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Generalization of unfolding operator for highly oscillating smooth boundary domains and homogenization

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Abstract Unfolding operators have been introduced and used to study homogenization problems. Initially, they were introduced for problems with rapidly oscillating coefficients and porous domains. Later, this has been developed for domains with oscillating boundaries, typically with rectangular or pillar type boundaries which are classified as non-smooth. In this article, we develop new unfolding operators, where the oscillations can be smooth and hence they have wider applications. We have demonstrated by developing unfolding operators for circular domains with rapid oscillations with high amplitude of $O(1)$ to study the homogenization of an elliptic problem.

Mathematics Subject Classification 80M35 · 80M40 · 35B27

1 Introduction

The main purpose of this article is to introduce certain unfolding operators in domains with highly oscillating boundaries which are smooth and periodic. Though our approach is quite general, we consider two types of domains as in Figs. 1 and 4. More details of these figures are available in later sections. The first domain consists of a fixed part with very general oscillations on a flat part of the boundary, whereas with the second case, we consider a circular

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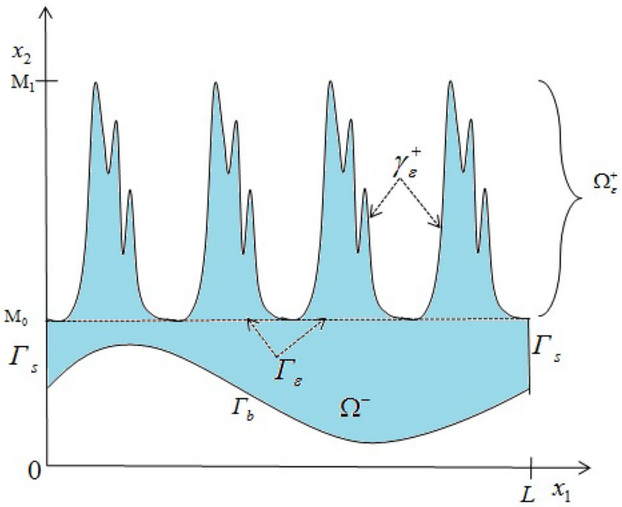


Fig. 1 Oscillating domain Ω_ε

domain with oscillating boundary. In both cases, we consider oscillations with amplitude of $O(1)$. We hope this work will also serve as a precursor to the study of arbitrary oscillating domains, that is of non-periodic type. So far in the literature, the unfolding operators are available only for non-smooth oscillatory boundaries. Presently, we give a novel approach in defining unfolding operator for smooth but periodic oscillations with $O(1)$ amplitude. This immediately helps us to study homogenization problems in circular domains with rapid oscillations at the boundary. So far, the study on circular domains is very limited. We hope, the unfolding operators defined on circular domains with oscillating boundaries will set new trend in carrying out limiting analysis in such complex domains.

Using the new unfolding operators, we give few examples from homogenization. We consider, an example of non-linear problem in the first case and an example of a linear homogenization problem in a circular domain. The optimal control problems in such domains is the topic of discussion in another forthcoming manuscript.

The unfolding method for homogenization problems in domains with (non-smooth) oscillating boundaries has been widely used by various authors in the last decade. Typically, the domains are of fixed type with slab-type (pillar-type) oscillating boundaries. Recently, the authors in [1] have also introduced unfolding operators in domains with multi-level oscillations, known as branched structure and have studied homogenization of optimal control problem. The authors, Blanchard et al. [9,10], have first introduced unfolding in oscillating boundary domain to study homogenization of elastic rods with 3D space as well as thin plate. The general unfolding operator was first introduced by D. Cioranescu, A. Damlamian, G. Griso in 2002 (see [12]) and later it is used by various authors (see [13,14,19]). Though the convergence using unfolding operators looks similar to the two scale convergence (see [2,38]) of homogenization theory, we have demonstrated the power of unfolding in the study of the optimal control problems (see [1,34,36]). In these papers, we have characterized the optimal controls using unfolding operators.

Thus, developing unfolding operators in smooth oscillating boundaries has paramount importance for our future study as well. In fact, the study of homogenization in domains with oscillating smooth boundaries is very limited. Homogenization of an elastic thin film problem using Γ-convergence has been studied in [4]. See [11], for boundary homogenization with Neumann boundary condition and in [29], a Poisson problem with curved rough boundaries in a circular domain has been investigated. Indeed, the amplitude of oscillations is of $O(\varepsilon)$, whereas we consider the oscillations with amplitude of $O(1)$. In [5,6], Arrieta and Villanueva-Pesqueira have also considered a thin domain with smooth oscillating boundary.

On the other hand, there is lot of activity on domain with non-smooth oscillating boundaries; more specifically, a domain with a fixed part and a lot of thin periodically distributed parts (like pillars) attached along certain part of the flat boundary. For example, see [3] for error estimates and in [7,8], the homogenization of PDEs has been investigated in oscillating domain using Tartar's oscillating test functions. In [28], strongly contrasting diffusivity problem in highly oscillating boundaries has been studied. The authors, Esposito et al. [23], studied the asymptotic analysis of a p-Laplacian operator using Γ-convergence. Homogenization of an elliptic problem with homogeneous Neumann data has been studied in [26]. Gaudiello [25], has investigated Laplace equation with inhomogeneous Neumann boundary condition posed on oscillating boundary domain and in [27], using extension operators the authors have studied the homogenization of Ginzburg–Landau equation. Exact controllability problems in oscillating domains have been investigated in [17,18]. For literature on homogenization of optimal control problems on this type of domains one can refer to [16,20–22,33,35,39,40].

The present domain can also be seen as a very good generalization of domains with thick junctions. In addition to our recent work [1] in thick junctions, such a domain has been considered by Mel'nyk [30], where asymptotic expansion method is used and asymptotic study of Robin problem has been investigated in [15]. Also, see [32].

2 Oscillating boundary domain

For $j = 1, 2, \dots n-1$, let $L_j > 0$ and consider a small parameter $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1})$, with $\varepsilon_i = \frac{L_i}{N_i}$, $N_i \in \mathbb{Z}^+$. Denote any element $x \in \mathbb{R}^n$ as $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Let $L = (L_1, L_2, \dots, L_{n-1})$ and $\mathbb{B} = (0, L_1) \times (0, L_2) \times \cdots (0, L_{n-1}) \subset \mathbb{R}^{n-1}$. We, now describe the domain $\Omega_\varepsilon \subset \mathbb{R}^n$ and its boundaries as follows.

Let $g\colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth and periodic function with period L. That is, $g(L_j e_j + x') = g(x')\ \forall x' \in \mathbb{R}^{n-1},\ j = 1, 2, \dots, n-1$, where $\{e_1, e_2, \dots, e_{n-1}\}$ is the standard basis of $\mathbb{R}^{n-1}$. Let η be a smooth real valued function defined on $\mathbb{B}$ such that it takes the minimum at the boundary, that is, $\eta(x'_b) = M_0 =: \min_{x' \in \mathbb{B}} \eta(x')$, where $x'_b \in \partial\mathbb{B}$. Also, assume that the function $\eta - M_0$ is compactly supported in $\mathbb{B}$. Now, extend η to the whole $\mathbb{R}^{n-1}$ periodically with period L.

Let $M_1 =: \max_{x' \in \mathbb{B}} \eta(x')$ and $\mathfrak{m} > \max_{x' \in \mathbb{B}} g(x')$, with $M_1 > M_0 > \mathfrak{m}$. We define the domain Ω_ε as

$$\Omega_\varepsilon = \left\{(x', x_n) \in \mathbb{R}^n\colon x' \in \mathbb{B},\ g(x') < x_n < \eta_\varepsilon(x') = \eta\left(\frac{x'}{\varepsilon}\right)\right\}.$$

Note that, $\frac{x'}{\varepsilon} = (\frac{x_1}{\varepsilon_1}, \frac{x_2}{\varepsilon_2}, \dots, \frac{x_{n-1}}{\varepsilon_{n-1}})$. For $x', y' \in \mathbb{R}^{n-1}$, we formally define $x'y'$ as $x'y' = (x_1 y_1, \dots, x_{n-1} y_{n-1})$. The top boundary of Ω_ε is denoted by γ_ε^+ and is defined by

$$\gamma_\varepsilon^+ = \left\{(x', x_n)\colon x' \in \mathbb{B},\ x_n = \eta_\varepsilon(x')\right\}.$$

The bottom boundary Γ_b of Ω_ε is defined by

$$\Gamma_b = \left\{(x', x_n)\colon x_n = g(x'),\ x' \in \overline{\mathbb{B}}\right\}.$$

Let Ω_ε^+ be the top part of the domain Ω_ε, which is defined by

$$\Omega_\varepsilon^+ = \left\{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{B},\ M_0 < x_n < \eta_\varepsilon(x')\right\}.$$

The reference set $Y(a)$, for $a \in (M_0, M_1)$, is defined as $Y(a) = \{y \in \mathbb{B} : \eta(y) > a\}$. Note that $Y(a)$ is Lebesgue measurable as η is assumed to be a smooth function.

Denote Ω^-, the fixed part of the domain Ω_ε, which is described by

$$\Omega^- = \left\{(x', x_n) : x' \in \mathbb{B},\ g(x') < x_n < M_0\right\}.$$

The lateral and top boundaries of Ω^- denoted by Γ_s and Γ_0 are defined as

$$\Gamma_s = \left\{(x', x_n) : g(x') \le x_n \le M_0, x' \in \partial\mathbb{B}\right\} \text{ and } \Gamma_0 = \left\{(x', M_0) : x' \in \overline{\mathbb{B}}\right\},$$

respectively. The common boundary Γ_ε is the lower boundary of Ω_ε^+ which is defined as

$$\Gamma_\varepsilon = \left\{(x', x_n) \in \Omega_\varepsilon : x_n = M_0\right\}$$

We can also write Ω_ε as

$$\Omega_\varepsilon = Int\left(\overline{\Omega_\varepsilon^+ \cup \Omega^-}\right).$$

Our full domain or the limiting domain Ω (see Fig. 2) is described by

$$\Omega = \left\{(x', x_n) : x' \in \mathbb{B},\ g(x') < x_n < M_1\right\}.$$

The upper part of the limit domain Ω^+ is defined by $\Omega^+ = \mathbb{B} \times (M_0, M_1)$. The lower boundary of Ω is same as that of Ω_ε, namely Γ_b. The upper boundary Γ_u and the lateral boundaries $\Gamma_{s'}$ are defined as follows.

$$\Gamma_u = \left\{(x', M_1) : x' \in \overline{\mathbb{B}}\right\} \text{ and } \Gamma_{s'} = \left\{(x', x_n) : g(x') \le x_n \le M_1, x' \in \partial\overline{\mathbb{B}}\right\}$$

Define a set $\mathbb{E}_\varepsilon = \{k \in \mathbb{Z}^{n-1} : \varepsilon kL + \varepsilon\mathbb{B} \subseteq \mathbb{B}\}$ and the reference cell Λ^+ is defined as, (see Fig. 3)

$$\Lambda^+ =: \left\{(y', y_n) : y' \in \mathbb{B}, M_0 < y_n < \eta(y')\right\}.$$

Let $H^1_\#(\Omega_\varepsilon) = \{f \in H^1_{loc}(\mathbb{R}^n) : f(x' + kL, x_n) = f(x', x_n)\ \forall k \in \mathbb{Z}^{n-1}\}$. Note that, by kL, we mean $kL = (k_1L_1, \ldots, k_{n-1}L_{n-1})$. We call a function as Γ_s-periodic if it takes the same value on the opposite lateral sides of the domain Ω_ε.

3 Unfolding operator and its properties

In this section, we define the periodic unfolding operator (T^ε) and the boundary unfolding operator ($T^\varepsilon_{M_0}$) to study the asymptotic behaviour of the solution of PDEs posed on a domain with highly oscillating smooth boundary which we have described in Sect. 2. This is one of the most important sections of the article. Also, we look into some of its important properties which are required for our analysis in the later sections. Let us define the unfolded (fixed) domain Ω_U, where the unfolded functions are defined on, as below.

Let $\mathcal{G} = \{(x_n, y) : x_n \in (M_0,\ M_1), y \in Y(x_n)\}$, then Ω_U is defined as $\Omega_U = \mathbb{B} \times \mathcal{G}$, which can be written as

$$\Omega_U =: \left\{(x', x_n, y) |\ x' \in \mathbb{B}, M_0 < x_n < M_1,\ y \in Y(x_n)\right\}.$$

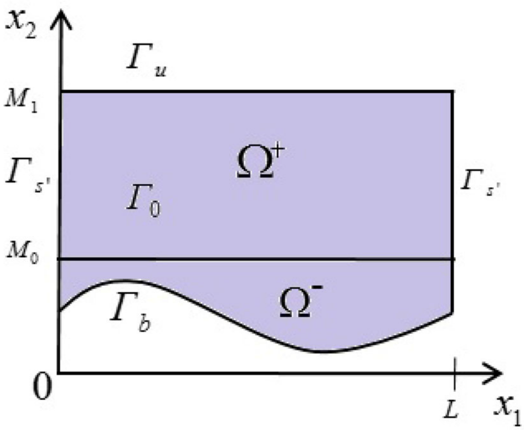


Fig. 2 Limiting domain

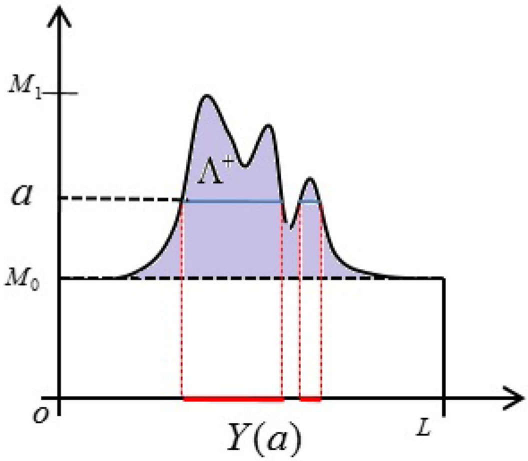


Fig. 3 Reference domain

We, now present an unfolding operator which is completely new and novel. For $x' \in \mathbb{R}^{n-1}$, we write $[x']_L$ as the integer part of x' with respect to L, that is, $[x']_L = kL$, where $k = (k_1, k_2, \ldots, k_{n-1})$ is the "largest" integer tuple such that $kL \leq x'$, that is, $k_j L_j \leq x_j$ for $j = 1, 2, \ldots, n-1$ and $\{x'\}_L = x' - [x']_L$.

