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Control Problem on a Rough Circular Domain and Homogenization

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Abstract. This paper is concerned with the asymptotic analysis of optimal control problems posed on a rough circular domain. The domain has two parts, namely a fixed outer part and an oscillating inner part. The period of the oscillation is of order $\varepsilon > 0$, a small parameter which approaches zero and the amplitude of the oscillation is fixed. We pose a periodic control on the oscillating part of the domain and study the homogenization of this problem using an unfolding operator suitably defined for this domain. One of the novelties of this paper is that we use the unfolding operator to characterize the optimal control in the non-homogenized level.

Keywords: Homogenization, Optimal control, Oscillating boundary, Unfolding operator, Rough circular domain

1. Introduction

In this article, we consider a rapidly oscillating circular domain which models domains from applications. For example, in jet engines, the fan models such a circular domain where the blades of the fan occupy the oscillatory part. Here, the base of each blade is small compared to the domain whereas the height of the fan is of $O(1)$ with respect to the domain. When the fan rotates at a high speed, turbulence can occur. This is one of the importance of studying control problems associated to fluid flow in varying oscillating domains. It leads to homogenization problems since the base of each blade is small, say of order ε. Another interesting example is the heat radiator where there are creases or folds made of conducting metal that heats up surrounding air. Attempting to do fluid flow problems is bit too ambitious so we start with a simple problem, but in the complex circular domains. To our knowledge, the study of homogenization problems in circular oscillating domains, in particular, those of order 1 amplitude, is very limited. But, there is a large number of literature in rectangular domains. Such problems are categorized as rough (rugous/oscillating) boundary problems and this attracts many fields of research such as aerodynamics, hemodynamics, and fluid dynamics, to name a few.

There is less research going on regarding the study of homogenization of problems in domains with oscillating smooth boundaries. For instance, Brizzi and Chalot [9] considered boundary homogenization with Neumann boundary condition. In [5, 6], Arrieta and Villanueva-Pesqueira posed a homogenization

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problem in a thin domain with smooth oscillating boundary. Recently, Aiyappan, Nandakumaran and Prakash [2] published a paper on a generalized unfolding method for highly oscillating smooth boundary domain and used it for homogenization.

On the other hand, there are lots of activities on domains with non-smooth oscillating boundaries, more specifically, domains with a fixed part and a lot of thin periodically distributed parts (like pillars) attached along certain part of the flat boundary. The study of the asymptotic analysis and error estimates of an elliptic problem posed on a rectangular rough domain was studied by Amirat, Bodart, De Maio and Gaudiello in [4] while the homogenization of PDEs in oscillating domain using Tartar's oscillating test functions has been investigated by Blanchard and Gaudiello in [7] and by Blanchard, Gaudiello and Mel'nyk in [8]. In [21], Gaudiello and Sili considered strongly contrasting diffusivity problem in highly oscillating boundaries. On the other hand, Corbo Esposito, Donato, Gaudiello and Picard have studied the asymptotic analysis of a p-Laplacian operator using Γ-convergence in [17]. Homogenization of an elliptic problem with homogeneous Neumann data has been studied by Gaudiello and Guibé in [19]. Gaudiello, in [18], investigated Laplace equation with inhomogeneous Neumann boundary condition posed on oscillating boundary domain and in [20], using extension operators, Gaudiello, Hadiji and Picard have studied the homogenization of Ginzburg-Landau equation. Exact controllability problems in oscillating domains have been investigated by De Maio and Nandakumaran in [12] and by De Maio, Nandakumaran and Perugia in [13]. For an introduction to homogenization, one can look into [10]. For literature on homogenization of optimal control problems on this type of domains one can refer to [3, 11, 14–16, 23, 24, 27, 28].

In our present work, we analyze a control problem posed on a domain whose oscillating boundary is given by arbitrary reference function η. By changing the reference function η we can get various rough domains.

More precisely, we consider a standard optimal control problem with two types of cost functionals, namely an L^2 cost functional

$$J_\varepsilon(y_\varepsilon, q) = \frac{1}{2}\int_{\mathcal{O}_\varepsilon} |y_\varepsilon - y_d|^2 r dr d\theta + \frac{\beta}{2}\int_{\mathcal{O}_\varepsilon^+} |q^\varepsilon|^2 r dr d\theta$$

and a Dirichlet cost functional

$$G_\varepsilon(y_\varepsilon, q) = \frac{1}{2}\int_{\mathcal{O}_\varepsilon} |\nabla y_\varepsilon - \nabla y_d|^2 r dr d\theta + \frac{\beta}{2}\int_{\mathcal{O}_\varepsilon^+} |q^\varepsilon|^2 r dr d\theta$$

where $q^\varepsilon(r,\theta) = q(r, \frac{\theta}{\varepsilon})$ and the state y_ε satisfies an elliptic problem posed on this oscillating domain $\mathcal{O}_\varepsilon$ given by

$$\begin{cases} -\Delta y_\varepsilon + y_\varepsilon = f_0 + \chi_{\mathcal{O}_\varepsilon^+} q^\varepsilon & \text{in } \mathcal{O}_\varepsilon, \\ \partial_\nu y_\varepsilon = 0 & \text{on } \partial\mathcal{O}_\varepsilon. \end{cases}$$

We apply a periodic control q^ε on the oscillating part of the domain $\mathcal{O}_\varepsilon^+$ which comes from q and study the homogenization of the optimal control problem by passing to the limit in the optimality system. For the asymptotic analysis, we use the unfolding operator for polar coordinates developed by Aiyappan, Nandakumaran and Prakash in [2] for these types of domains. The unfolding operator has been used cleverly to characterize the optimal control in non-homogenized level itself.

We now outline the contents of this paper. In section 2, we explain the oscillating domain $\mathcal{O}_\varepsilon$. In Section 3, we recall the unfolding operator and its properties for circular oscillating domains. The optimal control problem with the L^2 cost functional has been described in Section 4. One of our main result, namely the characterization of the optimal control via unfolding, has been derived in this section (see Theorem 4.2). The main convergence results of the optimal control problem (see Theorem 5.4) and discussion on the limit control problem are available in Section 5. Section 6 contains the convergence results corresponding to the Dirichlet cost functional.

