Appl Math Optim
https://doi.org/10.1007/s00245-018-9499-4

Homogenization of an Elliptic Equation in a Domain with Oscillating Boundary with Non-homogeneous Non-linear Boundary Conditions

Rajesh Mahadevan[1] · **A. K. Nandakumaran**[2] · **Ravi Prakash**[1]

Abstract While considering boundary value problems with oscillating coefficients or in oscillating domains, it is important to associate an asymptotic model which accounts for the average behaviour. This model permits to obtain the average behaviour without costly numerical computations implied by the fine scale of oscillations in the original model. The asymptotic analysis of boundary value problems in oscillating domains has been extensively studied and involves some key issues such as: finding uniformly bounded extension operators for function spaces on oscillating domains, the choice of suitable sequences of test functions for passing to the limit in the variational formulation of the model equations etc. In this article, we study a boundary value problem for the Laplacian in a domain, a part of whose boundary is highly oscillating (periodically), involving non-homogeneous non-linear Neumann or Robin boundary condition on the periodically oscillating boundary. The non-homogeneous Neumann condition or the Robin boundary condition on the oscillating boundary adds a further difficulty to the limit analysis since it involves taking the limits of surface integrals where the surface changes with respect to the parameter. Previously, some model problems have been studied successfully in Gaudiello (Ricerche Mat 43(2):239–292, 1994) and in Mel'nyk (Math Methods Appl Sci 31(9):1005–1027, 2008) by converting the surface term into a volume term using auxiliary boundary value problems. Some

✉ Rajesh Mahadevan
rmahadevan@udec.cl

A. K. Nandakumaran
nands@iisc.ac.in

Ravi Prakash
raviprakash.iisc@gmail.com

1 Depto. de Matemática, FCFM, Univ. de Concepción, Concepción, Chile

2 Department of Mathematics, Indian Institute of Science, Bengaluru, India

Published online: 10 May 2018

problems of this nature have also been studied using an extension of the notion of two-scale convergence (Allaire et al. in Proceedings of the international conference on mathematical modelling of flow through porous media, Singapore, 15–25, 1996, Neuss-Radu in C R Acad Sci Paris Sr I Math 322:899–904, 1996). In this article, we use a different approach to handle of such terms based on the unfolding operator.

Keywords Homogenization · Oscillating boundary · Unfolding operator

Mathematics Subject Classification 35J20 · 35J25 · 35B40 · 76M50 · 78M40 · 80M40

1 Introduction

In this article, we study the asymptotic behavior of a boundary value problem for the Laplacian in a domain, a part of whose boundary is highly oscillating in a periodic manner. The asymptotic analysis of boundary value problems involving oscillating coefficients or in oscillating domains, often called homogenization, has been the subject of extensive study. We refer to the following texts for an overview of several such problems [10,20,26]. In particular, homogenization problems involving oscillating boundaries also has been subject to a lot of study ever since the works of [14,15,27,38]. The principal motivations for studying boundary value problems in domains with oscillating boundaries come from modelisation issues involving heat radiators, flows over highly oscillating channels, absorption-diffusion in biological structures, acoustic vibrations in a medium with narrow channels etc. Other such models frequently arise in the context of thick junctions [34]. For a sampling of research on the asymptotic analysis of boundary value problems in domains with oscillating boundaries, we refer to [1,5,6,8,11–13,16,17,22,24,30,32].

The asymptotic analysis of boundary value problems in oscillating domains has been extensively studied and this involves some key issues such as: finding uniformly bounded extension operators for function spaces on oscillating domains, the choice of suitable sequences of test functions for passing to the limit in the variational formulation of the model equations etc. A simple choice of extension operators is the extension by zero to the limit domain but these extension operators are not uniformly bounded between the function spaces except for homogeneous Dirichlet boundary conditions on the oscillating boundary. Even for homogeneous Neumann boundary conditions on the oscillating boundary, normally, one needs to find other classes of extension operators which are uniformly bounded before one can pass to the limit in the equations. This, of course, will depend on the geometry of the oscillating domains and a discussion of the existence of such operators for Neumann problems in perforated or oscillating domains may be found, for instance, in [21,23,31]. In certain classes of problems, involving Neumann or Robin boundary conditions on the oscillating boundary, one can also handle this issue differently by extending separately the functions and their derivatives by zero to the limit domain. The passage to the limit is usually straightforward but the delicate part consists in correctly identifying the constitutive relations among the various limit terms. Such an approach has been used, for example, in [25,29] for obtaining the homogenized models.

In this article, we study a boundary value problem for the Laplacian in a domain involving non-homogeneous non-linear Neumann or Robin boundary condition on the periodically oscillating boundary. The non-homogeneous Neumann condition or the Robin boundary condition on the oscillating boundary are motivated by applications to control problems. This adds a further difficulty to the limit analysis since it involves taking the limits of surface integrals where the surface (oscillating boundary) changes with respect to the parameter. This issue has been handled successfully in Gaudiello [25] where a model problem involving non-homogeneous Neumann condition on the oscillating boundary has beed studied. Similar model problem involving non-linear Robin boundary condition on the oscillating boundary is studied in Mel'nyk [29]. They achieve it by converting the surface term into a volume term using auxiliary boundary value problems on suitable sections of the oscillating domain. This idea can be traced back to the works of Vanninathan [39,40] but the introduction of the auxiliary boundary value problems, is in some sense artificial, and has no final effect on the limit problem. Surface terms in periodic homogenization can also be handled using the extension of the two-scale convergence method proposed in [35] (see also [4]). The two-scale convergence method was originally introduced by Nguetseng [36] and subsequently, developed by Allaire [3]. Instead, in this article, we use a more natural and straightforward approach to handle such terms based on the unfolding operator. Previously, this method has been applied in the homogenization of a boundary value problem in an oscillating domain for handling homogeneous Neumann condition in Damlamian and Pettersson [22]. In this article, we show how it can be successfully applied in the asymptotic analysis of boundary value problems involving more complicated boundary terms as is the case in the case of non-homogeneous Neumann boundary condition or Robin boundary condition on the oscillating boundary. The basic idea of the unfolding operator is to effect a separation of scales by blowing up the function around any point with respect to the period. This means instead of the value at the point we keep complete information of the function around the point at the level of each cell.

The organization of the article is as follows. In Sect. 2, we present the setting of the problem and state the main results. In Sect. 3, we give a brief overview of the unfolding operator and some of its main properties. In Sect. 4, we prove some a priori estimates required for the asymptotic analysis. In Sect. 5, we provide the asymptotic analysis for the model problem with non-homogeneous Neumann condition on the oscillating boundary. This can be seen as a special case of the model involving the non-homogeneous non-linear Robin boundary condition considered in Sect. 6. But we prefer to present, separately, the simpler case involving non-homogeneous boundary condition first to fix the basic ideas.

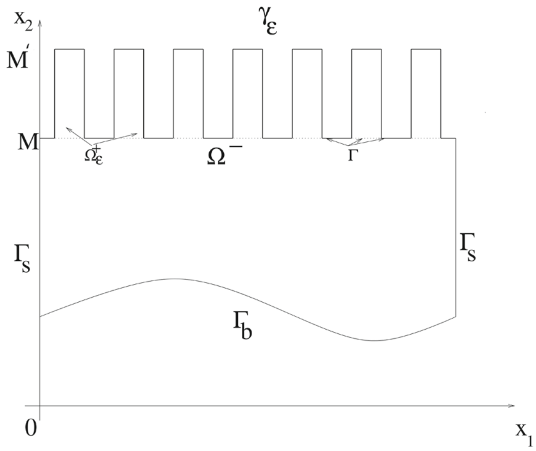

Fig. 1 Ω_ε

2 Problem Settings

2.1 Configuration of the Domain

For the simplicity of presentation, we consider a planar domain with regular edges on the sides and the bottom but whose upper boundary is highly oscillating with a periodic structure and is rectilinear like that which was considered in [5]. The amplitude of oscillations is of order 1 whereas the period of the oscillation is of order ε with $\varepsilon > 0$, a small parameter. This model corresponds to a 2:1:1 thick junction in the sense considered by Mel'nyk and Nazarov [31] which means that the body is two dimensional while the junction (interface with the body) is one dimensional with several periodic one dimensional protrusions. For the sake of convenience of the readers, we give a full description of it again.

Let $\varepsilon = \frac{1}{N}$, $N \in \mathbb{Z}^+$ where we let, eventually, $N \to +\infty$. We consider the two dimensional ε-dependent domain Ω_ε which consists of two parts, namely, Ω_ε^+ and Ω^-. As it is clear from notations, Ω^- is independent of ε but Ω_ε^+ has a highly oscillating ε-dependent boundary which we denote by γ_ε (see Fig. 1).

Let $L,\ M,\ M' > 0$ be positive constants and $g : \mathbb{R} \to \mathbb{R}$ be a smooth L-periodic function such that $M' > M > m$ where m is the maximum value of the smooth function g in $[0, L]$. We denote the fixed portion Ω^- of Ω_ε as

$$\Omega^- := \{(x_1, x_2) :\ 0 < x_1 < L, g(x_1) < x_2 < M\}$$

and the ε-dependent portion Ω_ε^+ of Ω_ε as

$$\Omega_\varepsilon^+ = \bigcup_{k=0}^{N-1} (k\varepsilon L + \varepsilon a, k\varepsilon L + \varepsilon b) \times (M, M')$$

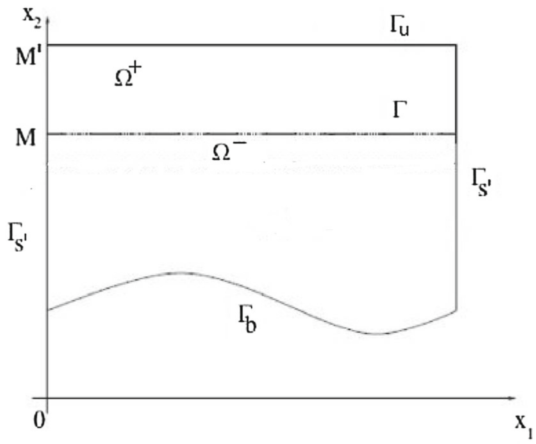

Fig. 2 Ω

where a, b are positive constants such that $0 < a < b < L$. We can represent Ω_ε formally as

$$\Omega_\varepsilon = \text{Interior}\{\overline{\Omega^- \cup \Omega_\varepsilon^+}\}.$$

Here, Interior$\{S\}$ and $\bar{S}$, respectively, denotes the interior and closure of the set S in $\mathbb{R}^2$ with respect to the standard topology.

The vertical, bottom and top boundaries of Ω^- are to be denoted by Γ_s, Γ_b and Γ, respectively. More precisely,

$$\Gamma_s = \{(0, x_2) : \; g(0) \le x_2 \le M\} \cup \{(L, x_2) : \; g(L) \le x_2 \le M\},$$
$$\Gamma_b = \{(x_1, x_2) : \; 0 \le x_1 \le L, \; x_2 = g(x_1)\}$$

and

$$\Gamma = \{(x_1, M) : \; 0 \le x_1 \le L\}.$$

The highly oscillating boundary γ_ε of Ω_ε can be written as

$$\gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \{\Gamma_s \cup \Gamma_b\}$$

where $\partial\Omega_\varepsilon$ denotes the boundary of Ω_ε.

In the later part of Sect. 2.3, we shall see that our limit problem is posed in a fixed domain Ω which consists of two parts (see Fig. 2).

One part is Ω^- which we described earlier and another part can be denoted as Ω^+. We can write Ω^+ mathematically as

$$\Omega^+ = \{(x_1, x_2) : \; 0 < x_1 < L, M < x_2 < M'\}$$

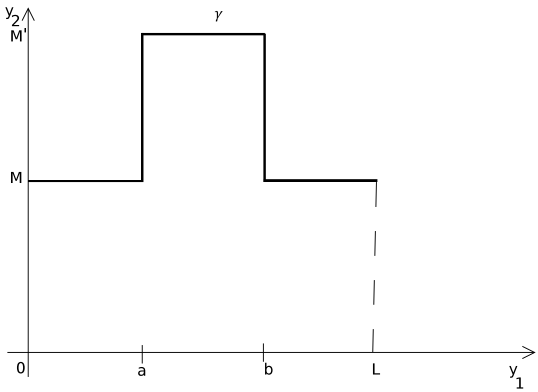

Fig. 3 Reference boundary

and Ω as

$$\Omega = \{(x_1, x_2) :\ 0 < x_1 < L, g(x_1) < x_2 < M'\}.$$

Notice that the interface of Ω^+ and Ω^- is, once again, Γ. It can also be pointed out that the bottom boundary of Ω already has a notation as Γ_b. The vertical and top boundaries of Ω can be denoted by $\Gamma_{s'}$ and Γ_u respectively. More precisely,

$$\Gamma_{s'} = \{(0, x_2) :\ g(0) \le x_2 \le M'\} \cup \{(L, x_2) :\ g(L) \le x_2 \le M'\},$$

and

$$\Gamma_u = \{(x_1, M') :\ 0 \le x_1 \le L\}.$$

The problem under consideration has Neumann data on the oscillating boundary γ_ε of Ω_ε. This data arises from a L^2 function defined on the reference boundary γ (see Fig. 3) of the cell domain which is defined as

$$\gamma := \gamma_{bl} \cup \gamma_{sl} \cup \gamma_u \cup \gamma_{sr} \cup \gamma_{br}$$

where

$$\begin{aligned}
\gamma_{bl} &= \{(z_1, M) : 0 \le z_1 \le a\}, \\
\gamma_{sl} &= \{(a, y_2) : M \le y_2 \le M'\}, \\
\gamma_u &= \{(z_1, M') : a \le z_1 \le b\}, \\
\gamma_{sr} &= \{(b, y_2) : M \le y_2 \le M'\} \text{ and} \\
\gamma_{br} &= \{(z_1, M) : b \le z_1 \le L\}.
\end{aligned}$$

2.2 Notations and the Function Space Setting

For $y \in L^2(\Omega)$ (or $y \in L^2(\Omega_\varepsilon)$) we shall denote by y^+ the restriction of y to Ω^+ (or Ω_ε^+ respectively) and by y^- the restriction of y to Ω^-. Let $H^1_{per}(\Omega_\varepsilon)$ represents the $H^1(\Omega_\varepsilon)$ functions which are periodic in the x_1-direction with period L. Similarly, $C^\infty_{per}\left(\bar{\Omega}\right)$ and $L^2_{per}(\Omega)$, respectively, represents the $C^\infty\left(\bar{\Omega}\right)$ and $L^2(\Omega)$ functions which are periodic in the x_1-direction with period L. In a similar way, we denote $L^2_{per}(\gamma)$ and $H^{1/2}_{per}(\gamma)$, respectively, the space of $L^2(\gamma)$ and $H^{1/2}(\gamma)$ functions which are periodic on γ in the x_1-direction with period L.

