $Y_{2,m} =]b_{2m-1}, b_{2m}[$ for $m = 1, 2, 3, 4$ and $h_{i,m} = |Y_{i,m}|$ is the Lebesgue measure of $Y_{i,m}$.

Now, we define $\Omega_{i,\epsilon}$ for $i = 0, 1, 2$ as,

$$\Omega_{i,\epsilon} = \{(x_1, x_2) \in \Omega_\epsilon^+ : M_i < x_2 < M_{i+1}\}.$$

In other words, $\Omega_{i,\epsilon}$ can be considered as the union of thin sticks at level i, namely

$$\Omega_{i,\epsilon} = \bigcup_{m=1}^{2^i} \Omega_{i,m}^\epsilon, \text{ where } \Omega_{i,m}^\epsilon = \bigcup_{j=0}^{N-1} D_{i,m}^{j,\epsilon} \quad \text{where} \quad D_{i,m}^{j,\epsilon} = (j\epsilon L + \epsilon Y_{i,m}) \times]M_i, M_{i+1}[.$$

We can also write Ω_ϵ as $\Omega_\epsilon = Int\left(\overline{\Omega^- \cup \Omega_{0,\epsilon} \cup \Omega_{1,\epsilon} \cup \Omega_{2,\epsilon}}\right)$.

Our full domain denoted by Ω (Figure 2) is given by $\Omega = \{(x_1, x_2) : 0 < x_1 < L, g(x_1) < x_2 < M_3\}$. The domain Ω consists of four parts $\Omega^-, \Omega_0, \Omega_1, \Omega_2$, where $\Omega_i = \{(x_1, x_2) : 0 < x_1 < L, M_i < x_2 < M_{i+1}\}$ for $i = 0, 1, 2$. In fact $\Omega = Int\left(\overline{\Omega^- \cup \Omega_0 \cup \Omega_1 \cup \Omega_2}\right)$.

The bottom boundary Γ_b of Ω_ϵ is defined as $\Gamma_b = \{(x_1, x_2) : x_2 = g(x_1), x_1 \in [0, L]\}$. The vertical boundary of Ω^- denoted by Γ_s is given by $\Gamma_s = \{(0, x_2) : g(0) \le x_2 \le M_0\} \cup \{(L, x_2) : g(L) \le x_2 \le M_0)\}$ and define the boundary Γ_ϵ as $\partial\Omega_\epsilon \setminus \Gamma_s$, which the oscillating boundary. The common boundaries $\Gamma_{i,\epsilon}$ are the lower boundaries of $\Omega_{i,\epsilon}$ which are defined as $\Gamma_{i,\epsilon} = \{(x_1, x_2) \in \overline{\Omega_\epsilon} : x_2 = M_i\}$ *for* $i = 0, 1, 2$.

The bottom part of the boundary of Ω is same as Ω_ϵ which is Γ_b. The top boundary of Ω is denoted by $\Gamma_u = \{(x_1, M_3) : 0 \le x_1 \le L\}$. The boundaries Γ_i are defined as $\Gamma_i = \{(x_1, M_i) : 0 \le x_1 \le L\}$ for $i = 0, 1, 2$. The vertical boundary of Ω is denoted by $\Gamma_{s'}$ and can be written as $\Gamma_{s'} = \{(0, x_2) : g(0) \le x_2 \le M_3\} \cup \{(L, x_2) : g(L) \le x_2 \le M_3\}$ and define the boundary Γ as $\partial\Omega \setminus \Gamma_{s'}$.

The reference domain Λ^+ (see second figure in Figure 2) is defined as $\Lambda^+ = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < L, M_0 < x_2 < \eta(x_1)\}$, and we denote $\Lambda_T^+ =]0, T[\times \Lambda^+$, $\Lambda_i =]0, L[\times]M_i, M_{i+1}[$ and $\Lambda_{i,m} = Y_{i,m} \times]M_i, M_{i+1}[$.

Let $H_\#^1(\Omega_\epsilon) = \{u \in H_\#^1(\Omega_\epsilon) : u(x_1 + kL, x_2) = u(x_1, x_2)\ \forall\, k \in \mathbb{Z}\}$. We call a function Γ_s-periodic if it takes the same value on both sides of Γ_s. We denote
$Q_\epsilon =]0, T[\times \Omega_\epsilon$, $Q_\epsilon^+ =]0, T[\times \Omega_\epsilon^+$, $Q_{i,m}^\epsilon =]0, T[\times \Omega_{i,m}^\epsilon$, $Q_{i,\epsilon} =]0, T[\times \Omega_{i,\epsilon}$, $Q_i =]0, T[\times \Omega_i$ and $Q =]0, T[\times \Omega$.

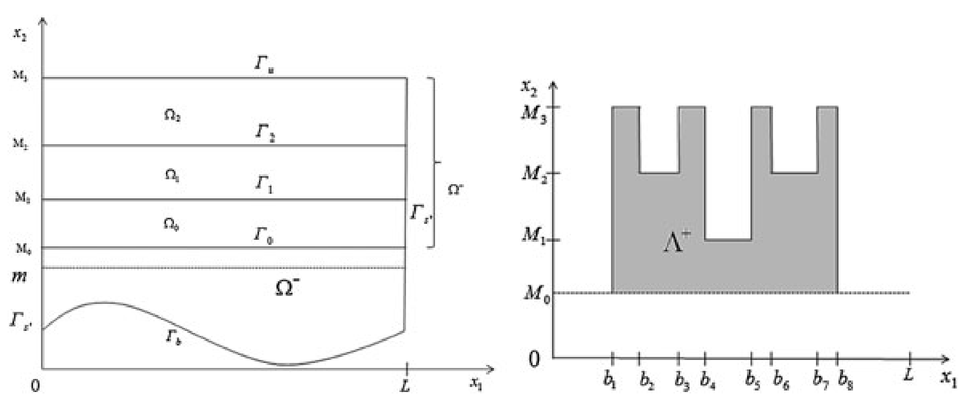

Figure 2. Ω and Λ^+.

3. Problem description

3.1. Linear parabolic problem

First, we consider the following linear parabolic problem with given $f_\epsilon \in L^2(Q_\epsilon)$;

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon + u_\epsilon = f_\epsilon \quad \text{in }]0,T[\times\Omega_\epsilon, \\ \partial_\nu u_\epsilon = 0 \quad \text{on }]0,T[\times\Gamma_\epsilon, \quad u_\epsilon(0,x) = 0 \text{ in } \Omega_\epsilon, \quad u_\epsilon \textit{ is } \Gamma_s - \textit{periodic}. \end{cases} \tag{3.1}$$

Remark 3.1
Here after, whenever we use the subscripts i and m, it will always mean that the corresponding statement will hold for $i = 0, 1, 2$ and $m = 1, \cdots, 2^i$ unless otherwise specified.

Recall that a function $u_\epsilon \in W(0,T; H^1_\#(\Omega_\epsilon), H^1_\#(\Omega_\epsilon)^*) =: \{\varphi \in L^2(0,T; H^1_\#(\Omega_\epsilon)) |\ \varphi' \in L^2(0,T;(H^1_\#(\Omega_\epsilon))^*)\}$, is a weak solution to the problem (3.1) if

$$\int_{Q_\epsilon} u'_\epsilon \psi + \int_{Q_\epsilon} \nabla_x u_\epsilon \cdot \nabla_x \psi + \int_{Q_\epsilon} u_\epsilon \psi = \int_{Q_\epsilon} f_\epsilon \psi, \quad \forall\ \psi \in L^2(0,T; H^1_\#(\Omega_\epsilon)). \tag{3.2}$$

Equivalently (ref [38]), $-\int_{Q_\epsilon} u_\epsilon \psi' + \int_{Q_\epsilon} \nabla_x u_\epsilon \cdot \nabla_x \psi + \int_{Q_\epsilon} u_\epsilon \psi = \int_{Q_\epsilon} f_\epsilon \psi$, for all $\psi \in C^1([0,T]; H^1_\#(\Omega_\epsilon))$ with $\psi(T) = 0$. For fixed $\epsilon > 0$, there exists a unique solution u_ϵ, and it is known that $u_\epsilon \in C([0,T]; L^2(\Omega_\epsilon))$ and thus the equality $u_\epsilon|_{t=0} = 0$ makes sense. A priori estimate of the linear parabolic problem gives

$$\|u_\epsilon{}'\|_{L^2(0,T;H^1_\#(\Omega_\epsilon)^*)} + \|u_\epsilon\|_{L^2(0,T;H^1_\#(\Omega_\epsilon))} \leq C\|f_\epsilon\|_{L^2(]0,T[\times\Omega_\epsilon)}. \tag{3.3}$$

In fact, it is not difficult to see that the constant C is independent of ϵ.

3.2. An optimal control problem

In Section 6, we consider a corresponding optimal control problem which is described in the succeeding text. We consider the controls coming from the fixed reference cell Λ_T^+ and periodically distributed. This may be useful in applications. Let us consider the control problem:

$$(P_\epsilon) \quad \inf\{J_\epsilon(u_\epsilon,\theta) | (u_\epsilon,\theta) \in W(0,T; H^1(\Omega_\epsilon), H^1(\Omega_\epsilon)^*) \times L^2_\#(\Lambda_T^+),\ (u_\epsilon, \theta^\epsilon) \text{ satisfies (3.5)}\}$$

where, $W(0,T; H^1_\#(\Omega_\epsilon), H^1_\#(\Omega_\epsilon)^*) =: \{\varphi \in L^2(0,T; H^1_\#(\Omega_\epsilon)) |\ \varphi' \in L^2(0,T;(H^1_\#(\Omega_\epsilon))^*)\}$. The cost functional $J_\epsilon(u_\epsilon,\theta)$ is defined as

$$J_\epsilon(u_\epsilon,\theta) = \frac{1}{2}\int_{Q_\epsilon} |u_\epsilon - u_d|^2 + \frac{1}{2}\int_{\Omega_\epsilon} |u_\epsilon(T) - u_d(T)|^2 + \frac{\beta}{2}\int_{Q_\epsilon} \chi_{Q_\epsilon^+} |\theta^\epsilon|^2 \tag{3.4}$$

with $\theta^\epsilon(t,x_1,x_2) = \theta(t,\frac{x_1}{\epsilon},x_2)$ and the desired state $u_d \in L^2(Q)$. Given $\theta \in L^2(\Lambda_T^+)$ and $f_0 \in L^2(Q_\epsilon)$ with $f_0 = 0$ in Q_ϵ^+, the function u_ϵ satisfies the state equation:

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon + u_\epsilon = f_0 + \chi_{Q_\epsilon^+}\theta^\epsilon \quad \text{in }]0,T[\times\Omega_\epsilon, \\ \partial_\nu u_\epsilon = 0 \quad \text{on }]0,T[\times\Gamma_\epsilon, \quad u_\epsilon(0,x) = 0 \quad \text{in } \Omega_\epsilon, \quad u_\epsilon \quad \textit{is } \Gamma_s - \textit{periodic}. \end{cases} \tag{3.5}$$

4. Unfolding operators and its properties

In this section, we define the periodic unfolding operators and the boundary unfolding operators and study some of their properties. Unfolding operator is the main tool which we use to study the asymptotic behavior of the solution. The idea of introducing unfolding operator in 2^i periodic components separately for the i^{th} branch is to derive the limiting contribution separately from each component. It is, however, possible to introduce a single unfolding operator for each branch. For $x \in \mathbb{R}$, we write $[x]_L$ as the integer part of x with respect to L, that is, $[x]_L = kL$, where k is the largest integer such that $kL \leq x$ and $\{x\}_L = x - [x]_L$.

Definition 4.1
Define $\phi^\epsilon_{i,m} : Q_i \times Y_{i,m} \to Q_{i,\epsilon}$ by $\phi(t,x_1,x_2,y) = \left(t, \epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y, x_2\right)$, where $i = 0, 1, 2$ and $m = 1, \cdots, 2^i$. The ϵ-unfolding of a function $u : Q_{i,\epsilon} \to \mathbb{R}$ is the function $u \circ \phi^\epsilon_{i,m} : Q_i \times Y_{i,m} \to \mathbb{R}$. The operator which maps every function $u : Q_{i,\epsilon} \to \mathbb{R}$ to its ϵ-unfolding is called the unfolding operator. Let the unfolding operator be denoted by $T^\epsilon_{i,m}$, that is,

$$T^\epsilon_{i,m} : \{u : Q_{i,\epsilon} \to \mathbb{R}\} \to \{v : Q_i \times Y_{i,m} \to \mathbb{R}\} \quad \text{defined by } T^\epsilon_{i,m} u(t,x_1,x_2,y) = u\left(t, \epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y, x_2\right).$$

Here $T^\epsilon_{i,m}$ is an unfolding operator corresponding to the m^{th} component of the i^{th} branch (or i^{th} level). If U is an open subset of $\mathbb{R}^3$ containing $Q^\epsilon_{i,m}$ and u is real valued function on U, $T^\epsilon_{i,m} u$ will mean $T^\epsilon_{i,m}$ acting on the restriction of u to $Q^\epsilon_{i,m}$. We derive the following properties of the unfolding operator $T^\epsilon_{i,m}$. We only sketch the proofs as it is similar to the one in [27, 28].