2. The Oscillating Circular Boundary Domain

In this section, we explain a circular domain whose boundary is highly oscillating. Literature regarding homogenization problems on circular domains is limited (see [22, 26, 30]). In [22], Madureira and Valentin considered a Poisson problem where the amplitude of the oscillations is of order ε while in [26, 30], studied homogenization problems on a domain with highly oscillating interfaces. In our case, we consider oscillations of $O(1)$.

For a small parameter $\varepsilon = \frac{2\pi}{N}$, $N \in \mathbb{Z}^+$, we consider an oscillating boundary domain $\mathcal{O}_\varepsilon$ as given in Figure 1.

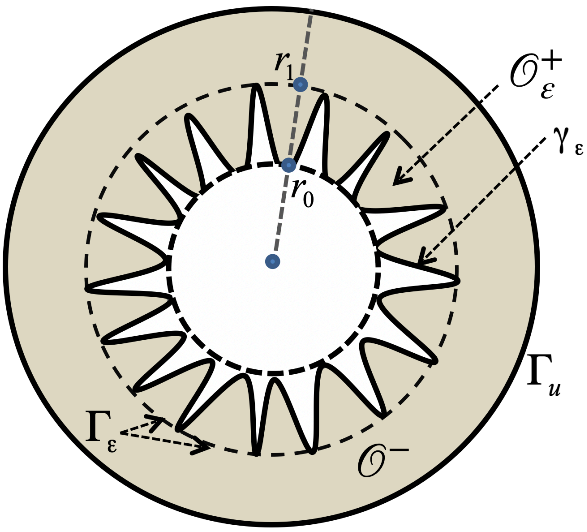

Fig. 1. Circular Oscillating Domain $\mathcal{O}_\varepsilon$

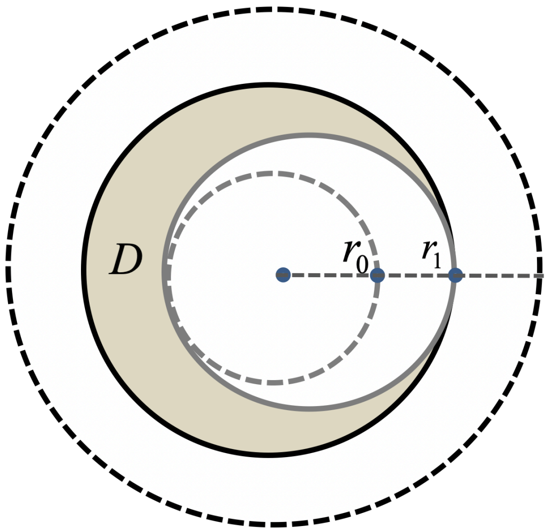

Fig. 2. Reference Domain D

We describe the domain $\mathcal{O}_\varepsilon$ and its boundaries as follows. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth and periodic function with period 2π and η be a smooth real valued function defined on $[0, 2\pi]$ such that it takes the maximum at the end points, that is, $\eta(0) = \eta(2\pi) = r_1 =: \max_{\theta\in[0,2\pi]} \eta(\theta)$. Also assume that the function $r_1 - \eta$ is compactly supported in $(0, 2\pi)$. Now extend η to the whole real line periodically with period 2π.

Let $r_0 =: \min_{\theta\in[0,2\pi]} \eta(\theta)$ and $\mathfrak{m} < \min_{\theta\in[0,2\pi]} g(\theta)$ with $0 < r_0 < r_1 < \mathfrak{m}$. Now, define the domain $\mathcal{O}_\varepsilon$ as

$$\mathcal{O}_\varepsilon = \{(r,\theta) \in \mathbb{R}^2 : \ 0 < \theta \leqslant 2\pi, \ \eta_\varepsilon(\theta) = \eta\left(\tfrac{\theta}{\varepsilon}\right) < r < g(\theta)\}.$$

Typically, $\mathcal{O}_\varepsilon$ consists of an annulus type region bounded by the inner circle of radius r_1 and outer boundary given by g and an oscillating region bounded by the outer circle of radius r_1 and the oscillating inner boundary defined by η_ε. The oscillating inner boundary of $\mathcal{O}_\varepsilon$ denoted by γ_ε is given by $\gamma_\varepsilon = \{(r,\theta) : \ \theta \in [0, 2\pi], \ r = \eta_\varepsilon(\theta)\}$. The fixed outer boundary Γ_u of $\mathcal{O}_\varepsilon$ is defined by $\Gamma_u = \{(r,\theta) : \ r = g(\theta), \theta \in [0, 2\pi]\}$. Let $\mathcal{O}_\varepsilon^+$ be the oscillating part of the domain $\mathcal{O}_\varepsilon$, which is $\mathcal{O}_\varepsilon^+ = \{(r,\theta) \in \mathbb{R}^2 \ : 0 < \theta \leqslant 2\pi, \eta_\varepsilon(\theta) < r < r_1\}$. The reference cell D is defined as $D =: \{(r,\theta) : \eta(\theta) < r < r_1, 0 < \theta \leqslant 2\pi\}$ and the reference set $Y(a)$ for $a \in [r_0, r_1]$, is defined as

$$Y(a) = \{\theta \in (0, 2\pi) : (a\cos\theta, a\sin\theta) \in \overline{D}\}.$$