Definition 3.1 (*The unfolding operator*) Let $\phi^\varepsilon : \Omega_U \to \Omega_\varepsilon^+$ be defined by $(x', x_n, y) \to \left(\varepsilon \left[\frac{x'}{\varepsilon}\right]_L + \varepsilon y, x_n\right)$, that is, $\phi^\varepsilon(x', x_n, y) = \left(\varepsilon \left[\frac{x'}{\varepsilon}\right]_L + \varepsilon y, x_n\right)$. The ε-unfolding of a function $u: \Omega_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \Omega_U \to \mathbb{R}$. The operator which maps every function $u: \Omega_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. Let the unfolding operator is denoted by T^ε, that is,

$$T^\varepsilon : \left\{u : \Omega_\varepsilon^+ \to \mathbb{R}\right\} \to \{v : \Omega_U \to \mathbb{R}\}$$

defined by

$$T^\varepsilon u(x', x_n, y) = u \circ \phi^\varepsilon(x', x_n, y) = u\left(\varepsilon \left[\frac{x'}{\varepsilon}\right]_L + \varepsilon y, x_n\right).$$

If U is an open subset of $\mathbb{R}^n$ containing Ω_ε^+ and u is real valued function on U, $T^\varepsilon u$ will mean T^ε acting on the restriction of u to Ω_ε^+. Some of the properties that T^ε enjoys are given below.

Proposition 3.2 *For each fixed $\varepsilon > 0$, T^ε is linear. Further, if $u, v \colon \Omega_\varepsilon^+ \to \mathbb{R}$, then, $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.*

The proof follows directly from the definition.

Proposition 3.3 *Let $u \in L^1(\Omega_\varepsilon^+)$. Then,*

$$\int_{\Omega_U} T^\varepsilon u \, dxdy = |\mathbb{B}| \int_{\Omega_\varepsilon^+} u \, dx.$$

Proof

$$\begin{aligned}
\int_{\Omega_U} T^\varepsilon u \, dxdy &= \int_{x_n=M_0}^{M_1} \int_{y\in Y(x_n)} \int_{\mathbb{B}} u\left(\varepsilon\left[\frac{x'}{\varepsilon}\right]_L + \varepsilon y,\ x_n\right) dx'dydx_n \\
&= \int_{M_0}^{M_1} \int_{y\in Y(x_n)} \sum_{k\in\mathbb{E}_\varepsilon} \int_{x'\in\varepsilon kL+\varepsilon\mathbb{B}} u(k\varepsilon L + \varepsilon y, x_n) \, dx'dydx_n \\
&= \sum_{k\in\mathbb{E}_\varepsilon} \int_{x'\in\varepsilon kL+\varepsilon\mathbb{B}} dx' \int_{M_0}^{M_1} \int_{y\in Y(x_n)} u(k\varepsilon L + \varepsilon y, x_n) \, dydx_n \\
&= \varepsilon^{n-1}|\mathbb{B}| \sum_{k\in\mathbb{E}_\varepsilon} \int_{M_0}^{M_1} \int_{y\in Y(x_n)} u(k\varepsilon L + \varepsilon y, x_n) \, dydx_n \\
&= |\mathbb{B}| \sum_{k\in\mathbb{E}_\varepsilon} \int_{M_0}^{M_1} \int_{z\in k\varepsilon L+\varepsilon Y(x_n)} u(z, x_n) \, dzdx_n \\
&= |\mathbb{B}| \int_{\Omega_\varepsilon^+} u(x) \, dx.
\end{aligned}$$

□

Proposition 3.4 *Let $u \in L^2(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2(\Omega_U)$ and $\|T^\varepsilon u\|_{L^2(\Omega_U)} = \sqrt{|\mathbb{B}|}\|u\|_{L^2(\Omega_\varepsilon^+)}$.*

Proof Given that $u \in L^2(\Omega_\varepsilon^+)$ implies $|u|^2 \in L^1(\Omega_\varepsilon^+)$. From, the above proposition, we get

$$\int_{\Omega_U} |T^\varepsilon u|^2 dxdy = \int_{\Omega_U} T^\varepsilon |u|^2 \, dxdy = |\mathbb{B}| \int_{\Omega_\varepsilon^+} |u|^2 \, dx.$$

This implies

$$\|T^\varepsilon u\|_{L^2(\Omega_U)} = \sqrt{|\mathbb{B}|}\|u\|_{L^2(\Omega_\varepsilon^+)}.$$

□

Proposition 3.5 *Let* $u \in H^1(\Omega_\varepsilon^+)$*. Then,* $T^\varepsilon u \in L^2(\mathbb{B}; H^1(\mathcal{G})$*. Moreover,* $\dfrac{\partial}{\partial x_n} T^\varepsilon u = T^\varepsilon \dfrac{\partial u}{\partial x_n}$ *and* $\dfrac{\partial}{\partial y_j} T^\varepsilon u = \varepsilon_j T^\varepsilon \dfrac{\partial u}{\partial x_j}$*, for* $j = 1, 2, \ldots, n-1$.

Proof Since there is no oscillation in x_n direction (and hence no unfolding), we obtain that $\dfrac{\partial}{\partial x_n} T^\varepsilon u = T^\varepsilon \dfrac{\partial u}{\partial x_n}$. Now, we have

$$
\begin{aligned}
\|T^\varepsilon u\|^2_{L^2(\mathbb{B}; H^1(\mathcal{G}))} &= \int\limits_{\mathbb{B}} \|T^\varepsilon u\|^2_{H^1(\mathcal{G})}\, dx' \\
&= \int\limits_{\Omega_U} \left(\sum_{j=1}^{n-1} \varepsilon_j^2 T^\varepsilon \left| \frac{\partial u}{\partial x_j} \right|^2 + T^\varepsilon \left| \frac{\partial u}{\partial x_n} \right|^2 + T^\varepsilon |u|^2 \right) dxdy \\
&= \int\limits_{\Omega_U} T^\varepsilon \left(\sum_{j=1}^{n-1} \varepsilon_j^2 \left| \frac{\partial u}{\partial x_i} \right|^2 + \left| \frac{\partial u}{\partial x_n} \right|^2 + |u|^2 \right) dxdy \\
&= |\mathbb{B}| \int\limits_{\Omega_\varepsilon^+} \left(\sum_{j=1}^{n-1} \varepsilon_j^2 \left| \frac{\partial u}{\partial x_j} \right|^2 + \left| \frac{\partial u}{\partial x_n} \right|^2 + |u|^2 \right) dx \\
&\leq |\mathbb{B}| \|u\|^2_{H^1(\Omega_\varepsilon^+)} < \infty.
\end{aligned}
$$

□

Proposition 3.6 *Let* $u \in L^2(\Omega^+)$*. Then,* $T^\varepsilon u \to u$ *in* $L^2(\Omega_U)$*. More generally, let* $u_\varepsilon \to u$ *in* $L^2(\Omega^+)$*. Then,* $T^\varepsilon u_\varepsilon \to u$ *in* $L^2(\Omega_U)$.

Proof To see the first part of the theorem, consider $\phi \in \mathcal{D}(\Omega^+)$.

$$
\begin{aligned}
\sup_{(x,y)\in\Omega_U} |(T^\varepsilon \phi)(x', x_n, y) - \phi(x', x_n)| &= \sup_{(x,y)\in\Omega_U} \left|\phi\left(\varepsilon \left[\tfrac{x'}{\varepsilon}\right]_L + \varepsilon y, x_n\right) - \phi(x', x_n)\right| \\
&\leq m_\phi(|\varepsilon L|),
\end{aligned}
$$

where m_ϕ is the modulus of continuity of the function ϕ which is defined as

$$
m_\phi(\delta) = \sup_{z_1, z_2 \in \Omega^+} \{|\phi(z_1) - \phi(z_2)| \colon |z_1 - z_2| < \delta\}.
$$

Since ϕ is uniformly continuous in Ω^+, $m_\phi(|\varepsilon L|) \to 0$ as $\varepsilon \to 0$. Hence,

$$
\sup_{\Omega_U} |T^\varepsilon \phi - \phi| \to 0 \; as \; \varepsilon \to 0
$$

and thus,

$$
T^\varepsilon \phi \to \phi \; in \; L^2(\Omega_U) \quad \forall\, \phi \in \mathcal{D}(\Omega^+).
$$

The density of $\mathcal{D}(\Omega^+)$ in $L^2(\Omega^+)$ completes the proof of the first part. Now, suppose that $u_\varepsilon \to u$ in $L^2(\Omega^+)$. Then,

$$\begin{aligned}
\|T^\varepsilon u_\varepsilon - u\|_{L^2(\Omega_U)} &= \|T^\varepsilon u_\varepsilon - T^\varepsilon u + T^\varepsilon u - u\|_{L^2(\Omega_U)} \\
&\leq \|T^\varepsilon u_\varepsilon - T^\varepsilon u\|_{L^2(\Omega_U)} + \|T^\varepsilon u - u\|_{L^2(\Omega_U)} \\
&= \|T^\varepsilon (u_\varepsilon - u)\|_{L^2(\Omega_U)} + \|T^\varepsilon u - u\|_{L^2(\Omega_U)} \\
&= \sqrt{|\mathbb{B}|}\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon^+)} + \|T^\varepsilon u - u\|_{L^2(\Omega_U)}, \quad \text{by Proposition 3.4} \\
&\leq \sqrt{|\mathbb{B}|}\|u_\varepsilon - u\|_{L^2(\Omega^+)} + \|T^\varepsilon u - u\|_{L^2(\Omega_U)} \\
&\to 0 \;\; as \;\; \varepsilon \to 0.
\end{aligned}$$

□

Under weak convergence, we have the following result.

Proposition 3.7 *Let, for every ε, $u_\varepsilon \in L^2(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega_U)$. Then,*

$$\widetilde{u}_\varepsilon \rightharpoonup \frac{1}{|\mathbb{B}|} \int\limits_{y \in Y(x_n)} u(x', x_n, y)\, dy$$

weakly in $L^2(\Omega^+)$. Here, $\widetilde{u}_\varepsilon$ is the zero extension of u_ε to Ω^+.

Proof Let $\psi \in \mathcal{D}(\Omega^+)$, then,

$$\begin{aligned}
\int\limits_{\Omega^+} \widetilde{u}_\varepsilon \psi &= \frac{1}{|\mathbb{B}|} \int\limits_{\Omega_U} T^\varepsilon u_\varepsilon T^\varepsilon \psi \\
&\to \frac{1}{|\mathbb{B}|} \int\limits_{\Omega_U} u\psi \text{ as } \varepsilon \to 0, \text{ by Proposition 3.6} \\
&= \int\limits_{\mathbb{B}} \int\limits_{M_0}^{M_1} \left(\frac{1}{|\mathbb{B}|} \int\limits_{y \in Y(x_n)} u\, dy \right) \psi\, dx_n dx' \\
&= \int\limits_{\Omega^+} \left(\frac{1}{|\mathbb{B}|} \int\limits_{y \in Y(x_n)} u\, dy \right) \psi\, dx_n dx' \;\; \forall \psi \in \mathcal{D}(\Omega^+)
\end{aligned}$$

This completes the proof as $\mathcal{D}(\Omega^+)$ is dense in $L^2(\Omega^+)$. □

Proposition 3.8 *Let $u_\varepsilon \in H^1(\Omega_\varepsilon^+)$ for every $\varepsilon > 0$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\mathbb{B}; H^1(\mathcal{G}))$. Then, $\widetilde{u}_\varepsilon \rightharpoonup \frac{1}{|\mathbb{B}|} \int_{Y(x_n)} u \;\, dy$ and $\widetilde{\frac{\partial u_\varepsilon}{\partial x_n}} \rightharpoonup \frac{1}{|\mathbb{B}|} \int_{Y(x_n)} \frac{\partial u}{\partial x_n} \, dy$ weakly in $L^2(\Omega^+)$.*

Proof Given that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\mathbb{B}; H^1(\mathcal{G}))$, which implies

$$T^\varepsilon u_\varepsilon \rightharpoonup u \text{ and } \frac{\partial}{\partial x_n} T^\varepsilon u_\varepsilon \rightharpoonup \frac{\partial u}{\partial x_n} \text{ weakly in } L^2(\Omega_U).$$

That is,

$$T^{\varepsilon}u_{\varepsilon} \rightharpoonup u \text{ and } T^{\varepsilon}\frac{\partial u_{\varepsilon}}{\partial x_n} \rightharpoonup \frac{\partial u}{\partial x_n} \text{ weakly in } L^2(\Omega_U).$$

Using Proposition 3.7, we get $\widetilde{u}_{\varepsilon} \rightharpoonup \dfrac{1}{|\mathbb{B}|}\displaystyle\int_{Y(x_n)} u \; dy$ in $L^2(\Omega^+)$ and $\widetilde{\dfrac{\partial u_{\varepsilon}}{\partial x_n}} \rightharpoonup \dfrac{1}{|\mathbb{B}|}\displaystyle\int_{Y(x_n)} \dfrac{\partial u}{\partial x_n}\, dy$ in $L^2(\Omega^+)$. □

3.1 Unfolding on the boundary

We, now define the boundary unfolding operator on Γ_{ε}, that is, on the common boundary of Ω_{ε}^{+} and Ω^{-}.