Proposition 4.1
Let $T^\epsilon_{i,m}$ be the unfolding operators given as in the Definition 4.1. Then

1. $T^\epsilon_{i,m}$ is linear and $T^\epsilon_{i,m}(uv) = T^\epsilon_{i,m}(u) T^\epsilon_{i,m}(v)$, where u,v : $Q_{i,\epsilon} \to \mathbb{R}$.

2. $\int\limits_{Q_i\times Y_{i,m}} T^{\epsilon}_{i,m}u \, dxdydt = L \int\limits_{Q^{\epsilon}_{i,m}} u \, dxdt$, where $u \in L^1(Q^{\epsilon}_{i,m})$.

Proposition 4.2

Let $u \in L^2(Q_{i,\epsilon})$. Then $T^{\epsilon}_{i,m}u \in L^2(Q_i \times Y_{i,m})$ and $\|T^{\epsilon}_{i,m}u\|_{L^2(Q_i\times Y_{i,m})} = \sqrt{L}\|u\|_{L^2(Q^{\epsilon}_{i,m})}$

Proof

Given that $u \in L^2(Q_{i,\epsilon})$ implies $|u|^2 \in L^1(Q_{i,\epsilon})$. From the Proposition 4.1, we obtain

$$\int\limits_{Q_i\times Y_{i,m}} |T^{\epsilon}_{i,m}u|^2 = \int\limits_{Q_i\times Y_{i,m}} T^{\epsilon}_{i,m}|u|^2 = L\int\limits_{Q^{\epsilon}_{i,m}} |u|^2.$$

□

Notation: Define the spaces $\mathcal{X}_i, \mathcal{X}, \mathcal{Z}_{i,m}$ as $\mathcal{X}_i = L^2(0,L;L^2(0,T;H^1_{\#}(M_i,M_{i+1})))$, $\mathcal{X} = L^2(0,L;L^2(0,T;H^1_{\#}(\Omega^-)))$ and $\mathcal{Z}_{i,m} = L^2(0,L;L^2(0,T;H^1((M_i,M_{i+1})\times Y_{i,m})))$ for $i = 0,1,2$ and $m = 1,\cdots,2^i$.

Proposition 4.3

Let $u \in L^2(0,T;H^1(\Omega^{\epsilon}_{i,m}))$. Then $T^{\epsilon}_{i,m}u \in \mathcal{Z}_{i,m}$. Moreover $\partial_{x_2}(T^{\epsilon}_{i,m}u) = T^{\epsilon}_{i,m}(\partial_{x_2}u)$ and $\partial_y(T^{\epsilon}_{i,m}u) = \epsilon T^{\epsilon}_{i,m}(\partial_{x_1}u)$.

Proof

$$\begin{aligned}\|T^{\epsilon}_{i,m}u\|^2_{\mathcal{Z}_{i,m}} &= \int_0^L\int_0^T \|T^{\epsilon}_{i,m}u\|^2_{H^1((M_i,M_{i+1})\times Y_{i,m}}\, dx_1dt \\ &= \int\limits_{Q_i\times Y_{i,m}} \left(\epsilon^2 T^{\epsilon}_{i,m}|\partial_{x_1}u|^2 + T^{\epsilon}_{i,m}|\partial_{x_2}u|^2 + T^{\epsilon}_{i,m}|u|^2\right)\, dxdydt \\ &= L\int\limits_{Q^{\epsilon}_{i,m}} \left(\epsilon^2|\partial_{x_1}u|^2 + |\partial_{x_2}u|^2 + |u|^2\right)\, dxdt \;\le\; L\|u\|^2_{L^2(0,T;H^1(\Omega^{\epsilon}_{i,m}))} < \infty.\end{aligned}$$

□

The following propositions are trivial or easy to prove.

Proposition 4.4

Let $u \in L^2(Q_i)$. Then $T^{\epsilon}_{i,m}u \to u$ in $L^2(Q_i \times Y_{i,m})$. More generally, if $u_\epsilon \to u$ in $L^2(Q_i)$, then $T^{\epsilon}_{i,m}u_\epsilon \to u$ in $L^2(Q_i \times Y_{i,m})$.

Proposition 4.5

Let, for every $\epsilon > 0$, $u_\epsilon \in L^2(Q^{\epsilon}_{i,m})$ be such that $T^{\epsilon}_{i,m}u_\epsilon \rightharpoonup u$ weakly in $L^2(Q_i \times Y_{i,m})$. Then $\widetilde{u}_\epsilon \rightharpoonup \frac{1}{L}\int\limits_{Y_{i,m}} u\, dy$ weakly in $L^2(Q_i)$. Here $\widetilde{u}_\epsilon$ is the zero extension of u_ϵ to Q_i.

Proposition 4.6

Let $u_\epsilon \in L^2(0,T;H^1(\Omega^{\epsilon}_{i,m}))$ for every $\epsilon > 0$ be such that $T^{\epsilon}_{i,m}u_\epsilon \rightharpoonup u$ weakly in $\mathcal{Z}_{i,m}$. Then $\widetilde{u}_\epsilon \rightharpoonup \frac{1}{L}\int\limits_{Y_{i,m}} u\, dy$ weakly in $\mathcal{X}_i$.

Proof

Given that $T^{\epsilon}_{i,m}u_\epsilon \rightharpoonup u$ weakly in $\mathcal{Z}_{i,m}$ implies $T^{\epsilon}_{i,m}u_\epsilon \rightharpoonup u$ and $\partial_{x_2}(T^{\epsilon}_{i}u_\epsilon) \rightharpoonup \partial_{x_2}u$ weakly in $L^2(Q_i \times Y_{i,m})$. That is, $T^{\epsilon}_{i,m}(\partial_{x_2}u_\epsilon) \rightharpoonup \partial_{x_2}u$ weakly in $L^2(Q_i\times Y_{i,m})$. Using Proposition 4.5 we obtain $\widetilde{u}_\epsilon \rightharpoonup \frac{1}{L}\int\limits_{Y_{i,m}} u\, dy$ in $L^2(Q_i)$ and $\widetilde{\partial_{x_2}u_\epsilon} \rightharpoonup \frac{1}{L}\int\limits_{Y_{i,m}} \partial_{x_2}u\, dy$ in $L^2(Q_i)$. But notice that $\partial_{x_2}\widetilde{u}_\epsilon = \widetilde{\partial_{x_2}u_\epsilon}$, because the extension by 0 does not affect the derivative along the x_2-direction. Hence, $\tilde{u}_\epsilon \rightharpoonup \frac{1}{L}\int\limits_{Y_{i,m}} u\, dy$ weakly in $\mathcal{X}_i$. □

4.1. *Unfolding on the boundary*

In this section, we define the boundary unfolding operators $T^{\epsilon}_{M_i,m}$ on functions defined on the boundary $\Gamma^{\epsilon}_{i,m}$. We now state the properties of the boundary unfolding operators and proofs are given.

Definition 4.2

The ϵ-unfolding of a function $u :]0,T[\times\Gamma^{\epsilon}_{i,m} \to \mathbb{R}$ denoted by $T^{\epsilon}_{M_i,m} : \{u :]0,T[\times\Gamma^{\epsilon}_{i,m} \to \mathbb{R}\} \to \{v :]0,T[\times]0,L[\times Y_{i,m} \to \mathbb{R}\}$ is defined as $T^{\epsilon}_{M_i,m}u(t,x_1,y) = u\left(t, \epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y\right)$.

If U is an open subset of $\mathbb{R}^2$ such that $\Gamma^{\epsilon}_{i,m} \subset U$ and $u :]0,T[\times U \to \mathbb{R}$ then $T^{\epsilon}_{M_i,m}u = T^{\epsilon}_{M_i,m}\left(u|_{\Gamma^{\epsilon}_{i,m}}\right)$.

Proposition 4.7

Let $T^{\epsilon}_{M_i,m}$ be the boundary unfolding operator defined as in Definition 4.2. Then, we have $T^{\epsilon}_{M_i,m}$ is linear and for u,v $:]0,T[\times\Gamma^{\epsilon}_{i,m} \to \mathbb{R}$, we have $T^{\epsilon}_{M_i,m}(uv) = T^{\epsilon}_{M_i,m}(u)T^{\epsilon}_{M_i,m}(v)$.

The proof follows directly from the definition.

Proposition 4.8
Let $u \in L^2(]0,T[\times\Gamma^\epsilon_{i,m})$. Then $T^\epsilon_{M_i,m}u \in L^2(]0,T[\times]0,L[\times Y_{i,m})$. Moreover $\|T^\epsilon_{M_i,m}u\|_{L^2(]0,T[\times]0,L[\times Y_{i,m})} = \sqrt{L}\|u\|_{L^2(]0,T[\times\Gamma^\epsilon_{i,m})}$

Proof

$$\begin{aligned}
\int_0^T\int_0^L\int_{Y_{i,m}} |T^\epsilon_{M_i,m}u|^2\,dx_1\,dt\,dy &= \int_0^T\int_0^L\int_{Y_{i,m}}\left|u\left(t,\epsilon\left[\frac{x_1}{\epsilon}\right]_L+\epsilon y\right)\right|^2\,dx_1dtdy\\
&= \int_{t=0}^{T}\int_{y\in Y_{i,m}}\sum_{k=0}^{N-1}\int_{x_1=k\epsilon L}^{(k+1)\epsilon L}|u(t,k\epsilon L+\epsilon y)|^2\,dx_1dtdy\\
&= \sum_{k=0}^{N-1}\int_{x_1=k\epsilon L}^{(k+1)\epsilon L}dx_1\int_{t=0}^{T}\int_{y\in Y_{i,m}}|u(t,k\epsilon L+\epsilon y)|^2\,dtdy\\
&= \epsilon L\sum_{k=0}^{N-1}\int_{t=0}^{T}\int_{y\in Y_{i,m}}|u(t,k\epsilon L+\epsilon y)|^2\,dx_1dtdy\\
&= L\sum_{k=0}^{N-1}\int_{t=0}^{T}\int_{x_1\in k\epsilon L+\epsilon Y_{i,m}}|u(t,x_1)|^2\,dx_1dt = L\int_0^T\int_{\Gamma^\epsilon_{i,m}}|u(t,x_1)|^2\,dx_1dt.
\end{aligned}$$

□

Proposition 4.9
Let $u \in L^2(0,T;H^1(\Gamma^\epsilon_{i,m}))$. Then $T^\epsilon_{M_i,m}u \in L^2(]0,T[\times]0,L[;H^1(Y_{i,m}))$and $\partial_y(T^\epsilon_{M_i,m}u) = \epsilon T^\epsilon_{M_i,m}(\partial_{x_1}u)$

Proof
It follows from the Definition that $\partial_y(T^\epsilon_{M_i,m}u) = \epsilon T^\epsilon_{M_i,m}(\partial_{x_1}u)$.

$$\begin{aligned}
\|T^\epsilon_{i,m}u\|^2_{L^2(]0,L[\times]0,T[;H^1(Y_{i,m}))} &= \int_0^L\int_0^T \|T^\epsilon_{M_i,m}u\|^2_{H^1(Y_{i,m}}\,dx_1dt\\
&= \int_0^L\int_0^T\int_{Y_{i,m}}\left(\epsilon^2 T^\epsilon_{M_i,m}|\partial_{x_1}u|^2 + T^\epsilon_{M_i,m}|u|^2\right)\,dx_1dydt\\
&= L\int_0^T\int_{\Gamma^\epsilon_{i,m}}\left(\epsilon^2|\partial_{x_1}u|^2+|\partial_{x_2}u|^2+|u|^2\right)\,dx_1dt \le L\|u\|^2_{L^2(]0,T[;H^1(\Gamma^\epsilon_{i,m}))} < \infty.
\end{aligned}$$

□

Proposition 4.10
Let $u \in L^2(]0,T[\times]0,L[)$. Then $T^\epsilon_{M_i,m}u \to u$ in $L^2(]0,T[\times]0,L[\times Y_{i,m})$.