In otherwords, for $a \in (r_0, r_1]$, $Y(a) = \{\theta \in (0, 2\pi) : \eta(\theta) < a\}$ and $Y(r_0) = \{\theta \in (0, 2\pi) : \eta(\theta) = r_0\}$. Note that $Y(a)$ is Lebesgue measurable as η is assumed to be a smooth function and $h(a) = |Y(a)|$, where $|Y(a)|$ is the Lebesgue measure of the set $Y(a)$. Defining $Y(a)$ in this novel way is crucial in the definition of the unfolding operators in circular oscillating domains. We choose η in such a way that h is strictly positive in $[r_0, r_1]$. Denote $\mathcal{O}^-$, the fixed part of the domain $\mathcal{O}_\varepsilon$, which is described by $\mathcal{O}^- = \{(r,\theta) : 0 < \theta \leqslant 2\pi, \ r_1 < r < g(\theta)\}$. The inner boundary of $\mathcal{O}^-$ denoted by Γ_0 is defined as $\Gamma_0 = \{(r_1,\theta) : \ 0 \leqslant \theta \leqslant 2\pi\}$. The common boundary Γ_ε is defined as $\Gamma_\varepsilon = \{(r,\theta) \in \mathcal{O}_\varepsilon : r = r_1\}$. We can also write $\mathcal{O}_\varepsilon$ as $\mathcal{O}_\varepsilon = Int\left(\overline{\mathcal{O}_\varepsilon^+ \cup \mathcal{O}^-}\right)$. The full domain or the limiting domain $\mathcal{O}$ is described by $\mathcal{O} = \{(r,\theta) : 0 < \theta \leqslant 2\pi, \ r_0 < r < g(\theta)\}$ and the inner limit domain $\mathcal{O}^+$ is given by $\mathcal{O}^+ = \{(r,\theta) : 0 < \theta \leqslant 2\pi, \ r_0 < r < r_1\}$. The boundaries of $\mathcal{O}$ are Γ_u and Γ_b, where $\Gamma_b = \{(r_0,\theta) : \ 0 \leqslant \theta \leqslant 2\pi\}$ and Γ_u is the same as defined earlier.

3. Unfolding Operator and its Properties

We now recall the relevant periodic unfolding operator T^ε and the boundary unfolding operator $T^\varepsilon_{r_1}$ which was developed by Aiyappan, Nandakumaran and Prakash in [2] for circular oscillating domain to study homogenization problems posed on this type of domain. We also present some of its important properties which are required for our analysis in Sections 5 and 6. Let us define the unfolded (fixed) domain $\mathcal{O}_u$, where the unfolded functions are defined, as below.

Let $\mathcal{G} = \{(r,\tau) : r \in (r_0, \ r_1), \tau \in Y(r)\}$, then $\mathcal{O}_u$ is defined as $\mathcal{O}_u = (0, 2\pi) \times \mathcal{G}$ and it can be written as

$$\mathcal{O}_u =: \{(r,\theta,\tau)| \ 0 < \theta < 2\pi, r_0 < r < r_1, \ \tau \in Y(r)\}.$$

For $x \in \mathbb{R}$, we write $[x]_{2\pi}$ as the integer part of x with respect to 2π, that is, $[x]_{2\pi} = 2k\pi$, where k is the largest integer such that $2k\pi \leqslant x$ and $\{x\}_{2\pi} = x - [x]_{2\pi}$. We now give the definition of unfolding operator for our domain. The unfolding operator has been used to understand the various scales present in a function u. Thus the relevance of T^ε comes into effect when T^ε is applied on functions u^ε with different scales x and $\frac{x}{\varepsilon}$, like $u^\varepsilon(x) = u(x, \frac{x}{\varepsilon})$. This is already available in the literature for rectangular type domains. In our case, the domain is circular and hence we define the unfolding operator in polar coordinates.

Definition 3.1. *(The unfolding operator) For each fixed $\varepsilon > 0$, the unfolding operator T^ε unfolds any function u defined on the oscillating domain $\mathcal{O}_\varepsilon^+$ into another function $(T^\varepsilon u)$ defined on the fixed domain $\mathcal{O}_u$. More precisely, the unfolding operator*

$$T^\varepsilon : \{u \mid u : \mathcal{O}_\varepsilon^+ \to \mathbb{R}\} \to \{v \mid v : \mathcal{O}_u \to \mathbb{R}\}$$

is defined by

$$(T^\varepsilon u)(r, \theta, \tau) = u\left(r, \varepsilon \left[\tfrac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right) \quad \forall\, u \in \mathcal{O}_\varepsilon^+.$$

If U is an open subset of $\mathbb{R}^2$ containing $\mathcal{O}_\varepsilon^+$ and u is a real valued function on U, $T^\varepsilon u$ will mean T^ε acting on the restriction of u to $\mathcal{O}_\varepsilon^+$. Some of the properties of T^ε are given below. Though the proofs can be found in [2], we recall them in the appendix for completeness.

Proposition 3.2. *For each fixed $\varepsilon > 0$, T^ε is linear and $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$, where $u, v : \mathcal{O}_\varepsilon^+ \to \mathbb{R}$.*

Proposition 3.3. *Let $u \in L^1(\mathcal{O}_\varepsilon^+)$. Then*

$$\int_{\mathcal{O}_u} T^\varepsilon u \; r dr d\theta d\tau = 2\pi \int_{\mathcal{O}_\varepsilon^+} u \; r dr d\theta.$$

Proposition 3.4. *Let $u \in L^2(\mathcal{O}_\varepsilon^+)$. Then $T^\varepsilon u \in L^2(\mathcal{O}_u)$ and $\|T^\varepsilon u\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\|u\|_{L^2(\mathcal{O}_\varepsilon^+)}$.*

Proposition 3.5. *Let $u, \dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta} \in L^2(\mathcal{O}_\varepsilon^+)$. Then $T^\varepsilon u, \dfrac{\partial}{\partial r}T^\varepsilon u, \dfrac{\partial}{\partial \tau}T^\varepsilon u \in L^2(\mathcal{O}_u)$. Moreover, $\dfrac{\partial}{\partial r}T^\varepsilon u = T^\varepsilon \dfrac{\partial u}{\partial r}$ and $\dfrac{\partial}{\partial \tau}T^\varepsilon u = \varepsilon T^\varepsilon \dfrac{\partial u}{\partial \theta}$.*

Proposition 3.6. *Let $u \in L^2(\mathcal{O}^+)$. Then $T^\varepsilon u \to u$ in $L^2(\mathcal{O}_u)$. More generally, if $y_\varepsilon \to y$ in $L^2(\mathcal{O}^+)$, then $T^\varepsilon y_\varepsilon \to y$ in $L^2(\mathcal{O}_u)$.*

Proposition 3.7. *For every $\varepsilon > 0$, let $y_\varepsilon \in L^2(\mathcal{O}_\varepsilon^+)$ such that $T^\varepsilon y_\varepsilon \rightharpoonup y$ weakly in $L^2(\mathcal{O}_u)$. Then*

$$\widetilde{y}_\varepsilon \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} y(r, \theta, \tau)\, d\tau$$

weakly in $L^2(\mathcal{O}^+)$. Here, $\widetilde{y}_\varepsilon$ is the zero extension of y_ε to $\mathcal{O}^+$.