2.3 Statement of the Problem

Let $u \in L^2_{per}(\gamma)$. For scaling parameters $\alpha \geq 0, \beta \geq 0$, define $u_\varepsilon \in L^2_{per}(\gamma)$ as follows

$$u_\varepsilon := (\varepsilon^\beta \chi_{\gamma_{bl}} + \varepsilon^\alpha \chi_{\gamma_{sl}} + \varepsilon^\beta \chi_{\gamma_u} + \varepsilon^\alpha \chi_{\gamma_{sr}} + \varepsilon^\beta \chi_{\gamma_{br}})u \tag{2.1}$$

where χ_A denotes the characteristic function of any set A. Using (2.1), we define $u_\varepsilon^\varepsilon \in L^2(\gamma_\varepsilon)$ such that

$$u_\varepsilon^\varepsilon(x_1, x_2) = u_\varepsilon\left(\frac{x_1}{\varepsilon}, x_2\right) \tag{2.2}$$

where we abuse notation to denote the fractional part of $\frac{x_1}{\varepsilon}$ with respect to L by $\frac{x_1}{\varepsilon}$ itself. Observe that $u_\varepsilon^\varepsilon$ defined here is εL periodic.

Consider a function $\mu(x_2, s)$ with $x_2 \in [M, M']$ and $s \in \mathbb{R}$ which is smooth in its arguments. $\mu|_{s=0} = 0$ and there exists constants $C_0, 0 \leq\ C_1,\ C_2$ such that for all its arguments

$$\mid \partial_{x_2}\mu(x_2, s) \mid \leq C_0, \quad C_1 \leq \partial_s \mu(x_2, s) \leq C_2. \tag{2.3}$$

The second assumption in the above implies the strong monotonicity

$$(\mu(x_2, s) - \mu(x_2, s'))(s - s') \geq C_1(s - s')^2 \quad \text{for all } x_2 \text{ and for all } s\ , s'\ . \tag{2.4}$$

For $f \in L^2_{per}(\Omega)$, a given data function μ with the above mentioned properties and a positive constant η, our aim is to study the asymptotic behavior of the following boundary value problem with a non-linear Robin boundary condition on the oscillating boundary

$$\begin{cases} -\Delta y_\varepsilon + y_\varepsilon = f \quad in \ \Omega_\varepsilon, \\ \dfrac{\partial y_\varepsilon}{\partial \nu} + \varepsilon^\eta \mu(x_2, y_\varepsilon) = u_\varepsilon^\varepsilon \quad on \ \gamma_\varepsilon, \\ y_\varepsilon = 0 \quad on \ \Gamma_b, \\ y_\varepsilon \quad is \ \Gamma_s - periodic\,, \end{cases} \tag{2.5}$$

Note that, if $\mu \equiv 0$ we have a non-homogeneous Neumann boundary condition

$$\frac{\partial y_\varepsilon}{\partial \nu} = u_\varepsilon^\varepsilon \quad on \ \gamma_\varepsilon \tag{2.6}$$

instead of the non-linear Robin boundary condition on the oscillating boundary.

A solution to (2.5) is intended in the weak sense for which we consider the Hilbert space

$$\mathcal{V}_\varepsilon \ := \{y \in H^1_{per}\,(\Omega_\varepsilon) : \ y|_{\Gamma_b} = 0\}$$

which is a closed subspace of $H^1_{per}\,(\Omega_\varepsilon)$. The weak formulation of (2.5) consists in finding a y_ε in the Hilbert space $\mathcal{V}_\varepsilon$ such that

$$\begin{aligned} &\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \phi + \int_{\Omega_\varepsilon} y_\varepsilon \phi + \varepsilon^\eta \int_{\gamma_\varepsilon} \mu(x_2, y_\varepsilon)\phi \\ &= \int_{\Omega_\varepsilon} f\phi + \int_{\gamma_\varepsilon} u_\varepsilon^\varepsilon \phi, \quad \forall \phi \in \mathcal{V}_\varepsilon \end{aligned} \tag{2.7}$$

where we have taken Γ_s-periodic to mean, periodic on Ω_ε in the x_1 direction with period L. The existence and uniqueness of the weak solution of (2.7) can be proved by standard methods of the theory of monotone operators (for e.g. using Corollary 2.2 and Proposition 2.3, Chapter 2 [37]).

Our main result is the following.

Theorem 2.1 (Main Theorem) *Consider $\alpha \geq 1$, $\beta \geq 0$ and $\eta \geq 1$, let $y_\varepsilon \in \mathcal{V}_\varepsilon$ be the unique weak solution of (2.7). Then*

$$\widetilde{y_\varepsilon^+} \rightharpoonup \left(\frac{b-a}{L}\right) y^+ \ \ weakly \ in \ L^2((0,L); H^1((M, M')))$$

and

$$y_\varepsilon^- \rightharpoonup y^- \ \ weakly \ in \ \ H^1\left(\Omega^-\right)$$

where $\widetilde{y}_\varepsilon$ is the extension of y_ε on Ω_ε^+ by 0 to the whole of Ω^+. Moreover $y = (y^+, y^-)$ belongs to $\mathcal{V}_0$ where

$$\mathcal{V}_0 := \Big\{ y = (y^+, y^-) : \ y \in L^2_{per}(\Omega), \ \frac{\partial y^+}{\partial x_2} \in L^2(\Omega^+), \ y^- \\ \in H^1(\Omega^-), \ y^+ = y^- \text{ on } \Gamma, \ y^-|_{\Gamma_b} = 0 \Big\}$$

and satisfies the coupled system of partial differential equations in a weak sense

$$\begin{cases} -\dfrac{\partial^2 y^+}{\partial x_2^2} + y^+ + \left(\dfrac{2}{b-a}\right) \delta_{\eta 1}\, \mu(x_2, y^+(x_1, x_2)) = f + \delta_{\alpha 1}\theta \ \ in\ \Omega^+, \\ \dfrac{\partial y^+}{\partial x_2} = \dfrac{\delta_{\beta 0}}{b-a} \displaystyle\int_a^b u(z_1, M')\, dz_1 \ \ on\ \Gamma_u, \\ y^+ = y^- \ \ \text{on } \Gamma, \\ \dfrac{\partial y^-}{\partial x_2} - \left(\dfrac{b-a}{L}\right) \dfrac{\partial y^+}{\partial x_2} = \dfrac{\delta_{\beta 0}}{L} \displaystyle\int_{(0,a)\cup(b,L)} u(z_1, M)\, dz_1 \ \ on\ \Gamma, \\ -\Delta y^- + y^- = f \ \ in\ \Omega -, \\ y^- = 0 \ \ on\ \Gamma_b, \\ y \ \ is\ \ \Gamma_{s'} - periodic\,. \end{cases} \tag{2.8}$$

In the above, $\Gamma_{s'}$-periodic means periodic on Ω in the x_1 direction with period L and $\delta_{\gamma\kappa}$ is the Kronecker delta function. Here θ which appears in the first equation is given by

$$\theta(x_1, x_2) = \left(\frac{u(a, x_2) + u(b, x_2)}{b-a} \right) \ a.e.\ (x_1, x_2) \in \Omega^+. \tag{2.9}$$

Remark 2.2 The space $\mathcal{V}_0$ with the inner product

$$\langle u\,,\ v \rangle = \int_{\Omega^+} \frac{\partial u^+}{\partial x_2} \frac{\partial v^+}{\partial x_2} + \int_{\Omega^+} u^+ v^+ + \int_{\Omega^-} \nabla u^- \cdot \nabla v^- + \int_{\Omega^-} u^- v^-$$

can be seen to be a Hilbert space and the canonical weak formulation of (2.8) can be shown to have existence and uniqueness in $\mathcal{V}_0$ in the setting of monotone operators by arguing similarly as in Theorem 5.1 [30].

3 Unfolding Operator and Its Properties

In this section, we recall the definition of the periodic unfolding operator and present some fundamental properties.

The idea of using an unfolding operator method to handle periodic oscillations in the coefficients or of the domains can be seen in the works by Arbogast et al. [7]. and Cioranescu et al. in [18]. It's usefulness in deciphering weak convergence and simplifying the asymptotic analysis of boundary value problems with periodically oscillating coefficients or in periodically oscillating domains has been shown in [19]. Recently this method has been extended to cover the case of locally periodic oscillations [9].

We refer to [19] for an overview of the unfolding operator and to [22] and [33] for the proofs of several properties outlined below.

Let $[0, L]$ be a reference cell. For $x \in \mathbb{R}$, we write $[x]_L$ as the integer part of x with respect to L, that is $[x]_L = kL$, where k is the largest integer such that $kL \leq x$. We also denote the fractional part of x with respect to L by $\{x\}_L$. So, we have $x = [x]_L + \{x\}_L$.

Definition 3.1 *(The unfolding operator)* The operator which maps any function $y : \overline{\Omega_\varepsilon^+} \to \mathbb{R}$ to its ε-unfolding $T^\varepsilon y : \overline{\Omega^+} \times [0, L] \to \mathbb{R}$ defined through

$$T^\varepsilon y(x_1, x_2, z_1) = \widetilde{y}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]_L + \varepsilon z_1, x_2\right) \tag{3.1}$$

is called the unfolding operator. Here $\widetilde{y}$ is the extension of y on $\overline{\Omega_\varepsilon^+}$ by 0 to the whole of $\overline{\Omega^+}$.

Note that, for $M < x_2 < M'$, $T^\varepsilon y(x_1, x_2, z_1)$ is piecewise constant with respect to x_1 and has it's support in $[a, b]$ with respect to the local variable z_1 since $\widetilde{y}$ is zero outside $\overline{\Omega_\varepsilon^+}$ in $\overline{\Omega^+}$. Given $U \subset \mathbb{R}^2$ open containing $\overline{\Omega_\varepsilon^+}$ and a function y on U, $T^\varepsilon y$ will mean T^ε acting on the restriction of y to $\overline{\Omega_\varepsilon^+}$.

Proposition 3.1 *(a) The unfolding operator T^ε is linear. If $y_1,\ y_2 \in L^1(\Omega_\varepsilon^+)$ are such that $y_1\, y_2 \in L^1(\Omega_\varepsilon^+)$, then $T^\varepsilon(y_1 y_2) = T^\varepsilon(y_1)T^\varepsilon(y_2)$.*
(b) If $y \in L^1(\Omega_\varepsilon^+)$ then we have

$$\int\limits_{\Omega^+\times(0,L)} T^\varepsilon y\ dx_1dx_2dz_1 = \int\limits_{\Omega^+\times(a,b)} T^\varepsilon y\ dx_1dx_2dz_1 = L\int\limits_{\Omega_\varepsilon^+} y\ dx_1dx_2\,.$$

(c) If $w \in L^1(\gamma_\varepsilon)$ then we have

$$L\int_{\gamma_\varepsilon\cap\Gamma} w = \int_0^L \int_{(0,a)\cup(b,L)} T^\varepsilon w(x_1, M, z_1)\ dx_1dz_1 \tag{3.2}$$

$$L\int_{\gamma_\varepsilon\cap\Gamma_u} w = \int_0^L \int_{(a,b)} T^\varepsilon w(x_1, M', z_1)\ dx_1dz_1 \tag{3.3}$$

$$L\int_{\gamma_\varepsilon\setminus\{\Gamma\cup\Gamma_u\}} w = \frac{1}{\varepsilon}\left(\int_0^L\int_M^{M'} T^\varepsilon w(x_1, x_2, a)\ dx_1dx_2 + \int_0^L\int_M^{M'} T^\varepsilon w(x_1, x_2, b)\ dx_1dx_2\right) \tag{3.4}$$

Proposition 3.2 *(a) If $y \in L^2(\Omega_\varepsilon^+)$, then $T^\varepsilon y \in L^2(\Omega^+ \times (0, L))$ and*

$$\|T^\varepsilon y\|_{L^2(\Omega^+\times(0,L))} = \sqrt{L}\|y\|_{L^2(\Omega_\varepsilon^+)}\,.$$

(b) *For every* $y \in L^2(\Omega^+)$, *we have* $T^\varepsilon\left(y_{|\Omega_\varepsilon^+}\right) \overset{\varepsilon\to 0}{\longrightarrow} y$ *strongly in* $L^2(\Omega^+ \times (0, L))$.

(c) *If* $y_\varepsilon \in L^2(\Omega_\varepsilon^+)$ *be such that* $T^\varepsilon y_\varepsilon \rightharpoonup y$ *weakly in* $L^2(\Omega^+ \times (0, L))$ *as* $\varepsilon \to 0$, *then*

$$\widetilde{y}_\varepsilon \rightharpoonup \frac{1}{L}\int_a^b y \, dz_1 \text{ weakly in } L^2(\Omega^+)\,.$$

(d) *If* $y \in H^1(\Omega_\varepsilon^+)$, *then* $T^\varepsilon y \in L^2((0, L); H^1((M, M') \times (a, b)))$. *Moreover*

$$\frac{\partial}{\partial x_2} T^\varepsilon y = T^\varepsilon \frac{\partial y}{\partial x_2}, \quad \frac{\partial}{\partial z_1} T^\varepsilon y = \varepsilon T^\varepsilon \frac{\partial y}{\partial x_1} \quad \text{and}$$

$$\|T^\varepsilon y\|_{L^2((0,L);H^1((M,M')\times(a,b)))} \le C\|y\|_{H^1(\Omega_\varepsilon^+)} \tag{3.5}$$

where $C > 0$ *is a positive constant independent of* ε.