Proof
Consider $\phi \in \mathcal{D}(]0,T[\times]0,L[)$.

$$\begin{aligned}
\sup_{(t,x_1)\in]0,T[\times]0,L[\times Y_{i,m}}|(T^\epsilon\phi)(t,x_1,y)-\phi(t,x_1)| &= \sup_{(t,x_1,y)\in]0,T[\times]0,L[\times Y_{i,m}}|\phi(t,\epsilon[\tfrac{x_1}{\epsilon}]_L+\epsilon y)-\phi(t,x_1)|\\
&\le m_\phi(\epsilon L)
\end{aligned}$$

where m_ϕ is the modulus of continuity of the function ϕ which is defined as

$$m_\phi(\delta) = \sup_{z_1,z_2\in]0,T[\times]0,L[}\{|\phi(z_1)-\phi(z_2)| : |z_1-z_2| < \delta\}.$$

Because ϕ is uniformly continuous in $]0,T[\times]0,L[$, $m_\phi(\epsilon L) \to 0$ as $\epsilon L \to 0$. Hence,

$$\sup_{]0,T[\times]0,L[\times Y_{i,m}}|T^\epsilon\phi-\phi| \to 0 \; as\; \epsilon \;\to 0$$

And so,

$$T^\epsilon_{M_i,m}\phi \to \phi \;in\; L^2(]0,T[\times]0,L[\times Y_{i,m}) \quad \forall\,\phi\in\mathcal{D}(]0,T[\times]0,L[).$$

The density of $\mathcal{D}(]0,T[\times]0,L[)$ in $L^2(]0,T[\times]0,L[)$ completes the proof. □

Proposition 4.11
Suppose that $u_\epsilon \to u$ in $L^2(]0,T[\times]0,L[)$. Then $T^\epsilon_{M_i,m}u_\epsilon \to u$ in $L^2(]0,T[\times]0,L[\times Y_{i,m})$.

Proof
Suppose that $u_\epsilon \to u$ in $L^2(]0,T[\times]0,L[)$. Then

$$\begin{aligned}\|T^\epsilon_{M_i,m}u_\epsilon - u\|_{L^2(]0,T[\times]0,L[\times Y_{i,m})} &= \|T^\epsilon_{M_i,m}u_\epsilon - T^\epsilon_{M_i,m}u + T^\epsilon_{M_i,m}u - u\|_{L^2(]0,T[\times]0,L[\times Y_{i,m})}\\ &\le \|T^\epsilon_{i,m}u_\epsilon - T^\epsilon_{i,m}u\|_{L^2(]0,T[\times]0,L[\times Y_{i,m})} + \|T^\epsilon_{i,m}u - u\|_{L^2(]0,T[\times]0,L[\times Y_{i,m})}\\ &= \sqrt{L}\|u_\epsilon - u\|_{L^2(]0,T[\times \Gamma^\epsilon_{i,m})} + \|T^\epsilon_{M_i,m}u - u\|_{L^2(]0,T[\times]0,L[\times Y_{i,m})}\\ &\le \sqrt{L}\|u_\epsilon - u\|_{L^2(]0,T[\times]0,L[)} + \|T^\epsilon_{i,m}u - u\|_{L^2(]0,T[\times]0,L[\times Y_{i,m})} \to 0 \ \ as \ \ \epsilon \to 0.\end{aligned}$$

□

Proposition 4.12
Suppose that u_ϵ is a sequence in $L^2(]0,T[\times\Gamma^\epsilon_{i,m})$ such that $T^\epsilon_{M_i,m}u_\epsilon \rightharpoonup u$ weakly in $L^2(]0,T[\times]0,L[\times Y_{i,m})$.
Then $\widetilde{u_\epsilon} \rightharpoonup \frac{1}{L}\int_{Y_{i,m}} u\, dy$weakly in $L^2(]0,T[\times]0,L[)$.

Proof
Note that, for $\psi \in L^2(]0,T[\times]0,L[)$, we have

$$\int_0^T\int_0^L\int_{Y_{i,m}} T^\epsilon_{M_i,m}u_\epsilon T^\epsilon_{M_i,m}\psi \ dx_1dydt = L\int_0^T\int_{\Gamma^\epsilon_{i,m}} u_\epsilon\psi dx_1dt$$

Now,

$$\begin{aligned}\int_0^T\int_0^L \widetilde{u_\epsilon}\psi dx_1dt &= \int_0^T\int_{\Gamma^\epsilon_{i,m}} u_\epsilon\psi dx_1dt\\ &= \frac{1}{L}\int_0^T\int_0^L\int_{Y_{i,m}} T^\epsilon_{M_i,m}u_\epsilon T^\epsilon_{M_i,m}\psi \ dx_1dydt\\ &\to \frac{1}{L}\int_0^T\int_0^L\int_{Y_{i,m}} u\psi \ dx_1dydt\\ &= \int_0^T\int_0^L\left(\frac{1}{L}\int_{Y_{i,m}} udy\right)\psi \ dx_1dt.\end{aligned}$$

□

5. Convergence analysis

5.1. Function spaces

To give an appropriate meaning to the weak solution of homogenized problem, let us introduce the function space $\mathcal{H}$. We use the ideas introduced in [6]. We say, a multi-sheeted function of the form

$$\varphi := \begin{cases}\{\varphi_{2,1},\varphi_{2,2},\varphi_{2,3},\varphi_{2,4}\} \ if \ x\in\Omega_2,\\ \{\varphi_{1,1},\varphi_{1,2}\} \ if \ x\in\Omega_1; \quad \varphi_{0,1} \ if \ x\in\Omega_0, \quad \varphi^- \ if \ x\in\Omega^-\end{cases} \tag{5.1}$$

belongs to $\mathcal{H}$, if $\varphi^-\in H^1(\Omega^-)$, for each $i=0,1,2$, the functions $\varphi_{i,m}\in L^2(0,L;H^1(M_i,M_{i+1})$ for $m=1,\cdots,2^i$ and on the boundaries (in the sense of trace), they satisfy

$$\begin{aligned}\varphi^-|_{\Gamma_0} = \varphi_{0,1}|_{\Gamma_0}, \quad \varphi_{0,1}|_{\Gamma_1} = \varphi_{1,1}|_{\Gamma_1} = \varphi_{1,2}|_{\Gamma_1},\\ \varphi_{1,1}|_{\Gamma_2} = \varphi_{2,1}|_{\Gamma_2} = \varphi_{2,2}|_{\Gamma_2}, \quad \varphi_{1,2}|_{\Gamma_2} = \varphi_{2,3}|_{\Gamma_2} = \varphi_{2,4}|_{\Gamma_2}.\end{aligned} \tag{5.2}$$

Note that the space $\mathcal{H}$ is continuously and densely embedded in the Hilbert space V of functions whose components belongs to the corresponding L^2-spaces with the inner products, respectively, given by

$$(\varphi,\psi)_V := (\varphi^-,\psi^-)_{L^2(\Omega^-)} + \sum_{i=0}^2\sum_{m=1}^{2^i}(\varphi_{i,m},\psi_{i,m})_{L^2(\Omega_i)}$$

$$(\varphi,\psi)_{\mathcal{H}} := (\varphi,\psi)_V + (\nabla\varphi^-,\nabla\psi^-)_{L^2(\Omega^-)} + \sum_{i=0}^2\sum_{m=1}^{2^i}(\partial_{x_2}\varphi_{i,m},\partial_{x_2}\psi_{i,m})_{L^2(\Omega_i)}.$$

Now, introduce an operator $\mathcal{A}: L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H}^*)$ by the formula

$$\langle\mathcal{A}\varphi,\psi\rangle := \int_{Q^-}(\nabla\varphi\cdot\nabla\psi + \varphi\psi)dxdt + \sum_{i=0}^2\sum_{m=1}^{2^i}\frac{h_{i,m}}{L}\int_{Q_i}(\partial_{x_2}\varphi_{i,m}\,\partial_{x_2}\psi_{i,m} + \varphi_{i,m}\,\psi_{i,m})\,dxdt \tag{5.3}$$

for all $\varphi, \psi \in L^2(0,T;\mathcal{H})$. Define a linear functional $F \in L^2(0,T;\mathcal{H}^*)$ as

$$\langle F, \psi \rangle := \int_{Q^-} f_0 \psi^- \, dxdt + \sum_{i=0}^{2} \sum_{m=1}^{2^i} \frac{h_{i,m}}{L} \int_{Q_i} f_0 \, \psi_{i,m} \, dxdt.$$

Definition 5.1
A multi-sheeted function $u \in L^2(0,T;\mathcal{H})$ with $u' \in L^2(0,T;\mathcal{H}^*)$ is said to be a solution to the homogenized problem if it satisfies

$$\langle u', \psi \rangle + \langle \mathcal{A}u, \psi \rangle = \langle F, \psi \rangle, \ \ \forall \ \psi \in L^2(0,T;\mathcal{H}) \text{ and } u|_{t=0} = 0. \tag{5.4}$$

The *multi-sheeted* function u can be represented as

$$u(t,x) = \begin{cases} u_{i,m} & \text{if } (t,x) \in Q_i, \ \text{for } i = 0,1,2; \ m = 1,\cdots,2^i \\ u^- & \text{if } (t,x) \in Q^-. \end{cases} \tag{5.5}$$

Such problems are considered in Mel'nyk [6] in the context of problems in branched structures. The author has studied homogenization problems, based on asymptotic expansion. Regarding the existence of (5.4), we refer to [6]. The strong form of the above weak formulation is given by (5.16).

5.2. Homogenization

The main result of this section is stated in the following theorem. The final result is similar to the one in [6], where the author studied a semi-linear problem, but we give a proof using unfolding operators. This also will allow us to study other problems in branched structures. In fact, in the next section, we study an optimal control problem. This may not be able to study easily using asymptotic expansion. Further, one of our main contribution is that the characterization of the control via unfolding operators.

Theorem 5.1
Let u_ϵ and u be the solution of (3.1) and of (5.4), respectively, and f_ϵ is uniformly bounded in $L^2(Q_\epsilon)$ such that $\widetilde{f_\epsilon|_{Q^\epsilon_{i,m}}} \rightharpoonup \frac{h_{i,m}}{L} f_0$ in $L^2(Q_i)$, for some $f_0 \in L^2(Q)$. Then

$$\widetilde{u_{\epsilon\,|Q^\epsilon_{0,1}}} \rightharpoonup \frac{h_{0,1}}{L}\, u_{0,1} \quad \text{weakly in } \mathcal{X}_0, \qquad \widetilde{u_{\epsilon\,|Q^\epsilon_{1,m}}} \rightharpoonup \frac{h_{1,m}}{L}\, u_{1,m} \quad \text{weakly in } \mathcal{X}_1, \ \text{for } m = 1,2,$$
$$\widetilde{u_{\epsilon\,|Q^\epsilon_{2,m}}} \rightharpoonup \frac{h_{2,m}}{L}\, u_{2,m} \quad \text{weakly in } \mathcal{X}_2, \ \text{for } m = 1,2,3,4, \quad u_\epsilon|_{Q^-} \rightharpoonup u|_{Q^-} \quad \text{weakly in } \mathcal{X}.$$

Here $\widetilde{u_{\epsilon\,|Q^\epsilon_{i,m}}}$ is the zero extension of $u_{\epsilon\,|Q^\epsilon_{i,m}}$ to Q_i.

Proof
The proof follows in several steps.

Step 1 (A priori Estimate): Using a priori estimate (3.3), we have

$$\|u_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))} + \|u'_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon)^*)} \leq C\|\tilde{f}_\epsilon\|_{L^2(Q)}$$

where C is independent of ϵ. Because $\tilde{f}_\epsilon$ is bounded, we derive

$$\|u_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))} \leq C \tag{5.6}$$

where C is a constant independent of ϵ. Let us denote $u_{i,\epsilon}$ is the restriction of u_ϵ in $Q_{i,\epsilon}$ and u^-_ϵ is the restriction of u_ϵ in Q^-. To find the bound of the sequence $T^\epsilon_{i,m} u_{i,\epsilon}$ in the space $\mathcal{Z}_{i,m}$, we proceed as

$$\begin{aligned}
\|T^\epsilon_{i,m} u_{i,\epsilon}\|^2_{\mathcal{Z}_{i,m}} &= \int_0^L \int_0^T \|T^\epsilon_{i,m} u_{i,\epsilon}(t,x_1)\|^2_{H^1((M_i,M_{i+1})\times Y_{i,m})} \, dx_1 dt \\
&= \int_{Q_i \times Y_{i,m}} \left(\epsilon^2 T^\epsilon_{i,m} |\partial_{x_1} u_{i,\epsilon}|^2 + T^\epsilon_{i,m} |\partial_{x_2} u_{i,\epsilon}|^2 + T^\epsilon_{i,m} |u_{i,\epsilon}|^2\right) dxdtdy \\
&= L \int_{Q_{i,\epsilon}} \left(\epsilon^2 |\partial_{x_1} u_{i,\epsilon}|^2 + |\partial_{x_2} u_{i,\epsilon}|^2 + |u_{i,\epsilon}|^2\right) dxdt \ \leq \ L\, \|u_\epsilon\|^2_{L^2(0,T;H^1(\Omega_\epsilon))}.
\end{aligned}$$