We, now derive the convergence of unfolding for H^1 functions.

Proposition 3.8. *Let $y_\varepsilon \in H^1(\mathcal{O}_\varepsilon^+)$ for every $\varepsilon > 0$ such that $T^\varepsilon y_\varepsilon \rightharpoonup y$ and $\frac{\partial}{\partial r} T^\varepsilon y_\varepsilon \rightharpoonup \frac{\partial y}{\partial r}$ weakly in $L^2(\mathcal{O}_u)$. Then $\widetilde{y}_\varepsilon \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} y\, d\tau$ and $\widetilde{\frac{\partial y_\varepsilon}{\partial r}} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} \frac{\partial y}{\partial r}\, d\tau$ weakly in $L^2(\mathcal{O}^+)$.*

3.1. Unfolding on the Boundary

We, now define the boundary unfolding operator on Γ_ε, that is, on the common boundary of $\mathcal{O}_\varepsilon^+$ and $\mathcal{O}^-$.

Definition 3.9. *Let $\phi_{r_1}^\varepsilon : (0, 2\pi) \times Y(r_1) \to \Gamma_\varepsilon$ be defined by $(\theta, \tau) \to \varepsilon \left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau$. The ε-unfolding of a function $u : \Gamma_\varepsilon \to \mathbb{R}$ is the function $u \circ \phi_{r_1}^\varepsilon : (0, 2\pi) \times Y(r_1) \to \mathbb{R}$ denoted by $T_{r_1}^\varepsilon$. That is,*

$$T_{r_1}^\varepsilon : \{u \mid u : \Gamma_\varepsilon \to \mathbb{R}\} \to \{v \mid v : (0, 2\pi) \times Y(r_1) \to \mathbb{R}\}$$

by

$$T_{r_1}^\varepsilon u = u \circ \phi_{r_1}^\varepsilon = u\left(\varepsilon \left[\tfrac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right).$$

If U is an open subset of $\mathbb{R}^2$ such that $\Gamma_\varepsilon \subset U$ and $u : U \to \mathbb{R}$ then $T_{r_1}^\varepsilon u = T_{r_1}^\varepsilon (u|_{\Gamma_\varepsilon})$.

The properties of the boundary unfolding operator are given below without proof. In fact, all of them can be proved analogously as above.

Proposition 3.10. *(1) $T_{r_1}^\varepsilon$ is linear.*

(2) Let u,v be functions from $\Gamma_\varepsilon \to \mathbb{R}$. Then $T_{r_1}^\varepsilon(uv) = T_{r_1}^\varepsilon(u) T_{r_1}^\varepsilon(v)$.

(3) Let $u \in L^2(\Gamma_\varepsilon)$. Then $T_{r_1}^\varepsilon u \in L^2((0, 2\pi) \times Y(r_0))$. Moreover, $\|T_{r_1}^\varepsilon u\|_{L^2((0,2\pi)\times Y(r_0))} = \sqrt{2\pi}\|u\|_{L^2(\Gamma_\varepsilon)}$.

(4) Let $u \in H^1(\Gamma_\varepsilon)$. Then $T_{r_1}^\varepsilon u \in L^2(0, L; H^1(Y(r_0)))$ and $\frac{\partial}{\partial \tau} T_{r_1}^\varepsilon u = \varepsilon T_{r_1}^\varepsilon \frac{\partial u}{\partial \theta}$.

(5) Let $u \in L^2(0, 2\pi)$. Then $T_{r_1}^\varepsilon u \to u$ in $L^2((0, 2\pi) \times Y(r_1))$.

(6) Suppose that $u_\varepsilon \to u$ in $L^2(0, 2\pi)$. Then $T_{r_1}^\varepsilon u_\varepsilon \to u$ in $L^2((0, 2\pi) \times Y(r_1))$.

(7) Suppose that u_ε is a sequence in $L^2(\Gamma_\varepsilon)$ such that $T_{r_1}^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0, 2\pi) \times Y(r_1))$. Then $\widetilde{u}_\varepsilon \rightharpoonup \frac{1}{2\pi} \int_{Y(r_1)} u\, d\tau$ weakly in $L^2(0, 2\pi)$.

In the next section, we describe an optimal control problem posed on this oscillating domain and study the existence and uniqueness of its solution. Also, we use the unfolding operator which we have developed, to characterize the optimal control. This is one of the main contributions of this article.

