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ORIGINAL ARTICLE

Asymptotic Analysis of a Boundary Optimal Control Problem on a General Branched Structure

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Abstract

We consider an optimal control problem posed on a domain with a highly oscillating smooth boundary where the controls are applied on the oscillating part of the boundary. There are many results on domains with oscillating boundaries where the oscillations are pillar-type (non-smooth) while the literature on smooth oscillating boundary is very few. In this article, we use appropriate scaling on the controls acting on the oscillating boundary leading to different limit control problems; namely, boundary optimal control and interior optimal control problem. In the last part of the article, we visualize the domains as a branched structure and we introduce unfolding operators to get contributions from each level at every branch.

KEYWORDS:
Homogenization; Optimal control; Asymptotic analysis; Unfolding operator; Oscillating boundary domain

1 | INTRODUCTION

The study of boundary-value problems in domains with rough boundaries or interfaces was started to understand various physical models such as scattering of waves on rough boundaries, transmission and reflection of waves on rough interfaces, mechanical problems concerning plates with densely spaced stiffeners, flows over rough walls, vibrations of strongly inhomogeneous elastic bodies, the fluid-structure interaction problems in which the displacement of the oscillating boundary is governed by equations of elastic structures, viz., beam equations in one dimension, plate and shell equations in higher dimensions, etc. In order to understand these kind of physical problems, we need to understand the asymptotic behavior of the governing equation as it involves different scales. Also materials with rough boundary are used in several industrial applications like microstrip radiator and nano technologies, fractal type construction, etc. This motivates us to study homogenization of various problems including optimal control problems in oscillating domains.

In this article, we study the asymptotic behavior of an optimal control problem associated with the Laplace equation in a 3-dimensional domain with highly oscillating boundary which are smooth and periodic. This can be generalized to any dimension $n \geqslant 3$ though we restrict to the case $n = 3$. Though plenty of research articles on oscillating domain have appeared in the last 10-15 years, most of them consider pillar type oscillating domains and some generalization of the same. But recently S. Aiyappan, A. K. Nandakumaran and Ravi Prakash[3] have considered an oscillatory domain with smooth oscillatory part. There was a novel approach by these authors in defining new unfolding operators which they have used to study a homogenization problem, in fact, a non-linear problem. The domain, we denote as Ω_ε, has two parts namely the oscillating part Ω_ε^+ and the fixed part Ω^- (see Figure 1 for a 2-D pictorial representation). Consideration of smooth domains will allow us to consider more practical problems like circular domains with oscillatory boundaries (application to circular type radiators), domains with branched structure as in[2]. In fact, we present a result on branched structures in the last part of this manuscript which is more general than that considered in[2].

In this article, we study an optimal control problem with periodic controls acting on the oscillating boundary of the domain. Using the unfolding operator to be introduced later, we characterize the optimal control at every ε stage by introducing an adjoint system. We have also studied limiting behavior of the optimal control problem. The characterization of the optimal control is also used to study the homogenization of the optimal control problem.

The study of homogenization on oscillating boundary domain was started long back. For example, one can look at the work of V. P. Kotljarov and E. Ja. Hruslov (see [28]). They have considered Helmholtz equation on oscillating domain to study the limiting behavior. In 1978, R. Brizzi and J. P. Chalot analysed Laplace equation with Neumann boundary condition in [15] in various oscillating domains. There are plenty of articles concerning homogenization of boundary value problems posed on pillar-type, that is non-smooth and and amplitude of O(1), domains. T. A. Mel'nyk studied homogenization of Laplace equation with non-linear mixed boundary conditions [31]. In [5,6], the authors have considered Laplace equation and studied asymptotic behavior and derived error estimates. For homogenization of non-linear problems in oscillating domains, one can look into the work of A. Gaudiello and T. A. Mel'nyk in [26], where they have studied a homogenization of a monotone problem with nonlinear signorini boundary conditions. In [32], T. A. Mel'nyk studied non-linear parabolic problem using asymptotic expansion. The authors, U. De Maio, A. K. Nandakumaran and C. Perugia have studied exact controllabillity problems in oscillating domains in [21,20]. For more literatures in this direction, we refer to [1,12,13,14,24,25,27] and the references there in. In all the above works the amplitude of the oscillation is of O(1). There has been several works on oscillating thin domain where the amplitudeof oscillation is of $O(\varepsilon)$. For example, one can look into the work of J. Arrieta and M. Pereira in [7,8], where they have considered an elliptic PDE on thin oscillating domain with homogenious Numann boundary condition. J. Arrieta and M. Pereira in [9] have also studied homogenization of an elliptic PDE in a locallay periodic thin domain. Further in [10], the authors have developed unfolding operator for locally periodic thin domain to study the homogenization of an elliptic PDE and obtained corrector results.

In [35,34,38] control problems where the controls are acting away from the oscillating part of the domain have been investigated with certain error estimates. Recently, A. K. Nandakumaran, Ravi Prakash and Bidhan Chandra Sardar in [36] considered a second order elliptic interior optimal control problem. In their work, they have used the method of periodic unfolding to characterize the control and studied asymptotic behaviour. The periodic unfolding operators to study homogenization problems was introduced in 2002 by D. Cioranescu, A. Damlamian and G. Griso [16]. These all are interior control problems. For a boundary optimal control problem in a pillar-type oscillating domain, we refer to the work of A. K. Nandakumaran, Ravi Prakash and Bidhan Chanda Sardar (see [37]), where they have considered Neumann boundary control problem on the oscillating part of the boundary. In [4], S. Aiyappan and Bidhan Chandra Sardar have studied a biharmonic boundary optimal control problem. For more literature on homogenization of optimal control problems, one can look into [3,2,23,22,19,33,35,34,37] and the references there in. For general periodic homogenization theory we refer to [11,17,40] and the references there in. For optimal control problems and derivation of optimality system one can look into [30,39].

In this paper, we consider Neumann controls acting on the oscillating part of the boundary which are smooth. Since we are considering smooth oscillating domains, we can workout all the details to circular oscillating domains, which we believe has far reaching applications, using unfolding operators in polar coordinates [3]. We obtain the optimality system at every ε stage and then, we have analyzed the asymptotic behavior of this optimality system as $\varepsilon \to 0$. The controls are defined with different scaling factors. Thus, we get two different limit optimal control problem depending on the scaling factor.

The arrangement of this article is as follows. In Section 2, we present the domain and problem description. In Section 3, the unfolding operator and its properties are stated. Some preliminary results and estimates are discussed in Section 4. The main results and their proofs have been presented in Section 5. In the last section, we briefly discuss a similar optimal control problem in a branched structure domain and derived the homogenized problem on a multi-sheeted function space. viewing the oscillating domain as branched structure as in Section 6 will have more applications as it provides information at any level we wish.

2 | DOMAIN AND PROBLEM DESCRIPTION

For a fixed parameter $\varepsilon = \frac{1}{m}$ with $m \in \mathbb{N}$, we consider the domain $\Omega_\varepsilon \subset \mathbb{R}^3$. Through out this article, we denote $x' = (x_1, x_2)$, $y' = (y_1, y_2)$, $z' = (z_1, z_2)$. Let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be a smooth 1-periodic function (that is η periodic in x_1 and x_2 direction with period 1). Let $M_1 = \sup\{\eta(x') |\ x' \in [0,1]^2\}$, and $M_0 = \inf\{(\eta(x') |\ x' \in [0,1]^2\}$. Define for $x' \in \mathbb{R}^2$, $\eta^\varepsilon(x') = \eta(\frac{x'}{\varepsilon})$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth 1−periodic function such that $\sup\{g(x') |\ x' \in [0,1]^2\} < M_0$. Note that the graph of η_ε will define the smooth, periodic oscillating boundary of the domain Ω_ε to be defined below whereas the graph of g will define the bottom

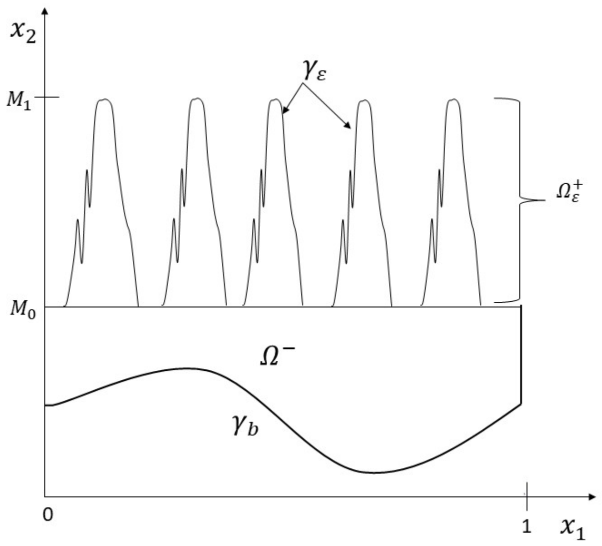

FIGURE 1 Oscillating domain (2D)

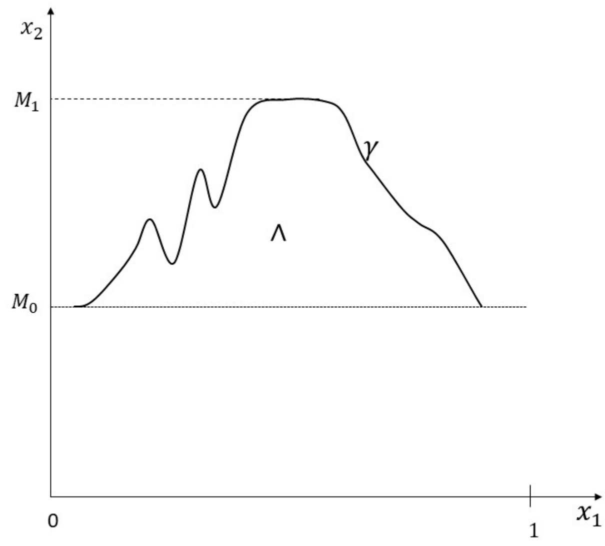

FIGURE 2 Reference cell

boundary. The domain Ω_ε can be written as

$$\Omega_\varepsilon = \{(x', x_3) |\ x' \in (0,1)^2,\ g(x') < x_3 < \eta^\epsilon(x')\}.$$

The oscillating part of the domain Ω_ε^+ is given by

$$\Omega_\varepsilon^+ = \left\{ (x', x_3) |\ M_0 < x_3 < \eta\left(\frac{x'}{\varepsilon}\right) \text{ for all } x' \in (0,1)^2 \right\}$$

and the fixed part Ω^- is defined as

$$\Omega^- = \{(x', x_3) |\ x' \in (0,1)^2,\ g(x') < x_3 < M_0\}.$$

The boundary of Ω_ε, namely $\partial\Omega_\varepsilon$, can be written as $\gamma_b \cup \gamma_s \cup \gamma_\varepsilon$, where

$$\begin{aligned}
\gamma_b &= \{(x', g(x')) |\ x' \in [0,1]^2\},\\
\gamma_s &= \{(g(x'), x_3) |\ x' \in \partial(0,1)^2,\ g(x') < x_3 < M_0\}\\
\gamma_\varepsilon &= \partial\Omega_\varepsilon \backslash (\gamma_b \cup \gamma_s).
\end{aligned}$$

The reference cell Λ (Fig. 2) is defined as $\Lambda = \{(x', x_3) |\ x' \in [0,1]^2, M_0 < x_3 < \eta(x')\}$ and the reference boundary γ is given by $\gamma = \{(x', \eta(x')) |\ x' \in [0,1]^2\}$. We can write $\gamma = S \cup F$, where

$$S = \{(x', \eta(x')) |\ \nabla_{x'}\eta(x') \neq 0\} \text{ and } F = \gamma \backslash S.$$

More precisely, F can be written as

$$F = F_{M_0} \cup F_{M_1} \cup \bigcup_{k=1}^{N} F_{t_k}$$

where $F_t = \{(x', \eta(x')) |\ \eta(x') = t,\ \nabla_{x'}\eta(x') = 0\}$ and $M_0 < t_1 < t_2 < ... < t_N < M_1$.
Define the sets

$$\begin{aligned}
Y_s &= \{x' \in [0,1]^2 |\ (x', \eta(x')) \in S\}\\
Y_{M_i} &= \{x' \in [0,1]^2 |\ (x', \eta(x')) \in F_{M_i}\ i = 0, 1\}\\
Y_{t_k} &= \{x' \in [0,1]^2 |\ (x', \eta(x')) \in F_{t_k}\ k = 1, 2, ..., N\}
\end{aligned}$$

Using these notations, we can write γ_ε as $\gamma_\varepsilon = S^\varepsilon \cup F^\varepsilon$, where

$$S^\varepsilon = \left\{ (x', \eta^\varepsilon(x')) \middle| \, x' \in \left(\sum_{i,j=0}^{m-1} (\varepsilon(i,j) + \varepsilon Y_s) \right) \right\} \text{ and } F^\varepsilon = F^\varepsilon_{M_0} \cup F^\varepsilon_{M_1} \cup \bigcup_{k=1}^{N} F^\varepsilon_{t_k},$$

where

$$F^\varepsilon_{M_l} = \left\{ (x', \eta^\varepsilon(x')) \middle| \, x' \in \left(\sum_{i,j=0}^{m-1} (\varepsilon(i,j) + \varepsilon Y_{M_l}) \right) \right\}, \text{ for } l = 0, 1$$

and

$$F^\varepsilon_{t_k} = \left\{ (x', \eta^\varepsilon(x')) \middle| \, x' \in \left(\sum_{i,j=0}^{m-1} (\varepsilon(i,j) + \varepsilon Y_{t_k}) \right) \right\}, \text{ for } k = 1, 2, ..., N.$$

The full domain Ω is defined as

$$\Omega = \{(x', x_3) | \; x' \in (0,1)^2, \; g(x') < x_3 < M_1\}.$$

In other words, Ω can also be written as $\Omega = \text{interior}\left(\overline{(\Omega^+ \cup \Omega^-)}\right)$, where

$$\Omega^+ = \{(x', x_3) | \; x' \in (0,1)^2, \; M_0 < x_3 < M_1\}.$$

The bottom boundary of Ω is same as that of Ω_ε, that is γ_b, and the side boundaries $\gamma_{s'}$ are defined as

$$\gamma_{s'} = \{(g(x'), x_3) | \; x' \in \partial(0,1)^2, \; g(x') \leqslant x_3 \leqslant M_1\}.$$

The interface boundary is defined as $\gamma_c = \{(x', M_0) | \;\; x' \in (0,1)^2\}$ and top boundary part, denoted by γ_u is given by $\gamma_u = \{(x', M_1) | \; x' \in (0,1)^2\}$. Let us define the space $L^2_{per}(\gamma)$ as

$$L^2_{per}(\gamma) = \{\theta \in L^2(\gamma) \, : \, \theta((i,j) + y', \eta(y')) = \theta(y', \eta(y')), \; \forall (i,j) \in \mathbb{Z}^2, \text{ almost every } y' \in (0,1)^2\}.$$

Now we describe the control problem posed on this domain.

2.1 | Problem Description

Here we are considering a boundary optimal control problem on the oscillating domain Ω_ε. As the oscillating parameter $\varepsilon \to 0$, the Lebesgue measure of the boundary increases in the $O(\frac{1}{\varepsilon})$. So we multiply the control by the factor ε^α with $\alpha \geq 1$ to get the uniform bound for the optimal solution. The other factor with ε^α which is of $O(1)$ are multiplied to get a nice characterization of the optimal control via unfolding operator.