(e) *If* $T^\varepsilon y_\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} y$ *weakly in* $L^2((0, L); H^1((M, M') \times (a, b)))$ *for a sequence* $y_\varepsilon \in H^1(\Omega_\varepsilon^+)$, *then*

$$\widetilde{y}_\varepsilon \rightharpoonup \frac{1}{L}\int_a^b y \, dz_1 \text{ weakly in } L^2((0, L); H^1((M, M')))\,.$$

Remark 3.3 The property (d) of Proposition 3.2 shows that a Sobolev function loses regularity in the direction of unfolding with respect to the macroscopic variable (x_1 in this discussion) but this regularity is recovered in the local variable (z_1 in this discussion).

4 A Priori Norm-Estimates

In the first two lemmas, we show how to obtain some boundary estimates.

Lemma 4.1 *There exists a positive constant C independent of* ε *such that*

$$\|y\|_{L^2(\gamma_\varepsilon\cap\Gamma)} \le C\|y\|_{H^1(\Omega_\varepsilon)} \quad \forall y \in \mathcal{V}_\varepsilon \text{ and,} \tag{4.1}$$

$$\|y\|_{L^2(\gamma_\varepsilon\cap\Gamma_u)} \le C\|y\|_{H^1(\Omega_\varepsilon)} \quad \forall y \in \mathcal{V}_\varepsilon. \tag{4.2}$$

Proof By classical trace theorem, applied to the restriction of $H^1(\Omega_\varepsilon)$ functions to $H^1(\Omega^-)$, there exists a positive constant $C > 0$ independent of ε such that

$$\int_{\gamma_\varepsilon\cap\Gamma} y^2 \le \int_\Gamma y^2 \le C\|y\|^2_{H^1(\Omega^-)} \le C\|y\|^2_{H^1(\Omega_\varepsilon)} \quad \forall y \in \mathcal{V}_\varepsilon.$$

For the second estimate, first we consider smooth y in $\mathcal{V}_\varepsilon$. We decompose the boundary integral and apply the fundamental theorem of calculus on each vertical section, to get

$$
\begin{aligned}
\int_{\gamma_\varepsilon \cap \Gamma_u} y^2 &= \sum_{k=0}^{N-1} \int_{k\varepsilon L+\varepsilon a}^{k\varepsilon L+\varepsilon b} (y(x_1, M'))^2 \, dx_1 \\
&= \sum_{k=0}^{N-1} \int_{k\varepsilon L+\varepsilon a}^{k\varepsilon L+\varepsilon b} \left(\int_{g(x_1)}^{M'} \frac{\partial y}{\partial x_2}(x_1, x_2) \, dx_2 \right)^2 \, dx_1 \\
&\leq C \left(\sum_{k=0}^{N-1} \int_{k\varepsilon L+\varepsilon a}^{k\varepsilon L+\varepsilon b} \int_{g(x_1)}^{M'} \left| \frac{\partial y}{\partial x_2} \right|^2 \right) \\
&= C \left(\left\| \frac{\partial y}{\partial x_2} \right\|_{L^2(\Omega_\varepsilon)}^2 \right)
\end{aligned}
$$

by applying Cauchy–Schwarz inequality on the vertical sections with $C > 0$ depending only on the vertical diameter. This gives (4.2) for smooth y, and by density, we obtain the desired inequality for all y in $\mathcal{V}_\varepsilon$. □

Lemma 4.2 *Let $u_\varepsilon^\varepsilon$ be the boundary data. Then*

$$
\|u_\varepsilon^\varepsilon\|_{L^2(\gamma_\varepsilon \cap \Gamma)}^2 = \varepsilon^{2\beta} \int_{(0,a)\cup(b,L)} |u(z_1, M)|^2 dz_1 \tag{4.3}
$$

$$
\|u_\varepsilon^\varepsilon\|_{L^2(\gamma_\varepsilon \cap \Gamma_u)}^2 = \varepsilon^{2\beta} \int_{(a,b)} |u(z_1, M')|^2 dz_1 \, . \tag{4.4}
$$

Proof This readily follows from the properties (3.2) and (3.3) of Proposition 3.1 and using the fact that $T^\varepsilon u_\varepsilon^\varepsilon(x_1, x_2, z_1) = \varepsilon^\beta u(z_1, x_2)$. □

Now, let $y_\varepsilon \in \mathcal{V}_\varepsilon$ be the unique weak solution of (2.5) so that (2.7) holds.

Proposition 4.3 *For $\alpha, \eta \geq 1$ and $\beta \geq 0$, there exist a positive constant $C > 0$, independent of ε, such that*

$$
\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \textit{ for all } \varepsilon \, .
$$

Proof Let $y_\varepsilon \in \mathcal{V}_\varepsilon$ be a solution of (2.7). Choosing $\phi = y_\varepsilon$ in (2.7), we have

$$
\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}^2 + \varepsilon^\eta \int_{\gamma_\varepsilon} \mu(x_2, y_\varepsilon) y_\varepsilon = \int_{\Omega_\varepsilon} f y_\varepsilon + \int_{\gamma_\varepsilon} u_\varepsilon^\varepsilon y_\varepsilon.
$$

Then, by the monotonicity (2.4) and the fact that $\mu(x_2, 0) = 0$ for all x_2, we obtain

$$
\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}^2 \leq \int_{\Omega_\varepsilon} f y_\varepsilon + \int_{\gamma_\varepsilon} u_\varepsilon^\varepsilon y_\varepsilon. \tag{4.5}
$$

By Cauchy–Schwarz inequality, we get

$$
\int_{\Omega_\varepsilon} f y_\varepsilon \leq \|f\|_{L^2(\Omega_\varepsilon)} \|y_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \|f\|_{L^2(\Omega)} \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \tag{4.6}
$$

We have, using (4.1) and (4.3) and the fact that $u \in L^2_{per}(\gamma)$,

$$\int_{\gamma_\varepsilon \cap \Gamma} u^\varepsilon_\varepsilon y_\varepsilon \leq \|u^\varepsilon_\varepsilon\|_{L^2(\gamma_\varepsilon\cap\Gamma)} \|y_\varepsilon\|_{L^2(\gamma_\varepsilon\cap\Gamma)} \leq C\varepsilon^\beta \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} . \tag{4.7}$$

Similarly, using (4.2) and (4.4) and the fact that $u \in L^2_{per}(\gamma)$, we have

$$\int_{\gamma_\varepsilon \cap \Gamma_u} u^\varepsilon_\varepsilon y_\varepsilon \leq \|u^\varepsilon_\varepsilon\|_{L^2(\gamma_\varepsilon\cap\Gamma_u)} \|y_\varepsilon\|_{L^2(\gamma_\varepsilon\cap\Gamma_u)} \leq C\varepsilon^\beta \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} . \tag{4.8}$$

Lastly, splitting the boundary integral on $\gamma_\varepsilon \setminus \{\Gamma \cup \Gamma_u\}$ into it's different periodic components, using the definition of $u^\varepsilon_\varepsilon$ and the fact that $T^\varepsilon y_\varepsilon$ is piecewise constant with respect to x_1, we get

$$\begin{aligned}
&\int_{\gamma_\varepsilon\setminus\{\Gamma\cup\Gamma_u\}} u^\varepsilon_\varepsilon y_\varepsilon \\
&= \varepsilon^\alpha \sum_{k=0}^{N-1} \left(\int_M^{M'} u(a,x_2)\, y_\varepsilon(k\varepsilon L + \varepsilon a, x_2) dx_2 + \int_M^{M'} u(b,x_2)\, y_\varepsilon(k\varepsilon L + \varepsilon b, x_2) dx_2 \right) \\
&= \frac{\varepsilon^\alpha}{\varepsilon L} \left(\int_0^L \int_M^{M'} u(a,x_2) T^\varepsilon y_\varepsilon(x_1,x_2,a)\, dx_1 dx_2 + \int_0^L \int_M^{M'} u(b,x_2) T^\varepsilon y_\varepsilon(x_1,x_2,b)\, dx_1 dx_2 \right)
\end{aligned}$$

This, using the fact that $u \in L^2_{per}(\gamma)$, implies that

$$\begin{aligned}
\int_{\gamma_\varepsilon\setminus\{\Gamma\cup\Gamma_u\}} u^\varepsilon_\varepsilon y_\varepsilon &\leq \frac{C\,\varepsilon^{\alpha-1}}{L} \left(\|T^\varepsilon y_\varepsilon|_{z_1=a}\|_{L^2(\Omega^+)} + \|T^\varepsilon y_\varepsilon|_{z_1=b}\|_{L^2(\Omega^+)} \right) \\
&\leq \frac{C\,\varepsilon^{\alpha-1}}{L} \|T^\varepsilon y_\varepsilon\|_{L^2(0,L;H^1((M,M')\times(0,L)))} \\
&\leq \frac{C\,\varepsilon^{\alpha-1}}{L^2} \|y_\varepsilon\|_{H^1(\Omega^+_\varepsilon)}
\end{aligned} \tag{4.9}$$

where at the end we have used trace estimate with respect to the z_1 variable following the fact that $T^\varepsilon y_\varepsilon$ is in $L^2(0,L;H^1((M,M')\times(0,L)))$ and the property (3.5). Then by (4.7), (4.8) and (4.9). we have

$$\int_{\gamma_\varepsilon} u^\varepsilon_\varepsilon y_\varepsilon \leq C(\varepsilon^\beta + \varepsilon^{\alpha-1}) \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} . \tag{4.10}$$

The required uniform bound of y_ε in $H^1(\Omega_\varepsilon)$ can be obtained by combining (4.5), (4.6) and (4.10). $\square$

We now give the proof of the main theorem.

5 Proof of Theorem 2.1: Case $\mu \equiv 0$

We shall prove the Theorem 2.1 in the case of non-homogeneous Neumann condition on the oscillating boundary. This case is easier to handle but shows the essential difficulties of the more general Robin boundary condition. Before we prove Theorem 2.1, we make some observations and prove some preliminary results in the form of some lemmas.

We consider $y_\varepsilon \in \mathcal{V}_\varepsilon$ solution of (2.7) with $\mu \equiv 0$, that is, with Neumann condition (2.6). The bound for y_ε in Proposition 4.3 together with the inequality (3.5) of Proposition 3.2(d) allows us to conclude that

$$\|y_\varepsilon^-\|_{H^1(\Omega^-)} \le C \tag{5.1}$$

and

$$\|T^\varepsilon(y_\varepsilon^+)\|_{L^2(0,L;H^1((M,M')\times(0,L)))} \le C \tag{5.2}$$

where $C > 0$ is a positive constant independent of ε. This implies the existence of $y^- \in H^1\left(\Omega^-\right)$ and $y^+ \in L^2((0, L); H^1((M, M') \times (0, L)))$ such that

$$y_\varepsilon^- \rightharpoonup y^- \text{ weakly in } H^1\left(\Omega^-\right) \tag{5.3}$$

and

$$T^\varepsilon(y_\varepsilon^+) \rightharpoonup y^+ \text{ weakly in } L^2((0, L); H^1((M, M') \times (0, L))) \tag{5.4}$$

up to a subsequence still denoted by ε. In what follows, for simplicity, we shall suppress the super-indices $^+$ and $^-$ wherever it is clear that we are looking at the restriction of y_ε to Ω_ε^+ or Ω^- respectively.

Lemma 5.1 *y^+ is independent of the z_1 variable.*

Proof From Proposition 3.2(d), we know

$$\frac{\partial(T^\varepsilon y_\varepsilon)}{\partial z_1} = \varepsilon T^\varepsilon\left(\frac{\partial y_\varepsilon}{\partial x_1}\right). \tag{5.5}$$

Now, by Proposition 3.2(a) and Proposition 4.3, we observe that

$$\left\|T^\varepsilon\left(\frac{\partial y_\varepsilon}{\partial x_1}\right)\right\|_{L^2(\Omega^+\times(0,L))} = \sqrt{L}\left\|\frac{\partial y_\varepsilon}{\partial x_1}\right\|_{L^2(\Omega_\varepsilon^+)} \le C\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C \tag{5.6}$$

showing that the sequence $T^\varepsilon\left(\dfrac{\partial y_\varepsilon}{\partial x_1}\right)$ is bounded in $L^2(\Omega^+ \times (0, L))$.