4. Optimal Control Problem

We consider an interior optimal control problem where the controls are coming from the fixed reference cell D and periodically distributed over $\mathcal{O}_\varepsilon^+$. Suppose we have the elliptic system:

$$\begin{cases} -\Delta y_\varepsilon + y_\varepsilon = f_0 + \chi_{\mathcal{O}_\varepsilon^+} q^\varepsilon & \text{in } \mathcal{O}_\varepsilon, \\ \partial_\nu y_\varepsilon = 0 & \text{on } \partial\mathcal{O}_\varepsilon, \end{cases} \tag{4.1}$$

where $q \in L^2(D)$ with $q^\varepsilon(r,\theta) = q(r, \frac{\theta}{\varepsilon})$ and the source term $f_0 \in L^2(\mathcal{O})$. It is known that equation (4.1) admits a unique weak solution y_ε in $H^1(\mathcal{O}_\varepsilon)$ by applying the Lax-Milgram theorem. The solution operator is linear and continuous from $L^2(\mathcal{O}_\varepsilon) \times L^2(D)$ into $H^1(\mathcal{O}_\varepsilon)$, i.e.

$$\|y_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} \leqslant C \left(\|f_0\|_{L^2(\mathcal{O})} + \|q\|_{L^2(D)} \right), \tag{4.2}$$

where $C > 0$ is independent of ε. We define the L^2-cost functional J_ε as

$$J_\varepsilon(y_\varepsilon, q) = \frac{1}{2} \int_{\mathcal{O}_\varepsilon} |y_\varepsilon - y_d|^2 r dr d\theta + \frac{\beta}{2} \int_{\mathcal{O}_\varepsilon^+} |q^\varepsilon|^2 r dr d\theta \tag{4.3}$$

with the desired state $y_d \in L^2(\mathcal{O})$. Now, we define the optimal control problem as follows.

Find $(\overline{y}_\varepsilon, \overline{q}_\varepsilon) \in H^1(\mathcal{O}_\varepsilon) \times L^2(D)$ such that

$$(P_\varepsilon) \quad J_\varepsilon(\overline{y}_\varepsilon, \overline{q}_\varepsilon) = \inf \{ J_\varepsilon(y_\varepsilon, q) | (y_\varepsilon, q) \in H^1(\mathcal{O}_\varepsilon) \times L^2(D),\ (y_\varepsilon, q) \text{ satisfies } (4.1) \}.$$

In this section, we analyze the control problem with the L^2-cost functional. In Section 6, we study the homogenization of the control problem with a Dirichlet cost functional. We have the following existence result for each fixed $\varepsilon > 0$ (see Raymond [29]).

Theorem 4.1. *For each $\varepsilon > 0$, the minimization problem (P_ε) admits a unique solution.*

The focus of the following subsection is the derivation of the optimality system and characterization of the optimal control.

4.1. Optimality System

One of our main results is the derivation of the following optimality system and characterization of optimal control via the unfolding operator which is given in the following theorem.

Theorem 4.2. *Let $(\overline{y}_\varepsilon, \overline{q}_\varepsilon) \in H^1(\mathcal{O}_\varepsilon) \times L^2(D)$ be the optimal solution to (P_ε), then the optimal control is characterized by*

$$\overline{q}_\varepsilon(r,\tau)|_D = \frac{-1}{2\pi\beta} \int_0^{2\pi} (T^\varepsilon \overline{p}_\varepsilon^+)(r,\theta,\tau)\, d\theta \tag{4.4}$$

where $\overline{y}_\varepsilon$ satisfies (4.1) with $q^\varepsilon = \overline{q}_\varepsilon(r, \frac{\theta}{\varepsilon})$ and the adjoint state $\overline{p}_\varepsilon$ satisfies the problem

$$\begin{cases} -\Delta \overline{p}_\varepsilon + \overline{p}_\varepsilon = \overline{y}_\varepsilon - y_d & \text{in } \mathcal{O}_\varepsilon, \\ \partial_\nu \overline{p}_\varepsilon = 0 & \text{on } \partial\mathcal{O}_\varepsilon. \end{cases} \tag{4.5}$$

Conversely, if a pair $(\hat{y}_\varepsilon, \hat{p}_\varepsilon)$ satisfies the following system

$$\begin{cases} -\Delta \hat{y}_\varepsilon + \hat{y}_\varepsilon = f_0 + \chi_{\mathcal{O}_\varepsilon^+} \hat{Q}_\varepsilon; \ -\Delta \hat{p}_\varepsilon + \hat{p}_\varepsilon = \hat{y}_\varepsilon - y_d & \text{in } \mathcal{O}_\varepsilon, \\ \partial_\nu \hat{y}_\varepsilon = 0; \ \partial_\nu \hat{p}_\varepsilon = 0 & \text{on } \partial\mathcal{O}_\varepsilon, \\ \hat{q}_\varepsilon(r,\tau) = \dfrac{-1}{2\pi\beta} \displaystyle\int_0^{2\pi} T^\varepsilon \hat{p}_\varepsilon^+ \, d\theta & \text{on } D, \end{cases}$$

then the pair $(\hat{y}_\varepsilon, \hat{q}_\varepsilon)$ is the optimal solution to (P_ε). Here $\hat{Q}_\varepsilon(r,\theta) = \hat{q}_\varepsilon(r, \frac{\theta}{\varepsilon})$.

Proof. Given $q \in L^2(D)$, let $F_\varepsilon(q) = J_\varepsilon(y_\varepsilon(q), q)$, where $y_\varepsilon(q)$ is the solution to the equation (4.1). Set $\overline{Q}_\varepsilon(r,\theta) = \overline{q}_\varepsilon(r, \frac{\theta}{\varepsilon})$.

Using appropriate computation on $\frac{1}{\lambda}\left(F_\varepsilon(\overline{q}_\varepsilon + \lambda q) - F_\varepsilon(\overline{q}_\varepsilon)\right)$ and taking limit as $\lambda \to 0$, we get

$$F'_\varepsilon(\overline{q}_\varepsilon)q = \int_{\mathcal{O}_\varepsilon} (\overline{y}_\varepsilon - y_d) w_\varepsilon r dr d\theta + \beta \int_{\mathcal{O}_\varepsilon} \chi_{\mathcal{O}_\varepsilon^+} \overline{Q}_\varepsilon q^\varepsilon r dr d\theta,$$

where $q^\varepsilon(r,\theta) = q(r, \frac{\theta}{\varepsilon})$ and w_ε is the solution of the following equation

$$\begin{cases} -\Delta w_\varepsilon + w_\varepsilon = \chi_{\mathcal{O}_\varepsilon^+} q^\varepsilon & \text{in } \mathcal{O}_\varepsilon, \\ \partial_\nu w_\varepsilon = 0 & \text{on } \partial\mathcal{O}_\varepsilon. \end{cases} \tag{4.6}$$

We skip the computations involved and refer the reader to [1, 25] for detailed computations.