For $\theta \in L^2_{per}(\gamma)$ define $\theta^\varepsilon(y', \eta(y')) = \tau_\varepsilon(y', \eta(y'))\theta(y', \eta(y'))$ where

$$\tau_\varepsilon(y', \eta(y')) = \left(\chi_{F_{M_0}} + \chi_{F_{M_1}} + \varepsilon^\alpha \frac{\sqrt{|\nabla \eta|}\sqrt{1 + |\nabla \eta|^2}}{\sqrt{\varepsilon^2 + |\nabla \eta|^2}} \chi_S + \varepsilon^\alpha \chi_{F_{in}} \right)(y', \eta(y')), \; \alpha \geqslant 1. \tag{2.1}$$

Here $F_{in} = \bigcup_{k=1}^{N} F_{t_k}$ and χ_A is the characteristic function of a set A. Further, α is a parameter and we get into two different limit problems depending on $\alpha > 1$ and $\alpha = 1$. The critical case $\alpha = 1$ is more interesting. Now for θ^ε, define $\hat{\theta}^\varepsilon \in L^2(\gamma_\varepsilon)$ as

$$\hat{\theta}^\varepsilon(y', \eta(y')) = \theta^\varepsilon\left(\frac{y'}{\varepsilon}, \eta\left(\frac{y'}{\varepsilon} \right) \right).$$

A function defined on Ω_ε is said to be γ_s periodic, if they take the same value on the opposite sides of γ_s. For $\theta \in L^2_{per}(\gamma)$, we consider the following L^2-cost functional

$$J_\varepsilon(u_\varepsilon, \theta) = \frac{1}{2} \int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 dx + \frac{\beta}{2} \int_{\gamma} |\theta|^2.$$

where the state u_ϵ satisfies the following PDE

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f \;\; in \;\; \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \hat{\theta}_\varepsilon \;\; on \;\; \gamma_\varepsilon, \\ u_\varepsilon = 0 \;\; on \; \gamma_b, \\ u_\varepsilon \;\; is \;\; \gamma_s \;\; periodic. \end{cases} \tag{2.2}$$

where $f \in L^2(\Omega)$ and $\hat{\theta}^\varepsilon \in L^2(\gamma_\varepsilon)$ defined as above. Now consider the following optimal control problem: find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1_\varepsilon \times L^2_{per}(\gamma)$, satisfies PDE (2.2) such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = inf\{J_\varepsilon(u_\varepsilon, \theta)| \;\; \theta \in L^2_{per}(\gamma)\}. \tag{2.3}$$

Here, H^1_ε is a Hilbert space defined by $H^1_\varepsilon = \{\phi \in H^1(\Omega_\varepsilon)| \;\; \phi|_{\gamma_b} = 0, \; \phi$ is γ_s periodic$\}$. Given f and θ as above, the variational formulation for the PDE (2.2) is given by: find $u_\varepsilon \in H^1_\varepsilon$ such that

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \phi dx + \int\limits_{\Omega_\varepsilon} u_\varepsilon \phi dx = \int\limits_{\Omega_\varepsilon} f \phi dx + \int\limits_{\gamma_\varepsilon} \hat{\theta}_\varepsilon \phi, \;\; \forall \phi \in H^1_\varepsilon. \tag{2.4}$$

The existence and uniqueness of weak solution of (2.2) follows from classical elliptic equation theory. Now, we state the existence and uniqueness of optimal solution to (2.3) whose proof follows from the classical theory (see [30]).

Theorem 2.1. For each $\varepsilon > 0$, the minimization problem (2.3) admits a unique solution $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1_\varepsilon \times L^2_{per}(\gamma)$.

The characterization of the optimal control using unfolding operator is given in Section 5, Theorem 5.1. Our aim is to analyze the asymptotic behavior of $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ as $\varepsilon \to 0$.

Remark 2.2. As the oscillating boundary is γ_ϵ, it is natural to consider the control contribution on γ_ϵ in the cost functional J_ϵ, instead of γ. One of the justification is that the control is coming from the fixed boundary of the reference domain which is more easy to apply. Nevertheless, it is also possible to consider the L^2 cost functional as

$$J^1_\varepsilon(u_\varepsilon, \theta) = \frac{1}{2} \int\limits_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 dx + \int\limits_{\gamma_\varepsilon} |\hat{\theta}^\varepsilon|^2.$$

Now consider the optimal control problem : find $(\bar{u}^1_\varepsilon, \bar{\theta}^1_\varepsilon)$ such that

$$J^1_\varepsilon(\bar{u}^1_\varepsilon, \bar{\theta}^1_\varepsilon) = inf\{J^1_\varepsilon(u_\varepsilon, \theta)| \;\; \theta \in L^2(\gamma), \; (u_\varepsilon, \theta) \; obeys \; (2.2)\}, \tag{2.5}$$

The optimal control problem (2.5) and (2.3) are equivalent. This is due to the following equality

$$\int\limits_{\gamma_\varepsilon} |\hat{\theta}^\varepsilon|^2 = \int\limits_{F_{M_0} \cup F_{M_1}} |\theta|^2 + \sum_{k=1}^{N} \int\limits_{F_{t_k}} \varepsilon^{2\alpha} |\theta|^2 + \int\limits_{S} \frac{\varepsilon^{2\alpha-1} |\nabla_{y'}\eta|(y') \sqrt{1 + |\nabla_{y'}\eta|^2(y')}}{\sqrt{\varepsilon^2 + |\nabla_{y'}\eta|^2(y')}} \left|\theta\left(y_1, \eta(y_1)\right)\right|^2. \tag{2.6}$$

The verification is straight forward.

3 | UNFOLDING OPERATOR AND ITS PROPERTIES

Here we will recall the definition of newly developed unfolding operator for domain with smooth oscillation and its properties without proof. For proof, one can see [3]. First, we will define the unfolded domain Ω_U in which the unfolded functions are defined. Before that we will introduce some notations. For $z \in [M_0, M_1]$, define the reference set

$$Y(z) = \{x' \in (0,1)^2| \;\; \eta(x') > z\},$$

in other words $Y(z) = \{x' \in (0,1)^2| \;\; (x', z) \in \Lambda\}$ and $h(z) = |Y(z)|$, Lebesgue measure of $Y(z)$. This is very crucial in the development of unfolding operator. The unfolded domain Ω_U is defined as follows,

$$\Omega_U = \{(x', x_3, y')| \;\; x' \in (0,1)^2, \; M_0 < x_3 < M_1, \; y' \in Y(x_3)\}.$$

Let $\mathscr{G} = \{(x_3, y')|\ x_3 \in (M_0, M_1),\ y' \in Y(x_3)\}$, then one can write, $\Omega_U = (0,1)^2 \times \mathscr{G}$. Let $\partial\mathscr{G} = \{(\eta(y'), y')|\ y' \in Y_s\}$. For $x' \in \mathbb{R}^2$, we write $[x'] = ([x_1],[x_2])$ and $\{x'\} = (\{x_1\},\{x_2\})$, here $[x_i]$ is the greatest integer part of x_i and $\{x_i\}$ is the fractional part of x_i.

Definition 3.1. (The unfolding operator) Let $\phi^\varepsilon : \Omega_U \to \Omega_\varepsilon^+$ be defined as $\phi^\varepsilon(x', x_3, y') = \left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y', x_3\right)$. The $\varepsilon-$ unfolding of a function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \Omega_U \to \mathbb{R}$. The operator which maps every function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. Let the unfolding operator be denoted by T^ε, that is,

$$T^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{T^\varepsilon(u) : \Omega_U \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon(u)(x', x_3, y') = u\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y', x_3\right).$$

If $U \subset \mathbb{R}^3$ containing Ω_ε^+ and u is a real valued function on U, $T^\varepsilon(u)$ will mean, T^ε acting on the restriction of u to Ω_ε^+. Some important properties of this unfolding operator are stated below.

Proposition 3.2. For each $\varepsilon > 0$,

(i) T^ε is linear. Further, if $u,\ v : \Omega_\varepsilon^+ \to \mathbb{R}$, then, $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.

(ii) Let $u \in L^1(\Omega_\varepsilon^+)$. Then,

$$\int_{\Omega_U} T^\varepsilon(u)\,dx\,dy' = \int_{\Omega_\varepsilon^+} u\,dx.$$

(iii) Let $u \in L^2(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2(\Omega_U)$ and $\|T^\varepsilon u\|_{L^2(\Omega_U)} = \|u\|_{L^2(\Omega_\varepsilon^+)}$.

(iv) For $u \in H^1(\Omega_\varepsilon^+)$, we have $T^\varepsilon u \in L^2((0,1)^2, H^1(\mathscr{G}))$. Moreover, $\frac{\partial}{\partial x_3}T^\varepsilon u = T^\varepsilon\frac{\partial u}{\partial x_3}$ and $\frac{\partial}{\partial y_i}T^\varepsilon u = \varepsilon T^\varepsilon\frac{\partial u}{\partial x_i}$, for $i = 1, 2$.

(v) For any $u \in L^2(\Omega_\varepsilon^+)$, $T^\varepsilon u \to u$ strongly in $L^2(\Omega_U)$. More generally, if $u_\varepsilon \to u$ strongly in $L^2(\Omega^+)$, then, $T^\varepsilon u \to u$ strongly in $L^2(\Omega_U)$.

(vi) For any ϕ defined on Ω_ε, we denote $\tilde{\phi}$, an extension by 0 to the full domain Ω. Let, for every ε, $u_\varepsilon \in L^2(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega_U)$. Then,

$$\tilde{u}_\varepsilon \rightharpoonup \int_{y' \in Y(x_3)} u(x', x_3, y')\,dy' \quad \text{weakly in } L^2(\Omega^+).$$

(vii) Let, for every $\varepsilon > 0$, $u_\varepsilon \in H^1(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0,1)^2, H^1(\mathscr{G}))$. Then, $\widetilde{u}_\varepsilon \rightharpoonup \int_{Y(x_3)} u\,dy'$ and $\widetilde{\frac{\partial u_\varepsilon}{\partial x_3}} \rightharpoonup \int_{Y(x_3)} \frac{\partial u}{\partial x_3}\,dy'$ weakly in $L^2(\Omega^+)$.

3.1 | Boundary Unfolding Operator and Its Properties

Here we will state the boundary unfolding operator and some of its properties . Proof will follow on similar lines as in[18].

Definition 3.3. For $i = 0, 1$, the ε-unfolding of a function $u : F_{M_i}^\varepsilon \to \mathbb{R}$ is the function $(T_i^\varepsilon u) : (0,1)^2 \times Y_{M_i} \to \mathbb{R}$ defined as $(T_i^\varepsilon u)(x', M_i, y') = u\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y', M_i\right)$.

If $U \subset \mathbb{R}^3$ containing $F_{M_i}^\varepsilon$ and u is a real valued function on U, $T^\varepsilon(u)$ will mean, T_i^ε acting on the restriction of u to $F_{M_i}^\varepsilon$. Some of its important properties that will require in our analysis are given below.

Proposition 3.4. For $i = 0, 1$.

(i) T_i^ε is linear, and if $u, v : F_{M_i}^\varepsilon \to \mathbb{R}$, then, $T_i^\varepsilon(uv) = T_i^\varepsilon(u)T_i^\varepsilon(v)$.

(ii) If $u \in L^2(F^\varepsilon_{M_i})$, then, $T^\varepsilon_i u \in L^2((0,1)\times Y_{M_i})$ and $\|T^\varepsilon u\|_{L^2((0,1)\times Y_{M_i})} = \|u\|_{L^2(F^\varepsilon_{M_i})}$.

(iii) If $u_\varepsilon \to u$ in $H^1((0,1)^2\times(M_0,M_1))$, then, $T^\varepsilon_i u \to u$ in $L^2((0,1)^2\times Y_{M_i})$.

(iv) If u_ε be a sequence in $L^2(F^\varepsilon_{M_i})$ such that $T^\varepsilon_i u_\varepsilon \rightharpoonup u$ weakly in $L^2((0,1)^2\times Y_{M_i})$, then, $\widetilde{u_\varepsilon} \rightharpoonup \int\limits_{Y_{M_i}} u dy'$ weakly in $L^2((0,1)^2)$

For $t_k \in (M_0, M_1)$, $k = 1,2,...,N$, we define similar kind of boundary unfolding operators that enjoy Proposition 3.4 as follows,

Definition 3.5. For $k = 1,2,...,N$, the ε-unfolding of a function $u : F^\varepsilon_{t_k} \to \mathbb{R}$ is the function $(T^\varepsilon_{t_k} u) : (0,1)^2\times Y_{t_k} \to \mathbb{R}$ defined as $(T^\varepsilon_{t_k} u)(x', t_k, y') = u\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y', t_k\right)$.

4 | SOME PRELIMINARY RESULTS AND *A PRIORI* ESTIMATE

In this section, we will prove some preliminary results which are the main ingredients for proving the main results.

Lemma 4.1. Let $v \in C^\infty(\overline{\Omega^+})$ and η be the reference function on $(0,1)^2$ as given in Section 2. Then,

$$(T^\varepsilon v)(x', \eta(y'), y') \to v(x', \eta(y')), \text{ as } \varepsilon \to 0$$

strongly in $L^\infty((0,1)^2\times Y_s)$.

Proof. Note that the function v is Lipschitz as $v \in C^\infty(\overline{\Omega^+})$. Now consider

$$\begin{aligned}\left|(T^\varepsilon v)(x', \eta(y'), y') - v(x', \eta(y'))\right| &= \left|v\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y', \eta(y')\right) - v(x', \eta(y'))\right| \\ &\leqslant C\left|\varepsilon\left\{\frac{x'}{\varepsilon}\right\} + \varepsilon y'\right| \leqslant C\epsilon,\end{aligned}$$

where C is a constant independent of ε. Thus we have

$$\sup_{(x',y')\in(0,1)^2\times Y_s} \{|T^\varepsilon v(x', \eta(y'), y') - v(x', \eta(y'))|\} \leqslant C\epsilon.$$

Passing to the limit as $\varepsilon \to 0$, we get the required result. □

Lemma 4.2. Let $\phi \in C^\infty(\overline{\Omega^+})$, then $(T^\varepsilon\phi)(x', \eta(y'), y')$ converges strongly to $\phi(x', \eta(y'))$ in $L^2((0,1)^2\times\partial\mathscr{G})$.

Proof. : By Lemma 4.1, we have

$$\int\limits_{(0,1)^2\times Y_s} |(T^\varepsilon\phi)(x', \eta(y'), y') - \phi(x', \eta(y'))|^2 dx'dy' \to 0,$$

as $\varepsilon \to 0$. Now,

$$\begin{aligned}&\int\limits_{(0,1)^2\times Y_s} |(T^\varepsilon\phi)(x', \eta(y'), y') - \phi(x', \eta(y'))|^2\sqrt{1+|\nabla_{y'}\eta(y')|^2}dx'dy' \\ &\leqslant k \int\limits_{(0,1)^2\times Y_s} |(T^\varepsilon\phi)(x', \eta(y'), y') - \phi(x', \eta(y'))|^2 dx'dy'\end{aligned}$$

where $k = sup\{\sqrt{1+|\nabla_{y'}\eta(y_1)|^2}| \ \ y' \in [0,1]^2\}$. In the last inequality if we pass to the limit as $\varepsilon \to 0$, we get the desired result. □

Lemma 4.3. (Convergence of Trace) Let $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(0,1;H^1(\mathscr{G}))$. Then, $T^\varepsilon u_\varepsilon|_{((0,1)^2\times\partial\mathscr{G})} \rightharpoonup u|_{((0,1)^2\times\partial\mathscr{G})}$ in $L^2((0,1)^2\times\partial\mathscr{G})$.