Definition 3.9 Let $\phi^{\varepsilon}_{M_0}: \mathbb{B} \times Y(M_0) \to \Gamma_{\varepsilon}$ be defined by $(x', y) \to \varepsilon\left[\frac{x'}{\varepsilon}\right]_L + \varepsilon y$. The ε-unfolding of a function $u: \Gamma_{\varepsilon} \to \mathbb{R}$ is the function $u \circ \phi^{\varepsilon}_{M_0}: \mathbb{B} \times Y(M_0) \to \mathbb{R}$ denoted by $T^{\varepsilon}_{M_0}$ is defined as

$$T^{\varepsilon}_{M_0}: \{u: \Gamma_{\varepsilon} \to \mathbb{R}\} \to \{v: \mathbb{B} \times Y(M_0) \to \mathbb{R}\}$$

by

$$T^{\varepsilon}_{M_0}u = u \circ \phi^{\varepsilon}_{M_0} = u\left(\varepsilon\left[\frac{x'}{\varepsilon}\right]_L + \varepsilon y\right).$$

If U is an open subset of $\mathbb{R}^{n-1}$ such that $\Gamma_{\varepsilon} \subset U$ and $u: U \to \mathbb{R}$ then $T^{\varepsilon}_{M_0}u = T^{\varepsilon}_{x_n = M_0}\left(u|_{\Gamma_{\varepsilon}}\right)$. The properties of boundary unfolding are given below. We omit the proofs here, as they are similar to that of unfolding operators.

Proposition 3.10 1. *$T^{\varepsilon}_{M_0}$ is linear.*
2. *Let u,v be functions from $\Gamma_{\varepsilon} \to \mathbb{R}$. Then, $T^{\varepsilon}_{M_0}(uv) = T^{\varepsilon}_{M_0}(u)T^{\varepsilon}_{M_0}(v)$.*
3. *Let $u \in L^2(\Gamma\varepsilon)$. Then, $T^{\varepsilon}_{M_0}u \in L^2(\mathbb{B} \times Y(M_0))$. Moreover $\|T^{\varepsilon}_{M_0}u\|_{L^2(\mathbb{B}\times Y(M_0))} = \sqrt{|\mathbb{B}|}\|u\|_{L^2(\Gamma\varepsilon)}$.*
4. *Let $u \in H^1(\Gamma_{\varepsilon})$. Then, $T^{\varepsilon}_{M_0}u \in L^2(\mathbb{B}; H^1(Y(M_0)))$ and $\dfrac{\partial}{\partial y_j}T^{\varepsilon}_{M_0}u = \varepsilon_j T^{\varepsilon}_{M_0}\dfrac{\partial u}{\partial x_j}$, for $j = 1, 2, \ldots, n-1$.*
5. *Let $u \in L^2(\mathbb{B})$. Then, $T^{\varepsilon}_{M_0}u \to u$ in $L^2(\mathbb{B} \times Y(M_0))$.*
6. *Suppose that $u_{\varepsilon} \to u$ in $L^2(\mathbb{B})$. Then, $T^{\varepsilon}_{M_0}u_{\varepsilon} \to u$ in $L^2(\mathbb{B} \times Y(M_0))$.*
7. *Suppose that u_{ε} is a sequence in $L^2(\Gamma_{\varepsilon})$ such that $T^{\varepsilon}_{M_0}u_{\varepsilon} \rightharpoonup u$ weakly in $L^2(\mathbb{B}\times Y(M_0))$. Then, $\widetilde{u}_{\varepsilon} \rightharpoonup \dfrac{1}{|\mathbb{B}|}\displaystyle\int_{Y(M_0)} u\; dy$ weakly in $L^2(\mathbb{B})$.*

In the next two sections, we study two homogenization problems, where one is a non-linear problem posed in the domain Ω_{ε} as in Fig. 1 and other is a linear problem described in a circular domain (see Fig. 4).

4 Homogenization

We consider a semi linear elliptic equation in the domain Ω_ε:

$$\begin{cases} -\Delta u_\varepsilon + k(u_\varepsilon) = f_\varepsilon \text{ in } \Omega_\varepsilon \\ \partial_\nu u_\varepsilon = 0 \text{ on } \gamma_\varepsilon^+ \cup \Gamma_b;\ u_\varepsilon \ is\ \Gamma_s\text{-}periodic. \end{cases} \tag{4.1}$$

Here, f_ε is a given function in $L^2(\Omega_\varepsilon)$; $k\colon \mathbb{R} \to \mathbb{R}$ is a smooth function such that $0 < C_1 < k'(t) < C_2\ \forall t \in \mathbb{R}$; ∂_ν is the outward normal derivative. By monotone operator theory (see [41,42]), it is known that if $f_\varepsilon \in L^2(\Omega_\varepsilon)$, then Eq. (4.1) admits a unique weak solution u_ε in $H^1_\#(\Omega_\varepsilon)$. Moreover, the solution satisfies the a priori estimate, that is,

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C\|f_\varepsilon\|_{L^2(\Omega)}, \tag{4.2}$$

where $C > 0$ is independent of ε. The non-linear problem is not new and it has been studied in [30], where the author considers the branched structure domain (rectangular type). Of course, they also do not use unfolding operators. We consider this non-linear problem in the general domain, we have discussed in the previous section and we use unfolding operators. From our earlier research, we have realized that unfolding operators are more suitable to such problems.

Limit problem Let $h(x_n) = |Y(x_n)|$, where $|Y(x_n)|$ is the Lebesgue measure of the set $Y(x_n)$ at $x_n \in (M_0, M_1)$. Here, we have chosen η such that h is a strictly positive function in $[M_0, M_1]$. Now, Consider the space

$$\widehat{W}(\Omega) = \left\{ \psi \in C^\infty(\overline{\Omega})\colon \psi \in L^2(\Omega),\ \frac{\partial \psi}{\partial x_n} \in L^2(\Omega),\ \psi^- \in H^1(\Omega^-) \right\}$$

with the inner product

$$\begin{aligned} \langle u, v\rangle_W = \langle hu, v\rangle_{L^2(\Omega^+)} + \langle h\partial_{x_n} u, \partial_{x_n} v\rangle_{L^2(\Omega^+)} \\ + \langle u, v\rangle_{H^1(\Omega^-)}. \end{aligned} \tag{4.3}$$

Now, define $W(\Omega)$ to be the completion of $\widehat{W}(\Omega)$ with respect to the norm defined by the above inner product. We can characterize the space $W(\Omega)$ as

$$W(\Omega) = \left\{ \psi \in L^2(\Omega)\colon \psi \in L^2(\Omega),\ \frac{\partial \psi}{\partial x_n} \in L^2(\Omega),\ \psi^- \in H^1(\Omega^-) \right\}.$$

Note that $W(\Omega)$ is a Hilbert space with the inner product defined as in (4.3).

We, now state the limit problem. Given $f_0 \in L^2(\Omega)$, consider the partial differential equation

$$\begin{cases} -\dfrac{\partial}{\partial x_n}\left(h(x_n)\dfrac{\partial u^+}{\partial x_n}\right) + h(x_n)k(u^+) = f_0^+ & in\ \Omega^+, \\ -\Delta u^- + k(u^-) = f_0^- & in\ \Omega^-, \\ \dfrac{\partial u^+}{\partial \nu} = 0 & on\ \Gamma_b \cup \Gamma_u, \\ u^+ = u^-,\quad \dfrac{h(M_0)}{|\mathbb{B}|}\dfrac{\partial u^+}{\partial x_n} = \dfrac{\partial u^-}{\partial x_n} & on\ \Gamma_0, \\ u\ \ is\ \Gamma_{s'}\text{-}periodic, \end{cases} \tag{4.4}$$

where

$$u = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-} \text{ and } f_0 = f_0^+\chi_{\Omega^+} + f_0^-\chi_{\Omega^-}. \tag{4.5}$$

The weak formulation of the above equation is: find $u \in W(\Omega)$ such that

$$\int_{\Omega^+} \frac{h(x_n)}{|\mathbb{B}|} \left(\frac{\partial u^+}{\partial x_n} \frac{\partial \psi}{\partial x_n} + k(u^+)\psi \right) + \int_{\Omega^-} \nabla u^- \cdot \nabla \psi + k(u^-)\psi$$
$$= \frac{1}{|\mathbb{B}|} \int_{\Omega^+} f_0^+ \psi + \int_{\Omega^-} f_0^- \psi \tag{4.6}$$

for all $\psi \in W(\Omega)$. The existence of the weak solution is guaranteed again by the monotonicity of k, similar to the case of (4.1). We have the following homogenization theorem.

Theorem 4.1 *Assume $f_\varepsilon \rightharpoonup f_0^-$ weakly in $L^2(\Omega^-)$ and $f_\varepsilon \equiv 0$ in Ω_ε^+ and let u_ε be the solution of the semi-linear elliptic equation* (4.1)*. Then,*

$$\widetilde{u_\varepsilon^+} \rightharpoonup \frac{h(x_n)}{|\mathbb{B}|} u_0^+ \text{ weakly in } L^2(\Omega^+),$$
$$\widetilde{\frac{\partial u_\varepsilon^+}{\partial x_n}} \rightharpoonup \frac{h(x_n)}{|\mathbb{B}|} \frac{\partial u_0^+}{\partial x_n} \text{ weakly in } L^2(\Omega^+),$$
$$u_\varepsilon^- \rightharpoonup u_0^- \text{ weakly in } H^1(\Omega^-),$$

where u_0 is the unique solution of the homogenized problem (4.4)*, with $f_0^+ \equiv 0$. Here, u_ε^+ and u_ε^- are the restrictions of u_ε to Ω_ε^+ and Ω^-, respectively.*

Remark 4.2 We can also consider non vanishing f_ε on Ω_ε^+, but in this case, we need to take $f_\varepsilon \in L^2(\Omega)$ and $f_\varepsilon \to f_0$ strongly in $L^2(\Omega)$.

Proof Recall the a priori estimate (4.2):

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \|f_\varepsilon\|_{L^2(\Omega)} \tag{4.7}$$

Since f_ε is given to be weakly convergent, there exists a constant $C > 0$ independent of ε such that $\|f_\varepsilon\|_{L^2(\Omega)} \leq C$. Hence, we have

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C. \tag{4.8}$$

Let us estimate $T^\varepsilon u_\varepsilon^+$ in the space $L^2(\mathbb{B}; H^1(\mathcal{G}))$ using the properties of the unfolding operator, which we have discussed in Sect. 3. By using Proposition 3.5, we get

$$\left\| T^\varepsilon u_\varepsilon^+ \right\|^2_{L^2(\mathbb{B}; H^1(\mathcal{G})} \leq |\mathbb{B}| \, \|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} . \tag{4.9}$$

The boundedness of the sequence $T^\varepsilon u_\varepsilon^+$ in $L^2(\mathbb{B}; H^1(\mathcal{G}))$ follows from estimate (4.8). By weak compactness, there exists a subsequence (still denoted by ε) such that

$$T^\varepsilon u_\varepsilon^+ \rightharpoonup u_0^+ \text{ weakly in } L^2(\mathbb{B}; H^1(\mathcal{G})), \tag{4.10}$$

which implies

$$T^\varepsilon u_\varepsilon^+ \rightharpoonup u_0^+ \text{ weakly in } L^2(\Omega_U), \tag{4.11}$$
$$\frac{\partial}{\partial x_n} T^\varepsilon u_\varepsilon^+ \rightharpoonup \frac{\partial u_0^+}{\partial x_n} \text{ weakly in } L^2(\Omega_U), \text{ that is,}$$
$$T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_n} \rightharpoonup \frac{\partial u_0^+}{\partial x_n} \text{ weakly in } L^2(\Omega_U) \tag{4.12}$$

and for $j = 1, 2, \ldots, n-1$

$$\frac{\partial}{\partial y_j} T^\varepsilon u_\varepsilon^+ \rightharpoonup \frac{\partial u_0^+}{\partial y_j} \text{ weakly in } L^2(\Omega_U), \text{ that is, } \varepsilon_j T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} \rightharpoonup \frac{\partial u_0^+}{\partial y_j} \text{ weakly in } L^2(\Omega_U). \tag{4.13}$$

From the Proposition 3.5, we have

$$\left\| T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} \right\|_{L^2(\Omega_U)} = \sqrt{|\mathbb{B}|} \left\| \frac{\partial u_\varepsilon^+}{\partial x_j} \right\|_{L^2(\Omega_\varepsilon^+)} \le \sqrt{|\mathbb{B}|} \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}.$$

Again estimate (4.8) implies the boundedness of the sequence $T^\varepsilon \dfrac{\partial u_\varepsilon^+}{\partial x_j}$ in the space $L^2(\Omega_U)$ for $1 \le j \le n-1$. Hence, from (4.13), we conclude that $\dfrac{\partial u_0^+}{\partial y_j} = 0$, for $1 \le j \le n-1$. Thus, we have

$$\widetilde{u_\varepsilon^+} \rightharpoonup \frac{1}{|\mathbb{B}|} \int_{Y(x_n)} u_0^+ dy \text{ and } \widetilde{\frac{\partial u_\varepsilon^+}{\partial x_n}} \rightharpoonup \frac{1}{|\mathbb{B}|} \int_{Y(x_n)} \frac{\partial u_0^+}{\partial x_n} dy \text{ weakly in } L^2(\Omega^+) \tag{4.14}$$

with the help of Proposition 3.8. Since u_0^+ is independent of y variable, we get

$$\int_{Y(x_n)} u_0^+ \, dy = h(x_n) u_0^+ \text{and} \int_{Y(x_n)} \frac{\partial u_0^+}{\partial x_n} \, dy = h(x_n) \frac{\partial u_0^+}{\partial x_n}. \tag{4.15}$$

Thus, (4.14) becomes

$$\widetilde{u_\varepsilon^+} \rightharpoonup \frac{h(x_n)}{|\mathbb{B}|} u_0^+ \text{ and } \widetilde{\frac{\partial u_\varepsilon^+}{\partial x_n}} \rightharpoonup \frac{h(x_n)}{|\mathbb{B}|} \frac{\partial u_0^+}{\partial x_n} \text{ weakly in } L^2(\Omega^+). \tag{4.16}$$

We know that $T^\varepsilon \dfrac{\partial u_\varepsilon^+}{\partial x_j}$ is bounded in $L^2(\Omega_U)$ for $1 \le j \le n-1$. Hence, by weak compactness, there is an element $P_j \in L^2(\Omega_U)$ such that up to subsequence (still denoted by ε),

$$T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} \rightharpoonup P_j \text{ weakly in } L^2(\Omega_U) \text{ for } 1 \le j \le n-1. \tag{4.17}$$

Since k satisfies $|k(t)| \le C(1 + |t|) \ \forall \ t \in \mathbb{R}$, we have $T^\varepsilon k(u_\varepsilon^+) = k(T^\varepsilon u_\varepsilon^+)$ is uniformly bounded in $L^2(\Omega_U)$ as $T^\varepsilon u_\varepsilon^+$ is bounded in $L^2(\Omega_U)$. Hence, there exists a ζ in $L^2(\Omega_U)$ such that

$$T^\varepsilon k(u_\varepsilon^+) \rightharpoonup \zeta \text{ in } L^2(\Omega_U) \tag{4.18}$$

Using the estimate of $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$, we have the boundedness of u_ε^- in the space $H^1(\Omega^-)$. Thus, up to a subsequence (still denoted by ε)

$$u_\varepsilon^- \rightharpoonup u_0^- \text{ weakly in } H^1(\Omega^-). \tag{4.19}$$

Define u_0 as

$$u_0(x) = u_0^+ \chi_{\Omega^+} + u_0^- \chi_{\Omega^-} \tag{4.20}$$

Claim 1 $u_0 \in W(\Omega)$ *and satisfies the limit problem* (4.4).