Since $(\overline{y}_\varepsilon, \overline{q}_\varepsilon)$ is an optimal solution to (P_ε), we have $F'_\varepsilon(\overline{q}_\varepsilon)q = 0$ for all $q \in L^2(D)$, it follows that

$$\int_{\mathcal{O}_\varepsilon} (\overline{y}_\varepsilon - y_d) w_\varepsilon r dr d\theta = -\beta \int_{\mathcal{O}_\varepsilon^+} \overline{Q}_\varepsilon \, q^\varepsilon r dr d\theta.$$

Using integration by parts in equations (4.5) and (4.6) with test functions w_ε and $\overline{p}_\varepsilon$, respectively, we get

$$\int_{\mathcal{O}_\varepsilon} (\overline{y}_\varepsilon - y_d) w_\varepsilon r dr d\theta = \int_{\mathcal{O}_\varepsilon^+} \overline{p}_\varepsilon^+ q^\varepsilon r dr d\theta \implies -\beta \int_{\mathcal{O}_\varepsilon^+} \overline{Q}_\varepsilon q^\varepsilon r dr d\theta = \int_{\mathcal{O}_\varepsilon^+} \overline{p}_\varepsilon^+ q^\varepsilon r dr d\theta, \quad \forall q \in L^2(D).$$

Now, note that $T^\varepsilon(\overline{Q}_\varepsilon)(r,\theta,\tau) = \overline{q}_\varepsilon(r,\tau)$ and $\overline{q}_\varepsilon|_D \rightharpoonup \overline{q}_0|_D$ weakly in $L^2(D)$. Applying the unfolding operator we get,

$$\int_{\mathcal{O}_\varepsilon^+} \overline{Q}_\varepsilon q^\varepsilon r dr d\theta = \frac{1}{2\pi} \int_{\mathcal{O}_u} T^\varepsilon \overline{Q}_\varepsilon T^\varepsilon q^\varepsilon r dr d\theta d\tau = \frac{1}{2\pi} \int_{\mathcal{O}_u} \overline{q}_\varepsilon q r dr d\theta d\tau = \int_D \overline{q}_\varepsilon q r dr d\tau.$$

Similarly,

$$\int_{\mathcal{O}_\varepsilon^+} \overline{p}_\varepsilon q^\varepsilon r dr d\theta = \frac{1}{2\pi}\int_{\mathcal{O}_u} T^\varepsilon \overline{p}_\varepsilon^+ T^\varepsilon q^\varepsilon r dr d\theta d\tau = \frac{1}{2\pi}\int_{\mathcal{O}_u} T^\varepsilon \overline{p}_\varepsilon^+ q r dr d\theta d\tau$$
$$= \int_D \left(\frac{1}{2\pi}\int_0^{2\pi} T^\varepsilon \overline{p}_\varepsilon^+ d\theta\right) q r dr d\tau.$$

Since this is valid for all $q \in L^2(D)$, we get

$$\overline{q}_\varepsilon = -\frac{1}{2\pi\beta}\left[\int_0^{2\pi} T^\varepsilon \overline{p}_\varepsilon^+ d\theta\right] \text{ a.e. in } D.$$

The converse can be proved easily with a reverse argument. □

The section that follows describes the homogenization of this control problem which is one of the main contributions of this paper.

5. Homogenization

Recall $h(r) = |Y(r)|$, where $|Y(r)|$ is the Lebesgue measure of the set $Y(r)$ at $r \in [r_0, r_1]$. Note that h is a strictly positive function in $[r_0, r_1]$. Let ψ be any function defined on $\mathcal{O}$, then ψ can be written as $\psi = \psi^+\chi_{\mathcal{O}^+} + \psi^-\chi_{\mathcal{O}^-}$ where $\psi^+ = \psi|_{\mathcal{O}^+}$ and $\psi^- = \psi|_{\mathcal{O}^-}$. Now, consider the anisotropic Sobolev space

$$W(\mathcal{O}) = \{\psi \in L^2(\mathcal{O}) : \ \frac{\partial\psi}{\partial r} \in L^2(\mathcal{O}),\ \psi^- \in H^1(\mathcal{O}^-)\ \}.$$

Note that $W(\mathcal{O})$ is a Hilbert space with the inner product

$$\langle u, v\rangle_W = \langle hu, v\rangle_{L^2(\mathcal{O}^+)} + \langle h\partial_r u, \partial_r v\rangle_{L^2(\mathcal{O}^+)} + \langle \partial_r u, \partial_r v\rangle_{L^2(\mathcal{O}^-)} + \langle \tfrac{1}{r^2}\partial_\theta u, \partial_\theta v\rangle_{L^2(\mathcal{O}^-)} + \langle u, v\rangle_{L^2(\mathcal{O}^-)},$$

where

$$\langle u, v\rangle_{L^2(A)} =: \int_A uv\, r dr d\theta \,\forall\, A \subseteq \mathcal{O}.$$

Given $f_0 \in L^2(\mathcal{O})$ and $q \in L^2(r_0, r_1)$, consider the limit state equation:

$$\begin{cases} -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial y^+}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial y^+}{\partial r} + h(r)y^+ = h(r)(f_0^+ + q\chi_{\mathcal{O}^+}) & \text{in } \mathcal{O}^+, \\ -\Delta y^- + y^- = f_0^- & \text{in } \mathcal{O}^-, \\ \dfrac{\partial y^+}{\partial \nu} = 0 & \text{on } \partial\mathcal{O}, \\ y^+ = y^-,\quad \dfrac{h(r_1)}{2\pi}\dfrac{\partial y^+}{\partial r} = \dfrac{\partial y^-}{\partial r} & \text{in } \Gamma_0. \end{cases} \tag{5.1}$$

The weak formulation of the above equation is given below.