Proof. Let $\phi \in \left(C^\infty((0,1)^2 \times \mathscr{G})\right)^3$. Consider the following

$$\int_{(0,1)^2\times\mathscr{G}} \nabla_{(x_3,y')}(T^\varepsilon u_\varepsilon)\cdot\phi = -\int_{(0,1)^2\times\mathscr{G}} T^\varepsilon u_\varepsilon \mathrm{div}\,\phi + \int_{(0,1)^2\times\partial\mathscr{G}} T^\varepsilon u_\varepsilon(\phi\cdot\nu).$$

We can pass the limit in the first two term of the above equation . We get the following as $\varepsilon \to 0$,

$$\int_{(0,1)^2\times\mathscr{G}} \nabla_{(x_3,y')} u\cdot\phi + \int_{(0,1)^2\times\mathscr{G}} u\mathrm{div}\,\phi = \lim_{\varepsilon\to 0}\int_{(0,1)^2\times\partial\mathscr{G}} T^\varepsilon u_\varepsilon(\phi\cdot\nu).$$

Using integration by parts in left side we see that

$$\lim_{\varepsilon\to 0}\int_{(0,1)^2\times\partial\mathscr{G}} T^\varepsilon u_\varepsilon(\phi\cdot\nu) = \int_{(0,1)^2\times\partial\mathscr{G}} u(\phi\cdot\nu).$$

As $\phi \in \left(C^\infty((0,1)^2 \times \mathscr{G})\right)^2$ is arbitrary, the result follows. □

We now give the uniform estimates on the optimal solution $\bar{\theta}_\varepsilon$ and $\bar{u}_\varepsilon$. Recall $\hat{\theta}_\varepsilon$ for a given $\theta \in L^2_{per}(\gamma)$. Let us calculate the $\|\hat{\theta}^\varepsilon\|_{\gamma_\varepsilon}$. Since, $\gamma_\varepsilon = S^\varepsilon \cup F^\varepsilon_{M_0} \cup F^\varepsilon_{M_1} \cup \bigcup_{k=1}^N F_{t_k}$, one can write

$$\|\hat{\theta}^\varepsilon\|^2_{L^2(\gamma_\varepsilon)} = \int_{F^\varepsilon_{M_0}} |\hat{\theta}^\varepsilon|^2 + \int_{F^\varepsilon_{M_1}} |\hat{\theta}^\varepsilon|^2 + \int_{S^\varepsilon} |\hat{\theta}^\varepsilon|^2 + \int_{\bigcup_{k=1}^N F^\varepsilon_{t_k}} |\hat{\theta}^\varepsilon|^2.$$

Now,

$$\int_{F^\varepsilon_{M_i}} |\hat{\theta}^\varepsilon|^2 = \sum_{i,j=0}^{m-1}\int_{\varepsilon(i,j)+\varepsilon Y_{M_i}} \left|\theta\left(\frac{x'}{\varepsilon}, M_0\right)\right|^2 dx'.$$

By change of variable and using the periodicity of θ, we have

$$\int_{F^\varepsilon_{M_i}} |\hat{\theta}^\varepsilon|^2 = \int_{F_{M_i}} |\theta|^2.$$

Similarly, we have

$$\int_{\bigcup_{k=1}^N F^\varepsilon_{t_k}} |\hat{\theta}^\varepsilon|^2 = \int_{\bigcup_{k=1}^N F_{t_k}} \varepsilon^{2\alpha}|\theta|^2.$$

Now,

$$\int_{S^\varepsilon} |\hat{\theta}^\varepsilon|^2 = \sum_{i,j=0}^{m-1}\int_{\varepsilon(i,j)+\varepsilon Y_s} \frac{\varepsilon^{2\alpha}|\nabla_{x'}\eta(\frac{x'}{\varepsilon})|(1+|\nabla_{x'}\eta(\frac{x'}{\varepsilon})|^2)}{\varepsilon^2+|\nabla_{x'}\eta(\frac{x'}{\varepsilon})^2}\theta\left(\frac{x'}{\varepsilon},\eta\left(\frac{x'}{\varepsilon}\right)\right)\sqrt{\varepsilon^2+|\nabla_{x'}\eta|^2\left(\frac{x'}{\varepsilon}\right)}dx'$$

By change of variable and periodicity of η and θ we have

$$\begin{aligned}\int_{S^\varepsilon} |\hat{\theta}^\varepsilon|^2 &= \sum_{i,j=0}^{m-1}\int_{Y_s} \frac{\varepsilon^{2\alpha}|\nabla_{y'}\eta(y')|(1+|\nabla_{y'}\eta(y')|^2)}{\varepsilon^2+|\nabla_{y'}\eta(y')|^2}\theta(y',\eta(y'))\sqrt{\varepsilon^2+|\nabla_{y'}\eta|^2(y')}\varepsilon dy' \\ &\leqslant C\int_S |\theta|^2\end{aligned}$$

where C is a generic constant independent of ε. Hence, we have the following lemma

Lemma 4.4. For any $\theta \in L^2_{per}(\gamma)$, we have $\|\hat{\theta}^\varepsilon\|^2_{L^2(\gamma\epsilon)} \leqslant C\|\theta\|^2_{L^2(\gamma)}$, where C is a constant independent of ε.

Lemma 4.5. (*A priori* estimates) The family of optimal solutions $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ of (2.3) is uniformly bounded in ε, that is, there exists $C > 0$ such that $\|\bar{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C$ and $\|\bar{\theta}_\varepsilon\|_{L^2(\gamma_\varepsilon)} \leqslant C$.

Proof. Boundedness of $\|\bar{\theta}_\varepsilon\|_{L^2(\gamma_\varepsilon)}$ is obvious from the definition of optimal solution and by Lemma 4.4. Now, choose $\phi = \bar{u}_\varepsilon$ in the weak formulation (2.4),

$$\int_{\Omega_\varepsilon} \nabla \bar{u}_\varepsilon \nabla \bar{u}_\varepsilon dx + \int_{\Omega_\varepsilon} \bar{u}_\varepsilon \bar{u}_\varepsilon dx = \int_{\Omega_\varepsilon} f \bar{u}_\varepsilon dx + \int_{\gamma_\varepsilon} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon dx, \tag{4.1}$$

where $\hat{\bar{\theta}}^\varepsilon_\varepsilon$ corresponds to $\bar{\theta}_\varepsilon$. This implies

$$\|\bar{u}_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \leqslant \|f\|_{L^2(\Omega)} \|\bar{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \left| \int_{\gamma_\varepsilon} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon \right|.$$

Boundedness of $\bar{u}_\varepsilon$ will be proved if we show that

$$\int_{\gamma_\varepsilon} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon \leqslant C \|\bar{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \tag{4.2}$$

Since $\gamma_\varepsilon = S^\epsilon \cup F^\varepsilon_{M_0} \cup F^\varepsilon_{M_1} \bigcup_{k=1}^N F_{t_k}$, we have

$$\int_{\gamma_\varepsilon} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon = \int_{F^\varepsilon_{M_0}} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon + \int_{F^\varepsilon_{M_1}} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon + \int_{\bigcup_{k=1}^N F^\varepsilon_{t_k}} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon + \int_{S^\epsilon} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon \tag{4.3}$$

Let us calculate the 4th integral of right hand side in (4.3)

$$\begin{aligned} &\int_{S^\epsilon} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon \\ &= \sum_{i,j=1}^{m-1} \int_{\varepsilon(i,j)+\varepsilon Y_s} \frac{\varepsilon^\alpha \sqrt{|(\nabla_{x'}\eta)\left(\frac{x'}{\varepsilon}\right)|} \sqrt{1 + |(\nabla_{x'}\eta)(\frac{x'}{\varepsilon})|^2}}{\sqrt{\varepsilon^2 + |(\nabla_{x'}\eta)\frac{x'}{\varepsilon}|^2}} \bar{\theta}_\varepsilon \left(\frac{x'}{\varepsilon}, \eta\left(\frac{x'}{\varepsilon}\right)\right) \bar{u}_\varepsilon \left(x', \eta\left(\frac{x'}{\varepsilon}\right)\right) \\ &\qquad \sqrt{1 + \frac{1}{\varepsilon^2} \left|(\nabla_{x'}\eta)\left(\frac{x'}{\varepsilon}\right)\right|^2} dx' \end{aligned}$$

By making change of variable $x' = \varepsilon(i,j) + \varepsilon y'$ the above expression is equal to

$$\begin{aligned} &\sum_{i,j=1}^{m-1} \int_{Y_s} \frac{\varepsilon^\alpha \sqrt{|\nabla_{y'}\eta(y')|} \sqrt{1 + |\nabla_{y'}\eta(y')|^2}}{\sqrt{\varepsilon^2 + |\nabla_{y'}\eta(y')|^2}} \bar{\theta}_\varepsilon \left(y', \eta(y')\right) \bar{u}_\varepsilon \left(\varepsilon(i,j) + \varepsilon y', \eta(y')\right) \\ &\qquad \sqrt{1 + \frac{1}{\varepsilon^2} |\nabla_{y'}\eta(y')|^2} \varepsilon^2 dy' \\ &= \int_{Y_s} \bar{\theta}_\varepsilon \left(y', \eta(y')\right) \left(\varepsilon^{\alpha-1} \sqrt{|\nabla_{y'}\eta(y')|} \int_{(0,1)^2} T^\varepsilon \bar{u}_\varepsilon \left(x', \eta(y'), y'\right) dx' \right) \sqrt{1 + |\nabla \eta(y')|^2} dy' \end{aligned}$$

By Hölder's inequality in the last equality, we have

$$\left| \int_{S^\epsilon} \hat{\bar{\theta}}^\varepsilon_\varepsilon \bar{u}_\varepsilon \right| \leqslant C \|\bar{\theta}_\varepsilon\|_{L^2(\gamma)} \|T^\varepsilon \bar{u}_\varepsilon\|_{L^2((0,1)^2, L^2(\partial \mathscr{G}))}.$$

From Proposition 3.2 (iv), we get $T^\varepsilon(\bar{u}_\varepsilon) \in L^2(0,1; H^1(\mathscr{G}))$. By the trace theorem, we have

$$\|T^\varepsilon \bar{u}_\varepsilon\|_{L^2((0,1)^2, L^2(\partial \mathscr{G}))} \leqslant C \|T^\varepsilon \bar{u}_\varepsilon\|_{L^2((0,1)^2, H^1(\mathscr{G}))}$$

where C is a constant independent of ε. By the continuity of the unfolding operator, we have

$$\|T^{\varepsilon}\bar{u}_{\varepsilon}\|_{L^2((0,1)^2,H^1(\mathscr{G}))} \leqslant \|\bar{u}_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})}.$$

Using the estimate (4.2), we get $\|\bar{\theta}_{\varepsilon}\|_{L^2(\gamma)} \leqslant C$. Hence we have

$$\left|\int_{S^{\epsilon}} \hat{\bar{\theta}}^{\epsilon}_{\varepsilon}\bar{u}_{\varepsilon}\right| \leqslant C\|\bar{u}_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})}. \tag{4.4}$$

From the Lemma 5.6 in[36], we have the following

$$\begin{cases} \left|\int_{F^{\epsilon}_{M_i}} \hat{\bar{\theta}}^{\epsilon}_{\varepsilon}\bar{u}_{\varepsilon}\right| \leqslant C\|u_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})}, \text{for i=0,1} \\ \left|\int_{\bigcup_{k=1}^{N} F^{\epsilon}_{t_k}} \hat{\bar{\theta}}^{\epsilon}_{\varepsilon}\bar{u}_{\varepsilon}\right| \leqslant C\|u_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})}. \end{cases} \tag{4.5}$$

From equation (4.1) and inequalities (4.4,4.5) give,

$$\|\bar{u}_{\varepsilon}\|^2_{H^1(\Omega_{\varepsilon})} \leqslant C\|\bar{u}_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})}.$$

This implies

$$\|\bar{u}_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})}\| \leqslant C$$

where C is a generic constant independent of ε. □

5 | MAIN RESULTS AND THEIR PROOFS

In this section, we present our main contributions, namely the characterization of optimal control, the derivation of limit optimality system and the main convergence results. The proofs will be given in the later sections. Let $(\bar{u}_{\varepsilon}, \bar{\theta}_{\varepsilon})$ be the optimal solution to the optimal control problem (2.3). The optimal control can be characterized using the adjoint problem with the help of newly developed unfolding operators defined earlier. Let us consider the following adjoint PDE,

$$\begin{cases} -\Delta\bar{v}_{\varepsilon} + \bar{v}_{\varepsilon} = \bar{u}_{\varepsilon} - u_d \ \text{ in } \ \Omega_{\varepsilon}, \\ \bar{v}_{\varepsilon} = 0 \ \text{ on } \gamma_b, \\ \dfrac{\partial\bar{v}_{\varepsilon}}{\partial\nu} = 0 \ \text{ on } \ \gamma_{\varepsilon}, \\ \bar{v}_{\varepsilon} \ \text{ is } \ \gamma_s \ \text{ periodic.} \end{cases} \tag{5.1}$$

The following theorem gives the characterization of the optimal control via unfolding.

Theorem 5.1. Given $f \in L^2(\Omega)$, and let $(\bar{u}_{\varepsilon}, \bar{\theta}_{\varepsilon})$ be the optimal solution to the optimal control problem (2.3) and $\bar{v}_{\varepsilon}$ satisfies (5.1). Then, the optimal control $\bar{\theta}_{\varepsilon} \in L^2_{per}(\gamma)$ is given by

$$\begin{cases} \bar{\theta}_{\varepsilon}\Big|_S = -\dfrac{\sqrt{|\nabla_{y'}\eta|(y')}}{\beta}\displaystyle\int_{(0,1)^2} T^{\epsilon}\bar{v}_{\varepsilon}(x',\eta(y'),y')dx' \\ \bar{\theta}_{\varepsilon}\Big|_{F_{M_i}} = -\dfrac{1}{\beta}\displaystyle\int_{(0,1)^2} T^{\epsilon}_i\bar{v}_{\varepsilon}(x',M_i,y')dx', \ i = 0,1 \\ \bar{\theta}_{\varepsilon}\Big|_{F_{t_k}} = -\dfrac{1}{\beta}\displaystyle\int_{(0,1)^2} T^{\epsilon}_{t_k}\bar{v}_{\varepsilon}(x',t_k,y')dx', \ k = 1,2,...,N \end{cases}$$

Conversely, assume that a pair $(\hat{u}_{\varepsilon}, \hat{v}_{\varepsilon}) \in H^1_{\varepsilon} \times H^1_{\varepsilon}$ solves the following system,

$$\begin{cases} -\Delta \hat{u}_\varepsilon + \hat{u}_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ -\Delta \hat{v}_\varepsilon + \hat{v}_\varepsilon = \hat{u}_\varepsilon - u_d \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial \hat{u}_\varepsilon}{\partial \nu} = \hat{\theta}^\varepsilon, \ \dfrac{\partial \hat{v}_\varepsilon}{\partial \nu} = 0 \text{ on } \gamma_\varepsilon; \ \hat{u}_\varepsilon = 0, \ \hat{v}_\varepsilon = 0 \text{ on } \gamma_b, \\ \theta\Big|_S = -\dfrac{\sqrt{|\nabla_{y'}\eta|(y')}}{\beta} \displaystyle\int_{(0,1)^2} T^\varepsilon \hat{v}_\varepsilon(x', \eta(y'), y')dx', \\ \theta\Big|_{F_{M_i}} = -\dfrac{1}{\beta} \displaystyle\int_{(0,1)^2} T_1^\varepsilon \hat{v}_\varepsilon(x', M_i, y')dx', \ i = 0, 1 \\ \theta\Big|_{F_{t_k}} = -\dfrac{1}{\beta} \displaystyle\int_{(0,1)^2} T_{t_k}^\varepsilon \hat{v}_\varepsilon(x', t_k, y')dx', \ k = 1, 2, ..., N \\ \hat{v}_\varepsilon, \hat{u}_\varepsilon \text{ are } \gamma_s \text{ periodic.} \end{cases} \tag{5.2}$$

Then, the pair $(\hat{u}_\varepsilon, \theta)$ is the optimal solution to (2.3).