So, the convergence (5.4) together with relation (5.5) and the bound (5.6) imply that

$$\frac{\partial y^+}{\partial z_1} = 0$$

in the distributional sense on $\Omega^+ \times (0, L)$. Since, $\Omega^+ \times (0, L)$ is a connected set we obtain that y^+ is independent of the z_1 variable on this region. □

As a corollary, using Proposition 3.2(e), we obtain the following convergence

$$\widetilde{y_\varepsilon^+} \rightharpoonup \left(\frac{b-a}{L}\right) y^+ \text{ weakly in } L^2((0, L); H^1((M, M'))). \tag{5.7}$$

Going back to the bound (5.6), we conclude that there exists a subsequence of $T^\varepsilon\left(\frac{\partial y_\varepsilon}{\partial x_1}\right)$, still indexed by ε, and $P \in L^2(\Omega^+ \times (0, L))$ such that

$$T^\varepsilon\left(\frac{\partial y_\varepsilon}{\partial x_1}\right) \rightharpoonup P \text{ weakly in } L^2(\Omega^+ \times (0, L)). \tag{5.8}$$

Lemma 5.2 *We claim that*

$$T^\varepsilon\left(\frac{\partial y_\varepsilon}{\partial x_1}\right) \rightharpoonup 0 \textit{ weakly in } L^2(\Omega^+ \times (0, L)).$$

Proof We take $\phi \in \mathcal{D}(\Omega^+)$ and $\psi \in C_c^\infty[0, L]$, and consider the following test functions on Ω_ε^+ which belong to $\mathcal{V}_\varepsilon$

$$\phi^\varepsilon(x) = \varepsilon\phi(x_1, x_2)\psi\left(\left\{\frac{x_1}{\varepsilon}\right\}_L\right). \tag{5.9}$$

From the definition of ε-unfolding of ϕ^ε and by Proposition 3.2(d), we get

$$\begin{aligned}
T^\varepsilon\phi^\varepsilon(x_1, x_2, z_1) &= \varepsilon\phi\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon z_1, x_2\right)\psi(z_1),\\
T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial x_1}(x_1, x_2, z_1) &= \frac{1}{\varepsilon}\frac{\partial}{\partial z_1}T^\varepsilon\phi^\varepsilon(x_1, x_2, z_1)\\
&= \varepsilon\frac{\partial\phi}{\partial x_1}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon z_1, x_2\right)\psi(z_1)\\
&\quad + \phi\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon z_1, x_2\right)\frac{d\psi}{dz_1}(z_1),\\
T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial x_2}(x_1, x_2, z_1) &= \varepsilon\frac{\partial\phi}{\partial x_2}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon z_1, x_2\right)\psi(z_1).
\end{aligned} \tag{5.10}$$

These equations give us the following convergences

$$T^\varepsilon \phi^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0 \text{ strongly in } L^2(\Omega^+ \times (0,L)), \tag{5.11}$$

$$T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_1} \overset{\varepsilon\to 0}{\longrightarrow} \phi(x_1,x_2)\frac{d\psi}{dz_1}(z_1) \text{ strongly in } L^2(\Omega^+ \times (0,L)), \tag{5.12}$$

$$T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_2} \overset{\varepsilon\to 0}{\longrightarrow} 0 \text{ strongly in } L^2(\Omega^+ \times (0,L))\,. \tag{5.13}$$

Using ϕ^ε as a test function in the variational formulation (2.7), we get

$$\begin{aligned}\int_{\Omega_\varepsilon^+} \nabla y_\varepsilon \cdot \nabla \phi^\varepsilon + \int_{\Omega_\varepsilon^+} y_\varepsilon \phi^\varepsilon = \int_{\Omega_\varepsilon^+} f\phi^\varepsilon \\ + \int_{\gamma_\varepsilon \backslash \{\Gamma \cup \Gamma_u\}} u_\varepsilon^\varepsilon \phi^\varepsilon\end{aligned} \tag{5.14}$$

since ϕ^ε vanishes on Γ and Γ_u due to the fact that ϕ is compactly supported in Ω^+. By the Proposition 3.1(b), we get

$$\begin{aligned}&\int_{\Omega_\varepsilon^+} \nabla y_\varepsilon \cdot \nabla \phi^\varepsilon + \int_{\Omega_\varepsilon^+} y_\varepsilon \phi^\varepsilon \\ &= \frac{1}{L}\left(\int_{\Omega^+\times(0,L)}\left(T^\varepsilon \frac{\partial y_\varepsilon}{\partial x_1}\, T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_1} + T^\varepsilon \frac{\partial y_\varepsilon}{\partial x_2}\, T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_2}\right) + \int_{\Omega^+\times(0,L)} T^\varepsilon y_\varepsilon\, T^\varepsilon \phi^\varepsilon\right)\end{aligned}$$

from which it follows, using the convergences (5.11)–(5.13), that

$$\int_{\Omega_\varepsilon^+} \nabla y_\varepsilon \cdot \nabla \phi^\varepsilon + \int_{\Omega_\varepsilon^+} y_\varepsilon \phi^\varepsilon \to \frac{1}{L}\int_{\Omega^+\times(0,L)} P\,\phi(x_1,x_2)\frac{d\psi}{dz_1}(z_1) \;\; as\ \varepsilon \to 0\,. \tag{5.15}$$

Similarly,after writing the boundary term using (3.4) taking into account (2.1) and (5.10) and then, using (5.11), we get,

$$\begin{aligned}&\int_{\Omega_\varepsilon^+} f\phi^\varepsilon + \int_{\gamma_\varepsilon \backslash \{\Gamma \cup \Gamma_u\}} u_\varepsilon^\varepsilon \phi^\varepsilon \\ &= \frac{1}{L}\int_{\Omega^+\times(0,L)} T^\varepsilon f\, T^\varepsilon \phi^\varepsilon + \frac{\varepsilon^\alpha}{L}\int_{\Omega^+} u(a,x_2)\phi(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon a, x_2)\psi(a) \\ &\quad + \frac{\varepsilon^\alpha}{L}\int_{\Omega^+} u(b,x_2)\phi(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon b, x_2)\psi(b)\, dx_1\, dx_2 \\ &\overset{\varepsilon\to 0}{\longrightarrow} 0\,.\end{aligned} \tag{5.16}$$

Combining (5.14)–(5.16), we get

$$\int_{\Omega^+\times(0,L)} P(x_1,x_2,z_1)\,\phi(x_1,x_2)\frac{d\psi}{dz_1}(z_1)\, dx_1 dx_2 dz_1 = 0$$

$$\forall \phi \in \mathcal{D}(\Omega^+),\ \forall \psi \in C_c^\infty[0, L]\,.$$

But given any $f \in C_c^\infty[a, b]$ one can find $\psi \in C_c^\infty[0, L]$ such that $\dfrac{d\psi}{dz_1} = f(z_1)$ on $[a, b]$ (for example, we can take $\psi = F\eta$ where F is a primitive of f and $\eta \in C_c^\infty[0, L]$ is a cut-off function which is identically equal to 1 on $[a, b]$). From this, it follows that

$$\int_a^b P(x_1, x_2, z_1)\,\phi(x_1, x_2) f(z_1)\, dx_1 dx_2 dz_1 = 0$$
$$\forall \phi \in \mathcal{D}(\Omega^+),\ \forall f \in C_c^\infty[a, b].$$

Hence, we get $P = 0$ a.e. in $\Omega^+ \times (0, L)$, that is, $T^\varepsilon \dfrac{\partial y_\varepsilon}{\partial x_1} \rightharpoonup 0$ weakly in $L^2(\Omega^+ \times (0, L))$. □

Proof of Theorem 2.1 We consider $\phi \in C_c^\infty(\Omega)$ and take its restriction to Ω_ε which is a test function belonging to $\mathcal{V}_\varepsilon$ in (2.7), and get

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \phi + \int_{\Omega_\varepsilon} y_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi + \int_{\gamma_\varepsilon} u_\varepsilon^\varepsilon \phi$$

which we write as

$$\int_{\Omega_\varepsilon^+} \frac{\partial y_\varepsilon}{\partial x_1}\frac{\partial \phi}{\partial x_1} + \int_{\Omega_\varepsilon^+} \frac{\partial y_\varepsilon}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega_\varepsilon^+} y_\varepsilon \phi + \int_{\Omega^-} \nabla y_\varepsilon \cdot \nabla \phi + \int_{\Omega^-} y_\varepsilon \phi$$
$$= \int_{\Omega_\varepsilon^+} f\phi + \int_{\Omega^-} f\phi + \int_{\gamma_\varepsilon} u_\varepsilon^\varepsilon \phi. \tag{5.17}$$

By the convergence (5.3), we have

$$\lim_{\varepsilon\to 0} \int_{\Omega^-} \nabla y_\varepsilon \cdot \nabla \phi = \int_{\Omega^-} \nabla y^- \cdot \nabla \phi, \tag{5.18}$$
$$\lim_{\varepsilon\to 0} \int_{\Omega^-} y_\varepsilon \phi = \int_{\Omega^-} y^- \phi. \tag{5.19}$$

In particular, since this holds for all $\phi \in \mathcal{D}(\Omega^-)$, we obtain the equation satisfied by y^- in Ω^-.

We continue to pass to the limit in the other terms. Using the property in Proposition 3.2(b), convergence (5.8) and the fact $P = 0$ proved in Lemma 5.2, we have

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon^+} \frac{\partial y_\varepsilon}{\partial x_1}\frac{\partial \phi}{\partial x_1} = \lim_{\varepsilon\to 0} \frac{1}{L} \int_{\Omega^+\times(0,L)} T^\varepsilon\left(\frac{\partial y_\varepsilon}{\partial x_1}\right) T^\varepsilon\left(\frac{\partial \phi}{\partial x_1}\right) = 0\,. \tag{5.20}$$

Now, using the properties of the unfolding operator in Propositions 3.1 and 3.2 and the convergence (5.4), and the fact y^+ is independent of the z_1 observed in Lemma 5.1, we conclude that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+}\frac{\partial y_\varepsilon}{\partial x_2}\frac{\partial \phi}{\partial x_2} = \lim_{\varepsilon\to 0}\frac{1}{L}\int_{\Omega^+\times(0,L)}\frac{\partial(T^\varepsilon y_\varepsilon)}{\partial x_2}T^\varepsilon\left(\frac{\partial\phi}{\partial x_2}\right) = \left(\frac{b-a}{L}\right)\int_{\Omega^+}\frac{\partial y^+}{\partial x_2}\frac{\partial\phi}{\partial x_2}, \tag{5.21}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+} y_\varepsilon\phi = \lim_{\varepsilon\to 0}\frac{1}{L}\int_{\Omega^+\times(0,L)} T^\varepsilon y_\varepsilon\, T^\varepsilon\phi = \left(\frac{b-a}{L}\right)\int_{\Omega^+} y^+\phi \tag{5.22}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+} f\phi = \lim_{\varepsilon\to 0}\frac{1}{L}\int_{\Omega^+\times(0,L)} T^\varepsilon f\, T^\varepsilon\phi = \left(\frac{b-a}{L}\right)\int_{\Omega^+} f\phi \tag{5.23}$$

where we have used the fact that f,ϕ are also independent of the z_1 variable.

Now we handle the boundary term. For this we apply (3.2), use the definition of the unfolding operator and u^ε to rewrite the integral and then pass to the limit to get

$$\begin{aligned}\int_{\gamma_\varepsilon\cap\Gamma} u_\varepsilon^\varepsilon\phi &= \frac{1}{L}\int_{(0,L)\times((0,a)\cup(b,L))} T^\varepsilon u_\varepsilon^\varepsilon T^\varepsilon\phi\\ &= \frac{\varepsilon^\beta}{L}\int_0^L\int_{(0,a)\cup(b,L)} u(z_1,M)T^\varepsilon\phi(x_1,M,z_1)\,dx_1\,dz_1,\\ &\xrightarrow{\varepsilon\to 0}\frac{\delta_{\beta 0}}{L}\int_0^L\left(\int_{(0,a)\cup(b,L)} u(z_1,M)\,dz_1\right)\phi(x_1,M)\,dx_1\end{aligned}$$

Similarly,

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon\cap\Gamma_u} u_\varepsilon^\varepsilon\phi = \frac{\delta_{\beta 0}}{L}\int_0^L\left(\int_{(a,b)} u(z_1,M')\,dz_1\right)\phi(x_1,M')\,dx_1.$$

Using (3.4) together with the definition of the boundary unfolding operator T^ε and that of $u_\varepsilon^\varepsilon$, we get

$$\begin{aligned}&\int_{\gamma_\varepsilon\setminus\{\Gamma\cup\Gamma_u\}} u_\varepsilon^\varepsilon\phi\\ &= \frac{\varepsilon^{\alpha-1}}{L}\left(\int_{\Omega^+} u(a,x_2)\phi(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon a,x_2)\,dx_1\,dx_2\right.\\ &\left.+\int_{\Omega^+} u(b,x_2)\phi(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon b,x_2)\,dx_1\,dx_2\right)\\ &\xrightarrow{\varepsilon\to 0}\frac{\delta_{\alpha 1}}{L}\left(\int_{\Omega^+} u(a,x_2)\phi(x_1,x_2)\,dx_1\,dx_2+\int_{\Omega^+} u(b,x_2)\phi(x_1,x_2)\,dx_1\,dx_2\right)\\ &= \delta_{\alpha 1}\frac{b-a}{L}\left(\int_{\Omega^+}\theta(x_1,x_2)\phi(x_1,x_2)\,dx_1\,dx_2\right)\end{aligned}$$

where θ has been defined in (2.9). So, we have

$$\begin{aligned}
\int_{\gamma_\varepsilon} u_\varepsilon^\varepsilon \phi &= \int_{\gamma_\varepsilon \cap \Gamma} u_\varepsilon^\varepsilon \phi + \int_{\gamma_\varepsilon \setminus \{\Gamma \cup \Gamma_u\}} u_\varepsilon^\varepsilon \phi + \int_{\gamma_\varepsilon \cap \Gamma_u} u_\varepsilon^\varepsilon \phi \\
&\xrightarrow{\varepsilon \to 0} \frac{\delta_{\beta 0}}{L} \int_0^L \left(\int_{(0,a)\cup(b,L)} u(z_1, M)\, dz_1 \right) \phi(x_1, M)\, dx_1 \\
&+ \delta_{\alpha 1} \frac{b-a}{L} \left(\int_{\Omega^+} \theta(x_1, x_2) \phi(x_1, x_2)\, dx_1\, dx_2 \right) \\
&+ \delta_{\alpha 1} \frac{b-a}{L} \left(\int_{\Omega^+} \theta(x_1, x_2) \phi(x_1, x_2)\, dx \right)
\end{aligned} \tag{5.24}$$

Combining (5.17)–(5.24), we get the following limit equation for

$$\begin{aligned}
&\left(\frac{b-a}{L}\right) \int_{\Omega^+} \frac{\partial y^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \left(\frac{b-a}{L}\right) \int_{\Omega^+} y^+ \phi + \int_{\Omega^-} \nabla y^- \cdot \nabla \phi + \int_{\Omega^-} y^- \phi \\
&= \left(\frac{b-a}{L}\right) \int_{\Omega^+} f\phi + \int_{\Omega^-} f\phi + \int_0^L \left(\frac{\delta_{\beta 0}}{L} \int_{(0,a)\cup(b,L)} u(z_1, M)\, dz_1 \right) \phi(x_1, M)\, dx_1 \\
&+ \left(\frac{b-a}{L}\right) \int_0^L \left(\frac{\delta_{\beta 0}}{b-a} \int_a^b u(z_1, M')\, dz_1 \right) \phi(x_1, M')\, dx_1 \\
&+ \delta_{\alpha 1} \left(\frac{b-a}{L}\right) \int_{\Omega^+} \theta(x_1, x_2) \phi(x_1, x_2)\, dx \quad \forall\, \phi \in \mathcal{V}_0
\end{aligned} \tag{5.25}$$

using the density of C_c^∞ in $\mathcal{V}_0$.