Find $y \in W(\mathcal{O})$ such that

$$\int_{\mathcal{O}^+} \frac{h(r)}{2\pi}\left(\frac{\partial y^+}{\partial r}\frac{\partial \psi^+}{\partial r} + y^+\psi^+\right) rdrd\theta + \int_{\mathcal{O}^-}\left(\frac{\partial y^-}{\partial r}\frac{\partial \psi^-}{\partial r} + \frac{1}{r^2}\frac{\partial y^-}{\partial \theta}\frac{\partial \psi^-}{\partial \theta} + y^-\psi^-\right) rdrd\theta$$
$$= \int_{\mathcal{O}^+}\frac{h(r)}{2\pi}(f_0^+ + q)\psi^+ rdrd\theta + \int_{\mathcal{O}^-} f_0^-\psi^- rdrd\theta,$$

for all $\psi \in W(\mathcal{O})$. By Lax-Milgram theorem, there exists a unique weak solution in $W(\mathcal{O})$ to problem (5.1) and the solution operator is linear and continuous. Also we have the following a priori estimate:

$$\|y\|_{W(\mathcal{O})} \leqslant C(\|f\|_{L^2(\mathcal{O})} + \|q\|_{L^2(r_0,r_1)}). \tag{5.2}$$

Now, we will state the limit optimal control problem.

Find $(\bar{y},\bar{q}) \in W(\mathcal{O}) \times L^2(r_0,r_1)$ such that

$$(P) \qquad J(\bar{y},\bar{q}) = \inf\{J(y,q) | (u,q) \in W(\mathcal{O}) \times L^2(r_0,r_1),\ (y,q) \text{ satisfies } (5.1)\},$$

where the cost functional J is defined as

$$J(y,q) = \frac{1}{2}\int_{\mathcal{O}^-} |y^- - y_d|^2 rdrd\theta + \frac{1}{4\pi}\int_{\mathcal{O}^+} h(r)|y^+ - y_d|^2 rdrd\theta + \frac{\beta}{2}\int_{r_0}^{r_1} h(r)|q|^2 rdr.$$

The following result can be easily verified as in the previous section.

Theorem 5.1. *The optimal control problem (P) has a unique solution.*

Now, we will establish the optimality system for the limit problem. The adjoint state $\overline{p} \in W(\mathcal{O})$ solves

$$\begin{cases} -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial \overline{p}^+}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial \overline{p}^+}{\partial r} + h(r)\overline{p}^+ = h(r)(\bar{y}^+ - y_d) & \text{in } \mathcal{O}^+, \\ -\Delta \overline{p}^- + \overline{p}^- = (\bar{y}^- - y_d) & \text{in } \mathcal{O}^-, \\ \dfrac{\partial \overline{p}}{\partial \nu} = 0 & \text{on } \partial\mathcal{O}, \\ \overline{p}^+ = \overline{p}^-, \quad \dfrac{h(r_1)}{2\pi}\dfrac{\partial \overline{p}^+}{\partial r} = \dfrac{\partial \overline{p}^-}{\partial r} & \text{on } \Gamma_0. \end{cases} \tag{5.3}$$

Theorem 5.2. *If $(\bar{y},\bar{q})$ is an optimal solution to (P), then*

$$\bar{q} = \frac{-1}{2\pi\beta}\int_0^{2\pi} \overline{p}^+ d\theta,$$

where $\overline{p} \in W(\mathcal{O})$ is the solution to the adjoint problem (5.3). Conversely, assume that the pair $(\hat{y}, \hat{p}) \in W(\mathcal{O}) \times W(\mathcal{O})$ solves the optimality system

$$\begin{cases} -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial \hat{y}^+}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial \hat{y}^+}{\partial r} + h(r)\hat{y}^+ = h(r)(f_0^+ + \hat{q}) & \text{in } \mathcal{O}^+, \\ -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial \hat{p}^+}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial \hat{p}^+}{\partial r} + h(r)\hat{p}^+ = h(r)(\hat{y}^+ - y_d) & \text{in } \mathcal{O}^+, \\ -\Delta\hat{y}^- + \hat{y}^- = f_0^- \; ; \; -\Delta\hat{p}^- + \hat{p}^- = (\hat{y}^- - y_d) & \text{in } \mathcal{O}^-, \\ \dfrac{\partial \hat{y}}{\partial \nu} = 0 \; ; \dfrac{\partial \hat{p}}{\partial \nu} = 0 & \text{in } \partial\mathcal{O}, \\ \hat{y}^+ = \hat{y}^-, \; \dfrac{h(r_1)}{2\pi}\dfrac{\partial \hat{y}^+}{\partial r} = \dfrac{\partial \hat{y}^-}{\partial r}; \; \hat{p}^+ = \hat{p}^-, \; \dfrac{h(r_1)}{2\pi}\dfrac{\partial \hat{p}^+}{\partial r} = \dfrac{\partial \hat{p}^-}{\partial r} & \text{on } \Gamma_0, \\ \hat{q} = \dfrac{-1}{2\pi\beta}\displaystyle\int_0^{2\pi} \hat{p}^+ d\theta, \end{cases} \tag{5.4}$$

then the pair $(\hat{y}, \hat{q})$ is the optimal solution to (P).

The theorem can be proved by following the similar steps of Theorem 4.2. For completeness, we give the proof in the appendix. Now, we will describe the main homogenization result in the following subsection.

5.1. Convergence Analysis

The homogenization of the optimal control problem P_ε is analyzed by changing the problem into polar coordinates and applying the unfolding operator. The weak form of the state equation is given in the following definition.

Definition 5.3. *We say the function y_ε is a weak solution of the state equation (4.1) if y_ε satisfies*

$$\int_{\mathcal{O}_\varepsilon} \left(\frac{\partial y_\varepsilon}{\partial r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial y_\varepsilon}{\partial \theta}\frac{\partial \psi}{\partial \theta} + y_\varepsilon \psi\right) r dr d\theta = \int_{\mathcal{O}_\varepsilon} (f_0 + \chi_{\mathcal{O}_\varepsilon^+}\overline{Q}_\varepsilon)\psi \, r dr d\theta,$$

for all $\psi \in H^1(\mathcal{O}_\varepsilon)$.