Proof. The optimal control problem (2.3) has a unique solution by Theorem 2.1. Denote the optimal solution of (2.3) by $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$. For $\lambda > 0$ and $\theta \in L^2(\gamma)$, we denote $u_{\varepsilon\lambda} = u_{\varepsilon\lambda}(f, \bar{\theta}_\varepsilon + \lambda\theta)$ which is the solution of state equation (2.2) with $\hat{\theta}_\varepsilon = \hat{\bar{\theta}}_\varepsilon^\varepsilon + \lambda\hat{\theta}^\varepsilon$. Let

$$F(\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} |\bar{u}_\varepsilon - u_d|^2 dx + \frac{\beta}{2}\int_\gamma |\theta|^2. \tag{5.3}$$

From the definition of optimal solution, it follows that for any $\lambda > 0$, we have

$$\frac{F(\bar{\theta}_\varepsilon + \lambda\theta) - F(\theta)}{\lambda} \geqslant 0. \tag{5.4}$$

One can derive the following by a simple calculation

$$F(\bar{\theta}_\varepsilon + \lambda\theta) - F(\theta) = \frac{1}{2\lambda}\int_{\Omega_\varepsilon} (u_{\varepsilon\lambda} + \bar{u}_\varepsilon - u_d)(u_{\varepsilon\lambda} - \bar{u}_\varepsilon)dx + \frac{\beta}{2}\int_\gamma (2\lambda\theta\bar{\theta}^\varepsilon + \lambda^2\theta^2).$$

Note that $(u_{\varepsilon\lambda} - \bar{u}_\varepsilon)$ satisfies,

$$\begin{cases} -\Delta(u_{\varepsilon\lambda} - \bar{u}_\varepsilon) + (u_{\varepsilon\lambda} - \bar{u}_\varepsilon) = 0 \text{ in } \Omega_\varepsilon \\ \frac{\partial(u_{\varepsilon\lambda} - \bar{u}_\varepsilon)}{\partial \nu} = \lambda\hat{\theta}_\varepsilon \text{ on } \gamma_\varepsilon \\ (u_{\varepsilon\lambda} - \bar{u}_\varepsilon) = 0 \text{ on } \gamma_b \\ (u_{\varepsilon\lambda} - \bar{u}_\varepsilon) - \gamma_s \text{ periodic.} \end{cases} \tag{5.5}$$

Let w_θ solves the following PDE

$$\begin{cases} -\Delta w_\theta + w_\theta = 0 \text{ in } \Omega_\varepsilon \\ \frac{\partial w_\theta}{\partial \nu} = \hat{\theta}_\varepsilon \text{ on } \gamma_\varepsilon \\ w_\theta = 0 \text{ on } \gamma_b \\ w_\theta, \ \gamma_s \text{ periodic.} \end{cases} \tag{5.6}$$

Then, $u_{\varepsilon\lambda} - \bar{u}_\varepsilon = \lambda w_\theta$. Using the *a priori* estimate, we get $\|u_{\varepsilon\lambda} - \bar{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \to 0$ as $\lambda \to 0$. Hence,

$$\lim_{\lambda\to 0} \frac{F(\bar{\theta}_\varepsilon + \lambda\theta) - F(\theta)}{\lambda} = \int_{\Omega_\varepsilon} (u_\varepsilon - u_d)w_\theta dx + \beta\int_\gamma \theta\bar{\theta}_\varepsilon. \tag{5.7}$$

From (5.4), we have

$$\int_{\Omega_\varepsilon} (\bar{u}_\varepsilon - u_d)w_\theta dx + \beta\int_\gamma \theta\bar{\theta}_\varepsilon \geqslant 0. \tag{5.8}$$

By using w_θ as a test function in (5.1) and $\bar{v}_\varepsilon$ in (5.6), we get

$$\int_{\Omega_\varepsilon} (\nabla \bar{v}_\varepsilon \nabla w_\theta + \bar{v}_\varepsilon w_\theta) dx = \int_{\Omega_\varepsilon} (\bar{u}_\varepsilon - u_d) w_\theta dx$$
$$\int_{\Omega_\varepsilon} (\nabla \bar{v}_\varepsilon \nabla w_\theta + \bar{v}_\varepsilon w_\theta) dx = \int_{\gamma_\varepsilon} \hat{\theta}^\varepsilon \bar{v}_\varepsilon$$

Then, we have

$$\int_{\gamma_\varepsilon} \hat{\theta}^\varepsilon \bar{v}_\varepsilon = \int_{\Omega_\varepsilon} (\bar{u}_\varepsilon - u_d) w_\theta dx$$

Hence, (5.8) becomes

$$\int_{\gamma_\varepsilon} \hat{\theta}^\varepsilon \bar{v}_\varepsilon + \beta \int_{\gamma} \theta \bar{\theta}_\varepsilon \geqslant 0. \tag{5.9}$$

Since, the above inequality is true for all $\theta \in L^2_{per}(\gamma)$, we obtain

$$\int_{\gamma_\varepsilon} \hat{\theta}^\varepsilon \bar{v}_\varepsilon + \beta \int_{\gamma} \theta \bar{\theta}_\varepsilon = 0, \ \ \forall \theta \in L^2_{per}(\gamma). \tag{5.10}$$

Since $\gamma_\varepsilon = S^\varepsilon \cup F^\varepsilon_{M_0} \cup F^\varepsilon_{M_1} \cup \bigcup_{k=1}^{N} F^\varepsilon_{t_k}$, the above equation becomes

$$\int_{F^\varepsilon_{M_0}} \hat{\theta}^\varepsilon \bar{v}_\varepsilon + \int_{F^\varepsilon_{M_0}} \hat{\theta}^\varepsilon \bar{v}_\varepsilon + \int_{\bigcup_{k=1}^N F^\varepsilon_{t_k}} \hat{\theta}^\varepsilon \bar{v}_\varepsilon + \int_{S^\varepsilon} \hat{\theta}^\varepsilon \bar{v}_\varepsilon + \beta \int_{\gamma} \theta \bar{\theta}_\varepsilon = 0.$$

We, now look at each of above integrals. Using boundary unfolding operator, the first is given by

$$\begin{aligned}
\int_{F^\varepsilon_{M_0}} \hat{\theta}^\varepsilon \bar{v}_\varepsilon &= \int_{(0,1)^2 \times Y_{M_0}} T^\varepsilon_1(\bar{v}_\varepsilon)(x', M_0, y') T^\varepsilon_1(\hat{\theta}^\varepsilon)(x', M_0, y') dx' dy' \\
&= \int_{(0,1)^2 \times Y_{M_0}} T^\varepsilon_1(\bar{v}_\varepsilon)(x', M_0, y') \hat{\theta} \left(\varepsilon \left[\frac{x'}{\varepsilon} \right] + \varepsilon y', M_0 \right) dx' dy' \\
&= \int_{Y_{M_0}} \left\{ \int_{(0,1)^2} T^\varepsilon_1(\bar{v}_\varepsilon)(x', M_0, y') dx' \right\} \theta(y', M_0) dy'.
\end{aligned} \tag{5.11}$$

Similarly, we have

$$\begin{aligned}
\int_{F^\varepsilon_{M_1}} \hat{\theta}^\varepsilon \bar{v}_\varepsilon &= \int_{Y_{M_1}} \left\{ \int_{(0,1)^2} T^\varepsilon_2(\bar{v}_\varepsilon)(x', M_1, y') dx' \right\} \theta(y', M_1) dy'. \\
\int_{F^\varepsilon_{t_k}} \hat{\theta}^\varepsilon \bar{v}_\varepsilon &= \int_{Y_{M_1}} \left\{ \varepsilon^\alpha \int_{(0,1)^2} T^\varepsilon_2(\bar{v}_\varepsilon)(x', M_1, y') dx' \right\} \theta(y', M_1) dy'.
\end{aligned} \tag{5.12}$$

Let us look at the integral on the lateral surface.

$$
\begin{aligned}
&\int_{S^\varepsilon} \hat{\theta}^\varepsilon \bar{v}_\varepsilon \\
&= \sum_{i,j=1}^{m-1} \int_{\varepsilon(i,j)+\varepsilon Y_s} \frac{\varepsilon^\alpha \sqrt{|\nabla_{x'}\eta(\frac{x'}{\varepsilon})|}\sqrt{1+|(\nabla_{x'}\eta)\left(\frac{x'}{\varepsilon}\right)|^2}}{\sqrt{\varepsilon^2+|\nabla_{x'}\eta\left(\frac{x'}{\varepsilon}\right)|^2}} \\
&\qquad \bar{\theta}_\varepsilon\left(\frac{x'}{\varepsilon},\eta\left(\frac{x'}{\varepsilon}\right)\right)\bar{v}_\varepsilon\left(x',\eta\left(\frac{x'}{\varepsilon}\right)\right)\sqrt{1+\frac{1}{\varepsilon^2}\left|\nabla_{x'}\eta\left(\frac{x'}{\varepsilon}\right)\right|^2}dx'
\end{aligned}
\tag{5.13}
$$

By making change of variable $x' = \varepsilon(i,j)+\varepsilon y'$, the above sums equals

$$
\begin{aligned}
&\sum_{i,j=1}^{m-1}\int_{Y_s} \frac{\varepsilon^\alpha\sqrt{|\nabla_{y'}\eta(y')|}\sqrt{1+|\nabla_{y'}\eta(y')|^2}}{\sqrt{\varepsilon^2+|\nabla_{y'}\eta(y')|^2}}\theta\left(y',\eta(y')\right)\bar{v}_\varepsilon\left(\varepsilon(i,j)+\varepsilon y',\eta(y')\right) \\
&\qquad \sqrt{1+\frac{1}{\varepsilon^2}|\nabla_{y'}\eta(y')|^2}\varepsilon^2 dy' \\
&= \int_{Y_s}\theta\left(y',\eta(y')\right)\left(\varepsilon^{\alpha-1}\sqrt{|\nabla_{y'}\eta(y')|}\sum_{i,j=0}^{m-1}\int_{(\varepsilon(i,j)+\varepsilon(0,1)^2)} T^\varepsilon\bar{v}_\varepsilon\left(x',\eta(y'),y'\right)dx'\right) \\
&\qquad \sqrt{1+|\nabla_{y'}\eta(y')|^2}dy' \\
&= \int_{Y_s}\theta\left(y',\eta(y')\right)\left(\varepsilon^{\alpha-1}\sqrt{|\nabla_{y'}\eta(y')|}\int_{(0,1)^2} T^\varepsilon\bar{v}_\varepsilon\left(x',\eta(y'),y'\right)dx'\right)\sqrt{1+|\nabla_{y'}\eta(y')|^2}dy'.
\end{aligned}
\tag{5.14}
$$

Substituting the the relations (5.11), (5.12), (5.13) in (5.10), we obtain,

$$
\begin{cases}
\displaystyle\int_S \theta(y',\eta(y'))\left\{\sqrt{|\nabla_{y'}\eta(y')|}\int_{(0,1)^2} T^\varepsilon\bar{v}_\varepsilon(x',\eta(y'),y')dx'\right\} \\
\displaystyle+\int_{Y_{M_0}}\theta(y',M_0)\left\{\int_{(0,1)^2} T^\varepsilon\bar{v}_\varepsilon(x',M_0,y')dx'\right\}dy' \\
\displaystyle+\int_{Y_{M_1}}\left\{\int_{(0,1)^2} T_1^\varepsilon(\bar{v}_\varepsilon)(x',M_1,y')dx'\right\}\theta(y',M_1)dy' \\
\displaystyle+\int_{\bigcup_k^N Y_{t_k}}\left\{\varepsilon^\alpha\int_{(0,1)^2} T_1^\varepsilon(\bar{v}_\varepsilon)(x',t_k,y')dx_1\right\}\theta(y',t_k)dy'+\beta\int_\gamma \bar{\theta}_\varepsilon\theta = 0
\end{cases}
$$

The above equality is true for all $\theta \in L^2_{per}(\gamma)$. By choosing $\theta \in L^2_{per}(\gamma)$ suitably, we get the desired result. □

Having obtained the characterization of the optimal control, we now proceed to find the limit optimal control problem. For the convenience of the reader, we explain the limit space and corresponding weak formulation of the limit problem as it is not very common. Finally, we give the main homogenization results.

5.0.1 | Limit Problem

Recall $h(x_3) = |Y(x_3)|,\ x_3 \in [M_0, M_1]$, where $|Y(x_3)|$ is the Lebesgue measure of the set $Y(x_3)$. Note that h is a strictly positive function. For each $x_3 \in [M_0, M_1]$, we denote $d\gamma(x_3)$ is the surface measure on the level curve

$$\gamma_{x_3}(x') = \{(x', x_3) \big| \ x' \in (0,1)^2,\ \eta(x') = x_3\}.$$

We define ω on (M_0, M_1) by

$$\omega(x_3) = \int\limits_{\gamma_{x_3}} \sqrt{1 + |\nabla_{x'}\eta(x')|^2} d\gamma(x_3).$$

Now consider the space

$$W(\Omega) = \left\{ \psi \in L^2(\Omega) | \ \frac{\partial \psi}{\partial x_3} \in L^2(\Omega),\ \ \psi \in H^1(\Omega^-) \right\}$$

with the following inner product

$$\langle u, v\rangle_{W(\Omega)} = \langle hu, v\rangle_{L^2(\Omega^+)} + \left\langle h\frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2} \right\rangle_{L^2(\Omega^+)} + \langle u, v\rangle_{H^1(\Omega^-)}.$$

Note that $W(\Omega)$ is a Hilbert space with respect to the given inner product. Throughout this article, we denote $\psi^+ = \psi|_{\Omega^+},\ \ \psi^- = \psi|_{\Omega^-}$, for $\psi \in W(\Omega)$.

The limit optimal control problem is described below:

For $f \in L^2_{per}(\Omega), \varrho_1, \varrho_2 \in \mathbb{R},\ \theta \in L^2(M_0, M_1)$, consider the following optimal control problem: find $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2) \in W(\Omega) \times L^2(M_0, M_1) \times \mathbb{R} \times \mathbb{R}$ such that

$$J(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2) = \inf \left\{ J(u, \theta, \varrho_1, \varrho_2) | \ \ \varrho_1, \varrho_2 \in \mathbb{R},\ \theta \in L^2(M_0, M_1), \right\} \tag{5.15}$$

where J is an L^2-cost functional defined as

$$\begin{aligned} J(u, \theta, \varrho_1, \varrho_2) = &\frac{1}{2} \int\limits_{\Omega^+} (h(x_3)\chi_{\Omega^+} + \chi_{\Omega^-})|u - u_d|^2 dx \\ &+ \frac{\beta}{2} \int\limits_{M_0}^{M_1} \frac{1}{|\omega(x_3)|} |\theta(x_3)|^2 dx_3 + \frac{\beta h(M_1)}{2|Y_{M_1}|}\varrho_2^2 + \frac{\beta}{2|Y_{M_0}|}\varrho_1^2. \end{aligned} \tag{5.16}$$

In the above definition, for given $\theta \in L^2(M_0, M_1),\ \varrho_1, \varrho_2 \in \mathbb{R}$, u solves the following PDE

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial u^+}{\partial x_3} \right) + h(x_3)u^+ = h(x_3)f^+ + \theta \ \text{ in } \Omega^+, \\ -\Delta u^- + u^- = f^- \text{ in } \Omega^- \\ \dfrac{\partial u^+}{\partial x_3} = \varrho_2, \\ u^+ = u^-,\ \dfrac{\partial u^-}{\partial x_3} - h(M_0)\dfrac{\partial u^+}{\partial x_3} = \varrho_1 \text{ on } \gamma_c, \\ u^- = 0 \ \text{ on } \ \gamma_b, \text{ and } u \text{ is } \ \gamma_s \text{ periodic.} \end{cases} \tag{5.17}$$

The existence and uniqueness of the limit optimal control problem is presented in the following theorem.