We know that $u_0 \in L^2(\Omega)$ and $\dfrac{\partial u_0^-}{\partial x_j} \in L^2(\Omega^-)$ for $1 \le j \le n$. To prove $u_0 \in W(\Omega)$, we need to show $\dfrac{\partial u_0}{\partial x_n} \in L^2(\Omega)$. Note that u_0^+ is independent of y variable and so is $\dfrac{\partial u_0^+}{\partial x_n}$. Hence, we have $\dfrac{\partial u_0^+}{\partial x_n} \in L^2(\Omega^+)$ and also $\dfrac{\partial u_0^-}{\partial x_n} \in L^2(\Omega^-)$. Thus, to show $\dfrac{\partial u_0}{\partial x_n} \in L^2(\Omega)$, it is enough to prove that the trace of u_0^+ and u_0^- are equal on Γ_0. Since $u_\varepsilon^+|_{\Gamma_\varepsilon} = u_\varepsilon^-|_{\Gamma_\varepsilon}$ implies the equality of trace for the boundary unfolding operator, we have $T^\varepsilon_{M_0}\left(u_\varepsilon^+|_{\Gamma_\varepsilon}\right) = T^\varepsilon_{M_0}\left(u_\varepsilon^-|_{\Gamma_\varepsilon}\right)$. That is,

$$\left(T^\varepsilon(u_\varepsilon^+)\right)|_{x_n=M_0} = T^\varepsilon_{M_0}\left(u_\varepsilon^-|_{\Gamma_\varepsilon}\right). \tag{4.21}$$

From the weak continuity of trace operator, we can write

$$\left(T^\varepsilon(u_\varepsilon^+)\right)|_{x_n=M_0} \rightharpoonup u_0^+|_{x_n=M_0} \text{ weakly in } L^2(\mathbb{B} \times Y(M_0))$$

and from (4.19), we get

$$u_\varepsilon{}^-|_{x_n=M_0} \to u^-|_{x_n=M_0} \text{ strongly in } L^2(\mathbb{B}).$$

This implies

$$T^\varepsilon_{M_0}\left(u_\varepsilon{}^-|_{x_n=M_0}\right) \to u_0^-|_{x_n=M_0} \text{ in } L^2\left(\mathbb{B} \times Y(M_0)\right).$$

Passing to the limit in (4.21) as $\varepsilon \to 0$, we get

$$u_0^+|_{x_n=M_0} = u_0^-|_{x_n=M_0} \text{ in } L^2(\mathbb{B}),$$

since u_0^+ and u_0^- are independent of y variable. This shows that $\dfrac{\partial u_0}{\partial x_n} \in L^2(\Omega)$. Thus, we proved the claim.

Identification of the limit P_j in (4.17).

For $\phi \in \mathcal{D}(\Omega^+)$, choose a test function $\phi_j^\varepsilon(x) = \varepsilon_j \phi(x)\{\frac{x_j}{\varepsilon_j}\}$, for $1 \le j \le n-1$, in such a way that ϕ_j^ε is continuous on Ω_ε^+. By applying unfolding operator on ϕ_j^ε and by Proposition 3.5, we get

$$\begin{aligned}
T^\varepsilon \phi_j^\varepsilon &= \varepsilon_j y_j T^\varepsilon \phi,\\
T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_j} &= \frac{1}{\varepsilon_j}\frac{\partial}{\partial y_j} T^\varepsilon \phi_j^\varepsilon = \varepsilon_j y_j T^\varepsilon \frac{\partial \phi}{\partial x_j} + T^\varepsilon \phi,\\
T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_i} &= \frac{1}{\varepsilon_i}\frac{\partial}{\partial y_i} T^\varepsilon \phi_j^\varepsilon = \varepsilon_j y_j T^\varepsilon \frac{\partial \phi}{\partial x_i}, i \ne j,\ 1 \le i \le n-1\\
T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_n} &= \varepsilon_j y_j T^\varepsilon \frac{\partial \phi}{\partial x_n}.
\end{aligned}$$

Let us recall the variational formulation of (4.1) with the test function $\widetilde{\phi_j^\varepsilon}$.

$$\int_{\Omega_\varepsilon^+} \nabla u_\varepsilon \cdot \nabla \phi_j^\varepsilon + \int_{\Omega_\varepsilon^+} k(u_\varepsilon)\phi_j^\varepsilon = 0, \text{ as } \phi_j^\varepsilon = 0 \text{ in } \Omega^- \tag{4.22}$$

Now notice,

$$\int\limits_{\Omega_\varepsilon^+} \nabla u_\varepsilon^+ \cdot \nabla \phi_j^\varepsilon = \frac{1}{|\mathbb{B}|} \int\limits_{\Omega_U} \sum_{i=0, i\neq j}^{n-1} \left(T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_i} T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_i} \right) + T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_j} + T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_n} T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_n}$$

$$= \frac{1}{|\mathbb{B}|} \int\limits_{\Omega_U} \sum_{i=0, i\neq j}^{n-1} T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_i} \varepsilon_j y_j T^\varepsilon \frac{\partial \phi}{\partial x_i} + T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} \left(\varepsilon_j y_j T^\varepsilon \frac{\partial \phi}{\partial x_j} + T^\varepsilon \phi \right)$$

$$+ T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_n} \varepsilon_j y_j T^\varepsilon \frac{\partial \phi}{\partial x_n}, \int\limits_{\Omega_\varepsilon^+} k(u_\varepsilon) \phi_j^\varepsilon = \frac{1}{|\mathbb{B}|} \int\limits_{\Omega_U} T^\varepsilon k(u_\varepsilon^+)\, T^\varepsilon \phi_j^\varepsilon$$

$$= \frac{1}{|\mathbb{B}|} \int\limits_{\Omega_U} \varepsilon_j y_j T^\varepsilon k(u_\varepsilon^+)\, T^\varepsilon \phi.$$

Equation (4.22) gives,

$$\int\limits_{\Omega_U} T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} T^\varepsilon \phi = -\int\limits_{\Omega_U} \sum_{i=0, i\neq j}^{n-1} \varepsilon_j y_j T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_i} T^\varepsilon \frac{\partial \phi}{\partial x_i} + \varepsilon_j y_j T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} T^\varepsilon \frac{\partial \phi}{\partial x_j}$$

$$+ \varepsilon_j y_j T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_n} T^\varepsilon \frac{\partial \phi}{\partial x_n} - \int\limits_{\Omega_U} \varepsilon_j y_j T^\varepsilon k(u_\varepsilon^+)\, T^\varepsilon \phi.$$

This implies,

$$\left| \int\limits_{\Omega_U} T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} T^\varepsilon \phi \right| \le \varepsilon_j C \| T^\varepsilon \phi \|_{L^2(\mathbb{B}; H^1(\mathcal{G}))}.$$

Hence,

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega_U} T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} T^\varepsilon \phi = \int\limits_{\Omega_U} P_j \phi = 0 \; \forall \phi \in \mathcal{D}(\Omega^+).$$

Thus, we conclude that $\int_{Y(x_n)} P_j dy \equiv 0$ on Ω^+ for $j = 1, 2, \ldots, n-1$.

Next, we derive the limit of the non-linear term.

Claim 2 $\int_{Y(x_n)} \zeta \, dy = hk(u_0^+)$ *in* $L^2(\Omega_U)$: let $\phi \in C^1(\overline{\Omega})$. Here, we use Browder–Minty method using the monotonicity of k (see [24,31]). We have

$$\int_{\Omega^-} |\nabla u_\varepsilon^- - \nabla \phi|^2 + \int_{\Omega_\varepsilon^+} |\partial_{x_n} u_\varepsilon^+ - \partial_{x_n} \phi|^2 + \int_{\Omega_\varepsilon^+} |\nabla_{x'} u_\varepsilon^+|^2$$

$$+ \int_{\Omega^-} (k(u_\varepsilon) - k(\phi))(u_\varepsilon - \phi) + \int_{\Omega_\varepsilon^+} (k(u_\varepsilon) - k(\phi))(u_\varepsilon - \phi) \ge 0.$$

By applying unfolding, we get

$$\int_{\Omega^-} |\nabla u_\varepsilon^- - \nabla \phi|^2 + \frac{1}{|\mathbb{B}|} \int_{\Omega_U} |T^\varepsilon \partial_{x_n} u_\varepsilon - T^\varepsilon \partial_{x_n} \phi|^2 + |T^\varepsilon \nabla_{x'} u_\varepsilon|^2$$

$$+ \int_{\Omega^-} (k(u_\varepsilon) - k(\phi))(u_\varepsilon - \phi) + \frac{1}{|\mathbb{B}|} \int_{\Omega_U} (T^\varepsilon k(u_\varepsilon) - T^\varepsilon k(\phi))(T^\varepsilon u_\varepsilon^+ - T^\varepsilon \phi) \ge 0.$$

Upon expanding the above inequality, we obtain

$$\int_{\Omega^-}\left(|\nabla u_\varepsilon|^2+k(u_\varepsilon)u_\varepsilon\right)+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}\left(|T^\varepsilon\nabla u_\varepsilon^+|^2+T^\varepsilon k(u_\varepsilon^+)T^\varepsilon u_\varepsilon^+\right)$$
$$+\int_{\Omega^-}\left(|\nabla\phi|^2-2\nabla u_\varepsilon\nabla\phi+k(\phi)\phi-k(u_\varepsilon)\phi-k(\phi)u_\varepsilon\right)$$
$$+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}\left(|T^\varepsilon\partial_{x_n}\phi|^2-2T^\varepsilon\partial_{x_n}u_\varepsilon^+T^\varepsilon\partial_{x_n}\phi+T^\varepsilon k(\phi)T^\varepsilon\phi\right.$$
$$\left.-T^\varepsilon k(u_\varepsilon^+)T^\varepsilon\phi-T^\varepsilon k(\phi)T^\varepsilon u_\varepsilon^+\right)\geq 0. \tag{4.23}$$

Now, let us recall the variational formulation:

$$\int_{\Omega^-}\nabla u_\varepsilon\nabla\phi+k(u_\varepsilon)\phi+\int_{\Omega_\varepsilon^+}\nabla u_\varepsilon\nabla\phi+k(u_\varepsilon)\phi=\int_{\Omega^-}f_\varepsilon\phi.$$

On applying unfolding,

$$\int_{\Omega^-}\nabla u_\varepsilon^-\nabla\phi+k(u_\varepsilon^-)\phi+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}\left(T^\varepsilon\nabla u_\varepsilon^+T^\varepsilon\nabla\phi+T^\varepsilon k(u_\varepsilon^+)T^\varepsilon\phi\right)=\int_{\Omega^-}f_\varepsilon\phi.$$

Using the convergence of $T^\varepsilon u_\varepsilon$ and u_ε^-, we can pass to the limit in the above equation and obtain

$$\int_{\Omega^-}\nabla u_0^-\nabla\phi+k(u_0^-)\phi+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}(\partial_{x_n}u_0^+\partial_{x_n}\phi+\zeta\phi)=\int_{\Omega^-}f_0\phi\ \forall\phi\in C^1(\overline{\Omega}).$$

As $C^1(\overline{\Omega})$ is dense in $W(\Omega)$, we have (choosing $\phi=u_0$)

$$\int_{\Omega^-}|\nabla u_0^-|^2+k(u_0^-)u_0^-+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}\left(|\partial_{x_n}u_0^+|^2+\zeta u_0^+\right)=\int_{\Omega^-}f_0u_0^-. \tag{4.24}$$

Note that

$$\lim_{\varepsilon\to 0}\left[\int_{\Omega^-}|\nabla u_\varepsilon|^2+k(u_\varepsilon)u_\varepsilon+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}\left(|T^\varepsilon\nabla u_\varepsilon^+|^2+T^\varepsilon k(u_\varepsilon^+)T^\varepsilon u_\varepsilon^+\right)\right]$$
$$=\lim_{\varepsilon\to 0}\int_{\Omega^-}f_\varepsilon u_\varepsilon$$
$$=\int_{\Omega^-}f_0u_0^-$$
$$=\int_{\Omega^-}|\nabla u_0^-|^2+k(u_0^-)u_0^-+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}\left(|\partial_{x_n}u_0^+|^2+\zeta u_0^+\right) \tag{4.25}$$

Now, we utilize (4.25) in passing to the limit in the first line of the equality (4.23). In the other components, we just apply weak convergence of $T^\varepsilon u_\varepsilon^-$ and u_ε^-, in respective spaces. Thus,

$$\int_{\Omega^-}|\nabla u_0^--\nabla\phi|^2+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}|\partial_{x_n}u_0^+-\partial_{x_n}\phi|^2+\int_{\Omega^-}(k(u_0^-)-k(\phi))(u_0^--\phi)$$
$$+\frac{1}{|\mathbb{B}|}\int_{\Omega_U}(\zeta-k(\phi))(u_0^+-\phi)\geq 0.$$

The above inequality is true for all $\phi \in W(\Omega)$, as $C^1(\overline{\Omega})$ is dense in $W(\Omega)$. Now, for a fixed $\psi \in C^1(\overline{\Omega})$, choose $\phi = u_0 - \lambda\psi$, $\lambda > 0$, to get

$$\lambda \int_{\Omega^-} |\nabla\psi|^2 + \frac{\lambda}{|\mathbb{B}|}\int_{\Omega_U} |\partial_{x_n}\psi|^2 + \int_{\Omega^-} \left(k(u_0^-) - k(u_0^- - \lambda\psi)\right)\psi$$
$$+ \frac{1}{|\mathbb{B}|}\int_{\Omega_U} \left(\zeta - k(u_0^+ - \lambda\psi)\right)\psi \geq 0 \;\forall\, \psi \in C^1(\overline{\Omega}).$$

As $\lambda \to 0$,

$$\frac{1}{|\mathbb{B}|}\int_{\Omega_U} (\zeta - k(u_0^+))\psi \geq 0 \;\forall\, \psi \in C^1(\overline{\Omega}).$$

Hence, $\int_{Y(x_n)} \zeta \, dy = h(x_n)k(u_0^+)$.