Now, in the case $\mu \equiv 0$, it follows that the equation (5.25) is precisely the weak formulation of (2.8) by showing that $y = (y^+, y^-)$ belongs to $\mathcal{V}_0$. For this, we observe that (5.3) implies that y^- belongs to $H^1(\Omega^-)$ and from (5.21) and the bound (4.3) we have that $\dfrac{\partial y^+}{\partial x_2} \in L^2(\Omega^+)$. The equality of the traces of y^+ and y^- on Γ can be shown following [33, Theorem 5.3, Step 3]. For this it is enough to calculate the limit of $\int_\Gamma y_\varepsilon \phi$ in two ways where ϕ is an arbitrary smooth test function supported on Γ. On the one hand the weak convergence (5.3) and the continuity of the trace operator implies that

$$\lim_{\varepsilon \to 0} \int_\Gamma y_\varepsilon \phi = \lim_{\varepsilon \to 0} \int_\Gamma y_\varepsilon^- \phi = \int_\Gamma y^- \phi\, .$$

Similar to (3.2) and (3.3) we can see that, for any $w \in L^1(\Gamma)$, we have

$$L \int_\Gamma w = \int_0^L \int_0^L T^\varepsilon w(x_1, M, z_1)\, dx_1\, dz_1\, .$$

So, applying this to $\int_\Gamma y_\varepsilon \phi$ and using (5.4) along with the continuity of the trace operator (the trace with respect to the x_2 variable) in $L^2((0, L); H^1((M, M') \times (0, L)))$, we get

$$\lim_{\varepsilon\to 0}\int_\Gamma y_\varepsilon\phi = \lim_{\varepsilon\to 0}\int_\Gamma y_\varepsilon^+\phi = \lim_{\varepsilon\to 0}\frac{1}{L}\int_0^L\int_0^L T^\varepsilon(y_\varepsilon^+)T^\varepsilon\phi(x_1,M,z_1)\,dx_1\,dz_1$$
$$= \int_\Gamma y^+\phi$$

since y^+ is independent of z_1 and $T^\varepsilon\phi(x_1, M, z_1) \to \phi(x_1, M)$ uniformly. The equality of the two limits for all test functions ϕ supported on Γ allows us to reach the desired conclusion.
Uniqueness of the solution of (5.25) implies that the entire sequence y_ε converges, not only the different subsequences. □

Remark 5.3 We would like to comment that in Gaudiello [25] the boundary data is very particular, namely, constant of the form ε^α. This happens when $\alpha = \beta$ in our case and u_ε is constant. In our case we have, like in their work, a distributed term θ in Ω^+ which arises from the non-homogeneous data on the vertical part of the oscillating boundary. But we also get some new non-trivial contributions in the limit problem on the upper boundary and on the interface.

6 Proof of Theorem 2.1: The General Case

Proof We now turn our attention to the proof of the general case involving a non-linear non-homogeneous boundary condition on the oscillating boundary.
Step I As shown in Proposition 4.3, there is a positive constant C independent of ε such that

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C \tag{6.1}$$

Using (2.3) and (6.1), we can prove that there exists a positive constant C independent of ε such that

$$\|\mu(\cdot, T^\varepsilon(y_\varepsilon^+))\|_{L^2(0,L;H^1((M,M')\times(0,L)))} \le C. \tag{6.2}$$

We have, as in the case $\mu = 0$, the convergences like in (5.3), (5.4) and (5.7) the following convergences

$$y_\varepsilon^- \rightharpoonup y^- \text{ weakly in } H^1\left(\Omega^-\right) \tag{6.3}$$
$$T^\varepsilon(y_\varepsilon^+) \rightharpoonup y^+ \text{ weakly in } L^2(0, L; H^1((M, M') \times (0, L))) \tag{6.4}$$
$$\widetilde{y_\varepsilon^+} \rightharpoonup \left(\frac{b-a}{L}\right) y^+ \text{ weakly in } L^2((0, L); H^1((M, M'))). \tag{6.5}$$

The bound (6.2) implies that, up to a subsequence still denoted by ε, there exist a function ζ in $L^2(0, L; H^1((M, M') \times (0, L)))$ such that

$$\mu(\cdot, T^\varepsilon(y_\varepsilon^+)) \rightharpoonup \zeta \text{ weakly in } L^2(0, L; H^1((M, M') \times (0, L))). \tag{6.6}$$

Step II Arguing similarly as Lemma 5.2, we can prove that

$$T^{\varepsilon}\left(\frac{\partial y_{\varepsilon}^{+}}{\partial x_1}\right) \rightharpoonup 0 \text{ weakly in } L^2(\Omega^+ \times (0, L)). \tag{6.7}$$

In fact, we choose the sequence of test functions ϕ^{ε} given by (5.9) used in Lemma 5.2 in the variational formulation (2.7) and pass to the limit. We need to handle the additional term $\varepsilon^{\eta} \int_{\gamma_{\varepsilon}} \mu(x_2, y_{\varepsilon})\phi^{\varepsilon}$ but this can be handled in a similar way using (3.2)–(3.4) and like in the calculations leading to (5.24) we get

$$\begin{aligned}&\left|\varepsilon^{\eta} \int_{\gamma_{\varepsilon}} \mu(x_2, y_{\varepsilon})\phi^{\varepsilon}\right| \\ &\quad \leq C\varepsilon^{\eta-1} \|\mu(\cdot, T^{\varepsilon}(y_{\varepsilon}^{+}))\|_{L^2(0,L;H^1((M,M')\times(0,L)))} \|T^{\varepsilon}\phi^{\varepsilon}\|_{L^2(0,L;H^1((M,M')\times(0,L)))}.\end{aligned} \tag{6.8}$$

The bound (6.2) leads to the conclusion

$$\varepsilon^{\eta} \int_{\gamma_{\varepsilon}} \mu(x_2, y_{\varepsilon})\phi^{\varepsilon} \to 0\,.$$

The rest is as in Lemma 5.2. Then, like in Lemma 5.1, we can prove that ζ is independent of the z_1 variable.

Step III We consider $\phi \in C_c^{\infty}(\Omega)$ and take its restriction to Ω_{ε} which can be taken as a test function belonging to $\mathcal{V}_{\varepsilon}$ in (2.7), and get

$$\begin{aligned}&\int_{\Omega_{\varepsilon}^{+}} \frac{\partial y_{\varepsilon}}{\partial x_1}\frac{\partial \phi}{\partial x_1} + \int_{\Omega_{\varepsilon}^{+}} \frac{\partial y_{\varepsilon}}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega_{\varepsilon}^{+}} y_{\varepsilon}\phi + \int_{\Omega^{-}} \nabla y_{\varepsilon}\cdot\nabla\phi + \int_{\Omega^{-}} y_{\varepsilon}\phi \\ &\quad + \varepsilon^{\eta}\int_{\gamma_{\varepsilon}} \mu(x_2, y_{\varepsilon})\phi = \int_{\Omega_{\varepsilon}^{+}} f\phi + \int_{\Omega^{-}} f\phi + \int_{\gamma_{\varepsilon}} u_{\varepsilon}^{\varepsilon}\phi.\end{aligned} \tag{6.9}$$

We have the convergences of the terms (5.18)–(5.24) as in Section 5. It remains to identify the limit of $\varepsilon^{\eta} \int_{\gamma_{\varepsilon}} \mu(x_2, y_{\varepsilon})\phi$ for passing to the limit in the above equation. Let us introduce the notation

$$\Xi_{\varepsilon}(x_1, x_2) = \mu(x_2, y_{\varepsilon}(x_1, x_2)) \text{ a.e. } (x_1, x_2) \in \Omega_{\varepsilon}^{+}$$

by which

$$T^{\varepsilon}\Xi_{\varepsilon}(x_1, x_2, z_1) = \mu(x_2, T^{\varepsilon}y_{\varepsilon}^{+}(x_1, x_2, z_1)).$$

From (6.2), we can write

$$\|T^{\varepsilon}\Xi_{\varepsilon}\|_{L^2(0,L;H^1((M,M')\times(0,L)))} \leq C.$$

Continuity of the trace operator and the convergence (6.6) and the independence of ζ of the z_1 variable provide

$$T^{\varepsilon}\Xi_{\varepsilon}(\cdot,a)\rightharpoonup\zeta \text{ weakly in } L^2(\Omega^+)$$

and

$$T^{\varepsilon}\Xi_{\varepsilon}(\cdot,b)\rightharpoonup\zeta \text{ weakly in } L^2(\Omega^+)\,.$$

This implies

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon\int_{\gamma_\varepsilon\setminus\{\Gamma\cup\Gamma_u\}}\mu(x_2,y_\varepsilon)\phi\\
&=\lim_{\varepsilon\to 0}\frac{1}{L}\left(\int_{\Omega^+}T^{\varepsilon}\Xi_{\varepsilon}(x_1,x_2,a)T^{\varepsilon}\phi(x_1,x_2,a)dx_1dx_2\right.\\
&\left.+\int_{\Omega^+}T^{\varepsilon}\Xi_{\varepsilon}(x_1,x_2,b)T^{\varepsilon}\phi(x_1,x_2,b)dx_1dx_2\right)\\
&=\frac{2}{L}\int_{\Omega^+}\zeta\phi. \qquad (6.10)
\end{aligned}$$

First note that, by (2.3) and $\mu(x_2,0)=0$, we have $|\mu(x_2,s)t|\le C_2\,s\,t$. So, by the trace theorem and (6.1), we have

$$\begin{aligned}
\left|\varepsilon\int_{\gamma_\varepsilon\cap\{\Gamma\cup\Gamma_u\}}\mu(x_2,y_\varepsilon)\phi\right| &\le C\varepsilon\int_{\gamma_\varepsilon\cap\{\Gamma\cup\Gamma_u\}}|y_\varepsilon\phi|\le C\varepsilon\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}\|\phi\|_{H^1(\Omega_\varepsilon)}\\
&\le C\varepsilon. \qquad (6.11)
\end{aligned}$$

Convergences (6.10) to (6.11), together, imply

$$\lim_{\varepsilon\to 0}\varepsilon^{\eta}\int_{\gamma_\varepsilon}\mu(x_2,y_\varepsilon)\phi=\delta_{\eta 1}\frac{2}{L}\int_{\Omega^+}\zeta\phi. \qquad (6.12)$$

Combining the convergences (5.18)–(5.24) and (6.12) we get the following limit problem

$$\begin{aligned}
&\left(\frac{b-a}{L}\right)\int_{\Omega^+}\frac{\partial y^+}{\partial x_2}\frac{\partial\phi}{\partial x_2}+\left(\frac{b-a}{L}\right)\int_{\Omega^+}y^+\phi\\
&\quad+\delta_{\eta 1}\frac{2}{L}\int_{\Omega^+}\zeta\phi+\int_{\Omega^-}\nabla y^-\cdot\nabla\phi+\int_{\Omega^-}y^-\phi\\
&=\left(\frac{b-a}{L}\right)\int_{\Omega^+}f\phi\\
&\quad+\int_{\Omega^-}f\phi+\int_0^L\left(\frac{\delta_{\beta 0}}{L}\int_{(0,a)\cup(b,L)}u(z_1,M)\,dz_1\right)\phi(x_1,M)\,dx_1
\end{aligned}$$

$$+\left(\frac{b-a}{L}\right)\int_0^L\left(\frac{\delta_{\beta 0}}{b-a}\int_a^b u(z_1,M')\,dz_1\right)\phi(x_1,M')\,dx_1$$
$$+\delta_{\alpha 1}\left(\frac{b-a}{L}\right)\int_{\Omega^+}\theta(x_1,x_2)\phi(x_1,x_2)\,dx \quad \forall\,\phi\in\mathcal{V}_0\,. \tag{6.13}$$

This is precisely the weak formulation of (2.8) once we see that (y^+, y^-) belongs to $\mathcal{V}_0$ (which is like in the previous section) and identify ζ (for $\eta > 1$ this is not necessary) which is the final step.