Theorem 5.2. The optimal control problem (5.15) admits a unique solution $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2) \in W(\Omega) \times L^2(M_0, M_1) \times \mathbb{R} \times \mathbb{R}$.

Proof. Let $m = \inf\{J(u, \theta, \varrho_1, \varrho_2) | \ \ \theta \in L^2(M_0, M_1), \varrho_1, \varrho_2 \in \mathbb{R}\}$. Since $\{J(u, \theta, \varrho_1, \varrho_2) | \ \ \theta \in L^2(M_0, M_1), \varrho_1, \varrho_2 \in \mathbb{R}\} \subset \mathbb{R}$ is nonempty and bounded below, there exists a sequence $u_n \in W(\Omega)$ *and* $\varrho_1^n, \varrho_2^n \in \mathbb{R}$ such that $J(u_n, \theta_n, \varrho_1^n, \varrho_2^n) \to m$. Without loss of generality, we can assume that

$$J(u_n, \theta_n, \varrho_1^n, \varrho_2^n) \leqslant J(u_0, 0, 0, 0),\ \ \forall n \in \mathbb{N}.$$

From above inequality we conclude that $|\varrho_1^n|, |\varrho_2^n|, \|u_n\|_{L^2(\Omega)}$ and $\|\theta_n\|_{L^2(M_0,M_1)}$ are uniformly bounded. In order to get a bound for u_n in $W(\Omega)$, we look into the following weak formulation: find $u_n \in W(\Omega)$ such that

$$\int_{\Omega^+} h(x_3)\left(\frac{\partial u_n^+}{\partial x_3}\frac{\partial \phi^+}{\partial x_3} + u_n^+\phi^+\right)dx + \int_{\Omega^-}(\nabla u_n^- \nabla \phi^- + u_n\phi^-)dx = \int_{\Omega^+} h(x_3)f^+\phi^+ dx + \int_{\Omega^-} f^-\phi^- dx + \int_{\gamma_u}\varrho_2^n\phi + \int_{\Omega^+}\theta_n\phi^+ + \int_{\gamma_c}\varrho_1^n\phi \tag{5.18}$$

for all $\phi \in W(\Omega)$. By taking $\phi = u_n$ in (5.18), we get a uniform bound on $\|u_n\|_{W(\Omega)}$ as $|\varrho_1^n|, |\varrho_2^n|$ and $\|\theta_n\|_{L^2(M_0,M_1)}$. So, there exists a common sub-sequence such that

(i) $u_n \rightharpoonup u_0$ weakly in $W(\Omega)$

(ii) $\theta_n \rightharpoonup \hat{\theta}$ weakly in $L^2(M_0, M_1)$

(iii) $\varrho_1^n \to \hat{\varrho}_1$ *in* $\mathbb{R}$

(iv) $\varrho_2^n \to \hat{\varrho}_2$ *in* $\mathbb{R}$.

The existence of optimal control will be proved if we show that for some $u_0 \in W(\Omega)$ and $\hat{\theta} \in L^2(M_0, M_1)$, $u_0 = u_0(\hat{\theta}, \hat{\varrho}_1, \hat{\varrho}_2)$ satisfying the system (2.1). Now, we pass to the limit as $n \to \infty$, in the weak formulation (5.18), we get

$$\int_{\Omega^+} h(x_3)\left(\frac{\partial u_0^+}{\partial x_3}\frac{\partial \phi^+}{\partial x_3} + u_0^+\phi^+\right)dx + \int_{\Omega^-}\nabla u_0^- \nabla\phi^- dx = \int_{\Omega^+} h(x_3)f^+\phi^+ dx + \int_{\Omega^-} f^-\phi^- dx + \int_{\Omega^+}\hat{\theta}\phi^+ dx + \int_{\gamma_{u_0}}\hat{\varrho}_2\phi + \int_{\gamma_c}\hat{\varrho}_1\phi.$$

This shows that $u_0 = u_0(\hat{\theta}, \hat{\varrho}_1, \hat{\varrho}_2)$ is the solution to the weak formulation (5.17). As the L^2-norm is weakly lower semi-continuous, we have

$$J(u_0, \hat{\theta}, \hat{\varrho}_1, \hat{\varrho}_2) = \frac{1}{2}\int_{\Omega^+}(h(x_3)\chi_{\Omega^+} + \chi_{\Omega^-})|u_0 - u_d|^2 dx + \frac{\beta}{2}\int_{M_0}^{M_1}\frac{1}{\omega(x_3)}|\theta(x_3)|^2 dx_3 \\ \frac{\beta h(M_1)}{2|Y_{M_1}|}\hat{\varrho}_2^2 + \frac{\beta}{2|Y_{M_0}|}\hat{\varrho}_1^2 \leqslant m. \tag{5.19}$$

This implies that

$$J(u_0, \hat{\theta}, \hat{\varrho}_1, \hat{\varrho}_2) = m.$$

This proves the existence part of the theorem. Uniqueness follows from the convexity of cost functional. □

We denote the solution of problem (5.15) by $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2)$. Now we will describe the limit adjoint equation which is the key to characterize the optimal control.

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial v^+}{\partial x_3}\right) + h(x_3)v^+ = h(x_3)(\bar{u}^+ - u_d) \text{ in } \Omega^+, \\ -\Delta v^- + v^- = \bar{u}^- - u_d \text{ in } \Omega^-, \\ \dfrac{\partial v^+}{\partial x_3} = 0 \text{ on } \gamma_u, \\ v^+ = v^- \text{ on } \gamma_c, \\ \dfrac{\partial v^-}{\partial x_3} - h(M_0)\dfrac{\partial v^+}{\partial x_3} = 0 \text{ on } \gamma_c, \\ v^- = 0, \text{ on } \gamma_b \text{ and } v \text{ is } \gamma_s \text{ periodic.} \end{cases} \tag{5.20}$$

The characterization of the limit optimal control is given in the following theorem. We present the case separately for $\alpha = 1$ and $\alpha > 1$. Recall the parameter α from (2.1).

Theorem 5.3. (For $\alpha = 1$) Let $f \in L^2(\Omega)$ and $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2)$ be the solution of the problem (5.15) and v satisfies (5.20). Then, the optimal control is given by

$$\begin{aligned}
\bar{\theta} &= -\frac{1}{\beta}|\omega(x_3)|\left(\int_{[0,1]^2} v(z', x_3)dz'\right), \\
\bar{\varrho}_1 &= -\frac{|Y_{M_0}|}{\beta}\int_{[0,1]^2} v(z', M_0)dz', \\
\bar{\varrho}_2 &= -\frac{|Y_{M_1}|}{\beta}\int_{[0,1]^2} v(z', M_1)dz',
\end{aligned} \tag{5.21}$$

Conversely, assume that a pair $(\hat{u}, \hat{v}) \in W(\Omega) \times W(\Omega)$ solves the following system of PDE,

$$\begin{cases}
-\frac{\partial}{\partial x_3}\left(h(x_3)\frac{\partial \hat{u}^+}{\partial x_3}\right) + h(x_3)\hat{u}^+ = h(x_3)f^+ + \hat{\theta} \text{ in } \Omega^+, \\
-\frac{\partial}{\partial x_3}\left(h(x_3)\frac{\partial \hat{v}^+}{\partial x_3}\right) + h(x_3)\hat{v}^+ = h(x_3)(\hat{u}^+ - u_d) \text{ in } \Omega^+, \\
-\Delta\hat{u}^- + \hat{u}^- = f^-, \quad -\Delta\hat{v}^- + \hat{v}^- = \hat{u}^- - \hat{u}_d \text{ in } \Omega^-, \\
\frac{\partial \hat{u}^+}{\partial x_3} = \hat{\varrho}_2, \quad \frac{\partial \hat{v}^+}{\partial x_3} = 0 \text{ on } \gamma_u, \\
\hat{u}^+ = \hat{u}^-, \quad \frac{\partial \hat{u}^-}{\partial x_3} - h(M_0)\frac{\partial \hat{u}^+}{\partial x_3} = \hat{\varrho}_1 \text{ on } \gamma_c, \\
\hat{v}^+ = \hat{v}^-, \; h(M_0)\frac{\partial \hat{v}^+}{\partial x_3} - \frac{\partial \hat{v}^-}{\partial x_3} = 0 \text{ on } \gamma_c, \\
\hat{\theta} = -\frac{1}{\beta}|\omega(x_3)|\left(\int_{[0,1]^2} \hat{v}(z', x_3)dz'\right), \\
\hat{\varrho}_1 = -\frac{|Y_{M_0}|}{\beta}\int_{[0,1]^2} \hat{v}(z', M_0)dz', \quad \hat{\varrho}_2 = -\frac{|Y_{M_1}|}{\beta}\int_{[0,1]^2} \hat{v}(z', M_1)dz', \\
\hat{u}^-, \hat{v}^- = 0 \text{ on } \gamma_b, \text{ and } \hat{u}, \hat{v} \text{ are } \gamma_s \text{ periodic.}
\end{cases} \tag{5.22}$$

Then, $(\hat{u}, \hat{\theta}, \hat{\varrho}_1, \hat{\varrho}_2)$ is the solution of the minimization problem (5.15)

Proof. Let $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2)$ be the optimal solution of (5.15). For $\theta \in L^2(M_0, M_1)$, $\varrho_1, \varrho_2 \in \mathbb{R}$, and $\lambda > 0$, we denote $u_\lambda = u(\bar{\theta} + \lambda\theta, \bar{\varrho}_1 + \lambda\varrho_1, \bar{\varrho}_2 + \lambda\varrho_2)$, $J_\lambda = J(u_\lambda, \bar{\theta} + \lambda\theta, \bar{\varrho}_1 + \lambda\varrho_1, \bar{\varrho}_2 + \lambda\varrho_2)$ *and* $\bar{J} = J(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2)$. Then, we have $J_\lambda - \bar{J} \geqslant 0$.
By a simple calculation, we get

$$\begin{aligned}
J_\lambda - \bar{J} =& \frac{1}{2}\int_\Omega \left(\chi_{\Omega^-} + h(x_3)\chi_{\Omega^+}\right)(u_\lambda + \bar{u} - 2u_d)(u_\lambda - \bar{u})dx + \frac{\beta}{2}\int_{M_0}^{M_1} \frac{1}{\omega(x_3)}(2\lambda\bar{\theta}\theta + \lambda^2\theta^2)dx_3 \\
&+ \frac{\beta h(M_1)}{2|Y_{M_1}|}(2\lambda\bar{\varrho}_2\varrho_2 + \lambda^2\varrho_2^2) + \frac{\beta}{2|Y_{M_0}|}(2\lambda\varrho_1\bar{\varrho}_1 + \lambda^2\varrho_1^2).
\end{aligned}$$

Note that, we can write $u_\lambda - \bar{u} = \lambda w$, where $w \in W(\Omega)$ satisfies the following PDE,

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial w^+}{\partial x_3}\right) + h(x_3)w^+ = \theta \text{ in } \Omega^+ \\ -\Delta w^- + w^- = 0 \text{ in } \Omega^- \\ \dfrac{\partial w^+}{\partial x_3} = \varrho_2 \text{ on } \gamma_u \\ w^+ = w^- \text{ on } \gamma_c \\ \dfrac{\partial w^-}{\partial x_3} - h(M_0)\dfrac{\partial w^+}{\partial x_3} = \varrho_1 \text{ on } \gamma_c \\ w^- = 0, \text{ on } \gamma_b \text{ and } w\ \gamma_s \text{ periodic.} \end{cases} \tag{5.23}$$

$$\lim_{\lambda\to 0} \frac{J_\lambda - \bar{J}}{\lambda} = \int_\Omega (\chi_{\Omega^-} + h(x_3)\chi_{\Omega^+})(\bar{u} - u_d) w\, dx + \beta \int_{M_0}^{M_1} \frac{1}{\omega(x_3)}\bar{\theta}\theta\, dx_3 + \frac{\beta h(M_1)}{|Y_{M_1}|}\varrho_2\bar{\varrho}_2 + \frac{\beta}{|Y_{M_0}|}\varrho_1\bar{\varrho}_1. \tag{5.24}$$

By choosing v as a test function in equation (5.23), we get

$$\int_{\Omega^+} h(x_3)\left(\frac{\partial w^+}{\partial x_3}\frac{\partial v^+}{\partial x_3} + w^+v^+\right)dx + \int_{\Omega^-} \nabla w^- \nabla v^- dx = \int_{\Omega^+} \theta v^+ dx + \int_{\gamma_c} \varrho_1 v + \int_{\gamma_u} h(M_1)\varrho_2 v.$$

And if we choose w as a test function in (5.20), we get

$$\int_{\Omega^+} h(x_3)\left(\frac{\partial w^+}{\partial x_3}\frac{\partial v^+}{\partial x_3} + w^+v^+\right)dx + \int_{\Omega^-} \nabla w^- \nabla v^- dx = \int_{\Omega} \left(\chi_{\Omega^+} + h(x_3)\chi_{\Omega^-}\right)(\bar{u} - u_d)dx.$$

Hence, we obtain

$$\int_{\Omega^+} \theta v^+ dx + \int_{\gamma_c} \varrho_1 v + \int_{\gamma_u} h(M_1)\varrho_2 v = \int_{\Omega} \left(h(x_2)\chi_{\Omega^+} + \chi_{\Omega^-}\right)(\bar{u} - u_d)dx. \tag{5.25}$$

Using the above relation (5.25) in (5.24), we get

$$\lim_{\lambda\to 0} \frac{J_\lambda - \bar{J}}{\lambda} = \int_{\Omega^+} \theta v^+ dx + \int_{\gamma_c} \varrho_1 v + \int_{\gamma_u} h(M_1)\varrho_2 v + \beta \int_{M_0}^{M_1} \frac{1}{\omega(x_3)}\bar{\theta}\theta\, dx_3 + \frac{\beta h(M_1)}{Y_{M_1}}\varrho_2\bar{\varrho}_2 + \frac{\beta}{Y_{M_0}}\varrho_1\bar{\varrho}_1. \tag{5.26}$$

Since, $\frac{J_\lambda - \bar{J}}{\lambda} \geqslant 0$ for every $\lambda > 0$. we get,

$$\int_{\Omega^+} \theta v^+ dx + \int_{\gamma_c} \varrho_1 v + \int_{\gamma_u} h(M_1)\varrho_2 v + \beta \int_{M_0}^{M_1} \frac{1}{\omega(x_3)}\bar{\theta}\theta\, dx_3 + \frac{\beta h(M_1)}{|Y_{M_1}|}\varrho_2\bar{\varrho}_2 + \frac{\beta}{|Y_{M_0}|}\varrho_1\bar{\varrho}_1 \geqslant 0,$$

for all $\theta \in L^2(M_0, M_1), \varrho_1, \varrho_2 \in \mathbb{R}$. Hence, we have

$$\int_{\Omega^+} \theta v^+ dx + \int_{\gamma_c} \varrho_1 v + \int_{\gamma_u} h(M_1)\varrho_2 v + \beta \int_{M_0}^{M_1} \frac{1}{\omega(x_3)}\bar{\theta}\theta\, dx_3 + \frac{\beta h(M_1)}{|Y_{M_1}|}\varrho_2\bar{\varrho}_2 + \frac{\beta}{|Y_{M_0}|}\varrho_1\bar{\varrho}_1 = 0, \tag{5.27}$$

for all $\theta \in L^2(M_0, M_1), \varrho_1, \varrho_2 \in \mathbb{R}$. Choosing $\theta, \varrho_1, \varrho_2$ suitably in the above equation, we get

$$\begin{aligned}
\bar{\theta} &= -\frac{1}{\beta}|\omega(x_3)|\left(\int_{[0,1]^2} v(z, x_3)dz\right), \\
\bar{\varrho_1} &= -\frac{|Y_{M_0}|}{\beta}\int_{[0,1]^2} v(x', M_0)dx'. \\
&\text{and} \\
\bar{\varrho_2} &= -\frac{|Y_{M_1}|}{\beta}\int_{[0,1]^2} v(z', M_1)dz'
\end{aligned} \tag{5.28}$$

This completes the proof of theorem (5.3). □

Remark 5.4. Note that, though we have started the optimal control problem with the boundary control, the limit control problem turns out to be a mixture of interior and boundary controls.