The boundedness of the sequence $T^\epsilon_{i,m} u_{i,\epsilon}$ in $\mathcal{Z}_{i,m}$ follows from (5.6). By weak compactness, there exists a subsequence (still denoted by ϵ) and $u_{i,m} \in \mathcal{Z}_{i,m}$ such that

$$T^\epsilon_{i,m} u_{i,\epsilon} \rightharpoonup u_{i,m} \text{ weakly in } \mathcal{Z}_{i,m}, \text{ which implies } T^\epsilon_{i,m} u_{i,\epsilon} \rightharpoonup u_{i,m} \text{ weakly in } L^2(Q_i \times Y_{i,m}) \tag{5.7}$$

$$\text{and}\quad T^{\epsilon}_{i,m}(\partial_{x_2}u_{i,\epsilon}) = \partial_{x_2}(T^{\epsilon}_{i,m}u_{i,\epsilon}) \rightharpoonup \partial_{x_2}u_{i,m} \text{ weakly in } L^2(Q_i \times Y_{i,m}). \tag{5.8}$$

Further,

$$\epsilon T^{\epsilon}_{i,m}(\partial_{x_1}u_{i,\epsilon}) = \partial_y(T^{\epsilon}_{i,m}u_{i,\epsilon}) \rightharpoonup \partial_y u_{i,m} \text{ weakly in } L^2(Q_i \times Y_{i,m}). \tag{5.9}$$

From the Proposition 4.2, we have $\|T^{\epsilon}_{i,m}(\partial_{x_1}u_{i,\epsilon})\|_{L^2(Q_i\times Y_{i,m})} = \sqrt{L}\,\|\partial_{x_1}u_{i,\epsilon}\|_{L^2(Q^{\epsilon}_{i,m})} \le \sqrt{L}\|u_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))}$. Thus, from (5.6), we obtain the boundedness of the sequence $T^{\epsilon}_{i,m}(\partial_{x_1}u_{i,\epsilon})$ in the space $L^2(Q_i \times Y_{i,m})$. Hence, from (5.9), it follows that $\partial_y u_{i,m} = 0$. Thus, $u_{i,m}$ is independent of y, and we conclude that

$$\widetilde{u_{i,\epsilon}} \rightharpoonup \frac{h_{i,m}}{L}u_{i,m} \text{ weakly in } \mathcal{X}_i \tag{5.10}$$

with the help of Proposition 4.6 and convergence (5.7). Moreover $\int_{Y_{i,m}} \partial_{x_2}u_{i,m}dy = h_{i,m}\partial_{x_2}u_{i,m}$. We know that $T^{\epsilon}_{i,m}(\partial_{x_1}u_{i,\epsilon})$ is bounded in $L^2(Q_i \times Y_{i,m})$. Hence by weak compactness, there is an element $P_{i,m} \in L^2(Q_i \times Y_{i,m})$ such that up to a subsequence (still denoted by ϵ),

$$T^{\epsilon}_{i,m}\partial_{x_1}u_{i,\epsilon} \rightharpoonup P_{i,m} \text{ weakly in } L^2(Q_i \times Y_{i,m}). \tag{5.11}$$

Using the estimate of $\|u_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))}$, we have the boundedness of u^-_ϵ in the space $L^2(0,T;H^1(\Omega^-))$. Thus, there exists $u^- \in L^2(0,T;H^1(\Omega^-))$ and a subsequence (still denoted by ϵ) such that

$$u^-_\epsilon \rightharpoonup u^- \text{ weakly in } L^2(0,T;H^1(\Omega^-)). \tag{5.12}$$

Define the multi-sheeted function u using u^- and $u_{i,m}$ as in (5.1).

Step 2 (Claim): The multi-sheeted function $u \in L^2(0,T;\mathcal{H})$**.** We have $u^- \in L^2(Q^-)$, $u_{i,m} \in L^2(Q_i)$ and $\partial_{x_1}u^-, \partial_{x_2}u^- \in L^2(Q^-)$. To prove $u \in L^2(0,T;\mathcal{H})$, we need to show that $\partial_{x_2}u_{i,m} \in L^2(Q_i)$ and also $u_{i,m}$ match on their corresponding common boundaries as in (5.2). Note that, u is independent of y variable and so is $\partial_{x_2}u$. Therefore, from the earlier discussion $\partial_{x_2}u_{i,m} \in L^2(Q_i)$ and $\partial_{x_2}u^- \in L^2(Q^-)$. Now, to prove $u \in L^2(0,T;\mathcal{H})$, it is enough to show that the trace of u^- and $u_{0,1}$ are equal on Γ_0, trace of $u_{0,1}$, $u_{1,1}$ and $u_{1,2}$ are equal on Γ_1, trace of $u_{1,1}, u_{2,1}$, and $u_{2,2}$ are equal on Γ_2 and trace of $u_{1,2}, u_{2,3}$, and $u_{2,4}$ are equal on Γ_2. Let us define an another unfolding operator T^{ϵ}_{-} as follows:

Let ϕ be a function defined on $Q^-_{\mathfrak{m}} =]0,T[\times]0,L[\times(\mathfrak{m},M_0)$ define $(T^{\epsilon}_{-}\phi)(t,x_1,x_2,y) = \phi\left(t,\epsilon\left[\frac{x_1}{\epsilon}\right]_L + \epsilon y, x_2\right)$ for $(t,x_1,x_2,y) \in]0,T[\times]0,L[\times(\mathfrak{m},M_0) \times Y_{0,1}$. Note that $u_{0,\epsilon}|_{\Gamma_{0,\epsilon}} = u_\epsilon{}^-|_{\Gamma_{0,\epsilon}}$. Now, apply the boundary unfolding operator on both sides, that is, $T^{\epsilon}_{M_0,1}\left(u_{0,\epsilon}|_{\Gamma_{0,\epsilon}}\right) = T^{\epsilon}_{M_0,1}\left(u_\epsilon{}^-|_{\Gamma_{0,\epsilon}}\right)$. Using the definition of unfolding operators, we can easily see that

$$T^{\epsilon}_{M_0,1}\left(u_{0,\epsilon}|_{\Gamma_{0,\epsilon}}\right) = T^{\epsilon}_{0,1}(u_{0,\epsilon})|_{x_2=M_0} \text{ and } T^{\epsilon}_{M_0,1}\left(u^-_\epsilon|_{\Gamma_{0,\epsilon}}\right) = T^{\epsilon}_{-}(u^-_\epsilon)|_{x_2=M_0}. \tag{5.13}$$

Because $T^{\epsilon}_{0,1}(u_{0,\epsilon})$ is differentiable in the x_2-direction, we can define the trace on Γ_0. From the weak continuity of the trace operator, we obtain the following convergence as $\epsilon \to 0$.

$$\left(T^{\epsilon}_{0,1}(u_{0,\epsilon})\right)|_{x_2=M_0} \rightharpoonup u_{0,1}|_{x_2=M_0} \text{ weakly in } L^2(]0,T[\times]0,L[\times Y_{0,1})$$

Note that, $T^{\epsilon}_{-}(u^-_\epsilon)$ converges weakly to u^- in $L^2(Q^-_{\mathfrak{m}} \times Y_{0,1})$. Here, we used the fact that u^-_ϵ is bounded in $L^2(0,T;H^1(]0,L[\times(\mathfrak{m},M_0)))$ and hence converges to u^- weakly in $L^2(0,T;H^1(]0,L[\times(\mathfrak{m},M_0)))$ and strongly in $L^2(Q^-_{\mathfrak{m}})$ by Lions–Aubin's Theorem ([38]). The weak continuity of trace operator gives us the following convergence.

$$\left(T^{\epsilon}_{-}(u^-_\epsilon)\right)|_{x_2=M_0} \rightharpoonup u^-|_{x_2=M_0} \text{ weakly in } L^2(]0,T[\times]0,L[\times Y_{0,1})$$

From the aforementioned convergences, we derive that $u_{0,1}|_{\Gamma_0} = u^-|_{\Gamma_0}$ in $L^2(]0,T[\times]0,L[)$, because $u_{0,1}$ and u^- are independent of the y variable. Similarly, we can show that $u_{1,1}|_{x_2=M_1} = u_{1,2}|_{x_2=M_1} = u_{0,1}|_{x_2=M_1}$, $u_{2,1}|_{x_2=M_2} = u_{2,2}|_{x_2=M_2} = u_{1,1}|_{x_2=M_2}$ and $u_{2,3}|_{x_2=M_2} = u_{2,4}|_{x_2=M_2} = u_{1,2}|_{x_2=M_2}$in $L^2(]0,T[\times]0,L[)$. Hence, we have shown that $u \in L^2(0,T;\mathcal{H})$.

Step 3: We claim that $P_{i,m} = 0$**.** To identify the limit $P_{i,m}$ in (5.11), we choose test functions as follows: For $\phi_i \in \mathcal{D}(Q_i)$ and $\psi_{i,m} \in \mathcal{D}(Y_{i,m})$ (extend it to whole $\mathbb{R}$ periodically with period L), choosing a test function

$$\phi^{\epsilon}_{i,m}(x) = \epsilon\phi_i(t,x)\psi_{i,m}\left(\left\{\frac{x_1}{\epsilon}\right\}\right),$$

in such a way that $\phi^{\epsilon}_{i,m}$ is continuous on $Q^{\epsilon}_{i,m}$. From the definition of ϵ-unfolding of $\phi^{\epsilon}_{i,m}$ and by Proposition 4.3, we obtain

$$\begin{aligned}
T^{\epsilon}_{i,m}\phi^{\epsilon}_{i,m} &= \epsilon\phi_i\left(t,\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon y, x_2\right)\psi_{i,m}(y),\\
T^{\epsilon}_{i,m}(\partial_{x_1}\phi^{\epsilon}_{i,m}) &= \frac{1}{\epsilon}\partial_y(T^{\epsilon}_{i,m}\phi^{\epsilon}_{i,m}) = \epsilon\partial_{x_1}\phi_i\left(t,\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon y, x_2\right)\psi_{i,m}(y) + \phi_i\left(t,\epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon y, x_2\right)\psi'_{i,m}(y),\\
T^{\epsilon}_{i,m}(\partial_{x_2}\phi^{\epsilon}_{i,m}) &= \epsilon\partial_{x_2}\phi_i\left(t,\epsilon\left[\frac{x_2}{\epsilon}\right] + \epsilon y, x_2\right)\psi_{i,m}(y).
\end{aligned}$$

The aforementioned equations gives us

$$T^\epsilon_{i,m}\phi^\epsilon_{i,m} \to 0 \text{ in } L^2(Q_i \times Y_{i,m}), \quad T^\epsilon_{i,m}(\partial_{x_1}\phi^\epsilon_{i,m}) \to \phi_i(t,x)\psi'_{i,m}(y) \text{ in } L^2(Q_i \times Y_{i,m})$$
$$T^\epsilon_{i,m}(\partial_{x_2}\phi^\epsilon_{i,m}) \to 0 \text{ in } L^2(Q_i \times Y_{i,m})$$

as $\epsilon \to 0$. From the variational formulation of (3.1) with the test function $\psi = \phi^\epsilon_{i,m}$, we obtain

$$-\int_{Q_\epsilon} u_\epsilon \partial_t \phi^\epsilon_{i,m}\, dxdt + \int_{Q_\epsilon} \nabla_x u_\epsilon \cdot \nabla_x \phi^\epsilon_{i,m}\, dxdt + \int_{Q_\epsilon} u_\epsilon \phi^\epsilon_{i,m}\, dxdt = \int_{Q_\epsilon} f_\epsilon \phi^\epsilon_{i,m}\, dxdt. \tag{5.14}$$

Now notice $-\int_{Q_\epsilon} u_\epsilon \partial_t \phi^\epsilon_{i,m} = \frac{-1}{L}\int_{Q_i\times Y_{i,m}} T^\epsilon_{i,m}u_\epsilon \; T^\epsilon_{i,m}\partial_t\phi^\epsilon_{i,m} \to 0$ as $\epsilon \to 0$ because $\partial_t \phi^\epsilon_{i,m} = (\partial_t \phi_{i,m})^\epsilon$ and $T^\epsilon_{i,m}\partial_t\phi^\epsilon_{i,m} = \epsilon \partial_t \phi(t, \epsilon[\frac{x_1}{\epsilon}] + \epsilon y, x_2)\psi_{i,m}(y) \to 0$ in $L^2(Q_i \times Y_{i,m})$. Using the properties of unfolding operators and convergence described earlier, it is easy to see that

$$\int_{Q_\epsilon} \nabla_x u_\epsilon \cdot \nabla_x \phi^\epsilon_{i,m}\, dxdt \to \frac{1}{L}\int_{Q_i\times Y_{i,m}} P_{i,m}\, \phi_i(x_1,x_2)\psi'_{i,m}(y) \text{ as } \epsilon \to 0.$$

This is more or less similar to the elliptic problems as in [7–9].