Step IV We are left to deal with the case $\eta = 1$. We claim that $\zeta(x_1,x_2) = \mu(x_2, y^+(x_1,x_2))$ for almost every $(x_1,x_2)\in\Omega^+$. Let us consider $\phi = y-\lambda\psi$ belonging to $\mathcal{V}_0$ where $y = y^+$ in Ω^+ and $y = y^-$ in Ω^- and for any given $\psi\in C_c^1(\Omega)$. Observe that, by the monotonicity of μ implied by the first of the assumptions on μ in (2.3), we have

$$\int_{\Omega^-}|\nabla y_\varepsilon-\nabla\phi|^2+\int_{\Omega^-}|y_\varepsilon-\phi|^2+\int_{\Omega_\varepsilon^+}\left|\partial_{x_2}y_\varepsilon-\partial_{x_2}\phi\right|^2$$
$$+\int_{\Omega_\varepsilon^+}\left|\partial_{x_1}y_\varepsilon\right|^2+\int_{\Omega_\varepsilon^+}|y_\varepsilon-\phi|^2$$
$$+\varepsilon\int_{\gamma_\varepsilon}(\mu(x_2,y_\varepsilon)-\mu(x_2,\phi))(y_\varepsilon-\phi)\geq 0$$

which we rewrite as

$$\int_{\Omega_\varepsilon^+}\left|\partial_{x_2}y_\varepsilon\right|^2+\int_{\Omega_\varepsilon^+}\left|\partial_{x_1}y_\varepsilon\right|^2+\int_{\Omega_\varepsilon^+}|y_\varepsilon|^2+\int_{\Omega^-}|\nabla y_\varepsilon|^2$$
$$+\int_{\Omega^-}|y_\varepsilon|^2+\varepsilon\int_{\gamma_\varepsilon}\mu(x_2,y_\varepsilon)y_\varepsilon$$
$$-2\int_{\Omega_\varepsilon^+}\partial_{x_2}y_\varepsilon\partial_{x_2}\phi+\int_{\Omega_\varepsilon^+}\left|\partial_{x_2}\phi\right|^2-2\int_{\Omega_\varepsilon^+}y_\varepsilon\phi+\int_{\Omega_\varepsilon^+}|\phi|^2$$
$$-2\int_{\Omega^-}\nabla y_\varepsilon\cdot\nabla\phi+\int_{\Omega^-}|\nabla\phi|^2+\int_{\Omega^-}|\phi|^2-2\int_{\Omega^-}y_\varepsilon\phi$$
$$-\varepsilon\int_{\gamma_\varepsilon}\mu(x_2,y_\varepsilon)\phi-\varepsilon\int_{\gamma_\varepsilon}\mu(x_2,\phi)y_\varepsilon+\varepsilon\int_{\gamma_\varepsilon}\mu(x_2,\phi)\phi\geq 0. \tag{6.14}$$

To obtain the limit expression of the terms in the first line of (6.14) we proceed by using y_ε as a test function in (2.7) and get

$$\int_{\Omega_\varepsilon^+}\left|\partial_{x_2}y_\varepsilon\right|^2+\int_{\Omega_\varepsilon^+}\left|\partial_{x_1}y_\varepsilon\right|^2+\int_{\Omega_\varepsilon^+}|y_\varepsilon|^2+\int_{\Omega^-}|\nabla y_\varepsilon|^2$$
$$+\int_{\Omega^-}|y_\varepsilon|^2+\varepsilon\int_{\gamma_\varepsilon}\mu(x_2,y_\varepsilon)y_\varepsilon=\int_{\Omega_\varepsilon}fy_\varepsilon+\int_{\gamma_\varepsilon}u_\varepsilon^\varepsilon y_\varepsilon$$
$$\xrightarrow{\varepsilon\to 0}\left(\frac{b-a}{L}\right)\int_{\Omega^+}fy^++\int_{\Omega^-}fy^-$$

$$
\begin{aligned}
&+\int_0^L \left(\frac{\delta_{\beta 0}}{L} \int_{(0,a)\cup(b,L)} u(z_1, M)\, dz_1 \right) y^+(x_1, M)\, dx_1 \\
&+\left(\frac{b-a}{L} \right) \int_0^L \left(\frac{\delta_{\beta 0}}{b-a} \int_a^b u(z_1, M')\, dz_1 \right) y^+(x_1, M')\, dx_1 \\
&+\delta_{\alpha 1} \left(\frac{b-a}{L} \right) \int_{\Omega^+} \theta(x_1, x_2) y^+(x_1, x_2)\, dx.
\end{aligned}
$$

Then, we use the variational Eq. (6.13) with $y = (y^+, y^-)$ replacing ϕ to obtain

$$
\begin{aligned}
&\int_{\Omega_\varepsilon^+} \left|\partial_{x_2} y_\varepsilon\right|^2 + \int_{\Omega_\varepsilon^+} \left|\partial_{x_1} y_\varepsilon\right|^2 + \int_{\Omega_\varepsilon^+} |y_\varepsilon|^2 \\
&\quad + \int_{\Omega^-} |\nabla y_\varepsilon|^2 + \int_{\Omega^-} |y_\varepsilon|^2 + \varepsilon \int_{\gamma_\varepsilon} \mu(x_2, y_\varepsilon) y_\varepsilon \\
&\quad \overset{\varepsilon\to 0}{\longrightarrow} \left(\frac{b-a}{L} \right) \int_{\Omega^+} \left| \frac{\partial y^+}{\partial x_2} \right|^2 + \left(\frac{b-a}{L} \right) \int_{\Omega^+} \left|y^+\right|^2 + \frac{2}{L} \int_{\Omega^+} \zeta y^+ \\
&\quad + \int_{\Omega^-} \left|\nabla y^-\right|^2 + \int_{\Omega^-} \left|y^-\right|^2 .
\end{aligned} \tag{6.15}
$$

Passing to the limit in the terms of the third line of (6.14) is straightforward using (6.3), whereas the limit of the main terms in the second line of (6.14) has already been obtained in (5.21)–(5.22). We get the following expression as the limit of these terms:

$$
\begin{aligned}
&\left(\frac{b-a}{L} \right) \left(-2 \int_{\Omega^+} \partial_{x_2} y^+\, \partial_{x_2} \phi + \int_{\Omega^+} \left|\partial_{x_2} \phi\right|^2 - 2 \int_{\Omega^+} y^+ \phi + \int_{\Omega^+} |\phi|^2 \right) \\
&-2 \int_{\Omega^-} \nabla y^- \cdot \nabla \phi + \int_{\Omega^-} |\nabla \phi|^2 + \int_{\Omega^-} |\phi|^2 - 2 \int_{\Omega^-} y^- \phi\, .
\end{aligned} \tag{6.16}
$$

Similar to (6.12), one can prove

$$
\lim_{\varepsilon\to 0} \varepsilon \int_{\gamma_\varepsilon} \mu(x_2, y_\varepsilon) \phi = \frac{2}{L} \int_{\Omega^+} \zeta \phi, \tag{6.17}
$$

$$
\lim_{\varepsilon\to 0} \varepsilon \int_{\gamma_\varepsilon} \mu(x_2, \phi) y_\varepsilon = \frac{2}{L} \int_{\Omega^+} \mu(x_2, \phi) y^+, \tag{6.18}
$$

$$
\lim_{\varepsilon\to 0} \varepsilon \int_{\gamma_\varepsilon} \mu(x_2, \phi) \phi = \frac{2}{L} \int_{\Omega^+} \mu(x_2, \phi) \phi. \tag{6.19}
$$

From (6.17)–(6.19), we will get the following limit for the terms in the last line of (6.14):

$$
-\frac{2}{L} \left(\int_{\Omega^+} \zeta \phi + \int_{\Omega^+} \mu(x_2, \phi) y^+ - \int_{\Omega^+} \mu(x_2, \phi) \phi \right) \tag{6.20}
$$

Combining (6.15), (6.16) and (6.20), we get

$$\int_{\Omega^-} \left|\nabla y^- - \nabla\phi\right|^2 + \int_{\Omega^-} \left|y^- - \phi\right|^2 + \left(\frac{b-a}{L}\right) \int_{\Omega^+} \left|\partial_{x_2} y^+ - \partial_{x_2}\phi\right|^2$$
$$+ \left(\frac{b-a}{L}\right) \int_{\Omega^+} \left|y^+ - \phi\right|^2 + \frac{2}{L} \int_{\Omega^+} (\zeta - \mu(x_2, \phi))(y^+ - \phi) \geq 0\,.$$

Since, $\phi = y - \lambda\psi$, we get

$$\lambda\left(\int_{\Omega^-} |\nabla\psi|^2 + \int_{\Omega^-} |\psi|^2 \right.$$
$$\left. + \left(\frac{b-a}{L}\right)\left(\int_{\Omega^+} \left|\partial_{x_2}\psi\right|^2 + \int_{\Omega^+} |\psi|^2\right)\right)$$
$$+ \frac{2}{L} \int_{\Omega^+} (\zeta - \mu(x_2, y^+ - \lambda\psi))\psi \geq 0\,.$$

By letting λ to go to 0, using the Dominated Convergence Theorem, we obtain

$$\int_{\Omega^+} (\zeta - \mu(x_2, y^+))\psi \geq 0.$$

Since ψ is an arbitrary element of $C_c^1(\Omega)$, we can conclude that

$$\zeta(x_1, x_2) = \mu(x_2, y^+(x_1, x_2)) \text{ a.e. } (x_1, x_2) \in \Omega^+\,.$$

□

7 A Three Dimensional Example

To show the efficacy of the method of unfolding operator in handling homogenization problems in domains with oscillating boundaries in any dimension, we consider a three dimension model from Mel'nyk [29] involving a $3:2:1$ thick junction but we consider non-homogeneous Robin boundary condition. We take $\Omega^- = (0,1) \times (0,1) \times [0, M]$ and Ω_ε of the form

$$\Omega_\varepsilon^+ = \left\{(x_1, x_2, x_3) : M \leq x_3 < g\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\right.$$
$$\left.\text{for all } (x_1, x_2) \in (0,1) \times (0,1)\right\} \tag{7.1}$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is a smooth piecewise smooth periodic function with period $[0,1]^2$. Let M' be the maximum of g in $[0,1]^2$. For any $t \in (M, M')$, we denote by $\omega(t)$ the open set

$$\omega(t) = \{(z_1, z_2) : g(z_1, z_2) > t\}$$

and introduce

$$\mathcal{G} = \{(x_3, z_1, z_2) : M < x_3 < M', (z_1, z_2) \in \omega(x_3)\}\,.$$

For simplicity, we assume that g has no flat portions and, for example, we can guarantee this by assuming that $g_{z_2} \neq 0$. The unfolding operator T^ε for the geometry of Ω_ε has the definition

$$T^\varepsilon y(x_1, x_2, x_3, z_1, z_2) = \widetilde{y}\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon z_1, \varepsilon\left[\frac{x_2}{\varepsilon}\right] + \varepsilon z_2, x_3\right) \tag{7.2}$$

for any given $y \in H^1(\Omega_\varepsilon)$. At fixed $x_3 \in (M, M')$, the support of $T^\varepsilon y$ with respect to the (z_1, z_2) variable is in $\overline{\omega(x_3)}$.

We have the following properties of the unfolding operator, similar to those stated in Propositions 3.1 and 3.2, and are standard to establish (see for example [2]).

Proposition 7.1 *(a) The unfolding operator T^ε is linear. Also, if $y_1,\ y_2 \in L^1(\Omega_\varepsilon^+)$ are such that $y_1 y_2 \in L^1(\Omega_\varepsilon^+)$, then $T^\varepsilon(y_1 y_2) = T^\varepsilon(y_1)T^\varepsilon(y_2)$.*

(b) If $y \in L^1(\Omega_\varepsilon^+)$, then we have

$$\int\limits_{\Omega^+\times(0,1)^2} T^\varepsilon y\; dx_1dx_2dx_3dz_1dz_2 = \int\limits_{(0,1)^2\times\mathcal{G}} T^\varepsilon y\; dx_1dx_2dx_3dz_1dz_2 = \int\limits_{\Omega_\varepsilon^+} y\; dx_1dx_2dx_3$$

and this gives, for every $y \in L^2(\Omega_\varepsilon^+)$, we have $T^\varepsilon y \in L^2(\Omega^+ \times (0, 1)^2)$ and

$$\|T^\varepsilon y\|_{L^2((0,1)^2\times\mathcal{G})} = \|y\|_{L^2(\Omega_\varepsilon^+)}.$$

(c) For every $y \in L^2(\Omega^+)$, we have $T^\varepsilon\left(y_{|\Omega_\varepsilon^+}\right) \overset{\varepsilon\to 0}{\longrightarrow} y$ strongly in $L^2(\Omega^+ \times (0, 1)^2)$.

(d) If $y_\varepsilon \in L^2(\Omega_\varepsilon^+)$ be such that $T^\varepsilon y_\varepsilon \rightharpoonup y$ weakly in $L^2(\Omega^+ \times (0, 1)^2)$ as $\varepsilon \to 0$, then

$$\widetilde{y}_\varepsilon \rightharpoonup \int_{\omega(x_3)} y\; dz_1dz_2 \text{ weakly in } L^2(\Omega^+)\,.$$

(e) If $y \in H^1(\Omega_\varepsilon^+)$, then $T^\varepsilon y \in L^2((0, 1)^2; H^1(\mathcal{G}))$. Moreover

$$\frac{\partial}{\partial x_3}T^\varepsilon y = T^\varepsilon\frac{\partial y}{\partial x_3},\ \ \frac{\partial}{\partial z_1}T^\varepsilon y = \varepsilon T^\varepsilon\frac{\partial y}{\partial x_1},\ \ \frac{\partial}{\partial z_2}T^\varepsilon y = \varepsilon T^\varepsilon\frac{\partial y}{\partial x_2}$$

and

$$\|T^\varepsilon y\|_{L^2((0,1)^2;H^1(\mathcal{G}))} \leq C\|y\|_{H^1(\Omega_\varepsilon^+)}$$

where $C > 0$ is a positive constant independent of ε.

(f) If $T^\varepsilon y_\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} y$ *weakly in* $L^2((0,1)^2; H^1(\mathcal{G}))$ *for a sequence* $y_\varepsilon \in H^1(\Omega_\varepsilon^+)$, *then*

$$\widetilde{y}_\varepsilon \rightharpoonup \int_{\omega(x_3)} y \, dz_1 dz_2 \text{ weakly in } L^2((0,1)^2; H^1((M, M'))) .$$

□

For boundary data of the form

$$u_\varepsilon^\varepsilon(x_1, x_2, x_3) = \varepsilon^\alpha u\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, g\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\right) \tag{7.3}$$

where u is such that

$$u \in L^2(\text{graph } g; d\sigma) \text{ that is}$$
$$\int_{(0,1)\times(0,1)} u^2(z_1, z_2, g(z_1, z_2))\sqrt{1 + (\partial_{z_1} g(z_1))^2 + (\partial_{z_1} g(z_2))^2}\, dz_1 dz_2 < \infty$$

and for $f \in L^2_{per}(\Omega)$, our aim is to study the asymptotic behavior of the following boundary value problem with a non-linear Robin boundary condition on the oscillating boundary S_ε

$$\begin{cases} -\Delta y_\varepsilon + y_\varepsilon = f \quad in \ \Omega_\varepsilon, \\ \dfrac{\partial y_\varepsilon}{\partial \nu} + \varepsilon^\eta \mu(y_\varepsilon) = u_\varepsilon^\varepsilon \ \ on \ S_\varepsilon, \\ y_\varepsilon = 0 \ \ on \ \Gamma_b, \\ y_\varepsilon \quad is \ \Gamma_s - periodic\, , \end{cases} \tag{7.4}$$

where we have taken Γ_s-periodic to mean, periodic on Ω_ε in the x_1, x_2 directions with period 1. The weak formulation of (2.5) is to find $y_\varepsilon \in \mathcal{V}_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} \nabla y_\varepsilon \cdot \nabla \phi + \int_{\Omega_\varepsilon} y_\varepsilon \phi + \varepsilon^\eta \int_{S_\varepsilon} \mu(y_\varepsilon)\phi = \int_{\Omega_\varepsilon} f\phi + \int_{S_\varepsilon} u_\varepsilon^\varepsilon \phi, \ \ \forall \phi \in \mathcal{V}_\varepsilon. \tag{7.5}$$

We can obtain the following result.