Now, we will write the optimal control problem for the case when $\alpha > 1$. Given $f \in L^2_{per}(\Omega)$, $\varrho_1, \varrho_2 \in \mathbb{R}$, consider the following control problem: find $(\bar{u}, \bar{\varrho}_1, \bar{\varrho}_2)$ such that

$$J(\bar{u}, \bar{\varrho}_1, \bar{\varrho}_2) = \inf\{J(u, \varrho_1, \varrho_2,)\Big|\, \varrho_1, \varrho_2 \in \mathbb{R}\} \tag{5.29}$$

where J is an L^2-cost functional defined as

$$J(u, \varrho_1, \varrho_2) = \int_{\Omega^+} (h(x_3)\chi_{\Omega+} + \chi_{\Omega^-})|u - u_d|^2 dx + \frac{\beta h(M_1)}{2|Y_{M_1}|}\varrho_2^2 + \frac{\beta}{2|Y_{M_0}|}\varrho_1^2. \tag{5.30}$$

and u satisfies the following PDE

$$\begin{cases}
-\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial u^+}{\partial x_3}\right) + h(x_3)u^+ = h(x_3)f^+ \text{ in } \Omega^+, \\
-\Delta u^- + u^- = f^- \text{ in } \Omega^- \\
\dfrac{\partial u^+}{\partial x_3} = \varrho_2 \text{ on } \gamma_u, \; u^+ = u^- \text{ on } \gamma_c, \\
\dfrac{\partial u^-}{\partial x_3} - h(M_0)\dfrac{\partial u^+}{\partial x_3} = \varrho_1 \text{ on } \gamma_c \\
u^- = 0, \text{ on } \gamma_b \text{ and } u\ \gamma_s \text{ periodic.}
\end{cases} \tag{5.31}$$

Theorem 5.5. (For $\alpha > 1$) Let $f \in L^2(\Omega)$ and $(\bar{u}, \bar{\varrho}_1, \bar{\varrho}_2)$ be the solution of the problem (5.29) and v satisfies (5.20). Then, the optimal control is given by

$$\begin{aligned}
\bar{\varrho_1} &= -\frac{|Y_{M_0}|}{\beta}\int_{[0,1]^2} v(z', M_0)dz', \\
\bar{\varrho_2} &= -\frac{|Y_{M_1}|}{\beta}\int_{[0,1]^2} v(z', M_1)dz',
\end{aligned} \tag{5.32}$$

Conversely, assume that a pair $(\hat{u}, \hat{v}) \in W(\Omega) \times W(\Omega)$ solves the following system of PDE,

$$
\begin{cases}
-\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial \hat{u}^+}{\partial x_3}\right) + h(x_3)\hat{u}^+ = h(x_3)f^+ + \hat{\theta} \ \text{ in } \Omega^+, \\
-\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial \hat{v}^+}{\partial x_3}\right) + h(x_3)\hat{v}^+ = h(x_3)(\hat{u}^+ - u_d) \text{ in } \Omega^+, \\
-\Delta \hat{u}^- + \hat{u}^- = f^- \text{ in } \Omega^- \\
-\Delta \hat{v}^- + \hat{v}^- = \hat{u}^- - \hat{u}_d \text{ in } \Omega^-, \\
\dfrac{\partial \hat{u}^+}{\partial x_3} = \hat{\varrho}_2, \ \dfrac{\partial \hat{v}^+}{\partial x_3} = 0 \ \text{ on } \gamma_u, \\
\hat{u}^+ = \hat{u}^-, \ \dfrac{\partial \hat{u}^-}{\partial x_3} - h(M_0)\dfrac{\partial \hat{u}^+}{\partial x_3} = \hat{\varrho}_1 \text{ on } \gamma_c, \\
\hat{v}^+ = \hat{v}^-, \ h(M_0)\dfrac{\partial \hat{v}^+}{\partial x_3} - \dfrac{\partial \hat{v}^-}{\partial x_3} = 0 \text{ on } \gamma_c, \\
\hat{\varrho}_1 = -\dfrac{|Y_{M_0}|}{\beta}\displaystyle\int_{[0,1]^2} \hat{v}(z', M_0)dz', \\
\hat{\varrho}_2 = -\dfrac{|Y_{M_1}|}{\beta}\displaystyle\int_{[0,1]^2} \hat{v}(z', M_1)dz', \\
\hat{u}^-, \hat{v}^- = 0, \ \text{ on } \gamma_b, \text{ and } \hat{u}, \hat{v} \text{ are } \gamma_s \text{ periodic.}
\end{cases}
\tag{5.33}
$$

Then, $(\hat{u}, \hat{\varrho}_1, \hat{\varrho}_2)$ is the solution of the minimization problem (5.29)

Proof of Theorem 5.5 follows in the same fashion as Theorem 5.3.
The main convergence results are stated below.

Theorem 5.6. (For $\alpha = 1$) Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ and $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2)$ be the solution of the problem (2.3) and (5.15) respectively, then,

$$
\begin{aligned}
&\widetilde{\bar{u}^+_\varepsilon} \rightharpoonup h(x_3)\bar{u}^+ \text{ weakly in } L^2((0,1)^2; H^1(M_0, M_1)), \\
&\widetilde{\bar{v}^+_\varepsilon} \rightharpoonup h(x_3)\bar{v}^+ \text{ weakly in } L^2((0,1)^2; H^1(M_0, M_1)), \\
&\bar{u}^-_\varepsilon \rightharpoonup \bar{u}^- \text{ weakly in } H^1(\Omega^-), \\
&\bar{v}^-_\varepsilon \rightharpoonup \bar{v}^- \text{ weakly in } H^1(\Omega^-), \\
&\langle \hat{\bar{\theta}}_\varepsilon, \phi \rangle_{L^2(\gamma_\varepsilon)} \to \langle \Theta, \phi \rangle \ \text{ for all } \ \phi \in H^1(\Omega^+),
\end{aligned}
\tag{5.34}
$$

where,

$$
\langle \Theta, \phi \rangle = \int_{(0,1)^2} \bar{\varrho}_1 \phi(x', M_0)dx' + \int_{(0,1)^2} \bar{\varrho}_2 \phi(x', M_1)dx' + \int_{\Omega^+} \bar{\theta}(x_3)\phi(x)dx.
$$

Here, for $\phi \in L^2(\Omega^+_\varepsilon)$, $\tilde{\phi}$ is the extension of ϕ by 0 to the domain Ω^+.

Proof. As $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ is the optimal solution for the optimal control problem (2.3). Using Lemma 4.5 we get the estimate $\|\bar{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C$, where $C > 0$ is a constant independent of ε.

Similarly, we bound the adjoint state, that is $\exists\, C > 0$ such that $\|\bar{v}_\varepsilon\| \leqslant C$. Then it follows from Theorem 4.1 in[3] that up to a sub-sequence $\exists\, u_0, v_0 \in W(\Omega)$ such that

$$
\begin{aligned}
&\widetilde{\bar{u}^+_\varepsilon} \rightharpoonup h(x_3)u_0^+, \ \widetilde{\bar{v}^+_\varepsilon} \rightharpoonup h(x_3)v_0^+ \text{ weakly in } L^2(\Omega^+), \\
&\widetilde{\frac{\partial \bar{u}^+_\varepsilon}{\partial x_3}} \rightharpoonup h(x_3)\frac{\partial u_0^+}{\partial x_3}, \ \widetilde{\frac{\partial \bar{v}^+_\varepsilon}{\partial x_3}} \rightharpoonup h(x_3)\frac{\partial v_0^+}{\partial x_2} \ \text{ weakly in } L^2(\Omega^+), \\
&\widetilde{\frac{\partial \bar{u}^+_\varepsilon}{\partial x_i}} \rightharpoonup 0, \ \widetilde{\frac{\partial \bar{v}^+_\varepsilon}{\partial x_i}} \rightharpoonup 0 \text{ weakly in } L^2(\Omega^+) \text{ for } i = 1, 2, \\
&\bar{u}^-_\varepsilon \rightharpoonup u_0^-, \ \bar{v}^-_\varepsilon \rightharpoonup v_0^- \text{ weakly in } \ H^1(\Omega^-).
\end{aligned}
\tag{5.35}
$$

Take $\phi \in C^\infty(\Omega) \cap W(\Omega)$ as a test function in the variational formulation of (2.2).

$$\int_{\Omega^-} \nabla \bar{u}_\varepsilon^- \nabla \phi dx + \int_{\Omega_\varepsilon^+} \nabla \bar{u}_\varepsilon^+ \nabla \phi dx + \int_{\Omega^-} \bar{u}_\varepsilon^- \phi dx + \int_{\Omega_\varepsilon^+} \bar{u}_\varepsilon^+ \phi dx = \int_{\Omega_\varepsilon^+} f \phi dx + \int_{\Omega_\varepsilon^-} f \phi dx + \int_{\gamma_\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi. \tag{5.36}$$

By the convergence results (5.35), we obtain

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\Omega^-} \nabla \bar{u}_\varepsilon^- \nabla \phi dx + \int_{\Omega_\varepsilon^+} \nabla \bar{u}_\varepsilon^+ \nabla \phi dx + \int_{\Omega^-} \bar{u}_\varepsilon^- \phi dx + \int_{\Omega_\varepsilon^+} \bar{u}_\varepsilon^+ \phi dx \\ = \int_{\Omega^+} h(x_3) \left(\frac{\partial u_0^+}{\partial x_3} \frac{\partial \phi^+}{\partial x_3} + u_0^+ \phi^+ \right) dx + \int_{\Omega^-} (\nabla u_0^- \nabla \phi^- + u_0^- \phi^-) dx \end{aligned} \tag{5.37}$$

and

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f \phi dx = \int_{\Omega_+} h(x_3) f^+ \phi^+ dx + \int_{\Omega^-} f^- \phi^- dx. \tag{5.38}$$

The last integral in (5.36) can be written as

$$\int_{\gamma_\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi = \int_{F_{M_0}^\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi + \int_{F_{M_1}^\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi + \int_{\bigcup_{k=1}^N F_{t_k}^\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi + \int_{S^\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi.$$

By a similar calculation as in Lemma 4.5, we have the following:

$$\begin{aligned} \int_{F_{M_0}^\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi &= \int_{Y_{M_0}} \left\{ \int_{[0,1]^2} T_1^\varepsilon(\phi)(x', M_0, y') dx' \right\} \bar{\theta}(y', M_0) dy' \\ \int_{F_{M_1}^\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi &= \int_{Y_{M_1}} \left\{ \int_{[0,1]^2} T_2^\varepsilon(\phi)(x', M_1, y') dx' \right\} \bar{\theta}(y', M_1) dy' \\ \int_{\bigcup_{k=1}^N F_{t_k}} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi &= \int_{\bigcup_{k=1}^N Y_{t_k}} \left\{ \varepsilon^\alpha \int_{[0,1]^2} T_{t_k}^\varepsilon(\phi)(x', t_k, y') dx' \right\} \bar{\theta}(y', t_k) dy'. \end{aligned} \tag{5.39}$$

We can write the integral $\int_{S^\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi$ using Theorem 5.1 as follows.

$$\begin{aligned} &\int_{S^\varepsilon} \hat{\bar{\theta}}_\varepsilon^\varepsilon \phi \\ &= - \frac{\varepsilon^{\alpha-1}}{\beta} \int_{[0,1]^2} \int_S |\nabla_{y'} \eta|(y') \left(\int_{[0,1]^2} T^\varepsilon \bar{v}_\varepsilon(z', x_3, y') dz' \right) T^\varepsilon \phi(x', x_3, y') ds dx', \end{aligned} \tag{5.40}$$

If we pass to the limit as $\varepsilon \to 0$, by Lemma 4.3, we have

$$\lim_{\varepsilon \to 0} \int_{S^\varepsilon} \hat{\bar{\theta}}_\varepsilon \phi = -\frac{\delta_{\alpha 1}}{\beta} \int_{[0,1]^2} \int_S |\nabla_{y'} \eta|(y') \left(\int_{[0,1]^2} v_0(z', x_3) dz' \right) \phi(x', x_3) ds dx', \tag{5.41}$$

where $\delta_{11} = 1$ and $\delta_{\alpha 1} = 0$ if $\alpha > 1$. Using the slicing lemma for Haudorff measure (see Lemma 7.6.1 in [29]), we can rewrite the above integral as,

$$-\frac{\delta_{\alpha 1}}{\beta}\int_{[0,1]^2}\int_{M_0}^{M_1}\int_{\gamma_{x_3}}\sqrt{1+|\nabla_{y'}\eta|^2(y')}\left(\int_{[0,1]^2} v_0(z',x_3)dz'\right)\phi(x',x_3)d\gamma_{x_3}(y')dx_3dx'$$
$$= -\frac{\delta_{\alpha 1}}{\beta}\int_{\Omega^+}|\omega(x_3)|\left(\int_{[0,1]^2} v_0(z',x_3)dz\right)\phi(x',x_3)dx_1dx_2dx_3 \tag{5.42}$$

where $\omega(x_3) = \int_{\gamma_{x_3}}\sqrt{1+|\nabla_{x'}\eta|^2(x')}d\gamma_{x_3}(x')$ and curve γ_{x_3} is given by $\{(y_1,y_2)\big|\ \eta(y_1,y_2)=x_3\}$.