Claim 3 u_0 *satisfies the limit equation*: Choose a test function $\psi \in C^\infty(\overline{\Omega})$ in the variational formulation of (4.1). Now, using the above limit analysis, we get

$$\begin{aligned}
\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla\psi + \int_{\Omega_\varepsilon} k(u_\varepsilon)\psi &= \int_{\Omega_\varepsilon^+} \nabla u_\varepsilon^+ \cdot \nabla\psi + \int_{\Omega_\varepsilon^+} k(u_\varepsilon^+)\psi + \int_{\Omega^-} \nabla u_\varepsilon^- \cdot \nabla\psi + k(u_\varepsilon^-)\psi \\
&= \frac{1}{|\mathbb{B}|}\int_{\Omega_U} \left(\sum_{j=1}^{n-1} T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_j} T^\varepsilon \frac{\partial \psi}{\partial x_j} + T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_n} T^\varepsilon \frac{\partial \psi}{\partial x_n}\right) \\
&\quad + \frac{1}{|\mathbb{B}|}\int_{\Omega_U} T^\varepsilon k(u_\varepsilon^+) T^\varepsilon \psi + \int_{\Omega^-} (\nabla u_\varepsilon^- \cdot \nabla\psi + k(u_\varepsilon^-)\psi) \\
&\to \frac{1}{|\mathbb{B}|}\int_{\Omega_U} \frac{\partial u_0^+}{\partial x_n}\frac{\partial \psi}{\partial x_n} + \frac{1}{|\mathbb{B}|}\int_{\Omega_U} k(u_0^+)\psi \\
&\quad + \int_{\Omega^-} \left(\nabla {u_0}^- \cdot \nabla\psi + k(u_0^-)\psi\right)
\end{aligned} \tag{4.26}$$

and

$$\int_{\Omega_\varepsilon} f_\varepsilon \psi \to \int_{\Omega^-} f_0^- \psi. \tag{4.27}$$

Hence,

$$\frac{1}{|\mathbb{B}|}\int_{\Omega_U} \frac{\partial u_0^+}{\partial x_n}\frac{\partial \psi}{\partial x_n} + \int_{\Omega^+} \frac{h(x_n)}{|\mathbb{B}|} k(u_0^+)\psi + \int_{\Omega^-} \nabla u_0^- \cdot \nabla\psi + k(u_0^-)\psi = \int_{\Omega^-} f_0^- \psi,$$

which implies

$$\int_{\Omega^+} \frac{h(x_n)}{|\mathbb{B}|}\left(\frac{\partial u_0^+}{\partial x_n}\frac{\partial \psi}{\partial x_n} + k(u_0^+)\psi\right) + \int_{\Omega^-} \nabla u_0^- \cdot \nabla\psi + k(u_0^-)\psi = \int_{\Omega^-} f_0^- \psi$$

$\forall\, \psi \in C^\infty(\overline{\Omega})$. Since $C^\infty(\overline{\Omega})$ is dense in $W(\Omega)$, the above equality is true for all ψ in $W(\Omega)$. Therefore, u_0 satisfies the differential equation (4.4) with $f_0^+ \equiv 0$. □

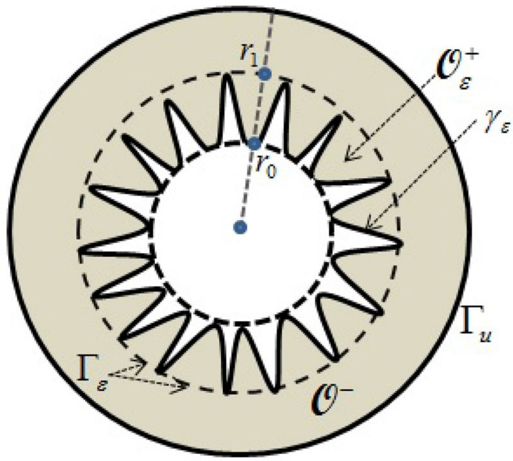


Fig. 4 Circular oscillating domain $\mathcal{O}_\varepsilon$

5 Oscillating circular boundary domain and problem description

In this section, we consider a circular domain whose boundary is highly oscillating. As discussed in the introduction, the literature in the circular domain is limited (see [29,37, 43]). In [29], the amplitude of the oscillations is of order ε and in [37,43], the authors have studied homogenization problem on a domain with highly oscillating interfaces. But, we consider, oscillations of $O(1)$. Our novelty in the study is the development of suitable unfolding operator. Using the unfolding operator, thus developed, we study the corresponding homogenization. The ideas of the unfolding operator comes from the previous section, but, we need to develop it in the set up of polar co-ordinates to apply it in circular domains. In fact, this is one of the major advantages of our approach of defining unfolding operators in smooth oscillating boundaries.

For a small parameter $\varepsilon = \frac{2\pi}{N}$, $N \in \mathbb{Z}^+$, we consider an oscillating boundary domain $\mathcal{O}_\varepsilon$ as given in the Fig. 4. We now describe the domain $\mathcal{O}_\varepsilon$ and its boundaries as follows. Let $g: \mathbb{R} \to \mathbb{R}$ be a smooth and periodic function with period 2π and η be a smooth real valued function defined on $[0, 2\pi]$ such that it takes the maximum at the end points, that is, $\eta(0) = \eta(2\pi) = r_1 =: \max_{\theta\in[0,2\pi]} \eta(\theta)$. Also assume that the function $r_1 - \eta$ is compactly supported in $(0, 2\pi)$. Now extend η to the whole real line periodically with period 2π.

Let $r_0 =: \min_{\theta\in[0,2\pi]} \eta(\theta)$ and $\mathfrak{m} < \min_{\theta\in[0,2\pi]} g(\theta)$, with $0 < r_0 < r_1 < \mathfrak{m}$. Now, define the domain $\mathcal{O}_\varepsilon$ as

$$\mathcal{O}_\varepsilon = \left\{(r, \theta) \in \mathbb{R}^2 : 0 < \theta \leq 2\pi,\ \eta_\varepsilon(\theta) = \eta\left(\frac{\theta}{\varepsilon}\right) < r < g(\theta)\right\}.$$

Typically, $\mathcal{O}_\varepsilon$ consists of an annulus type region bounded by the inner circle of radius r_1 and outer boundary given by g; and an oscillating region bounded by the outer circle of radius r_1 and the oscillating inner boundary defined by η_ε. The oscillating inner boundary of $\mathcal{O}_\varepsilon$ denoted by γ_ε is given by

$$\gamma_\varepsilon = \{(r, \theta) : \theta \in [0, 2\pi],\ r = \eta_\varepsilon(\theta)\}.$$

The fixed outer boundary Γ_u of $\mathcal{O}_\varepsilon$ is defined by

$$\Gamma_u = \{(r, \theta) : r = g(\theta),\ \theta \in [0, 2\pi]\}.$$

Let $\mathcal{O}_\varepsilon^+$ be the oscillating part of the domain $\mathcal{O}_\varepsilon$, which is

$$\mathcal{O}_\varepsilon^+ = \left\{(r, \theta) \in \mathbb{R}^2 : 0 < \theta \leq 2\pi,\ \eta_\varepsilon(\theta) < r < r_1\right\}.$$

The reference set $Y(a)$, for $a \in (r_0, r_1)$, is defined as $Y(a) = \{\tau \in [0, 2\pi] : \eta(\tau) < a\}$. Note that $Y(a)$ is Lebesgue measurable as η is assumed to be a smooth function. Denote $\mathcal{O}^-$, the fixed part of the domain $\mathcal{O}_\varepsilon$, which is described by

$$\mathcal{O}^- = \{(r, \theta) : 0 < \theta \leq 2\pi,\ r_1 < r < g(\theta)\}.$$

The inner boundary of $\mathcal{O}^-$ denoted by Γ_0 is defined as

$$\Gamma_0 = \{(r_1, \theta) : 0 \leq \theta \leq 2\pi\}.$$

The common boundary Γ_ε is defined as

$$\Gamma_\varepsilon = \{(r, \theta) \in \mathcal{O}_\varepsilon : r = r_1\}$$

We can also write $\mathcal{O}_\varepsilon$ as

$$\mathcal{O}_\varepsilon = Int\ \left(\overline{\mathcal{O}_\varepsilon^+ \cup \mathcal{O}^-}\right).$$

The full domain or the limiting domain $\mathcal{O}$ is described by

$$\mathcal{O} = \{(r, \theta) : 0 < \theta \leq 2\pi,\ r_0 < r < g(\theta)\}.$$

The inner part of the limit domain $\mathcal{O}^+$ is defined by

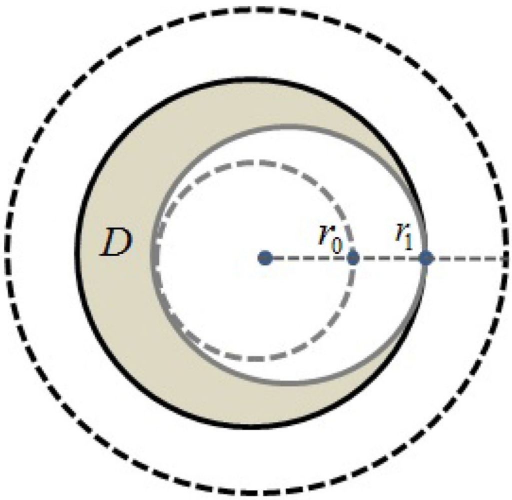
$$\mathcal{O}^+ = \{(r, \theta) : 0 < \theta \leq 2\pi,\ r_0 < r < r_1\}.$$

The boundaries of $\mathcal{O}$ are Γ_u and Γ_b, where

$$\Gamma_b = \{(r_0, \theta) : 0 \leq \theta \leq 2\pi\}$$

and Γ_u is same as defined earlier. The reference cell D is defined as, $D =: \{(r, \theta) : \eta(\theta) < r < r_1, 0 < \theta \leq 2\pi\}$ (Fig. 5).

Fig. 5 Reference domain D



6 Unfolding operator and its properties

We, now define the relevant periodic unfolding operator (T^{ε}) and the boundary unfolding operator ($T^{\varepsilon}_{r_1}$) in the circular domain with highly oscillating smooth boundary which we have described in Sect. 5, to study the asymptotic behaviour of the solution of PDEs. We, also look into some of its important properties which are required for our analysis in Sect. 7. Let us define the unfolded (fixed) domain $\mathcal{O}_u$, where the unfolded functions are defined, as below. Let $\mathcal{G} = \{(r, \tau) : r \in (r_0,\ r_1), \tau \in Y(r)\}$, then, $\mathcal{O}_u$ is defined as $\mathcal{O}_u = (0, 2\pi) \times \mathcal{G}$, which can be written as

$$\mathcal{O}_u =: \{(r, \theta, \tau) |\, 0 < \theta < 2\pi, r_0 < r < r_1,\ \tau \in Y(r)\}\,.$$

For $x \in \mathbb{R}$, we write $[x]_{2\pi}$ as the integer part of x with respect to 2π, that is, $[x]_{2\pi} = 2k\pi$, where k is the largest integer such that $2k\pi \leq x$ and $\{x\}_{2\pi} = x - [x]_{2\pi}$.

Definition 6.1 (*The unfolding operator*) Let $\phi^{\varepsilon} : \mathcal{O}_u \to \mathcal{O}_{\varepsilon}^{+}$ be defined by $(r, \theta, \tau) \to \left(r, \varepsilon \left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right)$. The ε-unfolding of a function $u : \mathcal{O}_{\varepsilon}^{+} \to \mathbb{R}$ is the function $u \circ \phi^{\varepsilon} : \mathcal{O}_u \to \mathbb{R}$. The operator which maps every function $u : \mathcal{O}_{\varepsilon}^{+} \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. Let the unfolding operator be denoted by T^{ε}, that is,

$$T^{\varepsilon} : \left\{u : \mathcal{O}_{\varepsilon}^{+} \to \mathbb{R}\right\} \to \left\{v : \mathcal{O}_u \to \mathbb{R}\right\}$$

defined by

$$(T^{\varepsilon}u)(r, \theta, \tau) = (u \circ \phi^{\varepsilon})(r, \theta, \tau) = u\left(r, \varepsilon\left[\tfrac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right).$$

If U is an open subset of $\mathbb{R}^2$ containing $\mathcal{O}_{\varepsilon}^{+}$ and u is a real valued function on U, $T^{\varepsilon}u$ will mean T^{ε} acting on the restriction of u to $\mathcal{O}_{\varepsilon}^{+}$. Some of the properties of T^{ε} are given below.

Proposition 6.2 *For each fixed* $\varepsilon > 0$, T^{ε} *is linear and* $T^{\varepsilon}(uv) = T^{\varepsilon}(u)T^{\varepsilon}(v)$, *where* $u, v : \mathcal{O}_{\varepsilon}^{+} \to \mathbb{R}$.