Further $\int_{Q_\epsilon} u_\epsilon \phi^\epsilon_{i,m}\, dxdt = \int_{Q^\epsilon_{i,m}} u_\epsilon \phi^\epsilon_{i,m}\, dxdt = \frac{1}{L}\int_{Q_i\times Y_{i,m}} T^\epsilon_{i,m}\, u_\epsilon \; T^\epsilon_{i,m}\phi^\epsilon_{i,m}\, dxdydt \to 0$ as $\epsilon \to 0$. Hence, the equation (5.14), as $\epsilon \to 0$ becomes,

$$\int_{Q_i\times Y_{i,m}} P_{i,m}\, \phi_i(t,x_1,x_2)\psi'_{i,m}(y) = 0 \quad \forall\, \phi_i \in \mathcal{D}(Q_i),\ \psi_{i,m} \in \mathcal{D}(Y_{i,m})$$

which implies $P_{i,m} = 0$.

Step 4: We, now establish that u satisfies the homogenized problem (5.4). Let $\mathcal{B}$ be the set of all multi-sheeted functions of the form:

$$\varphi(x_1,x_2) := \begin{cases} \varphi_{2,m}(x_1,x_2) & \text{if } x \in \Omega_2,\ \text{for } m = 1,2,3,4, \\ \varphi_{1,m}(x_1,x_2) & \text{if } x \in \Omega_1,\ \text{for } m = 1,2, \\ \varphi_0(x_1,x_2) & \text{if } x \in \Omega_0, \\ \varphi^-(x_1,x_2) & \text{if } x \in \Omega^-. \end{cases}$$

where $\varphi_{i,m} \in C^\infty(\Omega_i)$ and $\varphi^- \in C^\infty(\Omega^-)$ with $\varphi_{i,m}$ and $\varphi_{i+1,k}$ are equal on the interface as in (5.2) and also $\varphi^-(x_1,M_0) = \varphi_0(x_1,M_0)$.

Let $\psi_{i,m} \in \mathcal{D}]0,L[$ such that it equals one if $x_1 \in Y_{i,m}$ and zero outside a neighborhood of $Y_{i,m}$. Extend $\psi_{i,m}$ L-periodically to whole $\mathbb{R}$. Now, consider the test functions $\phi^\epsilon \in C([0,T];H^1(\Omega^\epsilon))$ with $\phi^\epsilon(T) = 0$ and ϕ^ϵ of the form

$$\phi^\epsilon(t,x_1,x_2) := \begin{cases} \varphi_{2,m}(t,x_1,x_2)\psi_{2,m}\left(\left\{\frac{x_1}{\epsilon}\right\}\right) & \text{if } x \in \Omega^\epsilon_{2,m},\ \text{for } m = 1,2,3,4, \\ \varphi_{1,m}(t,x_1,x_2)\psi_{1,m}\left(\left\{\frac{x_1}{\epsilon}\right\}\right) & \text{if } x \in \Omega^\epsilon_{1,m},\ \text{for } m = 1,2, \\ \varphi_0(t,x_1,x_2)\psi_0\left(\left\{\frac{x_1}{\epsilon}\right\}\right) & \text{if } x \in \Omega^\epsilon_0, \\ \varphi^-(t,x_1,x_2) & \text{if } x \in \Omega^-. \end{cases}$$

where $\varphi \in C(0,T;\mathcal{B})$. On applying unfolding operators, we obtain

$$T^\epsilon_{i,m}\phi^\epsilon_{i,m} = \varphi_{i,m}\left(t, \epsilon\left[\tfrac{x_1}{\epsilon}\right] + \epsilon y, x_2\right)\psi_{i,m}(y),$$
$$T^\epsilon_{i,m}(\partial_{x_1}\phi^\epsilon_{i,m}) = \tfrac{1}{\epsilon}\partial_y(T^\epsilon_{i,m}\phi^\epsilon_{i,m}) = \partial_{x_1}\varphi_{i,m}\left(t, \epsilon\left[\tfrac{x_1}{\epsilon}\right] + \epsilon y, x_2\right)\psi_{i,m}(y) + \tfrac{1}{\epsilon}\varphi_{i,m}\left(t, \epsilon\left[\tfrac{x_1}{\epsilon}\right] + \epsilon y, x_2\right)\psi'_{i,m}(y),$$
$$T^\epsilon_{i,m}(\partial_{x_2}\phi^\epsilon_{i,m}) = \partial_{x_2}\varphi^\epsilon_{i,m}\left(t, \epsilon\left[\tfrac{x_2}{\epsilon}\right] + \epsilon y, x_2\right)\psi_{i,m}(y).$$

Because $\psi_{i,m}(y) = 1$ and $\psi'_{i,m}(y) = 0$ for $y \in Y_{i,m}$, we infer that

$$T^\epsilon_{i,m}\phi^\epsilon_{i,m} \to \varphi_{i,m}(t,x) \text{ in } L^2(Q_i \times Y_{i,m}), \quad T^\epsilon_{i,m}(\partial_{x_1}\phi^\epsilon_{i,m}) \to \partial_{x_1}\varphi_{i,m}(t,x) \text{ in } L^2(Q_i \times Y_{i,m})$$
$$T^\epsilon_{i,m}(\partial_{x_2}\phi^\epsilon_{i,m}) \to \partial_{x_2}\varphi_{i,m}(t,x) \text{ in } L^2(Q_i \times Y_{i,m}).$$

Consider the left hand side of the variational formulation with the test function $\psi = \phi^\epsilon(t,x_1,x_2)$, namely

$$-\int_{Q_\epsilon} u_\epsilon \partial_t \phi^\epsilon\, dxdt + \int_{Q_\epsilon} \nabla_x u_\epsilon \cdot \nabla_x \phi^\epsilon\, dxdt + \int_{Q_\epsilon} u_\epsilon \phi^\epsilon\, dxdt \tag{5.15}$$

$$-\int_{Q^+_\epsilon} u_\epsilon \partial_t \phi^\epsilon\, dxdt = -\sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q^\epsilon_{i,m}} u_\epsilon \partial_t \phi^\epsilon\, dxdt = -\frac{1}{L}\sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q_i\times Y_{i,m}} T^\epsilon_{i,m}u_\epsilon\; T^\epsilon_{i,m}(\partial_t\phi^\epsilon_{i,m})\, dxdtdy$$
$$\to -\frac{1}{L}\sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q_i\times Y_{i,m}} u_{i,m}\,\varphi'_{i,m}\, dxdtdy = \sum_{i=0}^{2}\sum_{m=1}^{2^i}\frac{h_{i,m}}{L}\int_{Q_i} u_{i,m}\,\varphi'_{i,m}\, dxdt \quad \text{as } \epsilon \to 0;$$

Similarly as earlier, we obtain the following convergences as $\epsilon \to 0$.

$$\int_{Q_\epsilon^+} \nabla_x u_\epsilon \cdot \nabla_x \phi^\epsilon \, dxdt \to \sum_{i=0}^{2} \sum_{m=1}^{2^i} \frac{h_{i,m}}{L} \int_{Q_i} \partial_{x_2} u_{i,m} \, \partial_{x_2} \varphi_{i,m} \, dxdt$$

$$\int_{Q_\epsilon^+} u_\epsilon \phi^\epsilon \, dxdt \to \sum_{i=0}^{2} \sum_{m=1}^{2^i} \frac{h_{i,m}}{L} \int_{Q_i} u_{i,m} \, \varphi_{i,m} \, dxdt.$$

On the fixed domain Q^-, we have no issue in passing to the limit. Right hand side of the weak formulation, namely $\int_{Q^\epsilon_{i,m}} f_\epsilon \phi^\epsilon \, dxdt = \int_{Q_i} \widetilde{f_\epsilon|_{Q^\epsilon_{i,m}}} \, \phi^\epsilon_{i,m} \, dxdt = \int_{Q_i} \widetilde{f_\epsilon|_{Q^\epsilon_{i,m}}} \, \varphi_{i,m} \, dxdt \to \frac{h_{i,m}}{L} \int_{Q_i} f_0 \, \varphi_{i,m} \, dxdt$ converges to $\int_{Q^-} f_0 \varphi^- \, dxdt + \sum_{i=0}^{2} \sum_{m=1}^{2^i} \frac{h_{i,m}}{L} \int_{Q_i} f_0 \, \varphi_{i,m} \, dxdt$.

Finally, we obtain

$$\int_{Q^-} (\partial_t u^- \varphi^- + \nabla u^- \nabla \varphi^- + u^- \varphi^-) + \sum_{i=0}^{2} \sum_{m=1}^{2^i} \frac{h_{i,m}}{L} \left(\int_{Q_i} u'_{i,m} \, \varphi_{i,m} + \partial_{x_2}(u_{i,m}) \, \partial_{x_2}(\varphi_{i,m}) \right)$$
$$+ \sum_{i=0}^{2} \sum_{m=1}^{2^i} \frac{h_{i,m}}{L} \int_{Q_i} u_{i,m} \, \varphi_{i,m} = \int_{Q^-} f_0 \varphi^- + \sum_{i=0}^{2} \sum_{m=1}^{2^i} \frac{h_{i,m}}{L} \int_{Q_i} f_0 \, \varphi_{i,m}$$

$\forall \varphi \in C([0,T];\mathcal{B})$. Because $C([0,T];\mathcal{B})$ with $\psi(T) = 0$ is dense in $L^2(0,T;\mathcal{H})$, we have showed that u satisfies the homogenized problem (5.4). This completes the proof of Theorem 5.1.

□

Remark 5.1
If we consider $u_i = \sum_{m=1}^{2^i} h_{i,m} u_{i,m}$ and $f_i = \sum_{m=1}^{2^i} h_{i,m} f_0$, then $\bar{u}$ defined by

$$\bar{u} = \begin{cases} u_i & \text{if } (x,t) \in Q_i \text{ for } i = 0,1,2, \\ u^- & \text{if } (x,t) \in Q^- \end{cases}$$

satisfies the strong form

$$\begin{cases} \partial_t u_{i,m} - \frac{\partial^2}{\partial x_2^2} u_{i,m} + u_{i,m} = f_0 \text{ in }]0,T[\times\Omega_i \text{ for } i = 0,1,2, \text{ and } m = 1,\cdots,2^i \\ \partial_t u^- - \Delta u^- + u^- = f_0 \text{ in }]0,T[\times\Omega^-, \\ u^- = u_{0,1} \text{ on } \Gamma_0, \;\; u_{0,1} = u_{1,1} = u_{1,2} \text{ on } \Gamma_1, \;\; u_{1,1} = u_{2,1} = u_{2,2}; \;\; u_{1,2} = u_{2,3} = u_{2,4} \text{ on } \Gamma_2 \\ \sum_{m=1}^{2} h_{1,m} \partial_{x_2} u_{1,m} = \sum_{m=1}^{4} h_{2,m} \partial_{x_2} u_{2,m} \text{ on }]0,T[\times\Gamma_2 \;\; h_{0,1} \partial_{x_2} u_0 = \sum_{m=1}^{2} h_{1,m} \partial_{x_2} u_{1,m} \text{ on }]0,T[\times\Gamma_1 \\ \partial_{x_2} u^- = \frac{h_{0,1}}{L} \partial_{x_2} u_0 \text{ on }]0,T[\times\Gamma_0, \;\; \partial_\nu u_i = 0, \;\; \partial_\nu u^- = 0 \text{ on }]0,T[\times\Gamma, \\ u_i(0,x) = 0, \;\; u^-(0,x) = 0 \text{ in } \Omega, \;\; u_i, \; u^- \text{ are } \Gamma_{s'} - \text{periodic}. \end{cases} \tag{5.16}$$

From analysis using unfolding operators, we have obtained the limiting contribution from each branch at each stage. For example, at stage 1, namely from M_0 to M_1, there is only one branch which is periodically distributed (limiting contribution is $u_{0,1}$) whereas at stage 2, (M_1 to M_2) there are two branches which are periodically distributed (corresponding limiting contributions are $u_{1,1}$ and $u_{1,2}$). The last stage (M_2 to M_3) has four branches for which the limits are $u_{2,1}, u_{2,2}, u_{2,3}$ and $u_{2,4}$.

6. Optimal control

This section is devoted to the study of the homogenization of an optimal control problem posed on a branched structure as in the previous section.

6.1. *Nonhomogenized problem*

We, again recall the problem described in Section 3.2. We consider the controls coming from the fixed reference cell Λ_T^+ and periodically distributed. We have the following existence result for each fixed $\epsilon > 0$ (See Raymond [37]).

Theorem 6.1
For each $\epsilon > 0$, the minimization problem (P_ϵ) admits a unique solution.

One of our main result is the derivation of the following optimality system and characterization of optimal control via the unfolding operator.