From Theorem 5.1, using the characterization of optimal control, Proposition 3.4 and Lemma 4.3 we get

$$\lim_{\varepsilon\to 0}\int_{F^\varepsilon_{M_i}}\hat{\hat{\theta}}_\varepsilon\phi = -\frac{1}{\beta}\int_{Y_{M_i}}\int_{[0,1]^2}\left(\int_{[0,1]^2} v_0(z',M_i)dz'\right)\phi(x',M_i)dx'dy'$$
$$= -\frac{|Y_{M_i}|}{\beta}\int_{[0,1]^2}\left(\int_{[0,1]^2} v_0(z',M_i)dz'\right)\phi(x',M_i)dx' \tag{5.43}$$

There will be no contribution from the other flat parts because of the scaling factor. By combining the equations (5.37), (5.38), (5.41), (5.43), we get,

$$\int_{\Omega^+} h(x_3)\left(\frac{\partial u_0^+}{\partial x_3}\frac{\partial \phi^+}{\partial x_3} + u_0^+\phi^+\right)dx + \int_{\Omega^-}(\nabla u_0^-\nabla\phi^- + u_0^-\phi^-)dx$$
$$= \int_{\Omega_+} h(x_3)f^+\phi^+dx + \int_{\Omega^-} f^-\phi^-dx$$
$$- \frac{|Y_{M_0}|}{\beta}\int_{[0,1]^2}\left(\int_{[0,1]^2} v_0(z',M_0)dz'\right)\phi(x',M_0)dx'dy'$$
$$- \frac{|Y_{M_1}|}{\beta}\int_{[0,1]^2}\left(\int_{[0,1]^2} v_0(z',M_1)dz'\right)\phi(x',M_1)dx'dy'$$
$$- \frac{1}{\beta}\int_{\Omega^+}|\omega(x_3)|\left(\int_{[0,1]^2} v_0(z',x_3)dz\right)\phi(x',x_3)\,dx. \tag{5.44}$$

In a similar fashion, by passing to the limit as $\varepsilon \to 0$ in the variational formulation of PDE (5.1), we get

$$\int_{\Omega^+} h(x_3)\left(\frac{\partial v_0^+}{\partial x_3}\frac{\partial \phi^+}{\partial x_3} + v_0^+\phi^+\right)dx + \int_{\Omega^-}(\nabla v_0^-\nabla\phi^- + v_0^-\phi^-)\,dx = \int_{\Omega^+} h(x_3)(u_0^+ - u_d)\phi^+\,dx + \int_{\Omega^-}(u_0^- - u_d)\phi^-\,dx \tag{5.45}$$

Combining the relation (5.44), and (5.45), we get

$$
\begin{aligned}
&\int_{\Omega^+} h(x_3)\left(\frac{\partial u_0^+}{\partial x_3}\frac{\partial \phi^+}{\partial x_3} + u_0^+\phi^+\right)dx + \int_{\Omega^-}(\nabla u_0^- \nabla \phi^- + u_0^-\phi^-)dx \\
&\qquad + \int_{\Omega^+} h(x_3)\left(\frac{\partial v_0^+}{\partial x_3}\frac{\partial \phi^+}{\partial x_3} + v_0^+\phi^+\right)dx + \int_{\Omega^-}(\nabla v_0^- \nabla \phi^- + v_0^-\phi^-)dx \\
&\qquad = \int_{\Omega_+} h(x_3) f^+\phi^+ dx + \int_{\Omega^-} f^-\phi^- dx \\
&\qquad - \frac{|Y_{M_0}|}{\beta}\int_{[0,1]^2}\left(\int_{[0,1]^2} v_0(z', M_0)dz'\right)\phi(x', M_0)dx'dy' \\
&\qquad - \frac{|Y_{M_1}|}{\beta}\int_{[0,1]^2}\left(\int_{[0,1]^2} v_0(z', M_1)dz'\right)\phi(x', M_1)dx'dy' \\
&\qquad - \frac{1}{\beta}\int_{\Omega^+}|\omega(x_3)|\left(\int_{[0,1]^2} v_0(z', x_3)dz\right)\phi(x', x_3)dx \\
&\qquad + \int_{\Omega^+} h(x_3)(u_0^+ - u_d)\phi^+ dx + \int_{\Omega^-}(u_0^- - u_d)\phi^- dx,
\end{aligned}
\tag{5.46}
$$

for all $\phi \in C^\infty(\Omega) \cap W(\Omega)$. Thus, the above relation holds for all $\phi \in W(\Omega)$ as $C^\infty(\Omega)$ is dense in $W(\Omega)$ Hence, this shows that (u_0, v_0) is a solution of the system (5.33). As the optimal control problem admits unique solution, $u_0 = \bar{u},\ v_0 = \bar{v}$. Hence, we get the convergence of the whole sequence. This completes the proof. □

Theorem 5.7. (For $\alpha > 1$) Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ and $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2)$ be the solution of the problem (2.3) and (5.29) respectively, then,

$$
\begin{aligned}
&\widetilde{\bar{u}_\varepsilon^+} \rightharpoonup h(x_3)\bar{u}^+ \ weakly \text{ in } L^2((0,1)^2; H^1(M_0, M_1)), \\
&\widetilde{\bar{v}_\varepsilon^+} \rightharpoonup h(x_3)\bar{v}^+ \ weakly \text{ in } L^2((0,1)^2; H^1(M_0, M_1)), \\
&\bar{u}_\varepsilon^- \rightharpoonup \bar{u}^- \text{in } weakly\ H^1(\Omega^-), \\
&\bar{v}_\varepsilon^- \rightharpoonup \bar{v}^- \ weakly \text{ in } H^1(\Omega^-), \\
&\langle \hat{\bar{\theta}}_\varepsilon, \phi\rangle_{L^2(\gamma_\varepsilon)} \to \langle \Theta, \phi\rangle \ \text{ for all } \ \phi \in H^1(\Omega^+),
\end{aligned}
\tag{5.47}
$$

where,

$$
\langle \Theta, \phi\rangle = \int_{(0,1)^2} \bar{\varrho}_1 \phi(x', M_0)dx' + \int_{(0,1)^2} \bar{\varrho}_2 \phi(x', M_1)dx'.
$$

The proof of the above theorem is similar.

6 | OPTIMAL CONTROL PROBLEM IN BRANCHED STRUCTURE DOMAIN ($\alpha = 1$)

In this section, we will view our domain Ω_ε as a branched structure type domain as in Mel'nyk[32] and we derive the homogenized system on a multi-sheeted function space. Here, we follow the ideas introduced in S. Aiyappan and A. K. Nandakumaran[2], where they have considered an interior optimal control problem in a pillar type branched structure domain. Here we consider a boundary optimal control problem in a more general oscillating domain where as in[2] an interior control problem on a pillar type domain was analyzed. We divide the domain into number layers (sheets) and in each layer there are number of branches. By introducing unfolding operators in each branch at every level, we study the asymptotic behaviour of the optimal control problem and derive the limit problem in a multi-sheeted function space. Using multi-sheeted function space one can easily understand the contribution of the control applied on a specific branch at a particular level. Thus, it is useful in a need based control applications.

Here, we will fix some notations related to the branched structure. Let $\{M_r|\ 0 \le r \le k_0 + 1\}$ be the set of all local extremal values of η with $M_0 < M_1 < \cdots < M_{k_0} < M_{k_{0+1}}$. For $r = 0, 1, 2, \cdots, k_0$, we define $\Lambda_r = \{(x', x_3) \in \Lambda|\ \ M_r < x_3 < M_{r+1}\}$. Let

Λ_r has n_r number of connected components and are denoted as $\Lambda_{k,r}$ for $k \in \{1, 2, \cdots, n_r\}$. See Figure 3 for a model picture of the reference cell Λ with $k_0 = 4$ and $n_0 = 1$, $n_1 = 2$, $n_2 = 1$, $n_3 = 2$ and $n_4 = 1$. For $z \in (M_r, M_{r+1})$, the reference set $Y_{r,k}(z)$ is defined, analogous to $Y(z)$ as in Section 3, as $Y_{r,k}(z) = \{x' \in (0,1)^2 |\ (x', z) \in \Lambda_{r,k}\}$ and $h_{r,k}(z) = |Y_{r,k}(z)|$, the Lebesgue measure of $Y_{r,k}(z)$.

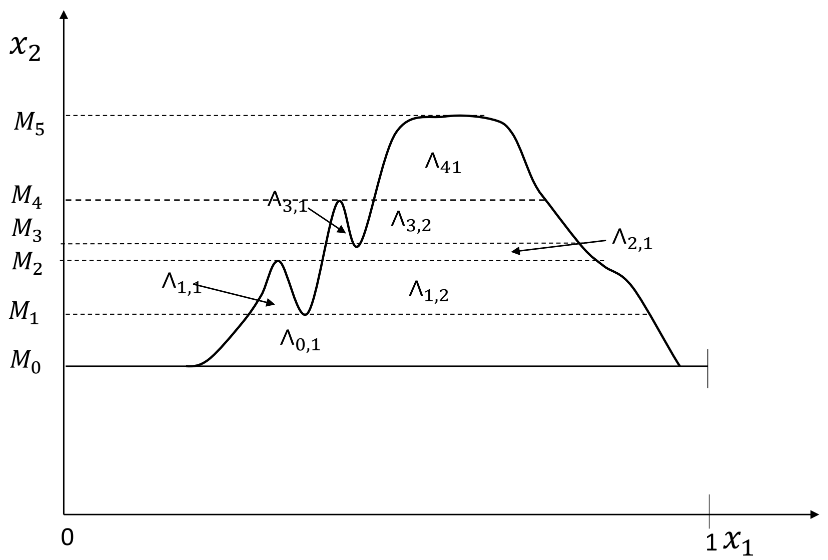

FIGURE 3 Reference cell Λ

We can write $\Lambda_r = \bigcup_{k=1}^{n_r} \Lambda_{k,r}$. Recall that $\varepsilon = \frac{1}{m}$ for $m \in \mathbb{N}$. Define $\Omega^\varepsilon_{r,k}$ for each level $r = 0, 1, 2, \cdots, k_0$ and every branch $k = 1, 2, \cdots, n_r$ as

$$\Omega^\varepsilon_{r,k} = \left\{ (\varepsilon(i,j) + \varepsilon y', x_3) \in \Omega^+_\varepsilon \middle| (y', x_3) \in \Lambda_{r,k},\ i, j = 0, 1, 2 \cdots m-1 \right\},$$

and $\Omega^\varepsilon_r = \bigcup_{k=1}^{n_r} \Omega^\varepsilon_{r,k}$. For each r, let $\Omega_r = \{(x', x_3) :\ \ x' \in (0,1)^2,\ x_3 \in (M_r, M_{r+1})\}$. The upper part of the limit domain Ω^+ can be written as $\Omega^+ = Interior \left\{ \bigcup_{r=0}^{k_0} \overline{\Omega}_r \right\}$ and for $r = 0, 1, \cdots, k_0 + 1$, $\gamma_r = \{(x', M_r) |\ x' \in (0,1)^2\}$ see Figure 4 .

To make the presentation simple, we assume that the surface $\partial\Lambda$ has no flat parts between $x_3 = M_0$ to $x_3 = M_{k_0+1}$. For each $r = 0, 1, 2, \cdots, k_0$, define $S_r = \{(x', \eta(x')) |\ x' \in (0,1)^2,\ M_r < \eta(x') < M_{r+1}\}$, $S_{r,k} = \partial\Lambda_{r,k} \cap S_r$, $S^\varepsilon_{r,k} = S^\varepsilon \cap \partial\Omega^\varepsilon_{r,k}$, and $\gamma^{r,k}_{x_3} = \gamma_{x_3}\big|_{S_{r,k}}$. Define the base flat boundary F_0 as $F_0 = \{(x', M_0) |\ x' \in (0,1)^2, \eta(x') = M_0\}$ and the top flat boundary $F_{k_0+1,k}$ as $F_{k_0+1,k} = \{x_3 = M_{k_0+1}\} \cap \partial\Lambda_{k_0,k}$. Denote Y_0 as the projection of F_0 on $(0,1)^2$ and $Y_{k_0+1,k}$, the projection of $F_{k_0+1,k}$ on $(0,1)^2$.

Now, we will define the unfolded domain corresponding to every branch at each level. For each pair (r,k), define $\mathscr{G}_{r,k} = \{(x_3, y') |\ x_3 \in (M_r, M_{r+1}),\ y' \in Y_{r,k}(x_3)\}$. The unfolded domains $\Omega^u_{r,k}$ are given by $\Omega^u_{r,k} = (0,1)^2 \times \mathscr{G}_{r,k}$. In other words, $\Omega^u_{r,k} = \{(x', x_3, y') |\ x' \in (0,1)^2,\ M_r < x_3 < M_{r+1}$ and $y' \in Y_{r,k}(x_3)\}$. Now, we define the unfolding operators for each level r and for each branch k where $r = 0, 1, 2, \cdots, k_0$, and $k = 1, 2, \cdots, n_r$.

Definition 6.1. The unfolding operator

$$T^\varepsilon_{r,k} : \{u : \Omega^\varepsilon_{r,k} \to \mathbb{R}\} \to \{T^\varepsilon_{r,k}(u) : \Omega^u_{r,k} \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon_{r,k}(u)(x', x_3, y') = u\left(\varepsilon \left[\frac{x'}{\varepsilon} \right] + \varepsilon y', x_3 \right).$$

We state some of the important properties without proof. The proofs are similar to that of Section 3.

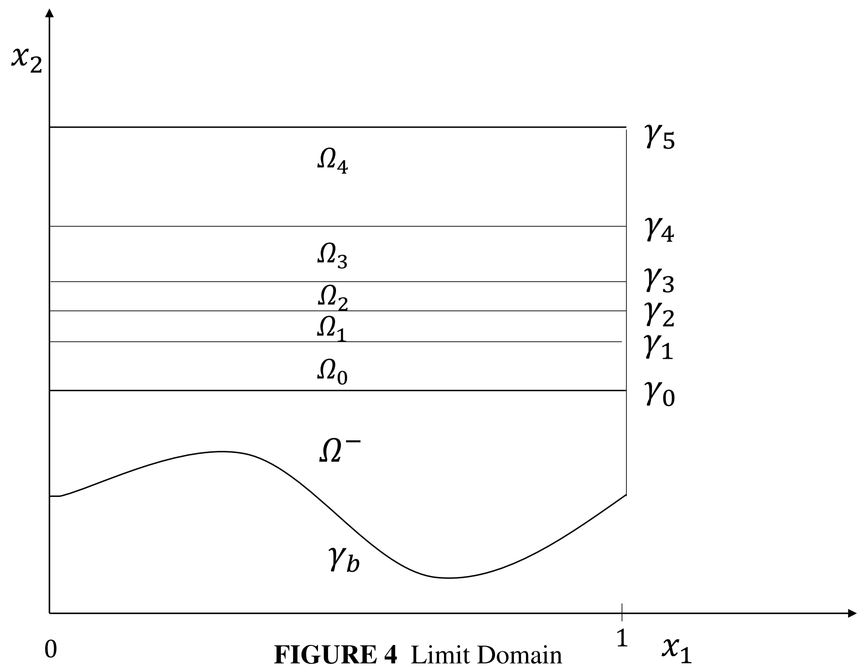

FIGURE 4 Limit Domain

Proposition 6.2. For each $\varepsilon > 0$, and r, k

(i) $T^{\varepsilon}_{r,k}$ is linear. Further, if $u,\ v : \Omega^{\varepsilon}_{r,k} \to \mathbb{R}$, then, $T^{\varepsilon}_{r,k}(uv) = T^{\varepsilon}_{r,k}(u)T^{\varepsilon}_{r,k}(v)$.

(ii) Let $u \in L^1(\Omega^{\varepsilon}_{r,k})$. Then,

$$\int_{\Omega^u_{r,k}} T^{\varepsilon}_{r,k}(u)dxdy' = \int_{\Omega^{\varepsilon}_{r,k}} udx.$$

(iii) Let $u \in L^2(\Omega^{\varepsilon}_{r,k})$. Then, $T^{\varepsilon}_{r,k}u \in L^2(\Omega^u_{r,k})$ and $\|T^{\varepsilon}_{r,k}u\|_{L^2(\Omega^u_{r,k})} = \|u\|_{L^2(\Omega^{\varepsilon}_{r,k})}$.