Theorem 7.2 *We have*

$$y_\varepsilon^- \rightharpoonup y^- \ \text{ weakly in } \ H^1\left(\Omega^-\right) \tag{7.6}$$
$$\widetilde{y_\varepsilon^+} \rightharpoonup |\omega(x_3)|\, y^+ \ \text{ weakly in } \ L^2((0,1)^2; H^1((M, M'))) \tag{7.7}$$

Moreover $y = (y^+, y^-)$ *satisfies the coupled system of partial differential equations*

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(|\omega(x_3)|\dfrac{\partial y^+}{\partial x_3}\right) + |\omega(x_3)|y^+ + \delta_{\eta 1}\Xi = |\omega(x_3)|f + \delta_{\alpha 1}\theta \ \ \textit{in } \Omega^+, \\ \dfrac{\partial y^+}{\partial x_2} = 0 \ \ \textit{on } \Gamma_u, \\ y^+ = y^- \ \ \text{on } \Gamma, \\ \dfrac{\partial y^-}{\partial x_2} - (|\omega(M)|)\dfrac{\partial y^+}{\partial x_2} = 0 \ \ \textit{on } \Gamma, \\ -\Delta y^- + y^- = f \ \ \textit{in } \Omega-, \\ y^- = 0 \ \ \textit{on } \Gamma_b, \\ y \ \ is \ \ \Gamma_{s'} - \textit{periodic}. \end{cases} \tag{7.8}$$

Here θ *and* Ξ *which appear in the first equation is given by*

$$\theta(x_1, x_2, x_3) = \left(\int_{\partial\omega(x_3)} u(z_1, z_2, x_3)d\gamma\right) \ a.e. \ (x_1, x_2, x_3) \in \Omega^+ \tag{7.9}$$

and

$$\Xi(x_1, x_2, x_3) = \text{per}(\partial\omega(x_3))\ \mu(y^+(x_1, x_2, x_3)) \ \ a.e. \ \ (x_1, x_2, x_3) \in \Omega^+. \tag{7.10}$$

where $\text{per}(\partial\omega(x_3))$ *denotes the perimeter of* $(\partial\omega(x_3))$. □

7.1 Sketch of Proof

For proving this result, we once again use the unfolding technique. Once again, for simplicity, we shall suppress the super-indices $^+$ and $^-$ wherever it is clear that we are looking at the restriction of y_ε to Ω_ε^+ or Ω^- respectively.

7.1.1 A Priori Estimates

The *a priori* bounds for y_ε follow similarly as in Proposition 4.3 as soon as we establish the following estimate on the oscillating surface S_ε:

$$\int_{S_\varepsilon} u_\varepsilon^\varepsilon y_\varepsilon \leq C\varepsilon^{\alpha-1}\|T^\varepsilon y_\varepsilon\|_{L^2((0,1)^2;H^1((M,M')\times(0,1)^2))} \tag{7.11}$$

To obtain this we first rewrite the surface integral using the slicing lemma for Hausdorff measures, Lemma 7.6.1 in [28],

$$\int_{S_\varepsilon} u_\varepsilon^\varepsilon y_\varepsilon = \int_M^{M'} \int_{S_\varepsilon \cap \pi^{-1}(t)} \frac{u_\varepsilon^\varepsilon y_\varepsilon}{|\nabla_{S_\varepsilon}\pi|}\, d\gamma\ dt$$

where π is the projection on the last coordinate, $\nabla_{S_\varepsilon}\pi$ is the tangential gradient of π and $d\gamma$ is the line element of the section of S_ε at the height t. Using the parametrization of the surface S_ε, we have explicitly

$$|\nabla_{S_\varepsilon}\pi| = \frac{\sqrt{\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)+\frac{1}{\varepsilon^2}g_{z_2}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}}{\sqrt{1+\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)+\frac{1}{\varepsilon^2}g_{z_2}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}}. \tag{7.12}$$

This can be obtained by projecting $\nabla\pi = (0,0,1)$ into the tangent space on S_ε equipped with the orthonormal basis

$$u_1 = \frac{1}{\sqrt{1+\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}}\left(1,0,\frac{1}{\varepsilon}g_{z_1}\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\right),$$

$$u_2 = \frac{\left(-\frac{1}{\varepsilon^2}g_{z_1}\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)g_{z_2}\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right),1+\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right),\frac{1}{\varepsilon}g_{z_2}\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\right)}{\sqrt{1+\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)+\frac{1}{\varepsilon^2}g_{z_2}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}\sqrt{1+\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}}.$$

Partitioning $(0,1)^2$ into cells of size ε, $Y_{ij}^\varepsilon = \{(x_1,x_2) : \varepsilon i \le x_1 \le \varepsilon(i+1), \varepsilon j \le x_1 \le \varepsilon(j+1)\}$ for $i,j = 0,1,\ldots,N-1$, and using (7.12), we obtain

$$\begin{aligned}
\int_{S_\varepsilon} u_\varepsilon^\varepsilon y_\varepsilon &= \int_M^{M'}\int_{S_\varepsilon\cap\pi^{-1}(t)}\frac{u_\varepsilon^\varepsilon y_\varepsilon}{|\nabla_{S_\varepsilon}\pi|}\,d\gamma_\varepsilon\,dt\\
&= \sum_{i,j=0}^{N-1}\int_M^{M'}\int_{S_\varepsilon\cap\pi^{-1}(t)\cap Y_{ij}^\varepsilon}\frac{u_\varepsilon^\varepsilon y_\varepsilon}{|\nabla_{S_\varepsilon}\pi|}\,d\gamma\,dt\\
&= \sum_{i,j=0}^{N-1}\int_M^{M'}\int_{S_\varepsilon\cap\pi^{-1}(t)\cap Y_{ij}^\varepsilon}\frac{u_\varepsilon^\varepsilon y_\varepsilon\sqrt{1+\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)+\frac{1}{\varepsilon^2}g_{z_2}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}}{\sqrt{\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)+\frac{1}{\varepsilon^2}g_{z_2}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}}\,d\gamma_\varepsilon\,dt\\
&= \sum_{i,j=0}^{N-1}\int_M^{M'}\int_{\varepsilon i}^{\varepsilon(i+1)}\frac{u_\varepsilon^\varepsilon y_\varepsilon\sqrt{1+\frac{1}{\varepsilon^2}g_{z_1}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)+\frac{1}{\varepsilon^2}g_{z_2}^2\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}}{\frac{1}{\varepsilon}g_{z_2}\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)}\,dx_1\,dt
\end{aligned}$$

where we have used the fact that the line element $d\gamma_\varepsilon$ of the curve $\left\{(x_1,x_2) : g\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right) = t\right\} \cap Y_{i,j}^\varepsilon$ with respect to the x_1 variable takes the form

$$d\gamma_\varepsilon = \frac{\sqrt{\frac{1}{\varepsilon^2} g_{z_1}^2\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) + \frac{1}{\varepsilon^2} g_{z_2}^2\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)}}{\frac{1}{\varepsilon} g_{z_2}\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)}\, dx_1 .$$

Then we change variables $x_1 = \varepsilon i + \varepsilon z_1$ in the interval $(\varepsilon i, \varepsilon(i+1))$, use the definition (in particular, the periodicity of u_ε) and the definition of the unfolding operator to obtain

$$\begin{aligned}
&\int_{S_\varepsilon} u_\varepsilon^\varepsilon y_\varepsilon \\
&= \varepsilon^{\alpha+1} \sum_{i,j=0}^{N-1} \int_M^{M'} \int_0^1 \frac{u(z_1, z_2, g(z_1, z_2)) T^\varepsilon y(\varepsilon i + \varepsilon z_1, \varepsilon j + \varepsilon z_2, g(z_1, z_2), z_1, z_2) \sqrt{\varepsilon^2 + g_{z_1}^2(z_1, z_2) + g_{z_2}^2(z_1, z_2)}}{g_{z_2}(z_1, z_2)}\, dz_1 dt \\
&= \varepsilon^{\alpha-1} \int_{(0,1)^2} \int_M^{M'} \int_0^1 \frac{u(z_1, z_2, g(z_1, z_2)) T^\varepsilon y(x_1, x_2, g(z_1, z_2), z_1, z_2) \sqrt{\varepsilon^2 + g_{z_1}^2(z_1, z_2) + g_{z_2}^2(z_1, z_2)}}{g_{z_2}(z_1, z_2)}\, dz_1 dt dx
\end{aligned} \tag{7.13}$$

where z_2 depends on z_1 and t and is determined by the curve $\{(z_1, z_2) : g(z_1, z_2) = t\}$ and dx stands for integration with respect to (x_1, x_2).

From this and the assumption on u, we obtain the estimate

$$\begin{aligned}
&\int_{S_\varepsilon} u_\varepsilon^\varepsilon y_\varepsilon \\
&\leq C\, \varepsilon^{\alpha-1} \left(\int_{(0,1)^2} \int_M^{M'} \int_0^1 \left(T^\varepsilon y(x_1, x_2, t, z_1, z_2)\right)^2 \frac{\sqrt{1 + g_{z_1}^2(z_1, z_2) + g_{z_2}^2(z_1, z_2)}}{g_{z_2}(z_1, z_2)}\, dz_1 dt dx \right)^{1/2} \\
&= C\, \varepsilon^{\alpha-1} \left(\int_{(0,1)^2} \int_S \left(T^\varepsilon y(x_1, x_2, z', z_1, z_2)\right)^2 \frac{\sqrt{1 + g_{z_1}^2(z_1, z_2) + g_{z_2}^2(z_1, z_2)}}{g_{z_2}(z_1, z_2)}\, dS(z', z_1, z_2) dx \right)^{1/2} \\
&\leq C\, \varepsilon^{\alpha-1} \|T^\varepsilon y_\varepsilon\|_{L^2((0,1)^2; H^1(\mathcal{G}))} \leq C\, \varepsilon^{\alpha-1} \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}
\end{aligned}$$

where $S = \{(z', z_1, z_2) : z' = g(z_1, z_2), (z_1, z_2) \in (0,1)^2\}$ and in the penultimate inequality, we have used trace estimate in $H^1(\mathcal{G})$.

7.1.2 Convergence Arguments

As a consequence of the a priori estimates, we shall have convergences similar to those in (6.3)–(6.7), upto a subsequence. To be precise, this time we shall have

$$y_\varepsilon^- \rightharpoonup y^- \ \text{ weakly in } \ H^1\left(\Omega^-\right) \tag{7.14}$$

$$T^\varepsilon y_\varepsilon^+ \rightharpoonup y^+ \ \text{ weakly in } \ L^2((0,1)^2; H^1(\mathcal{G})) \tag{7.15}$$

$$\widetilde{y_\varepsilon^+} \rightharpoonup |\omega(x_3)|\, y^+ \ \text{ weakly in } \ L^2((0,1)^2; H^1((M, M'))) \tag{7.16}$$

where $|\omega(x_3)|$ is the Lebesgue measure of the section $\omega(x_3)$. The convergence (7.16) follows from the convergence (7.15) by observing, similarly as in Lemma 5.1, that y^+ is independent of z_1, z_2. Arguing similarly as Lemma 5.2, we can prove that

$$T^{\varepsilon}\left(\frac{\partial y_{\varepsilon}^{+}}{\partial x_i}\right) \rightharpoonup 0 \text{ weakly in } L^2(\Omega^+ \times (0,1)^2) \quad i = 1, 2\,. \tag{7.17}$$

Using these we can obtain once again the convergences of the following integral terms

$$\lim_{\varepsilon\to 0}\int_{\Omega^-} \nabla y_{\varepsilon}^{-}\cdot\nabla\phi = \int_{\Omega^-}\nabla y^{-}\cdot\nabla\phi, \tag{7.18}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega^-} y_{\varepsilon}^{-}\phi = \int_{\Omega^-} y^{-}\phi \tag{7.19}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon}^{+}}\frac{\partial y_{\varepsilon}}{\partial x_1}\frac{\partial\phi}{\partial x_1} = \lim_{\varepsilon\to 0}\frac{1}{L}\int_{\Omega^+\times(0,1)^2} T^{\varepsilon}\left(\frac{\partial y_{\varepsilon}^{+}}{\partial x_1}\right)T^{\varepsilon}\left(\frac{\partial\phi}{\partial x_1}\right) = 0 \tag{7.20}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon}^{+}}\frac{\partial y_{\varepsilon}}{\partial x_2}\frac{\partial\phi}{\partial x_1} = \lim_{\varepsilon\to 0}\frac{1}{L}\int_{\Omega^+\times(0,1)^2} T^{\varepsilon}\left(\frac{\partial y_{\varepsilon}^{+}}{\partial x_2}\right)T^{\varepsilon}\left(\frac{\partial\phi}{\partial x_2}\right) = 0 \tag{7.21}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon}^{+}}\frac{\partial y_{\varepsilon}}{\partial x_3}\frac{\partial\phi}{\partial x_3} = \lim_{\varepsilon\to 0}\int_{\Omega^+\times(0,1)^2}\frac{\partial(T^{\varepsilon}y_{\varepsilon}^{+})}{\partial x_3}T^{\varepsilon}\left(\frac{\partial\phi}{\partial x_3}\right) = \int_{\Omega^+}|\omega(x_3)|\,\frac{\partial y^{+}}{\partial x_3}\frac{\partial\phi}{\partial x_3} \tag{7.22}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon}^{+}} y_{\varepsilon}\phi = \lim_{\varepsilon\to 0}\int_{\Omega^+\times(0,1)^2} T^{\varepsilon}y_{\varepsilon}^{+}\,T^{\varepsilon}\phi = \int_{\Omega^+}|\omega(x_3)|\,y^{+}\phi \tag{7.23}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon}^{+}} f\phi = \lim_{\varepsilon\to 0}\int_{\Omega^+\times(0,1)^2} T^{\varepsilon}f\,T^{\varepsilon}\phi = \int_{\Omega^+}|\omega(x_3)|f\phi \tag{7.24}$$

where we have used the fact that f, ϕ are also independent of the (z_1, z_2) variables. For details of these arguments, we refer to the preprint [2].