The proof follows directly from the Definition 6.1.

Proposition 6.3 *Let* $u \in L^1(\mathcal{O}_{\varepsilon}^{+})$. *Then,*

$$\int\limits_{\mathcal{O}_u} T^{\varepsilon}u \; r dr d\theta d\tau = 2\pi \int\limits_{\mathcal{O}_{\varepsilon}^{+}} u \; r dr d\theta.$$

Proof

$$\begin{aligned}
\int_{\mathcal{O}_u} T^\varepsilon u \; r dr d\theta d\tau &= \int_{r_0}^{r_1} \int_{\tau\in Y(r)} \int_0^{2\pi} u\left(r, \varepsilon\left[\tfrac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right) \; r d\theta d\tau dr \\
&= \int_{r_0}^{r_1} \int_{\tau\in Y(r)} \sum_{k=0}^{N-1} \int_{\theta=2k\varepsilon\pi}^{2(k+1)\varepsilon\pi} u(r, 2k\varepsilon\pi + \varepsilon\tau) \; r d\theta d\tau dr \\
&= \sum_{k=0}^{N-1} \int_{\theta=2k\varepsilon\pi}^{2(k+1)\varepsilon\pi} d\theta \int_{r_0}^{r_1} \int_{\tau\in Y(r)} u(r, 2k\varepsilon\pi + \varepsilon\tau) \; d\tau r dr \\
&= 2\pi\varepsilon \sum_{k=0}^{N-1} \int_{r_0}^{r_1} \int_{\tau\in Y(r)} u(r, 2k\varepsilon\pi + \varepsilon\tau) \; d\tau r dr \\
&= 2\pi \sum_{k=0}^{N-1} \int_{r_0}^{r_1} \int_{z\in 2k\varepsilon\pi+\varepsilon Y(r)} u(r, z) \; r dr d\theta \\
&= 2\pi \int_{\mathcal{O}_\varepsilon^+} u(r, \theta) \; r dr d\theta.
\end{aligned}$$

□

Now, for any $u \in L^2(\mathcal{O}_\varepsilon^+)$ implies $|u|^2 \in L^1(\mathcal{O}_\varepsilon^+)$. Hence, from the above proposition, we get

$$\int_{\mathcal{O}_u} |T^\varepsilon u|^2 \; r dr d\theta = \int_{\mathcal{O}_u} T^\varepsilon |u|^2 \; r dr d\theta \; = 2\pi \int_{\mathcal{O}_\varepsilon^+} |u|^2 \; r dr d\theta.$$

This implies that $\|T^\varepsilon u\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\|u\|_{L^2(\mathcal{O}_\varepsilon^+)}$ and thus we have the following proposition.

Proposition 6.4 *Let $u \in L^2(\mathcal{O}_\varepsilon^+)$. Then $T^\varepsilon u \in L^2(\mathcal{O}_u)$ and $\|T^\varepsilon u\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\|u\|_{L^2(\mathcal{O}_\varepsilon^+)}$.*

Proposition 6.5 *Let $u, \dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta} \in L^2(\mathcal{O}_\varepsilon^+)$. Then, $T^\varepsilon u, \dfrac{\partial}{\partial r}T^\varepsilon u, \dfrac{\partial}{\partial \tau}T^\varepsilon u \in L^2(\mathcal{O}_u)$. Moreover, $\dfrac{\partial}{\partial r}T^\varepsilon u = T^\varepsilon \dfrac{\partial u}{\partial r}$ and $\dfrac{\partial}{\partial \tau}T^\varepsilon u = \varepsilon T^\varepsilon \dfrac{\partial u}{\partial \theta}$.*

Proof By using the Definition 6.1, we can easily check that $\dfrac{\partial}{\partial r}T^\varepsilon u = T^\varepsilon \dfrac{\partial u}{\partial r}$ and $\dfrac{\partial}{\partial \tau}T^\varepsilon u = \varepsilon T^\varepsilon \dfrac{\partial u}{\partial \theta}$. By the Proposition 6.4, we have $\|T^\varepsilon u\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\|u\|_{L^2(\mathcal{O}_\varepsilon^+)}$, $\left\|T^\varepsilon \dfrac{\partial u}{\partial r}\right\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\left\|\dfrac{\partial u}{\partial r}\right\|_{L^2(\mathcal{O}_\varepsilon^+)}$ and $\left\|T^\varepsilon \dfrac{\partial u}{\partial \tau}\right\|_{L^2(\mathcal{O}_u)} = \varepsilon\sqrt{2\pi}\left\|\dfrac{\partial u}{\partial \theta}\right\|_{L^2(\mathcal{O}_\varepsilon^+)}$. Hence, the result follows from the hypothesis. □

Proposition 6.6 *Let $u \in L^2(\mathcal{O}^+)$. Then, $T^\varepsilon u \to u$ in $L^2(\mathcal{O}_u)$. More generally, if $y_\varepsilon \to y$ in $L^2(\mathcal{O}^+)$, then $T^\varepsilon y_\varepsilon \to y$ in $L^2(\mathcal{O}_u)$.*

Proof Consider $\phi \in \mathcal{D}(\mathcal{O}^+)$.

$$\sup_{(r,\theta,\tau)\in\mathcal{O}_u} |(T^\varepsilon\phi)(r,\theta,\tau) - \phi(r,\theta)| = \sup_{(r,\theta,\tau)\in\mathcal{O}_u} \left|\phi(r, \varepsilon[\tfrac{\theta}{\varepsilon}]_{2\pi} + \varepsilon\tau) - \phi(r,\theta)\right| \leq m_\phi(2\pi\varepsilon),$$

where m_ϕ is the modulus of continuity of the function ϕ which is defined as

$$m_\phi(\delta) = \sup_{z_1,z_2\in\mathcal{O}^+} \{|\phi(z_1) - \phi(z_2)| : |z_1 - z_2| < \delta\}.$$

Since, ϕ is uniformly continuous in $\mathcal{O}^+$, $m_\phi(2\pi\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence,

$$\sup_{\mathcal{O}_u} |T^\varepsilon\phi - \phi| \to 0 \ as \ \varepsilon \ \to 0.$$

Thus, $T^\varepsilon\phi \to \phi \ in \ L^2(\mathcal{O}_u) \ \ \forall\, \phi \in \mathcal{D}(\mathcal{O}^+)$. The density of $\mathcal{D}(\mathcal{O}^+)$ in $L^2(\mathcal{O}^+)$ completes the first part of the proposition. To see the second part, suppose that $y_\varepsilon \to y$ in $L^2(\mathcal{O}^+)$. Then,

$$\begin{aligned}
\|T^\varepsilon y_\varepsilon - y\|_{L^2(\mathcal{O}_u)} &= \|T^\varepsilon y_\varepsilon - T^\varepsilon y + T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&\leq \|T^\varepsilon y_\varepsilon - T^\varepsilon y\|_{L^2(\mathcal{O}_u)} + \|T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&= \|T^\varepsilon (y_\varepsilon - y)\|_{L^2(\mathcal{O}_u)} + \|T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&= \sqrt{2\pi}\|y_\varepsilon - y\|_{L^2(\mathcal{O}_\varepsilon^+)} + \|T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \text{ (by Proposition 6.4)} \\
&\leq \sqrt{2\pi}\|y_\varepsilon - y\|_{L^2(\mathcal{O}^+)} + \|T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&\to 0 \ \ as \ \ \varepsilon \to 0.
\end{aligned}$$

□

Proposition 6.7 *Let, for every ε, $y_\varepsilon \in L^2(\mathcal{O}_\varepsilon^+)$ be such that $T^\varepsilon y_\varepsilon \rightharpoonup y$ weakly in $L^2(\mathcal{O}_u)$. Then,*

$$\widetilde{y}_\varepsilon \rightharpoonup \frac{1}{2\pi} \int_{\tau\in Y(r)} y(r,\theta,\tau)\, d\tau$$

weakly in $L^2(\mathcal{O}^+)$. Here, $\widetilde{y}_\varepsilon$ is the zero extension of y_ε to $\mathcal{O}^+$.

Proof Let $\psi \in \mathcal{D}(\mathcal{O}^+)$. Then,

$$\begin{aligned}
\int_{\mathcal{O}^+} \widetilde{y}_\varepsilon \psi \ r dr d\theta &= \frac{1}{2\pi}\int_{\mathcal{O}_u} T^\varepsilon y_\varepsilon T^\varepsilon \psi \ r dr d\theta \\
&\to \frac{1}{2\pi}\int_{\mathcal{O}_u} y\psi \ r dr d\theta \ as \ \varepsilon \to 0, \text{ (by Propositions 6.2, 6.3 and 6.6)} \\
&= \int_0^{2\pi}\int_{r_0}^{r_1} \left(\frac{1}{2\pi}\int_{\tau\in Y(r)} y\, d\tau\right) \psi \ r dr d\theta \\
&= \int_{\mathcal{O}^+} \left(\frac{1}{2\pi}\int_{\tau\in Y(r)} y\, d\tau\right) \psi \ r dr d\theta \ \ \forall \psi \in \mathcal{D}(\mathcal{O}^+)
\end{aligned}$$

This completes the proof as $\mathcal{D}(\mathcal{O}^+)$ is dense in $L^2(\mathcal{O}^+)$. □

We, now derive the convergence of unfolding for H^1 functions.

Proposition 6.8 *Let $y_\varepsilon \in H^1(\mathcal{O}_\varepsilon^+)$ for every $\varepsilon > 0$ be such that $T^\varepsilon y_\varepsilon \rightharpoonup y$ and $\frac{\partial}{\partial r} T^\varepsilon y_\varepsilon \rightharpoonup \frac{\partial y}{\partial r}$ weakly in $L^2(\mathcal{O}_u)$. Then, $\widetilde{y_\varepsilon} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} y \, d\tau$ and $\widetilde{\frac{\partial y_\varepsilon}{\partial r}} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} \frac{\partial y}{\partial r}\, d\tau$ weakly in $L^2(\mathcal{O}^+)$.*

Proof Given that

$$T^\varepsilon y_\varepsilon \rightharpoonup y \text{ and } \frac{\partial}{\partial r} T^\varepsilon y_\varepsilon \rightharpoonup \frac{\partial y}{\partial r} \text{ weakly in } L^2(\mathcal{O}_u).$$

That is,

$$T^\varepsilon y_\varepsilon \rightharpoonup y \text{ and } T^\varepsilon \frac{\partial y_\varepsilon}{\partial r} \rightharpoonup \frac{\partial y}{\partial r} \text{ weakly in } L^2(\mathcal{O}_u).$$

Using Proposition 6.7, we get $\widetilde{y_\varepsilon} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} y \, d\tau$ in $L^2(\mathcal{O}^+)$ and $\widetilde{\frac{\partial y_\varepsilon}{\partial r}} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} \frac{\partial y}{\partial r}\, d\tau$ in $L^2(\mathcal{O}^+)$. □

6.1 Unfolding on the boundary

We, now define the boundary unfolding operator on Γ_ε, that is, on the common boundary of $\mathcal{O}_\varepsilon^+$ and $\mathcal{O}^-$.

Definition 6.9 Let $\phi_{r_1}^\varepsilon \colon (0, 2\pi) \times Y(r_1) \to \Gamma_\varepsilon$ be defined by $(\theta, \tau) \to \varepsilon \left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau$. The ε-unfolding of a function $u \colon \Gamma_\varepsilon \to \mathbb{R}$ is the function $u \circ \phi_{r_1}^\varepsilon \colon (0, 2\pi) \times Y(r_1) \to \mathbb{R}$ denoted by $T_{r_1}^\varepsilon$.

$$T_{r_1}^\varepsilon \colon \{u \colon \Gamma_\varepsilon \to \mathbb{R}\} \to \{v \colon (0, 2\pi) \times Y(r_1)\} \to \mathbb{R}$$

by

$$T_{r_1}^\varepsilon u = u \circ \phi_{r_1}^\varepsilon = u\left(\varepsilon \left[\tfrac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right).$$

If U is an open subset of $\mathbb{R}^2$ such that $\Gamma_\varepsilon \subset U$ and $u \colon U \to \mathbb{R}$ then $T_{r_1}^\varepsilon u = T_{r_1}^\varepsilon \left(u|_{\Gamma_\varepsilon}\right)$.

The properties of boundary unfolding are given below without proof. In fact, all of them can be proved analogously as above.

Proposition 6.10 *1. $T_{r_1}^\varepsilon$ is linear.*
2. *Let u, v be functions from $\Gamma_\varepsilon \to \mathbb{R}$. Then, $T_{r_1}^\varepsilon(uv) = T_{r_1}^\varepsilon(u) T_{r_1}^\varepsilon(v)$.*
3. *Let $u \in L^2(\Gamma\varepsilon)$. Then, $T_{r_1}^\varepsilon u \in L^2((0, 2\pi) \times Y(r_0))$. Moreover, $\|T_{r_1}^\varepsilon u\|_{L^2((0,2\pi)\times Y(r_0))} = \sqrt{2\pi}\|u\|_{L^2(\Gamma\varepsilon)}$.*
4. *Let $u \in H^1(\Gamma_\varepsilon)$. Then, $T_{r_1}^\varepsilon u \in L^2(0, L; H^1(Y(r_0)))$ and $\frac{\partial}{\partial \tau} T_{r_1}^\varepsilon u = \varepsilon T_{r_1}^\varepsilon \frac{\partial u}{\partial \theta}$.*
5. *Let $u \in L^2(0, 2\pi)$. Then, $T_{r_1}^\varepsilon u \to u$ in $L^2((0, 2\pi) \times Y(r_1))$.*
6. *Suppose that $u_\varepsilon \to u$ in $L^2(0, 2\pi)$. Then, $T_{r_1}^\varepsilon u_\varepsilon \to u$ in $L^2((0, 2\pi) \times Y(r_1))$.*
7. *Suppose that u_ε is a sequence in $L^2(\Gamma_\varepsilon)$ such that $T_{r_1}^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0, 2\pi) \times Y(r_1))$. Then, $\widetilde{u_\varepsilon} \rightharpoonup \frac{1}{2\pi}\int_{Y(r_1)} u \, d\tau$ weakly in $L^2(0, 2\pi)$.*

7 Homogenization

The homogenization of the following Laplace equation in the domain $\mathcal{O}_\varepsilon$ is the topic of discussion in this section. We use unfolding operator developed in the last section to carry out the analysis. Thus, consider

$$\begin{cases} -\Delta y_\varepsilon + y_\varepsilon = f_\varepsilon \text{ in } \mathcal{O}_\varepsilon \\ \partial_\nu y_\varepsilon = 0 \text{ on } \partial\mathcal{O}_\varepsilon. \end{cases} \tag{7.1}$$

Here, f_ε is a given function in $L^2(\mathcal{O}_\varepsilon)$; ∂_ν is the outward normal derivative. It is known that, if $f_\varepsilon \in L^2(\mathcal{O}_\varepsilon)$, then the Eq. (7.1) admits a unique weak solution y_ε in $H^1(\mathcal{O}_\varepsilon)$. The solution operator is linear and continuous from $L^2(\mathcal{O}_\varepsilon)$ into $H^1(\mathcal{O}_\varepsilon)$, that is,

$$\|y_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} \leq C\|f_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon)}, \tag{7.2}$$

where $C > 0$ is a constant independent of ε.