(iv) For $u \in H^1(\Omega^{\varepsilon}_{r,k})$, we have $T^{\varepsilon}_{r,k}(u) \in L^2((0,1)^2, H^1(\mathscr{G}_{r,k}))$. Moreover, $\frac{\partial}{\partial x_3}(T^{\varepsilon}_{r,k}u) = T^{\varepsilon}_{r,k}\left(\frac{\partial u}{\partial x_3}\right)$ and $\frac{\partial}{\partial y_i}T^{\varepsilon}_{r,k}u = \varepsilon T^{\varepsilon}_{r,k}\left(\frac{\partial u}{\partial x_i}\right)$, for $i = 1, 2$.

(v) For any $u \in L^2(\Omega^{\varepsilon}_{r,k})$, $T^{\varepsilon}_{r,k}u \to u$ strongly in $L^2(\Omega^u_{r,k})$. More generally, if $u_\varepsilon \to u$ strongly in $L^2(\Omega_r)$, then, $T^{\varepsilon}_{r,k}u \to u$ strongly in $L^2(\Omega^u_{r,k})$.

(vi) For any ϕ defined on $\Omega^{\varepsilon}_{r,k}$, we denote $\tilde{\phi}$, an extension by 0 to the domain Ω_r. Let, for every ε, $u_\varepsilon \in L^2(\Omega^{\varepsilon}_{r,k})$ be such that $T^{\varepsilon}_{r,k}u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega_{r,k}u)$. Then,

$$\tilde{u}_\varepsilon \rightharpoonup \int_{y' \in Y_{r,k}(x_3)} u(x', x_3, y')dy' \ \text{ weakly in } \ L^2(\Omega_r).$$

(vii) Let, for every $\varepsilon > 0$, $u_\varepsilon \in H^1(\Omega^{\varepsilon}_{r,k})$ be such that $T^{\varepsilon}_{r,k}u_\varepsilon \rightharpoonup u$ weakly in $L^2((0,1)^2, H^1(\mathscr{G}_{r,k}))$. Then, $\widetilde{u}_\varepsilon \rightharpoonup \int_{Y_{r,k}(x_3)} udy'$ and

$$\widetilde{\frac{\partial u_\varepsilon}{\partial x_3}} \rightharpoonup \int_{Y_{r,k}(x_3)} \frac{\partial u}{\partial x_3}dy' \text{ weakly in } L^2(\Omega_r).$$

Similarly, we can define the boundary unfolding operators as in earlier sections.

Here we will consider the same optimal control problem 2.3 as in Section 2 but F_{in} is an empty set, that is there is no flat surface between the planes $x_3 = M_0$ and $x_3 = M_{k_0+1}$. We describe the case for $\alpha = 1$. We skip here the other case $\alpha > 1$ as it is not interesting because there will be no contribution from the optimal control in the interior. The characterization of the optimal control will be given for each branch at every level using the corresponding unfolding operator.

The existence and uniqueness of the optimal control is given in Theorem 2.1. The following theorem gives the characterization of the optimal control via unfolding restricted to different branches at various level. We skip the proof as it is similar to that of Theorem 5.1.

Theorem 6.3. Given $f \in L^2(\Omega)$, and let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (2.3) and $\bar{v}_\varepsilon$ satisfies (5.1). Then, the optimal control $\bar{\theta}_\varepsilon \in L^2_{per}(\gamma)$ is given by

$$\begin{cases} \bar{\theta}_\varepsilon\Big|_{S_{r,k}} = -\dfrac{\sqrt{|\nabla_{y'}\eta|(y')}}{\beta} \displaystyle\int_{(0,1)^2} T^\varepsilon_{r,k}\bar{v}_\varepsilon(x',\eta(y'),y')dx' \; for\; 0 \le r \le k_0 \text{ and } 1 \le k \le n_r \\ \bar{\theta}_\varepsilon\Big|_{F_0} = -\dfrac{1}{\beta}\displaystyle\int_{(0,1)^2} T^\varepsilon_0 \bar{v}_\varepsilon(x', M_0, y')dx' \\ \bar{\theta}_\varepsilon\Big|_{F_{k_0+1,k}} = -\dfrac{1}{\beta}\displaystyle\int_{(0,1)^2} T^\varepsilon_{k_0,k}\bar{v}_\varepsilon(x', M_{k_0+1}, y')dx' \end{cases}$$

The converse as in Theorem 5.1 also holds.

6.1 | Limit Optimality System

Here we recall the following multi-sheeted function space $\mathcal{H}$ which was introduced by Mel'nyk in[32]. A function of the form

$$\phi = \begin{cases} \{\phi_{r,k} \;:\; k = 1,2,\cdots,n_r\} & \text{if } x \in \Omega_r, \;\; r = 0,1,2,\cdots,k_0, \\ \phi^- & \text{if } x \in \Omega^- \end{cases}$$

belongs to $\mathcal{H}$, if $\phi \in H^1(\Omega^-)$, for each r,k, $\phi_{r,k} \in L^2((0,1)^2, H^1(M_r, M_{r+1}))$, $\phi^- = \phi_{0,k}$ for $k = 1,2,\cdots,n_0$, and $\phi_{r,k'} = \phi_{r+1,k''}$ on γ_r if $\Lambda_{r,k'}$ and $\Lambda_{r+1,k''}$ shares an interface boundary γ_r, where $r \in \{1,2,\cdots,k_0\}$, $k' \in \{1,2,\cdots,n_r\}$, and $k'' \in \{1,2,\cdots,n_{r+1}\}$. Note that $\mathcal{H}$ is a Hilbert space with the following inner product[32,2].

$$(\phi,\psi)_{\mathcal{H}} = (\phi^-,\psi^-)_{L^2(\Omega^-)} + (\nabla\phi^-,\nabla\psi^-)_{L^2(\Omega^-)} + \sum_{r=0}^{k}\sum_{k=1}^{n_r}\left((\phi_{r,k},\psi_{r,k})_{L^2(\Omega_r)} + \left(\frac{\partial\phi_{r,k}}{\partial x_3}, \frac{\partial\psi_{r,k}}{\partial x_3}\right)_{L^2(\Omega_r)}\right)$$

For $f \in L^2(\Omega)$, for $r \in \{0,1,\cdots,k_0\}, k \in \{1,2,\cdots,n_r\}, \theta_{r,k} \in L^2(M_r, M_{r+1}), \varrho_{k_0+1,k}, \varrho_0 \in \mathbb{R}$ consider the following optimal control problem: find $(u, \bar{\theta}_{r,k}, \bar{\varrho}_{k_0+1,k}, \bar{\varrho}_0) \in \mathcal{H} \times L^2(M_r, M_{r+1}) \times \mathbb{R} \times \mathbb{R}$ such that

$$J(\bar{u}, \bar{\theta}_{r,k}, \bar{\varrho}_{k_0+1,k}, \bar{\varrho}_0) = \inf\{J(u, \theta_{r,k}, \varrho_{k_0+1,k}, \varrho_0)\} \tag{6.1}$$

where J is an L^2-cost functional given by

$$J(u,\theta) = \int_{\Omega^-} |u^- - u_d|^2 dx + \sum_{r=0}^{k_0}\sum_{k=1}^{n_r}\int_{\Omega_r} |h_{r,k}(x_3)u_{r,k} - u_d|^2 dx + \sum_{r=0}^{k_0}\sum_{k=1}^{n_r}\int_{\Omega_r} \frac{\beta}{2\omega_{r,k}(x_3)}|\theta_{r,k}|^2 dx$$
$$+ \sum_{k=1}^{n_{k_0}} \frac{\beta}{2|Y_{k_0+1},k|}\varrho^2_{k_0+1,k} + \frac{\beta}{2|Y_{M_0}|}\varrho_0^2.$$

and $u \in \mathcal{H}$ satisfies the following equation,

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(h_{r,k}(x_3)\dfrac{\partial u_{r,k}}{\partial x_3}\right) + u_{r,k} = f + \theta_{r,k} \ \text{ in } \Omega_r \\ -\Delta u^- + u^- = f \ \text{ in } \Omega^- \\ -\displaystyle\sum_{k=1}^{n_{k_0}} h_{r,k}(x_3)\dfrac{\partial u_{k_0,k}}{\partial x_3}(x', M_{k_0+1}) = \sum_{k=1}^{n_{k_0}} \varrho_{k_0+1,k} \ \text{ on } \gamma_{k_0+1} \\ \displaystyle\sum_{k=1}^{n_r} h_{r,k}(x_3)\dfrac{\partial u_{r,k}}{\partial x_3} - \sum_{k=1}^{n_{r+1}} h_{r,k}(x_3)\dfrac{\partial u_{r+1,k}}{\partial x_3} = 0 \ \text{ on } \gamma_{r+1} \text{ for } r \in \{0,1,\cdots,k_0-1\} \\ \displaystyle\sum_{k=1}^{n_0} h_{0,k}(x_3)\dfrac{\partial u_{0,k}}{\partial x_3} - \dfrac{\partial u^-}{\partial x_3} = \varrho_0 \ \text{ on } \gamma_0 \\ u^- = u_{0,k} \text{ on } \gamma_0 \text{ for } k = 1,2,\cdots,n_0, \text{ and } u_{r,k'} = u_{r+1,k''} \ \text{ on } \gamma_r \text{ if } \Lambda_{r,k'} \text{ and } \Lambda_{r+1,k''} \\ \text{shared interface boundary, where } r \in \{0,1,2,\cdots,k_0-1\}, \\ k' \in \{1,2,\cdots,n_r\},\ k'' \in \{1,2,\cdots,n_{r+1}\}. \\ u = 0 \text{ on } \gamma_b \text{ and } 1-\text{periodic in } x'. \end{cases} \tag{6.2}$$

The weak formulation to the above state equation is given as: find $u \in \mathcal{H}$ such that

$$\begin{aligned} &\sum_{r=0}^{k_0}\sum_{k=1}^{n_r}\int_{\Omega_r}\left(h_{r,k}(x_3)\frac{\partial u_{r,k}}{\partial x_3}\frac{\partial \phi_{r,k}}{\partial x_3} + u_{r,k}\phi_{r,k}\right)dx + \int_{\Omega^-}(\nabla u^-\nabla\phi^- + u^-\phi^-)dx \\ &\quad = \sum_{r=0}^{k_0}\sum_{k=1}^{n_r}\int_{\Omega_r} f\phi_{r,k}dx + \sum_{r=0}^{k_0}\sum_{k=1}^{n_r}\int_{\Omega_r}\theta_{r,k}\phi_{r,k}dx \\ &\quad + \int_{(0,1)^2}\varrho_0\phi^-(x', M_0)dx' + \sum_{k=1}^{n_{k_0}}\int_{(0,1)^2}\varrho_{k_0+1,k}\phi_{k_0,k}(x', M_{k_0+1,k})dx' \end{aligned}$$

for all $\phi \in \mathcal{H}$. Let $\bar{u} \in \mathcal{H}$ be the optimal state. Let us introduce the following adjoint equation

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(h_{r,k}(x_3)\dfrac{\partial \bar{v}_{r,k}}{\partial x_3}\right) + \bar{v}_{r,k} = \displaystyle\sum_{k=1}^{n_r}(\bar{u}_{r,k} - u_d) \ \text{ in } \Omega_r \\ -\Delta\bar{v}^- + \bar{v}^- = \bar{u}^- - u_d \ \text{ in } \Omega^- \\ -\displaystyle\sum_{k=1}^{n_{k_0}} h_{r,k}(x_3)\dfrac{\partial \bar{v}_{k_0,k}}{\partial x_3} = 0 \ \text{ on } \gamma_{k_0+1} \\ \displaystyle\sum_{k=1}^{n_r} h_{r,k}(x_3)\dfrac{\partial \bar{v}_{r,k}}{\partial x_3} - \sum_{k=1}^{n_{r+1}} h_{r,k}(x_3)\dfrac{\partial \bar{v}_{r+1,k}}{\partial x_3} = 0 \ \text{ on } \gamma_{r+1} \text{ for } r \in \{0,1,\cdots,k_0-1\} \\ \displaystyle\sum_{k=1}^{n_0} h_{0,k}(x_3)\dfrac{\partial \bar{v}_{0,k}}{\partial x_3} - \dfrac{\partial \bar{v}^-}{\partial x_3} = 0 \\ \bar{v}^- = \bar{v}_{0,k} \text{ on } \gamma_0 \text{ for } k = 1,2,\cdots,n_0, \text{ and } v_{r,k'} = v_{r+1,k''} \ \text{ on } \gamma_r \text{ if } \Lambda_{r,k'} \text{ and } \Lambda_{r+1,k''} \\ \text{shared interface boundary, where } r \in \{0,1,2,\cdots,k_0-1\}, \\ k' \in \{1,2,\cdots,n_r\},\ k'' \in \{1,2,\cdots,n_{r+1}\}. \\ \bar{u}^- = 0 \text{ on } \gamma_b \text{ and } 1-\text{periodic in } x' \end{cases} \tag{6.3}$$

Using the unfolding operators that we have defined at various levels and in different branches we characterize the optimal control in the following theorem. The proof can be written in a similar fashion as in[2].

Theorem 6.4. The optimal control problem 6.1 admits a unique solution and the optimal controls $(\bar{\theta}_{r,k}, \bar{\varrho}_{k_0+1,k}, \bar{\varrho}_0)$ can be characterize as

$$\bar{\theta}_{r,k} = -\frac{1}{\beta}\omega_{r,k}(x_3) \int_{(0,1)^2} \bar{v}_{r,k}(z', x_3)dz' \text{ where } r = 0, 1, 2, \cdots, k_0, \text{and } k = 1, 2, \cdots, n_r.$$

$$\bar{\varrho}_{k_0+1,k} = -\frac{|Y_{k_0+1,k}|}{\beta}\left[\int_{(0,1)^2} \bar{v}_{k_0,k}(z', M_{k_0+1})dz'\right] \text{ for } k = 1, 2, \cdots, n_{k_0}$$

$$\bar{\varrho}_0 = -\frac{|Y_{M_0}|}{\beta} \int_{(0,1)^2} \bar{v}_{0,k}(z', M_0)dz'.$$

The following theorem states the main convergence result. The proof is similar to that of Theorem 5.6.

Theorem 6.5. Let $\bar{u}_\varepsilon$, $\bar{v}_\varepsilon$ defined as earlier, then the following convergence result holds. For $r \in \{0, 1, 2, \cdots, k_0\}$ and corresponding $k \in \{1, 2, \cdots, n_r\}$,

$$\begin{aligned}
&\widetilde{\bar{u}_\varepsilon|_{\Omega^\varepsilon_{r,k}}} \rightharpoonup h_{r,k}(x_3)\bar{u}_{r,k} \text{ in } L^2((0,1)^2, H^1(M_r, M_{r+1})),\\
&\bar{u}^-_\varepsilon \rightharpoonup \bar{u}^- \text{ in } H^1(\Omega^-),\\
&\widetilde{\bar{v}_\varepsilon|_{\Omega^\varepsilon_{r,k}}} \rightharpoonup h_{r,k}(x_3)\bar{v}_{r,k} \text{ in } L^2((0,1)^2, H^1(M_r, M_{r+1})),\\
&\bar{v}^-_\varepsilon \rightharpoonup \bar{v}^- \text{ in } H^1(\Omega^-).
\end{aligned}$$

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