The bound (6.2) implies that, up to a subsequence still denoted by ε, there exist a function ζ in $L^2((0,1)^2; H^1(\mathcal{G}))$ such that

$$\mu(\cdot, T^{\varepsilon}(y_{\varepsilon}^{+})) \rightharpoonup \zeta \text{ weakly in } L^2((0,1)^2; H^1(\mathcal{G})) \tag{7.25}$$

and ζ can be shown to be independent of the local variables (z_1, z_2). Like in the previous section, it can be shown that

$$\zeta(x_1, x_2, x_3) = \mu(y^{+}(x_1, x_2, x_3)) \text{ a.e. } (x_1, x_2, x_3) \in \Omega^{+}. \tag{7.26}$$

Similarly as in (7.13), we obtain

$$\int_{S_{\varepsilon}} u_{\varepsilon}^{\varepsilon}\phi = \varepsilon^{\alpha-1}\int_{(0,1)^2}\int_{M}^{M'}\int_0^1 \frac{u(z_1,z_2,t)T^{\varepsilon}\phi(x_1,x_2,t,z_1,z_2)\sqrt{\varepsilon^2 + g_{z_1}^2(z_1,z_2) + g_{z_2}^2(z_1,z_2)}}{g_{z_2}(z_1,z_2)}\,dz_1dtdx$$

where z_2 depends on z_1 and t and is determined by the curve $\{(z_1, z_2) : g(z_1, z_2) = t\}$ and dx stands for integration with respect to (x_1, x_2). Now we can pass to the limit in the above and we have

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon} u_\varepsilon^\varepsilon \phi = \delta_{\alpha 1}\int_{(0,1)^2}\int_M^{M'}\int_0^1 u(z_1,z_2,t)\,\phi(x_1,x_2,t)\sqrt{1+\left(\frac{g_{z_1}(z_1,z_2)}{g_{z_2}(z_1,z_2)}\right)^2}\,dz_1dt\,dx_1dx_2$$
$$= \delta_{\alpha 1}\int_{\Omega^+}\left(\int_{\partial\omega(x_3)} u(z_1,z_2,x_3)d\gamma_{x_3}(z_1,z_2)\right)\phi(x_1,x_2,x_3)dx_1dx_2dx_3 \tag{7.27}$$

Finally

$$\lim_{\varepsilon\to 0}\varepsilon^\eta\int_{S_\varepsilon}\mu(y_\varepsilon)\phi$$
$$= \lim_{\varepsilon\to 0}\varepsilon^{\eta-1}\int_{(0,1)^2}\int_M^{M'}\int_0^1 \frac{\mu(T^\varepsilon y_\varepsilon^+(x_1,x_2,t,z_1,z_2))\sqrt{\varepsilon^2+g_{z_1}^2(z_1,z_2)+g_{z_2}^2(z_1,z_2)}}{g_{z_2}(z_1,z_2)}\,dz_1dtdx$$
$$= \delta_{\eta 1}\int_{\Omega^+}\left(\int_{\partial\omega(x_3)}\mu(y^+(x_1,x_2,x_3))d\gamma_{x_3}(z_1,z_2)\right)\phi(x_1,x_2,x_3)dx_1dx_2dx_3$$
$$= \delta_{\eta 1}\int_{\Omega^+}\mathrm{per}(\partial\omega(x_3))\mu(y^+(x_1,x_2,x_3))\,\phi(x_1,x_2,x_3)dx_1dx_2dx_3 \tag{7.28}$$

By the convergences (7.18)–(7.28), we conclude, by passing to the limit in (7.5), that

$$\int_{\Omega^-}\nabla y^-\cdot\nabla\phi+\int_{\Omega^+}|\omega(x_3)|\,\frac{\partial y^+}{\partial x_3}\frac{\partial\phi}{\partial x_3}+\delta_{\eta 1}\int_{\Omega^+}\mathrm{per}(\partial\omega(x_3))\mu(y^+(x_1,x_2,x_3))\,\phi(x_1,x_2,x_3)dx_1dx_2dx_3$$
$$+\int_{\Omega^+}|\omega(x_3)|\,y^+\phi=\int_{\Omega^+}|\omega(x_3)|f\phi+\delta_{\alpha 1}\int_{\Omega^+}\left(\int_{\partial\omega(x_3)}u(z_1,z_2,x_3)d\gamma_{x_3}(z_1,z_2)\right)\phi(x_1,x_2,x_3)dx_1dx_2dx_3$$

which is the weak formulation of (7.8). □

Remark 7.3 By taking $u_\varepsilon^\varepsilon = 0$ we recover the result proved in Mel'nyk [29] in the presence of homogeneous Robin boundary condition. The non-homogeneous Robin condition will permit us in the future to treat the problem of homogenization of control problems involving controls terms in the Robin boundary conditions. If the oscillating boundary has flat tops then one can expect a non-homogeneous non-linear Neumann or Robin condition on the upper boundary of Ω^+. To obtain this term it is enough to split the oscillating boundary into the flat portions and the oblique portions and handle the convergence of the flat portions similarly as in Sects. 5 and 6.

Acknowledgements The first author would like to thank the financial support of CONICYT-CHILE through the grant FONDECYT 1130595. The work was partially supported by the project (No. SR/S4/MS: 855/13 dtd 17.10.14) and the second author would like to acknowledge the support of Department of Science Technology (DST), India for the project. The third author wishes to thank CONICYT for the financial support through FONDECYT POSTDOCTORADO NO. 3140138. He would also like to thank the Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción (Chile) for their financial support through PROYECTOS VRID INICIACIÓN NO. 216.013.0.41-1.0IN.

References

1. Achdou, Y., Pironneau, O., Valentin, F.: Effective boundary conditions for laminar flows over periodic rough boundaries. J. Comput. Phys. **147**(1), 187–218 (1998)
2. Aiyappan, S., Nandakumaran, A. K., Prakash, R.: Generalization of unfolding operator for highly oscillatory smooth boundary domains and homogenization. Calc. Var. Partial. Differ. Equ. (to appear)

3. Allaire, G.: Homogenization and two-scale convergence. SIAM J. Math. Anal. **23**(6), 1482–1518 (1992)
4. Allaire, G., Damlamian, A., Hornung, U.: Two-scale convergence on periodic surfaces and applications. In: Proceedings of the International Conference on Mathematical Modelling of Flow through Porous Media, World Scientific Publication, Singapore , pp. 15–25 (1996)
5. Amirat, Y., Bodart, O., De Maio, U., Gaudiello, A.: Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary. SIAM J. Math. Anal **35**(no. 6), 1598–1616 (2004). (electronic)
6. Amirat, Y., Bodart, O., De Maio, U., Gaudiello, A.: Effective boundary condition for Stokes flow over a very rough surface. J. Differ. Equ. **254**(8), 3395–3430 (2013)
7. Arbogast, T., Douglas, J., Hornung, U.: Derivation of the double porosity model of single phase flow via homogenization theory. SIAM J. Math. Anal. **21**, 823–836 (1990)
8. Arrieta, J.M., Bruschi, S.M.: Rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a Lipschitz deformation. Math. Models Methods Appl. Sci. **17**(10), 1555–1585 (2007)
9. Arrieta, J.M., Villanueva-Pesqueira, M.: Unfolding operator method for thin domains with a locally periodic highly oscillatory boundary. SIAM J. Math. Anal. **48**(3), 1534–1671 (2016)
10. Bensoussan, A., Lions, J.-L., Papanicolaou, G.: Asymptotic Analysis for Periodic Structures. North Holland, Amsterdam (1978)
11. Blanchard, D., Carbone, L., Gaudiello, A.: Homogenization of a monotone problem in a domain with oscillating boundary. Mathe. Model. Numer. Anal. **33**(5), 1057–70 (1999)
12. Bonder, J.F., Orive, R., Rossi, J.D.: The best Sobolev trace constant in a domain with oscillating boundary. Nonlinear Anal. **67**(4), 1173–1180 (2007)
13. Bourgeat, A., Marusic-Paloka, E., Mikelic, A.: Effective fluid flow in a porous medium containing a thin fissure. Asymptotic Anal. **11**(3), 241–262 (1995)
14. Brizzi, R., Chalot, J.-P.: Homogénéisation de frontière. Université de Nice, Thèse d'Etat (1978)
15. Brizzi, R., Chalot, J.-P.: Boundary homogenization and neumann boundary value problem. Ricerche Mat. **46**(2), 341–387 (1997)
16. Bucur, D., Feireisl, E., Nečasová, Šárka, Wolf, J.: On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries. J. Differ. Equ. **244**(11), 2890–2908 (2008)
17. Chechkin, G.-A., Friedman, A., Piatnitski, A.-L.: The boundary value problem in a domain with very rapidly oscillating boundary. J. Math. Anal. Appl. **231**, 213–234 (1999)
18. Cioranescu, D., Damlamian, A., Griso, G.: Periodic unfolding and homogenization. C.R. Math. **335**(1), 99104 (2002)
19. Cioranescu, D., Damlamian, A., Griso, G.: The periodic unfolding method in homogenization. SIAM J. Math. Anal **40**(4), 15851620 (2008)
20. Cioranescu, D., Donato, P.: An Introduction to Homogenization. Oxford University Press, Oxford (1999)
21. Cioranescu, D., Paulin, J.Saint Jean: Homogenization in open sets with holes. J. Math. Anal. Appl. **71**, 590–607 (1979)
22. Damlamian, A., Pettersson, K.: Am. Inst. Math. Sci. Homogenization of oscillating boundaries. Discrete and continuous dynamical systems **23**(1), 197219 (2009)
23. De Maio, U., Mel'nyk, T.A.: Homogenization of the neumann problem in thick multi-structures of type 3 : 2 : 2. Math. Methods Appl. Sci. **28**(9), 865–879 (2005)
24. Esposito, A.C., Donato, P., Gaudiello, A., Picard, C.: Homogenization of the p-laplacian in a domain with oscillating boundary. Comm. Appl. Nonlinear Anal. **4**(4), 1–23 (1997)
25. Gaudiello, A.: Asymptotic behavior of non-homogeneous Neumann problems in domains with oscillating boundary. Ricerche Mat. **43**(2), 239–292 (1994)
26. Jikov, V.V., Kozlov, S.M., Oleĭnik, O.A.: Homogenization of Differential Operators and Integral Functionals. Springer-Verlag, Berlin (1994)
27. Kotliarov, V.P., Khruslov, E.Ya.: On a limit boundary condition of some neumann problem. Theor. Funkts. **10**, 83–96 (1970)
28. Krantz, S.G., Parks, H.R.: Geometric Integration Theory. Springer, Boston (2008)
29. MeI'nyk, T .A.: Homogenization of a boundary-value problem with a nonlinear boundary condition in a thick junction of type 3 : 2 : 1. Math. Methods Appl. Sci. **31**(9), 1005–1027 (2008). MR 2419087
30. Mel'nyk, T.A.: Asymptotic approximation for the solution to a semi-linear parabolic problem in a thick junction with the branched structure. J. Math. Anal. Appl. **424**(2), 1237–1260 (2015)

31. Mel'nyk, T.A., Nazarov, S.A.: Asymptotic structure of the spectrum of the neumann problem in a thin comb-like domain. CR Acad. Sci. **319**, 1343–1348 (1994)
32. Murat, F., Sili, Ali: Problèmes monotones dans des cylindres de faible diamètre formés de matériaux hétérogènes, (french) [monotonic problems in heterogeneous cylinders with vanishing diameter]. C. R. Acad. Sci. Paris Sr. I Math **320**(10), 1199–1204 (1995)
33. Nandakumaran, A.K., Prakash, Ravi, Sardar, B.C.: Homogenization of an optimal control problem in a domain with highly oscillating boundary using periodic unfolding method. Math. Eng. Sci. Aerosp. **4**(3), 281–303 (2013)
34. Nazarov, S.A.: Junctions of singularly degenerating domains with different limit dimensions. Trudy Seminara imeni I. G. Petrovskogo **18**, 3–79 (1995)
35. Neuss-Radu, M.: Some extensions of two-scale convergence. C. R. Acad. Sci. Paris Sr. I Math. **322**(9), 899–904 (1996)
36. Nguetseng, G.: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. **20**, 608–623 (1989)
37. Showalter, R.E.: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations. American Mathematical Society, Providence (1997)
38. Suzikov, G.V., Khruslov, E.Y.: On advancing sound waves through narrow channels in a reflecting layer. Theor. Funkts. **5**, 35–49 (1967)
39. Vanninathan, M.: Sur quelques problèmes de homogénéisation dans les équations aux dérivées partielles. Univérsité Pierre et Marie Curie, Thèse d'Etat (1979)
40. Vanninathan, M.: Homogenization of eigenvalue problems in perforated domains. Proc. Indian Acad. Sci. **90**(3), 239–271 (1981)