Theorem 6.2
Let $(\overline{u}_\epsilon, \overline{\theta}_\epsilon) \in W(0,T;H^1(\Omega_\epsilon),H^1(\Omega_\epsilon)^*) \times L^2_\#(\Lambda_T^+)$ be the optimal solution to (P_ϵ), then the optimal control is characterized by

$$\overline{\theta}_\epsilon(t,y_1,y_2)|_{]0,T[\times\Lambda_{i,m}} = \frac{-1}{\beta L}\int_0^L (T^\epsilon_{i,m}\,\overline{p}_\epsilon)(t,x_1,y_2,y_1)\,dx_1 \tag{6.1}$$

$\forall\ i = 0,1,2$ *and* $m = 1,\cdots,2^i$, where $\overline{u}_\epsilon$ satisfies (3.5) with $\theta^\epsilon = \overline{\theta}^\epsilon_\epsilon$, that is $\theta^\epsilon(t,x_1,x_2) = \overline{\theta}^\epsilon_\epsilon(t,x_1,x_2) = \overline{\theta}_\epsilon(t,\frac{x_1}{\epsilon},x_2)$ and the adjoint state $\overline{p}_\epsilon$ satisfies the problem

$$\begin{cases} -\partial_t\overline{p}_\epsilon - \Delta\overline{p}_\epsilon + \overline{p}_\epsilon = \overline{u}_\epsilon - u_d \quad \text{in }]0,T[\times\Omega_\epsilon, \\ \partial_\nu\overline{p}_\epsilon = 0 \quad \text{on }]0,T[\times\Gamma_\epsilon, \quad \overline{p}_\epsilon(T,x) = \overline{u}_\epsilon(T) - u_d(T) \quad \text{in } \Omega_\epsilon, \quad \overline{p}_\epsilon \ \ \textit{is}\ \Gamma_s - \textit{periodic}. \end{cases} \tag{6.2}$$

Conversely, if a pair $(\hat{u}_\epsilon,\hat{p}_\epsilon)$ satisfies the following system

$$\begin{cases} \partial_t\hat{u}_\epsilon - \Delta\hat{u}_\epsilon + \hat{u}_\epsilon = f_0 + \chi_{Q^+_\epsilon}\hat{\theta}^\epsilon_\epsilon;\ -\partial_t\hat{p}_\epsilon - \Delta\hat{p}_\epsilon + \hat{p}_\epsilon = \hat{u}_\epsilon - u_d \quad \text{in }]0,T[\times\Omega_\epsilon, \\ \partial_\nu\hat{u}_\epsilon = 0;\ \partial_\nu\hat{p}_\epsilon = 0 \quad \text{on }]0,T[\times\Gamma_\epsilon, \quad \hat{u}_\epsilon(0,x) = 0;\ \hat{p}_\epsilon(T,x) = \hat{u}_\epsilon(T) - u_d(T) \quad \text{in } \Omega_\epsilon, \\ \hat{\theta}_\epsilon(t,y_1,y_2) = \frac{-1}{\beta L}\int_0^L T^\epsilon_{i,m}\,\hat{p}_\epsilon dx_1 \ \ \textit{on}\ ]0,T[\times\Lambda_{i,m}\ \ \forall\, i = 0,1,2 \ \textit{and}\ \ m = 1,\cdots,2^i, \\ \hat{u}_\epsilon \ \textit{and}\ \hat{p}_\epsilon \ \ \textit{are}\ \Gamma_s - \textit{periodic}, \end{cases}$$

then the pair $(\hat{u}_\epsilon,\hat{\theta}_\epsilon)$ is the optimal solution to (P_ϵ). Here $\hat{\theta}^\epsilon_\epsilon(t,x_1,x_2) = \hat{\theta}_\epsilon(t,\frac{x_1}{\epsilon},x_2)$.

Proof
Given $\theta \in L^2_\#(\Lambda^+_T)$, set $F_\epsilon(\theta) = J_\epsilon(u_\epsilon(\theta),\theta)$, where $u_\epsilon(\theta)$ is the solution to the Eq. (3.5) with right hand side $f_0 + \chi_{Q^+_\epsilon}\theta^\epsilon$. Using appropriate computation on $\frac{1}{\lambda}\left(F_\epsilon(\overline{\theta}_\epsilon + \lambda\theta) - F_\epsilon(\overline{\theta}_\epsilon)\right)$ and taking limit as $\lambda \to 0$, we obtain (we skip the computations involved)

$$F'_\epsilon(\overline{\theta}_\epsilon)\theta = \int_{Q_\epsilon}(\overline{u}_\epsilon - u_d)w_\epsilon + \int_{\Omega_\epsilon}(\overline{u}_\epsilon(T) - u_d(T))w_\epsilon(T) + \frac{\beta}{L}\int_{Q_\epsilon}\chi_{Q^+_\epsilon}\,\overline{\theta}^\epsilon_\epsilon\theta^\epsilon,$$

where w_ϵ is the solution of the following equation

$$\begin{cases} \partial_t w_\epsilon - \Delta w_\epsilon + w_\epsilon = \chi_{Q^+_\epsilon}\,\theta^\epsilon \quad \text{in }]0,T[\times\Omega_\epsilon, \\ \partial_\nu w_\epsilon = 0 \quad \text{on }]0,T[\times\Gamma_\epsilon, \qquad w_\epsilon(0,x) = 0 \quad \text{in } \Omega_\epsilon, \quad w_\epsilon \ \ \textit{is}\ \Gamma_s - \textit{periodic}. \end{cases} \tag{6.3}$$

Because $(\overline{u}_\epsilon,\overline{\theta}_\epsilon)$ is an optimal solution to (P_ϵ), we have $F'_\epsilon(\overline{\theta}_\epsilon)\theta = 0$ for all $\theta \in L^2(\Lambda^+_T)$, it follows that

$$\int_{Q_\epsilon}(\overline{u}_\epsilon - u_d)w_\epsilon + \int_{\Omega_\epsilon}(\overline{u_\epsilon}(T) - u_d(T))w_\epsilon(T) = \frac{-\beta}{L}\int_{Q^+_\epsilon}\overline{\theta}^\epsilon_\epsilon\,\theta^\epsilon$$

Using integration by parts in the Eqs (6.2) and (6.3) with test functions w_ϵ and $\overline{p}_\epsilon$, respectively,
we obtain $\int_{Q_\epsilon}(\overline{u}_\epsilon - u_d)w_\epsilon + \int_{\Omega_\epsilon}(\overline{u}(T) - u_d(T))w_\epsilon(T) = \int_{Q^+_\epsilon}\overline{p}_\epsilon\theta^\epsilon$
Hence, we obtain $-\beta\int_{Q^+_\epsilon}\overline{\theta^\epsilon_\epsilon}\theta^\epsilon = \int_{Q^+_\epsilon}\overline{p_\epsilon}\theta^\epsilon,\ \ \forall\ \theta \in L^2(\Lambda^+_T)$
Now, note that $T^\epsilon_{i,m}(\overline{\theta}^\epsilon_\epsilon)(t,x,y) = \overline{\theta}_\epsilon(t,y,x_2)$ and $\overline{\theta}_\epsilon|_{]0,T[\times\Lambda_{i,m}} \rightharpoonup \overline{\theta}_0|_{]0,T[\times\Lambda_{i,m}}$ in $L^2(]0,T[\times\Lambda_{i,m})$. Applying the unfolding operator, we obtain,

$$\int_{Q^\epsilon_i}\overline{\theta}^\epsilon_\epsilon\theta^\epsilon = \frac{1}{L}\sum_{m=1}^{2^i}\int_{Q_i\times Y_{i,m}} T^\epsilon_{i,m}\overline{\theta}^\epsilon_\epsilon\,T^\epsilon_{i,m}\theta^\epsilon\,dxdydt = \frac{1}{L}\sum_{m=1}^{2^i}\int_{Q_i\times Y_{i,m}}\overline{\theta}_\epsilon\,\theta\,dxdydt = \sum_{m=1}^{2^i}\int_{]0,T[\times\Lambda_{i,m}}\overline{\theta}_\epsilon\,\theta$$

Similarly,

$$\begin{aligned}\int_{Q^\epsilon_i}\overline{p}_\epsilon\theta^\epsilon &= \frac{1}{L}\sum_{m=1}^{2^i}\int_{Q_i\times Y_{i,m}} T^\epsilon_{i,m}\overline{p}_\epsilon\,T^\epsilon_{i,m}\theta^\epsilon\,dxdydt = \frac{1}{L}\sum_{m=1}^{2^i}\int_{Q_i\times Y_{i,m}} T^\epsilon_{i,m}\overline{p}_\epsilon\,\theta\,dxdydt \\ &= \sum_{m=1}^{2^i}\int_{]0,T[\times\Lambda_{i,m}}\left(\frac{1}{L}\int_0^L T^\epsilon_{i,m}\overline{p}_\epsilon\,dx_1\right)\theta\end{aligned}$$

By choosing $\theta \in L^2(\Lambda^+_T)$ such that $\theta = 0$ on the complement of $]0,T[\times\Lambda_{i,m}$ (zero in $]0,T[\times\Lambda_{j,k}$ for $j \neq i,\ k \neq m$), we can show that $\overline{\theta}_\epsilon = -\frac{1}{\beta L}\left[\int_0^L T^\epsilon_{i,m}\overline{p}_\epsilon dx_1\right]$ a.e. in $]0,T[\times\Lambda_{i,m}$. □

Remark 6.1
Because we have introduced unfolding operators $T^{\epsilon}_{i,m}$ for each stage i and each branch m, we could obtain the characterization of the optimal control separately in $\Lambda_{i,m}$, $i = 0, 1, 2$, $m = 1, \cdots, 2^i$.

6.2. *Homogenized problem*

The limit problem is: find $(\overline{u}, \overline{\theta}) \in L^2(0, T; \mathcal{H}) \times L^2(]0, T[\times(M_0, M_3))$, such that

$$(P) \quad J(\overline{u}, \overline{\theta}) = \inf\{J(u, \theta) | (u, \theta) \in L^2(0, T; \mathcal{H}) \times L^2(]0, T[\times(M_0, M_3)),\ (u, \theta) \text{ satisfies (6.4)}\}.$$

The cost functional is defined as

$$J(u, \theta) = \frac{1}{2}\int_{Q^-} |u - u_d|^2 + \frac{1}{2}\sum_{i=0}^{2}\int_{Q_i}\left(\sum_{m=1}^{2^i}\frac{h_{im}}{L}|u_{i,m} - u_d|^2\right) + \frac{1}{2}\int_{Q^-}|u(T) - u_d(T)|^2$$
$$+ \frac{1}{2}\sum_{i=0}^{2}\int_{Q_i}\left(\sum_{m=1}^{2^i}\frac{h_{im}}{L}\,|u_{i,m}(T) - u_d(T)|^2\right) + \frac{\beta}{2}\sum_{i=0}^{2}\int_0^T\int_{M_i}^{M_{i+1}}\sum_{m=1}^{2^i} h_{im}|\theta_i|^2.$$

Here $\theta_i = \theta|_{]0,T[\times]M_i,M_{i+1}[}$. The state equation is defined by

$$\langle u'(t), \psi\rangle + \langle \mathcal{A}u, \psi\rangle = \langle F(\theta), \psi\rangle, u(0) = 0, \tag{6.4}$$

$\forall\ \psi\ \in\ W(0, T; \mathcal{H}, \mathcal{H}^*)$, where $\mathcal{A}$ and $F(\theta)$ are defined as follows:

$$\langle \mathcal{A}u, \psi\rangle := \int_{Q^-}(\nabla u^- \cdot \nabla\psi^- + u^-\psi^-)dxdt + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\frac{h_{i,m}}{L}\int_{Q_i}(\partial_{x_2}u_{i,m}\,\partial_{x_2}\psi_{i,m} + u_{i,m}\,\psi_{i,m})\,dxdt \tag{6.5}$$

for all $u, \psi \in L^2(0, T; \mathcal{H})$ and $\langle F(\theta), \psi\rangle := \int_{Q^-} f_0\psi^-\, dxdt + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\frac{h_{i,m}}{L}\int\limits_{Q_i}\theta_i\,\psi_{i,m}\,dxdt$,

$$\langle u'(t), \psi\rangle = -\int_{Q^-} u^-\,\partial_t\psi^- dxdt - \sum_{i=0}^{2}\sum_{m=1}^{2^i}\frac{h_{i,m}}{L}\int\limits_{Q_i} u_{i,m}\,\partial_t\psi_{i,m}\,dxdt$$
$$+ \int_{\Omega^-} u^-(T)\psi^-(T)dx + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\frac{h_{i,m}}{L}\int\limits_{\Omega_i} u_{i,m}(T)\,\psi_{i,m}(T)\,dx$$

The analogous strong form (Euler equation) can also be defined like in (5.16). The adjoint state $\overline{p} \in W(0, T; \mathcal{H}, \mathcal{H}^*)$ solves,

$$\begin{cases} -\langle \overline{p}'(t), \psi\rangle + \langle \mathcal{A}\overline{p}, \psi\rangle = \int_{Q^-}(\overline{u}^- - u_d)\psi^- + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q_i}\left(\frac{h_{im}}{L}(\overline{u}_{i,m} - u_d)\psi_{i,m}\right), \\ \overline{p}^-(T) = \overline{u}^-(T) - u_d(T) \text{ in } \Omega^-;\ \overline{p}_{i,m}(T) = \overline{u}_{i,m}(T) - u_d(T) \text{ in } \Omega_i, \end{cases} \tag{6.6}$$

$\forall\ \psi\ \in\ W(0, T; \mathcal{H}, \mathcal{H}^*)$. The following result can be easily verified as in the previous section.