The Eq. (7.1) can be written in Polar Coordinates as follows:

$$\begin{cases} -\dfrac{\partial^2 y_\varepsilon}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial y_\varepsilon}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 y_\varepsilon}{\partial \theta^2} + y_\varepsilon = f_\varepsilon \text{ in } \mathcal{O}_\varepsilon \\ \partial_\nu y_\varepsilon = 0 \text{ on } \partial\mathcal{O}_\varepsilon. \end{cases} \tag{7.3}$$

The variational formulation of the above equation is

$$\begin{aligned} &\int_{\mathcal{O}_\varepsilon} \frac{\partial y_\varepsilon}{\partial r}\frac{\partial \varphi}{\partial r} r dr d\theta + \int_{\mathcal{O}_\varepsilon} \frac{1}{r^2}\frac{\partial y_\varepsilon}{\partial \theta}\frac{\partial \varphi}{\partial \theta} r dr d\theta + \int_{\mathcal{O}_\varepsilon} y_\varepsilon \varphi \, r dr d\theta \\ &= \int_{\mathcal{O}_\varepsilon} f_\varepsilon \varphi \, r dr d\theta \ \ \forall \varphi \in H^1(\mathcal{O}_\varepsilon). \end{aligned} \tag{7.4}$$

Note that, $\nabla f(x_0) = \partial_r f(x_0)\widehat{e_r} + \frac{1}{r}\partial_\theta f(x_0)\widehat{e_\theta}$ where $\widehat{e_r}$ and $\widehat{e_\theta}$ are the unit normal vectors in radial and angular directions, respectively.

Limit problem Let $h(r) = |Y(r)|$, where $|Y(r)|$ is the Lebesgue measure of the set $Y(r)$ at $r \in (r_0, r_1)$. Here, we have chosen η such that h is a strictly positive function in $[r_0, r_1]$. Now, Consider the space

$$\widehat{W}(\mathcal{O}) = \left\{ \psi \in C^\infty(\overline{\mathcal{O}}) : \psi \in L^2(\mathcal{O}), \ \frac{\partial \psi^+}{\partial r} \in L^2(\mathcal{O}), \ \psi^- \in H^1(\mathcal{O}^-) \right\}$$

with the inner product

$$\begin{aligned} \langle u, v\rangle_W = &\langle hu, v\rangle_{L^2(\mathcal{O}^+)} + \langle h\partial_r u, \partial_r v\rangle_{L^2(\mathcal{O}^+)} + \langle \partial_r u, \partial_r v\rangle_{L^2(\mathcal{O}^-)} \\ &+ \left\langle \tfrac{1}{r^2}\partial_\theta u, \partial_\theta v\right\rangle_{L^2(\mathcal{O}^-)} + \langle u, v\rangle_{L^2(\mathcal{O}^-)}. \end{aligned} \tag{7.5}$$

where, $\langle u, v\rangle_{L^2(\mathcal{O}^+)} =: \int_{\mathcal{O}^+} uv \, r dr d\theta$. Now, define $W(\mathcal{O})$ to be the completion of $\widehat{W}(\mathcal{O})$ with respect to the norm defined by the above inner product. Hence, $W(\mathcal{O})$ is a Hilbert space with the inner product defined as in (7.5).

Given $f_0 \in L^2(\mathcal{O})$, consider the partial differential equation

$$\begin{cases} -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial y^+}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial y^+}{\partial r} + h(r)y^+ = f_0^+ & in\ \mathcal{O}^+, \\ -\Delta y^- + y^- = f_0^- & in\ \mathcal{O}^-, \\ \dfrac{\partial y^+}{\partial \nu} = 0 & on\ \Gamma \cup \Gamma_b, \\ y^+ = y^-, \quad \dfrac{h(r_1)}{2\pi}\dfrac{\partial y^+}{\partial r} = \dfrac{\partial y^-}{\partial r} & on\ \Gamma_0. \end{cases} \tag{7.6}$$

where,

$$y = y^+\chi_{\mathcal{O}^+} + y^-\chi_{\mathcal{O}^-} \text{ and } f_0 = f_0^+\chi_{\mathcal{O}^+} + f_0^-\chi_{\mathcal{O}^-}. \tag{7.7}$$

The weak formulation of the above equation is: Find $u \in W(\mathcal{O})$ such that

$$\int_{\mathcal{O}^+} \frac{h(r)}{2\pi}\left(\frac{\partial y^+}{\partial r}\frac{\partial \psi}{\partial r} + y^+\psi\right) r dr d\theta + \int_{\mathcal{O}^-}\left(\frac{\partial y^-}{\partial r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial y^-}{\partial \theta}\frac{\partial \psi}{\partial \theta} + y^-\psi\right) r dr d\theta$$
$$= \frac{1}{2\pi}\int_{\mathcal{O}^+} f_0^+\psi\, r dr d\theta + \int_{\mathcal{O}^-} f_0^-\psi\, r dr d\theta \tag{7.8}$$

$\forall\, \psi \in W(\mathcal{O})$. As $f_0 \in L^2(\mathcal{O})$, it defines a continuous linear functional on $W(\mathcal{O})$. Then, the Lax–Milgram lemma assists us in obtaining a unique weak solution of (7.6).

Theorem 7.1 *Assume that $\widetilde{f_\varepsilon}$ is uniformly bounded in $L^2(\mathcal{O})$ and let y_ε be the solution of the Laplace equation (7.1). Then,*

$$\widetilde{y_\varepsilon^+} \rightharpoonup \frac{h(r)}{2\pi} y_0^+, \quad \widetilde{\frac{\partial y_\varepsilon^+}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi}\frac{\partial y_0^+}{\partial r} \text{ weakly in } L^2(\mathcal{O}^+), \text{ and}$$
$$y_\varepsilon^- \rightharpoonup y_0^- \text{ weakly in } H^1(\mathcal{O}^-),$$

where y_0 is the unique weak solution of the homogenized problem (7.6), $f_0^+(r,\theta) = \int_{Y(r)} f_1(\theta, r, \tau)\, d\tau$ with $f_1(\theta, r, \tau)$ is the weak limit of $T^\varepsilon f_\varepsilon$ in $L^2(\mathcal{O}_u)$ and f_0^- is the L^2 weak limit of f_ε^- in $\mathcal{O}^-$. Here, y_ε^+ and y_ε^- are the restrictions of y_ε to $\mathcal{O}_\varepsilon^+$ and $\mathcal{O}^-$, respectively and $\widetilde{f_\varepsilon}$ is the zero extension of f_ε to $\mathcal{O}$.

Proof The continuity of the solution operator gives the following estimate.

$$\|y_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} \leq C\|\widetilde{f_\varepsilon}\|_{L^2(\mathcal{O})}. \tag{7.9}$$

Since, f_ε is uniformly bounded, there exists a constant $C > 0$ independent of ε, such that $\|f_\varepsilon\|_{L^2(\mathcal{O})} \leq C$. Hence, we have

$$\|y_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} \leq C. \tag{7.10}$$

Let us, estimate $T^\varepsilon y_\varepsilon^+$ and its derivatives in the space $L^2(\mathcal{O}_u)$ using the properties of the unfolding operator, which we have discussed in Sect. 6. By using Proposition 6.5, we get

$$\|T^\varepsilon y_\varepsilon\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\|y_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon^+)}, \quad \left\|T^\varepsilon\frac{\partial y_\varepsilon}{\partial r}\right\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\left\|\frac{\partial y_\varepsilon}{\partial r}\right\|_{L^2(\mathcal{O}_\varepsilon^+)} \text{ and}$$
$$\left\|T^\varepsilon\frac{\partial y_\varepsilon}{\partial \tau}\right\|_{L^2(\mathcal{O}_u)} = \varepsilon\sqrt{2\pi}\left\|\frac{\partial y_\varepsilon}{\partial \theta}\right\|_{L^2(\mathcal{O}_\varepsilon^+)}. \tag{7.11}$$

By weak compactness, there exists a subsequence (still denoted by ε) such that

$$T^{\varepsilon} y_{\varepsilon}^{+} \rightharpoonup y_0^{+} \text{ weakly in } L^2(\mathcal{O}_u), \tag{7.12}$$

$$\frac{\partial}{\partial r} T^{\varepsilon} y_{\varepsilon}^{+} \rightharpoonup \frac{\partial y_0^{+}}{\partial r}, \text{ that is, } T^{\varepsilon} \frac{\partial y_{\varepsilon}^{+}}{\partial r} \rightharpoonup \frac{\partial y_0^{+}}{\partial r} \text{ weakly in } L^2(\mathcal{O}_u) \tag{7.13}$$

and

$$\frac{\partial}{\partial \tau} T^{\varepsilon} y_{\varepsilon}^{+} \rightharpoonup \frac{\partial y_0^{+}}{\partial \tau}, \text{ that is, } \varepsilon T^{\varepsilon} \frac{\partial y_{\varepsilon}^{+}}{\partial \theta} \rightharpoonup \frac{\partial y_0^{+}}{\partial \tau} \text{ weakly in } L^2(\mathcal{O}_u). \tag{7.14}$$

From the Proposition 6.5, we have

$$\left\| T^{\varepsilon} \frac{\partial y_{\varepsilon}^{+}}{\partial \theta} \right\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi} \left\| \frac{\partial y_{\varepsilon}^{+}}{\partial \theta} \right\|_{L^2(\mathcal{O}_{\varepsilon}^{+})} \leq \sqrt{2\pi} \| y_{\varepsilon} \|_{H^1(\mathcal{O}_{\varepsilon})}.$$

The estimate (7.10) implies the boundedness of the sequence $T^{\varepsilon} \frac{\partial y_{\varepsilon}^{+}}{\partial \theta}$ in the space $L^2(\mathcal{O}_u)$. From the convergence (7.14), it follows that $\frac{\partial y_0^{+}}{\partial \tau} = 0$. Hence, we conclude that

$$\widetilde{y_{\varepsilon}^{+}} \rightharpoonup \frac{1}{2\pi} \int_{Y(r)} y_0^{+} d\tau \text{ and } \widetilde{\frac{\partial y_{\varepsilon}^{+}}{\partial r}} \rightharpoonup \frac{1}{2\pi} \int_{Y(r)} \frac{\partial y_0^{+}}{\partial r} d\tau \text{ weakly in } L^2(\mathcal{O}^{+}) \tag{7.15}$$

with the help of Proposition 6.8. Since y_0^{+} is independent of τ variable, we write

$$\int_{Y(r)} y_0^{+}\, d\tau = h(r) y_0^{+} \text{ and } \int_{Y(r)} \frac{\partial y_0^{+}}{\partial r}\, d\tau = h(r) \frac{\partial y_0^{+}}{\partial r}. \tag{7.16}$$

Thus, (7.15) becomes

$$\widetilde{y_{\varepsilon}^{+}} \rightharpoonup \frac{h(r)}{2\pi} y_0^{+} \text{ and } \widetilde{\frac{\partial y_{\varepsilon}^{+}}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi} \frac{\partial y_0^{+}}{\partial r} \text{ weakly in } L^2(\mathcal{O}^{+}). \tag{7.17}$$

We know that $T^{\varepsilon} \frac{\partial y_{\varepsilon}^{+}}{\partial \theta}$ is bounded in $L^2(\mathcal{O}_u)$. Hence, by weak compactness, there is an element $P \in L^2(\mathcal{O}_u)$ such that up to subsequence (still denoted by ε),

$$T^{\varepsilon} \frac{\partial y_{\varepsilon}^{+}}{\partial \theta} \rightharpoonup P \text{ weakly in } L^2(\mathcal{O}_u). \tag{7.18}$$

Using the estimate of $\| y_{\varepsilon} \|_{H^1(\mathcal{O}_{\varepsilon})}$, we have the boundedness of y_{ε}^{-} in the space $H^1(\mathcal{O}^{-})$. Thus, up to a subsequence (still denoted by ε)

$$y_{\varepsilon}^{-} \rightharpoonup y_0^{-} \text{ weakly in } H^1(\mathcal{O}^{-}). \tag{7.19}$$

Define, u_0 as

$$y_0(x) = \begin{cases} y_0^{+} & if\ x \in \mathcal{O}^{+}, \\ y_0^{-} & if\ x \in \mathcal{O}^{-}. \end{cases} \tag{7.20}$$

Claim $y_0 \in W(\mathcal{O})$ *and satisfies the limit problem* (7.6).