Theorem 6.3
The optimal control problem (P) has a unique solution.

Theorem 6.4
If $(\overline{u}, \overline{\theta})$ is an optimal solution to (P), then

$$\overline{\theta}_i =: \overline{\theta}|_{]0,T[\times]M_i,M_{i+1}[} = \frac{-1}{\beta r_i L}\int_0^L\sum_{m=1}^{2^i} h_{i,m}\overline{p}_{i,m}dx_1 \quad for\ \ i = 0, 1, 2$$

where $\overline{p}\ \in\ W(0, T; \mathcal{H}, \mathcal{H}^*)$ is the solution to the adjoint Eq. (6.6). Conversely, assume that the pair $(\hat{u}, \hat{p})\ \in\ W(0, T; \mathcal{H}, \mathcal{H}^*) \times W(0, T; \mathcal{H}, \mathcal{H}^*)$ solves the optimality system

$$\begin{cases} \langle \hat{u}'(t), \psi\rangle + \langle \mathcal{A}\hat{u}, \psi\rangle = \langle F(\hat{\theta}), \psi\rangle; \ \hat{u}(0) = 0, \\ -\langle \hat{p}'(t), \psi\rangle + \langle \mathcal{A}\hat{p}, \psi\rangle = \int_{Q^-}(\hat{u}^- - u_d)\psi^- + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q_i}\left(\frac{h_{im}}{L}(\hat{u}_{i,m} - u_d)\psi_{i,m}\right), \forall\ \psi\ \in\ W(0,T;\mathcal{H},\mathcal{H}^*), \\ \hat{p}^-(T) = (\hat{u}^-(T) - u_d(T))\ in\ \Omega^-;\ \hat{p}_{i,m}(T) = \hat{u}_{i,m}(T) - u_d(T)\ in\ \Omega_i, \\ \hat{\theta}_i =: \hat{\theta}|_{]0,T[\times]M_i,M_{i+1}[} = \frac{-1}{\beta r_i L}\int_0^L \sum_{m=1}^{2^i} h_{i,m}\hat{p}_{i,m}dx_1\ for\ i = 0,1,2. \end{cases} \tag{6.7}$$

Then, the pair $(\hat{u}, \hat{\theta})$ is the optimal solution to (P). Here $r_i = \sum_{m=1}^{2^i} h_{i,m}$.

Proof

Assume $(\overline{u}, \overline{\theta})$ is an optimal solution to (P) and $u(\theta)$ is a solution of (6.4) for a fixed arbitrary θ. Now, set $F(\theta) = J(u(\theta), \theta)$, then

$$\begin{aligned} F'(\overline{\theta})\theta = &\int_{Q^-}(\overline{u}^- - u_d)w^{\theta^-} + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q_i}\frac{h_{i,m}}{L}(\overline{u}_{i,m} - u_d)w^{\theta}_{i,m} \\ &+ \int_{\Omega^-}(\overline{u}^-(T) - u_d(T))w^{\theta^-}(T) + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{\Omega_i}\frac{h_{i,m}}{L}(\overline{u}_{i,m}(T) - u_d(T))w^{\theta}_{i,m}(T) \\ &+ \beta\sum_{i=0}^{2}\int_0^T\int_{M_i}^{M_{i+1}}\sum_{m=1}^{2^i} h_{i,m}\,\overline{\theta}_i\theta_i, \end{aligned}$$

where $\theta_i = \theta|_{]0,T[\times]M_i,M_{i+1}[}$ and the multi-sheeted function w^θ is the solution of the Eq. (6.8) given in the succeeding text.

$$\langle \partial_t w^\theta(t), \psi\rangle + \langle \mathcal{A}w^\theta, \psi\rangle = \sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q_i}\frac{h_{i,m}}{L}\theta_i\psi_{i,m}, \tag{6.8}$$

with $w^\theta(0) = 0,\ \forall\ \psi\ \in\ W(0,T;\mathcal{H},\mathcal{H}^*)$. Because $(\overline{u}, \overline{\theta})$ is a solution to (P), we have $F'(\overline{\theta})\theta = 0$ for all $\theta \in L^2(0,T;L^2(M_0,M_3))$. That is,

$$\begin{aligned} &\int_{Q^-}(\overline{u}^- - u_d)w^{\theta^-} + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q_i}\frac{h_{i,m}}{L}(\overline{u}_{i,m} - u_d)w^\theta_{i,m} + \int_{\Omega^-}(\overline{u}^-(T) - u_d(T))w^{\theta^-}(T) \\ &\quad + \sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{\Omega_i}\frac{h_{i,m}}{L}(\overline{u}_{i,m}(T) - u_d(T))w^\theta_{i,m}(T) = -\beta\sum_{i=0}^{2}\int_0^T\int_{M_i}^{M_{i+1}}\sum_{m=1}^{2^i} h_{i,m}\,\overline{\theta}_i\theta_i. \end{aligned} \tag{6.9}$$

Choosing w^θ and $\overline{p}$ as the test functions in the weak formulation of the Eqs (6.6) and (6.8) respectively, we obtain

$$\begin{cases} \langle \partial_t w^\theta(t), \overline{p}\rangle + \langle \mathcal{A}\overline{p}, w^\theta\rangle &= +\int_{\Omega^-}\overline{p}^-(T)w^{\theta^-}(T) + \sum_{i=0}^2\sum_{m=1}^{2^i}\int_{\Omega_i}\frac{h_{i,m}}{L}\overline{p}_{i,m}(T)w^\theta_{i,m}(T) \\ &\quad + \int_{Q^-}(\overline{u}^- - u_d)w^{\theta^-} + \sum_{i=0}^2\sum_{m=1}^{2^i}\int_{Q_i}\frac{h_{im}}{L}(\overline{u}_{i,m} - u_d)w^\theta_{i,m}, \\ \langle \partial_t w^\theta(t), \overline{p}\rangle + \langle \mathcal{A}\overline{p}, w^\theta\rangle &= \sum_{i=0}^2\sum_{m=1}^{2^i}\int_{Q_i}\frac{h_{i,m}}{L}\theta_i\overline{p}_{i,m}, \end{cases} \tag{6.10}$$

with $\overline{p}(T) = \overline{u}(T) - u_d(T)$. Now, we compare the Eqs (6.9) and (6.10) to obtain

$$\sum_{i=0}^{2}\sum_{m=1}^{2^i}\int_{Q_i}\frac{h_{i,m}}{L}\theta_i\overline{p}_{i,m} = -\beta\sum_{i=0}^{2}\int_0^T\int_{M_i}^{M_{i+1}}\sum_{m=1}^{2^i} h_{i,m}\,\overline{\theta}_i\theta_i.$$

That is, $\sum_{i=0}^2\int_0^T\int_{M_i}^{M_{i+1}}\left[\int_0^L\sum_{m=1}^{2^i}\frac{h_{i,m}}{L}\overline{p}_{i,m}\right]\theta_i = \sum_{i=0}^2\int_0^T\int_{M_i}^{M_{i+1}}\left[-\beta\sum_{m=1}^{2^i} h_{i,m}\,\overline{\theta}_i\right]\theta_i$.

Hence we obtain, $\overline{\theta}_i = \frac{-1}{\beta r_i L}\int_0^L\left[\sum_{m=1}^{2^i} h_{i,m}\overline{p}_{i,m}\right]$, where $r_i = \sum_{m=1}^{2^i} h_{i,m}$. □

6.3. Convergence analysis

Assume that $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ is the optimal solution to the problem (P_ϵ). Let $u_\epsilon(0)$ be the solution to the problem (3.5) corresponding to $\theta_\epsilon = 0$, then from (3.4), we obtain $\|u_\epsilon(0)\|_{L^2(0,T;H^1(\Omega_\epsilon))} \leq C$, where $C > 0$ is independent of ϵ. Using optimality of the solution $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$, we obtain

$$\int_{Q_\epsilon}(\overline{u}_\epsilon - u_d)^2 + \int_{\Omega_\epsilon}(\overline{u}_\epsilon(T) - u_d(T))^2 + \frac{\beta}{2}\int_{Q^+_\epsilon}|\overline{\theta}^\epsilon_\epsilon|^2 \leq C.$$

Thus, we have

$$\|\overline{\theta}_\epsilon^\epsilon\|_{L^2(Q_\epsilon^+)} = \|\overline{\theta}_\epsilon\|_{L^2(]0,T[\times\Lambda+)} \le C \quad \text{and } \|\overline{u}_\epsilon\|_{L^2(0,T;L^2(\Omega_\epsilon))} \le C. \tag{6.11}$$

Further, $\overline{p}_\epsilon$ satisfies $\|\overline{p}_\epsilon\|_{L^2(Q_\epsilon)} \le C$.

Theorem 6.5
Let $(\overline{u}_\epsilon, \overline{\theta}_\epsilon)$ and $(\overline{u}, \overline{\theta})$ be the solutions of (P_ϵ) and of (P), respectively. Then

$$\begin{aligned}
&\overline{\theta}_\epsilon|_{]0,T[\times\Lambda_i} \rightharpoonup \overline{\theta}_i \quad \textit{weakly in } L^2(\Lambda_i\times]0,T[), \textit{for } i=0,1,2,\\
&\widetilde{u_{\epsilon\,|\Omega^\epsilon_{0,1}}} \rightharpoonup \frac{h_{0,1}}{L}u_{0,1} \quad \textit{weakly in } \mathcal{X}_0, \quad \widetilde{u_{\epsilon\,|\Omega^\epsilon_{1,m}}} \rightharpoonup \frac{h_{1,m}}{L}u_{1,m} \quad \textit{weakly in } \mathcal{X}_1, \quad \textit{for } m=1,2,\\
&\widetilde{u_{\epsilon\,|\Omega^\epsilon_{2,m}}} \rightharpoonup \frac{h_{2,m}}{L}u_{2,m}\ \textit{weakly in } \mathcal{X}_2, \quad \textit{for } m=1,2,3,4, \quad u_\epsilon|_{\Omega^-} \rightharpoonup \overline{u}^- \quad \textit{weakly in } \mathcal{X}.
\end{aligned}$$

where $\overline{\theta}_i = \frac{-1}{\beta r_i L}\int_0^L \left[\sum_{m=1}^{2^i} h_{i,m}\overline{p}_{i,m}\right]$. Here $\widetilde{\ }$ represents the zero extension and $r_i = \sum_{m=1}^{2^i} h_{i,m}$ for $i = 0,1,2$.

Proof
Using the weak formulation of the state equation (3.5), we have

$$\|\overline{u}_\epsilon'\|_{L^2(0,T;H^1(\Omega_\epsilon)^*)} + \|\overline{u}_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))} \le C(\|f_0\|_{L^2(Q^-)} + \|\overline{\theta}_\epsilon\|_{L^2(]0,T[\times\Lambda)}),$$

where C is independent of ϵ. Using (6.11), we derive $\|\overline{u}_\epsilon'\|_{L^2(0,T;H^1(\Omega_\epsilon)^*)} \le C$, $\|\overline{u}_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))} \le C$. Because $\|\overline{\theta}_\epsilon\|_{L^2(\Lambda_T^+)}$ is bounded, there exists a subsequence (still denoted by ϵ) and a $\theta_0 \in L^2(\Lambda_T^+)$ such that

$$\overline{\theta}_\epsilon \rightharpoonup \theta_0 \quad \text{weakly in } L^2(\Lambda_T^+). \tag{6.12}$$

By using the similar analysis discussed in Section 4, we can easily derive the following convergences.

$$\begin{aligned}
&\widetilde{u_{\epsilon\,|Q^\epsilon_{0,1}}} \rightharpoonup \frac{h_{0,1}}{L}u_{0,1} \text{ weakly in } \mathcal{X}_0, \quad \widetilde{u_{\epsilon\,|Q^\epsilon_{1,m}}} \rightharpoonup \frac{h_{1,m}}{L}u_{1,m} \text{ weakly in } \mathcal{X}_1, \ \textit{for } m=1,2,\\
&\widetilde{u_{\epsilon\,|Q^\epsilon_{2,m}}} \rightharpoonup \frac{h_{2,m}}{L}u_{2,m} \text{ weakly in } \mathcal{X}_2, \ \textit{for } m=1,2,3,4, \quad \overline{u}_\epsilon|_{Q^-} \rightharpoonup u|_{Q^-} \text{ weakly in } \mathcal{X},
\end{aligned}$$

where u satisfies the limit Eq (6.4) with $\overline{\theta} = \theta_0$. Also we have the following convergence for $\overline{p}_\epsilon$

$$\begin{aligned}
&\widetilde{\overline{p}_{\epsilon\,|Q^\epsilon_{0,1}}} \rightharpoonup \frac{h_{0,1}}{L}p_{0,1} \quad \text{weakly in } \mathcal{X}_0, \quad \widetilde{\overline{p}_{\epsilon\,|Q^\epsilon_{1,m}}} \rightharpoonup \frac{h_{1,m}}{L}p_{1,m} \quad \text{weakly in } \mathcal{X}_1, \ \textit{for } m=1,2,\\
&\widetilde{\overline{p}_{\epsilon\,|Q^\epsilon_{2,m}}} \rightharpoonup \frac{h_{2,m}}{L}p_{2,m} \text{ weakly in } \mathcal{X}_2, \ \textit{for } m=1,2,3,4, \quad \overline{p}_\epsilon|_{Q^-} \rightharpoonup p|_{Q^-} \text{ weakly in } \mathcal{X}.
\end{aligned}$$

where p_0 satisfies the limit Eq (6.6) with $\overline{u} = u$.