We know that $y_0 \in L^2(\mathcal{O})$ and $\frac{\partial y_0}{\partial r} \in L^2(\mathcal{O}^{-})$. To prove $y_0 \in W(\mathcal{O})$, we need to show $\frac{\partial y_0}{\partial r} \in L^2(\mathcal{O})$. Note that y_0^{+} is independent of τ variable and so is $\frac{\partial y_0^{+}}{\partial r}$. Hence, we have

$\dfrac{\partial y_0^+}{\partial r} \in L^2(\mathcal{O}^+)$ and also $\dfrac{\partial y_0}{\partial r} \in L^2(\mathcal{O}^-)$. Thus, to show $\dfrac{\partial y_0}{\partial r} \in L^2(\mathcal{O})$, it is enough to prove the trace of y_0^+ and y_0^- are equal on Γ_0. Since $y_\varepsilon^+|_{\Gamma_\varepsilon} = y_\varepsilon^-|_{\Gamma_\varepsilon}$ implies the equality of trace for the boundary unfolding operator, we have $T_{r_1}^\varepsilon\left(y_\varepsilon^+|_{\Gamma_\varepsilon}\right) = T_{r_1}^\varepsilon\left(y_\varepsilon^-|_{\Gamma_\varepsilon}\right)$ i.e.

$$\left(T^\varepsilon(y_\varepsilon^+)\right)|_{r=r_1} = T_{r_1}^\varepsilon\left(y_\varepsilon^-|_{\Gamma_\varepsilon}\right). \tag{7.21}$$

From the weak continuity of trace operator, we can write

$$\left(T^\varepsilon(y_\varepsilon^+)\right)|_{r=r_1} \rightharpoonup y_0^+|_{r=r_1} \text{ weakly in } L^2((0,2\pi)\times Y(r_0))$$

and from (7.19), we get

$$y_\varepsilon{}^-|_{r=r_1} \to y_0^-|_{r=r_1} \text{ strongly in } L^2(0,2\pi).$$

This implies

$$T_{r_1}^\varepsilon\left(y_\varepsilon{}^-|_{r=r_1}\right) \to y_0^-|_{r=r_1} \text{ in } L^2((0,2\pi)\times Y(r_1)).$$

Passing to the limit in (7.21) as $\varepsilon \to 0$ we get

$$y_0^+|_{r=r_1} = y_0^-|_{r=r_1} \text{ in } L^2(0,2\pi),$$

since y_0^+ and y_0^- are independent of τ variable.
Identification of the limit P in (7.18): finally, we identify P which is identically 0.
For $\phi \in \mathcal{D}(\mathcal{O}^+)$ and $\zeta(z) \in \mathcal{D}(0,2\pi)$, choose $\psi \in \mathcal{D}(0,2\pi)$ such that $\psi'(z) = \zeta(z)$. Now choose a test function

$$\phi^\varepsilon(r,\theta) = \varepsilon\phi(r,\theta)\psi\left(\left\{\tfrac{\theta}{\varepsilon}\right\}\right),$$

in such a way that ϕ^ε is continuous on $\mathcal{O}_\varepsilon^+$. From the definition of ε-unfolding of ϕ^ε and by Proposition 6.5, we get

$$\begin{aligned}
T^\varepsilon\phi^\varepsilon &= \varepsilon\phi\left(r, \varepsilon\left[\tfrac{\theta}{\varepsilon}\right] + \varepsilon\tau\right)\psi(\tau),\\
T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial\theta} &= \frac{1}{\varepsilon}\frac{\partial}{\partial\tau}T^\varepsilon\phi^\varepsilon = \varepsilon\frac{\partial\phi}{\partial\theta}\left(r, \varepsilon\left[\frac{\theta}{\varepsilon}\right] + \varepsilon\tau\right)\psi(\tau) + \phi\left(r, \varepsilon\left[\frac{\theta}{\varepsilon}\right] + \varepsilon\tau\right)\psi'(\tau),\\
T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial r} &= \varepsilon\frac{\partial\phi}{\partial r}\left(r, \varepsilon\left[\frac{\theta}{\varepsilon}\right] + \varepsilon\tau\right)\psi(\tau).
\end{aligned}$$

From the above equations, we derive the following convergences, as $\varepsilon \to 0$

$$T^\varepsilon\phi^\varepsilon \to 0 \; in \; L^2(\mathcal{O}_u) \tag{7.22}$$

$$T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial\theta} \to \phi(r,\theta)\psi'(\tau) \; in \; L^2(\mathcal{O}_u) \tag{7.23}$$

$$T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial r} \to 0 \; in \; L^2(\mathcal{O}_u) \tag{7.24}$$

Let us recall the variational formulation of (7.1) with the test function ϕ^ε.

$$\int_{\mathcal{O}_\varepsilon}\left(\frac{\partial y_\varepsilon}{\partial r}\frac{\partial\phi^\varepsilon}{\partial r} + \frac{1}{r^2}\frac{\partial y_\varepsilon}{\partial\theta}\frac{\partial\phi^\varepsilon}{\partial\theta} + y_\varepsilon\phi^\varepsilon\right) r dr d\theta = \int_{\mathcal{O}_\varepsilon} f_\varepsilon\phi^\varepsilon \, r dr d\theta \; \forall\, \varphi \in H^1(\mathcal{O}_\varepsilon) \tag{7.25}$$

Now notice,

$$\int_{\mathcal{O}_\varepsilon}\left(\frac{\partial y_\varepsilon}{\partial r}\frac{\partial \phi^\varepsilon}{\partial r}+\frac{1}{r^2}\frac{\partial y_\varepsilon}{\partial \theta}\frac{\partial \phi^\varepsilon}{\partial \theta}\right)rdrd\theta=\int_{\mathcal{O}_\varepsilon^+}\left(\frac{\partial y_\varepsilon}{\partial r}\frac{\partial \phi^\varepsilon}{\partial r}+\frac{1}{r^2}\frac{\partial y_\varepsilon}{\partial \theta}\frac{\partial \phi^\varepsilon}{\partial \theta}\right)rdrd\theta$$
$$=\frac{1}{2\pi}\int_{\mathcal{O}_u}\left(T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial r}\,T^\varepsilon\frac{\partial \phi^\varepsilon}{\partial r}+\frac{1}{r^2}T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial \theta}\,T^\varepsilon\frac{\partial \phi^\varepsilon}{\partial \theta}\right)\,rdrd\theta d\tau$$
$$\to\frac{1}{2\pi}\int_{\mathcal{O}_u}P\,\phi(r,\theta)\psi'(\tau)\,rdrd\theta d\tau\ \ as\ \varepsilon\to 0; \tag{7.26}$$

$$\int_{\mathcal{O}_\varepsilon} y_\varepsilon\phi^\varepsilon\ rdrd\theta=\int_{\mathcal{O}_\varepsilon^+} y_\varepsilon^+\phi^\varepsilon\ rdrd\theta=\frac{1}{2\pi}\int_{\mathcal{O}_u}T^\varepsilon y_\varepsilon^+\,T^\varepsilon\phi^\varepsilon\ rdrd\theta d\tau\to 0\ \ as\ \varepsilon\to 0 \tag{7.27}$$

and

$$\int_{\mathcal{O}_\varepsilon} f_\varepsilon\phi^\varepsilon\ rdrd\theta=\int_{\mathcal{O}_\varepsilon^+} f_\varepsilon\phi^\varepsilon\ rdrd\theta=\frac{1}{2\pi}\int_{\mathcal{O}_u}T^\varepsilon f_\varepsilon\,T^\varepsilon\phi^\varepsilon\ rdrd\theta d\tau\to 0\ \ as\ \varepsilon\to 0. \tag{7.28}$$

Combining (7.26), (7.27) and (7.28), we get, $\int_{\mathcal{O}_u} P\,\phi(r,\theta)\psi'(\tau)=0$, which implies

$$\int_{\mathcal{O}^+}\left(\int_{Y(r)}P\psi'(\tau)\,d\tau\right)\phi(\theta,r)\,rdrd\theta=0\ \forall\,\phi\in\mathcal{D}(\mathcal{O}^+).$$

That is,

$$\int_{Y(r)}P(r,\theta,\tau)\psi'(\tau)=0.$$

Hence,

$$\int_{Y(r)}P(r,\theta,\tau)\zeta(\tau)\,d\tau=0\ a.e.\ (r,\theta)\in\mathcal{O}^+,\ \zeta\in\mathcal{D}(0,2\pi).$$

Thus, $P\equiv 0$ a.e. on $\mathcal{O}^+$

Claim y_0 *satisfies the limit equation*: Choose a test function $\psi\in C^\infty(\overline{\mathcal{O}})$ in the variational formulation of the Polar form of (7.3), which is given by

$$\int_{\mathcal{O}_\varepsilon}\left(\frac{\partial y_\varepsilon}{\partial r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial y_\varepsilon}{\partial \theta}\frac{\partial \psi}{\partial \theta}+y_\varepsilon\psi\right)rdrd\theta$$
$$:=\int_{\mathcal{O}_\varepsilon^+}\left(\frac{\partial y_\varepsilon}{\partial r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial y_\varepsilon}{\partial \theta}\frac{\partial \psi}{\partial \theta}+y_\varepsilon\psi\right)rdrd\theta$$
$$+\int_{\mathcal{O}^-}\left(\frac{\partial y_\varepsilon}{\partial r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial y_\varepsilon}{\partial \theta}\frac{\partial \psi}{\partial \theta}+y_\varepsilon\psi\right)rdrd\theta=\int_{\mathcal{O}_\varepsilon}f_\varepsilon\psi\ rdrd\theta$$

Consider the first 3 terms, to get

$$\int_{\mathcal{O}_\varepsilon^+} \left(\frac{\partial y_\varepsilon}{\partial r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial y_\varepsilon}{\partial \theta} \frac{\partial \psi}{\partial \theta} + y_\varepsilon \psi \right) r dr d\theta$$
$$= \frac{1}{2\pi} \int\limits_{\mathcal{O}_u} \left(T^\varepsilon \frac{\partial y_\varepsilon^+}{\partial r} T^\varepsilon \frac{\partial \psi}{\partial r} + \frac{1}{r^2} T^\varepsilon \frac{\partial y_\varepsilon^+}{\partial \theta} T^\varepsilon \frac{\partial \psi}{\partial \theta} + T^\varepsilon y_\varepsilon^+ T^\varepsilon \psi \right) r dr d\theta d\tau$$
$$\to \frac{1}{2\pi} \int\limits_{\mathcal{O}_u} \left(\frac{\partial y_0^+}{\partial r} \frac{\partial \psi}{\partial r} + y_0^+ \psi \right) r dr d\theta d\tau.$$

The next 3 terms satisfies

$$\int_{\mathcal{O}^-} \left(\frac{\partial y_\varepsilon}{\partial r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial y_\varepsilon}{\partial \theta} \frac{\partial \psi}{\partial \theta} + y_\varepsilon \psi \right) r dr d\theta$$
$$\to \int_{\mathcal{O}^-} \left(\frac{\partial y_0^-}{\partial r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial y_0^-}{\partial \theta} \frac{\partial \psi}{\partial \theta} + y_0^- \psi \right) r dr d\theta \tag{7.29}$$

and finally,

$$\int\limits_{\mathcal{O}_\varepsilon} f_\varepsilon \psi \; r dr d\theta = \int\limits_{\mathcal{O}_\varepsilon^+} f_\varepsilon \psi \; r dr d\theta + \int\limits_{\mathcal{O}^-} f_\varepsilon \psi \; r dr d\theta$$
$$= \frac{1}{2\pi} \int\limits_{\mathcal{O}_u} T^\varepsilon f_\varepsilon T^\varepsilon \psi \; r dr d\theta d\tau + \int\limits_{\mathcal{O}^-} f_\varepsilon \psi \; r dr d\theta$$
$$\to \frac{1}{2\pi} \int\limits_{\mathcal{O}_u} f_1 \psi \; r dr d\theta d\tau + \int\limits_{\mathcal{O}^-} f_0^- \psi \; r dr d\theta. \tag{7.30}$$

Hence, as $\varepsilon \to 0$, the limit variational formulation becomes

$$\frac{1}{2\pi} \int\limits_{\mathcal{O}_u} \left(\frac{\partial y_0^+}{\partial r} \frac{\partial \psi}{\partial r} + y_0^+ \psi \right) r dr d\theta d\tau$$
$$+ \int_{\mathcal{O}^-} \left(\frac{\partial y_0^-}{\partial r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial y_0^-}{\partial \theta} \frac{\partial \psi}{\partial \theta} + y_0^- \psi \right) r dr d\theta$$
$$= \frac{1}{2\pi} \int\limits_{\mathcal{O}_u} f_1 \psi \; r dr d\theta d\tau + \int\limits_{\mathcal{O}^-} f_0^- \psi \; r dr d\theta. \tag{7.31}$$

That is, y_0 satisfies the equation

$$\int\limits_{\mathcal{O}^+} \frac{h(r)}{2\pi} \left(\frac{\partial y_0^+}{\partial r} \frac{\partial \psi}{\partial r} + y_0^+ \psi \right) r dr d\theta$$
$$+ \int_{\mathcal{O}^-} \left(\frac{\partial y_0^-}{\partial r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial y_0^-}{\partial \theta} \frac{\partial \psi}{\partial \theta} + y_0^- \psi \right) r dr d\theta$$
$$= \frac{1}{2\pi} \int\limits_{\mathcal{O}^+} f_0^+ \psi \; r dr d\theta + \int\limits_{\mathcal{O}^-} f_0^- \; r dr d\theta \psi \tag{7.32}$$

$\forall\, \psi \in C^\infty(\overline{\mathcal{O}})$, where $f_0^+(r,\theta) = \int_{Y(r)} f_1(r,\theta,\tau)\, d\tau$. As we know that $C^\infty(\overline{\mathcal{O}})$ is dense in $W(\mathcal{O})$, the above equation is true for all ψ in $W(\mathcal{O})$. Therefore, y_0 satisfies (7.6). □

8 Conclusions

In this article, the major contribution is the development of unfolding operators in domains with smooth and highly oscillating boundaries. Then, the unfolding operators, thus developed have used to study the homogenization problems in such domains. Such unfolding operators with smooth oscillating boundaries is completely new to our knowledge. This novel approach is very handy to the study of homogenization problems. In the first part of this article, we have considered oscillations on a flat part of the boundary. In the second, we have modified our ideas to develop unfolding operators in curved domains and as a sample, we have studied it in a circular domain. We have used the corresponding Polar form of the Laplace equation.

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