To prove the convergence of the optimality system, now it is enough to prove $\overline{\theta} = \theta_0$. Recall the optimality condition (6.1):

$$\overline{\theta}_\epsilon(t,y_1,y_2)|_{]0,T[\times\Lambda_{i,m}} = \frac{-1}{\beta\, L}\int_0^L T^\epsilon_{i,m}\,\overline{p}_\epsilon(t,x_1,y_2,y_1)dx_1 \tag{6.13}$$

$\forall\ i = 0,1,2$ and $m = 1,\cdots,2^i$. By the convergence (6.12), we obtain

$$\overline{\theta}_\epsilon|_{]0,T[\times\Lambda_{i,m}} \rightharpoonup \theta_{i,m} \text{ in } L^2(]0,T[\times\Lambda_{i,m}). \tag{6.14}$$

Now, we apply the homogenization results of Section 5 on the adjoint state to obtain the convergence

$$T^\epsilon_{i,m}\,\overline{p}_\epsilon \rightharpoonup p_{i,m} \text{ weakly in } L^2(Q_i\times Y_{i,m}) \quad \text{as } \epsilon \to 0. \tag{6.15}$$

Now, as $\epsilon \to 0$, the Eq. (6.13) becomes

$$\theta_{i,m}(t,y_1,y_2) = \frac{-1}{\beta L}\int_0^L p_{i,m}(t,x_1,y_2,y_1)dx_1 \ \ \textit{for } i=0,1,2 \text{ and } m=1,\cdots,2^i.$$

By noting the fact that $p_{i,m}$ is independent of y_1, we conclude that $\theta_{i,m}$ also independent of the variable y_1.
Consider $\overline{\theta}_\epsilon|_{]0,T[\times\Lambda_i} = \frac{-1}{\beta\, L}\sum_{m=1}^{2^i}\int_0^L T^\epsilon_{i,m}\,\overline{p}_\epsilon dx_1$ for all $i = 0,1,2$. Choose a test function $\psi \in L^2(]0,T[\times\Lambda_i)$ such that ψ is independent of y_1, then

$$\int_{]0,T[\times\Lambda_i} \overline{\theta}_\epsilon \psi = \frac{-1}{\beta\, L} \sum_{m=1}^{2^i} \int_{]0,T[\times\Lambda_{i,m}} \int_0^L T^\epsilon_{i,m}\, \overline{p}_\epsilon \psi = \frac{-1}{\beta\, L} \int_0^T \int_0^L \int_{M_i}^{M_{i+1}} \sum_{m=1}^{2^i} \int_{Y_{i,m}} T^\epsilon_{i,m}\, \overline{p}_\epsilon \psi.$$

As $\epsilon \to 0$, we obtain $\int_0^T \int_{M_i}^{M_{i+1}} \sum_{m=1}^{2^i} \int_{Y_{i,m}} \theta_i \psi = \frac{-1}{\beta\, L} \int_0^T \int_0^L \int_{M_i}^{M_{i+1}} \sum_{m=1}^{2^i} \int_{Y_{i,m}} p_{i,m} \psi$ which in turn implies

$$\int_0^T \int_{M_i}^{M_{i+1}} \left(\sum_{m=1}^{2^i} h_{i,m} \theta_i \right) \psi = \int_0^T \int_{M_i}^{M_{i+1}} \left(\frac{-1}{\beta\, L} \int_0^L \sum_{m=1}^{2^i} h_{i,m} p_{i,m} \right) \psi.$$

Because θ_i is independent of y. Hence, $\theta_i = \frac{-1}{\beta r_i L} \int_0^L \sum_{m=1}^{2^i} h_{i,m} p_{i,m}$ for $i = 0, 1, 2$ and $r_i = \sum_{m=1}^{2^i} h_{i,m}$. Therefore, we obtain the optimality system corresponding to the minimization problem (P). The Theorem 6.4 says that, the optimal solution is (u_0, θ_0). Hence, by the uniqueness, we have, $\overline{u} = u$, $\overline{p} = p_0$ and $\overline{\theta} = \theta_0$ which completes the proof. □

7. Conclusions

We have considered an optimal control problem in a domain with branched structure, depending on a small parameter $\epsilon > 0$. We have applied control in the oscillating interior part of the domain. Here, we have made a special consideration, namely, we took the control in a fixed reference domain Λ_T^+, and we scaled and distributed it periodically to the oscillating part of the domain under consideration. We have derived the optimality condition for this special kind of control problem for which we have introduced and utilized the periodic unfolding operators. Finally, we have found the homogenization of the optimal control problem as the parameter ϵ goes to zero. In the limit problem, we obtain a multi-sheeted function which characterize the contribution of the restriction of the state solution u_ϵ at each branch of each level. This was achieved by introducing unfolding operators on the different branches at different levels.

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References

1. Lenczner M. Multiscale model for atomic force microscope array mechanical behavior. *Applied Physics Letters* 2007; **90**:091908.
2. Lyshevshi SE. *Mems and Nems: Systems, Devices, and Structures*. CRC Press: Boca Raton, FL, 2002.
3. Durante T, Mel'nyk TA. Homogenization of quasilinear optimal control problems involving a thick multilevel junction of type 3 :2 :1. *ESAIM Control, Optimisation and Calculus of Variations* 2012; **18**(2):583–610.
4. Durante T, Mel'nyk TA. Asymptotic analysis of an optimal control problem involving a thick two-level junction with alternate type of controls. *Journal of Optimization Theory and Applications* 2010; **144**:205–225.
5. De Maio U, Durante T, Mel'nyk TA. Asymptotic approximation for the solution to the Robin problem in a thick multi-level junction. *Mathematical Models and Methods in Applied Sciences* 2005; **15**(12):1897–1921.
6. Mel'nyk TA. Asymptotic approximation for the solution to a semi-linear parabolic problem in a thick junction with the branched structure. *Journal of Mathematical Analysis and Applications* 2015; **424**:1237–1260.
7. Nandakumaran AK, Prakash R, Sardar BC. Homogenization of an optimal control problem in a domain with highly oscillating boundary using periodic unfolding method. *Mathematics in Engineering, Science and Aerospace* 2013; **4**(3):281–303.
8. Nandakumaran AK, Prakash R, Sardar BC. Homogenization of an optimal control via unfolding method. *SIAM Journal on Control and Optimization* 2015; **53**(5):3245–3269.
9. Prakash R, Sardar BC. Homogenization of boundary optimal control problem in a domain with highly oscillating boundary via periodic unfolding method. *Nonlinear Studies- The International Journal* 2015; **22**(2):213–240.
10. Amirat Y, Bodart O, De Maio U, Gaudiello A. Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary. *SIAM Journal on Mathematical Analysis* 2004; **35**(6):1598–1616.
11. Blanchard D, Gaudiello A. Homogenization of highly oscillating boundaries and reduction of dimensionfor monotone problem. *ESAIM Control, Optimisation and Calculus of Variations* 2003; **9**:449–460.
12. Blanchard D, Gaudiello A, Mel'nyk TA. Boundary homogenization and reduction of dimension in a Kirchhoff-Love plate. *SIAM Journal on Mathematical Analysis* 2008; **39**(6):1764–1787.
13. Mel'nyk TA, Vashchuk PS. Homogenization of a boundary-value problem with varying type of boundary conditions in a thick two-level junction. *Nonlinear Oscillations* 2005; **8**(2):240–255.
14. Mel'nyk TA. Homogenization of the Poisson equation in a thick periodic junction. *Zeitschrift fur Analysis und ihre Anwendungen* 1999; **18**(4):953–975.
15. Mel'nyk TA. Homogenization of a singularly perturbed parabolic problem in thick periodic junction of the type 3:2:1. *Ukrainian Mathematical Journal* 2000; **52**(11):1737–1748.
16. De Maio U, Gaudiello A, Lefter C. Optimal control for a parabolic problem in a domain with highly oscillating boundary. *Applicable Analysis* 2004; **83**(12):1245–1264.
17. Nandakumaran AK, Prakash R, Raymond JP. Asymptotic analysis and error estimates for an optimal control problem with oscillating boundaries. *Annali dell'Universita di Ferrara* 2012; **58**:143–166.

18. Prakash R. Optimal control problem for the time-dependent Kirchhoff-Love plate in a domain with rough boundary. *Asympttic Analysis* 2013; **81**: 337–355.
19. Durante T, Faella L, Perugia C. Homogenization and behavior of optimal controls for the wave equation in domains with oscillating boundary. *Nonlinear Differential Equations and Applications* 2007; **14**:455–489.
20. Esposito AC, Donato P, Gaudiello A, Picard C. Homogenization of the p-Laplacian in a domain with oscillating boundary. *Communications on Applied Nonlinear Analysis* 1997; **4**(4):1–23.
21. Gaudiello A. Asymptotic behaviour of non-homogeneous Neumann problems in domains with oscillating boundary. *Ricerche Mat* 1994; **43**(2): 239–292.
22. Gaudiello A, Hadiji R, Picard C. Homogenization of the Ginzburg-Landau equation in a domain with oscillating boundary. *Communications on Pure and Applied Analysis* 2003; **7**(2-3):209–223.
23. Cioranescu D, Damlamian A, Griso G. Periodic unfolding and homogenization. *Comptes Rendus Mathematique* 2002; **335**(1):99–104.
24. Cioranescu D, Damlamian A, Griso G. The periodic unfolding method in homogenization. *SIAM Journal on Mathematical Analysis* 2008; **40**:1585–1620.
25. Damlamian A. An elementary introduction to periodic unfolding, in multi scale problems and asymptotic analysis. *GAKUTO International Series Mathematical Sciences and Applications* 2005; **24**:119–136.
26. Damlamian A, Pettersson K. Homogenization of oscillating boundaries. *Discrete and Continuous Dynamical Systems* 2009; **23**(1-2):197–210.
27. Blanchard D, Gaudiello A, Griso G. Junction of a periodic family of elastic rods with a 3d plate. Part I. *Journal de Mathématiques Pures et Appliquées* 2007; **88**(1):1–33.
28. Blanchard D, Gaudiello A, Griso G. Junction of a periodic family of elastic rods with a thin plate. Part II. *Journal de Mathématiques Pures et Appliquées* 2007; **88**(2):149–190.
29. Gaudiello A, Sili A. Homogenization of highly oscillating boundaries with strongly contrasting diffusivity. *SIAM Journal on Mathematical Analysis* 2015; **47**(3):1671–1692.
30. Bensoussan A, Lions J-L, Papanicolaou G. *Asymptotic Analysis for Periodic Structures*. North-Holland Pub. Co.: Amsterdam; New York:, 1978.
31. Cioranescu D, Donato P. *An Introduction to Homogenization*, Oxford Lecture Notes Series in Mathematics and its Applications, 17. Oxford University Press: Oxford, 1999.
32. Tartar L. The General theory of homogenization. In *A Personalized Introduction*. Lecture notes of the union mathematica Italian, Vol. 7. Springer-Verlag, Berlin, 2009.
33. Lions J-L. *Optimal Control of Systems Governed by Partial Differential Equations., Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170*. Springer-Verlag: New York, 1971.
34. Lions J-L. *Some Methods in the Mathematical Analysis of Systems and Their Control*. Kexue Chubanshe (Science Press): Beijing, 1981.
35. Lions J-L. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribues. Tome 1, Recherches en Mathéematiques Appliquees [Research in Applied Mathematics], vol. 8,Masson, Paris, Controlabilite exacte. (1988)[Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch*.
36. Lions J-L. *Controlabilite exacte, perturbations et stabilisation de syst'emes distribues. Tome 2, Recherches en Mathematiques Appliquees [Research in Applied Mathematics](1988) vol. 9, Masson, Paris, Perturbations*.
37. Raymond JP. *Optimal Control of Partial Differential Equations*. Institut de Mathatiques de Toulouse, University Paul Sabatier: 31062 Toulouse Cedex, France. http://www.math.univ-toulouse.fr/raymond/book-ficus.pdf.
38. Showalter RE. Monotone operators in Banach space and non-linear partial differential equations. In *Mathematical Surveys and Monographs*, Vol. 49. American Mathematical Society, 1997.