In this subsection, we describe the homogenization of the optimal control problem. Assume that $(\overline{y}_\varepsilon, \overline{q}_\varepsilon)$ is the optimal solution to problem (P_ε). Let $y_\varepsilon(0)$ be the solution to problem (4.1) corresponding to $q_\varepsilon = 0$, then from (4.2), we get $\|y_\varepsilon(0)\|_{H^1(\mathcal{O}_\varepsilon)} \leqslant C$, where $C > 0$ is independent of ε. Using optimality of the solution $(\overline{y}_\varepsilon, \overline{q}_\varepsilon)$, we get

$$\frac{1}{2}\int_{\mathcal{O}_\varepsilon} (\overline{y}_\varepsilon - y_d)^2 r dr d\theta + \frac{\beta}{2}\int_{\mathcal{O}_\varepsilon^+} |\overline{Q}_\varepsilon|^2 r dr d\theta \leqslant C.$$

Thus, we have

$$\|\overline{Q}_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon^+)} = \|\overline{q}_\varepsilon\|_{L^2(D)} \leqslant C \text{ and } \|\overline{y}_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon)} \leqslant C. \tag{5.5}$$

Further, $\overline{p}_\varepsilon$ satisfies $\|\overline{p}_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} \leqslant C$. The zero extension of a function φ defined on $\mathcal{O}_\varepsilon^+$ to $\mathcal{O}^+$ is denoted by $\widetilde{\varphi}$. That is,

$$\widetilde{\varphi} = \begin{cases} \varphi & \text{in } \mathcal{O}_\varepsilon^+ \\ 0 & \text{in } \mathcal{O}^+ \setminus \mathcal{O}_\varepsilon^+. \end{cases}$$

Now, we prove the main theorem of this section.

Theorem 5.4. *Let $(\overline{y}_\varepsilon, \overline{q}_\varepsilon)$ and $(\overline{y}, \overline{q})$ be the optimal solutions to (P_ε) and (P), respectively. Then*

$$\begin{aligned} &\widetilde{\overline{y}_\varepsilon^+} \rightharpoonup \frac{h(r)}{2\pi}\,\overline{y}^+, \quad \widetilde{\frac{\partial \overline{y}_\varepsilon^+}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi}\frac{\partial \overline{y}^+}{\partial r} \quad \textit{weakly in } L^2(\mathcal{O}^+), \\ &\widetilde{\overline{p}_\varepsilon^+} \rightharpoonup \frac{h(r)}{2\pi}\,\overline{p}^+, \quad \widetilde{\frac{\partial \overline{p}_\varepsilon^+}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi}\frac{\partial \overline{p}^+}{\partial r} \quad \textit{weakly in } L^2(\mathcal{O}^+), \textit{ and} \\ &\overline{y}_\varepsilon^- \rightharpoonup \overline{y}_0^-, \; \overline{p}_\varepsilon^- \rightharpoonup \overline{p}^- \quad \textit{weakly in } H^1(\mathcal{O}^-), \end{aligned} \tag{5.6}$$

where $\overline{q} = \dfrac{-1}{2\pi\beta}\displaystyle\int_0^{2\pi} \overline{p}^+ d\theta$ and $\overline{p}_\varepsilon$, $\overline{p}$ are the solutions of (4.5) and (5.3), respectively.

Proof. Since the sequence $\overline{q}_\varepsilon$ is bounded in $L^2(D)$ (by the estimate (5.5)), by weak compactness, there exists a subsequence (still denoted by ε) and q_0 such that $\overline{q}_\varepsilon \rightharpoonup q_0$ weakly in $L^2(D)$. Recall the continuity estimate (ref. (4.2)) of the state solution $\overline{y}_\varepsilon$, that is,

$$\|\overline{y}_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} \leqslant C\left(\|f_0\|_{L^2(\mathcal{O})} + \|\overline{q}_\varepsilon\|_{L^2(D)}\right), \tag{5.7}$$

where $C > 0$ is independent of ε. Let us estimate $T^\varepsilon \overline{y}_\varepsilon^+$ and its derivatives in the space $L^2(\mathcal{O}_u)$. By the Proposition 3.4, we get

$$\begin{aligned} \|T^\varepsilon y_\varepsilon\|_{L^2(\mathcal{O}_u)} &= \sqrt{2\pi}\|y_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon^+)}, \\ \left\|T^\varepsilon \frac{\partial y_\varepsilon}{\partial r}\right\|_{L^2(\mathcal{O}_u)} &= \sqrt{2\pi}\left\|\frac{\partial y_\varepsilon}{\partial r}\right\|_{L^2(\mathcal{O}_\varepsilon^+)}, \text{ and} \\ \left\|T^\varepsilon \frac{\partial y_\varepsilon}{\partial \tau}\right\|_{L^2(\mathcal{O}_u)} &= \varepsilon\sqrt{2\pi}\left\|\frac{\partial y_\varepsilon}{\partial \theta}\right\|_{L^2(\mathcal{O}_\varepsilon^+)}. \end{aligned}$$

By the weak compactness, there exists a subsequence (still denoted by ε) such that

$$T^\varepsilon \overline{y}_\varepsilon^+ \rightharpoonup y_0^+ \quad \text{weakly in } L^2(\mathcal{O}_u), \tag{5.8}$$

$$\frac{\partial}{\partial r}T^\varepsilon \overline{y}_\varepsilon^+ \rightharpoonup \frac{\partial y_0^+}{\partial r}, \text{ that is, } T^\varepsilon \frac{\partial \overline{y}_\varepsilon^+}{\partial r} \rightharpoonup \frac{\partial y_0^+}{\partial r} \quad \text{weakly in } L^2(\mathcal{O}_u), \text{ and} \tag{5.9}$$

$$\frac{\partial}{\partial \tau}T^\varepsilon \overline{y}_\varepsilon^+ \rightharpoonup \frac{\partial y_0^+}{\partial \tau}, \text{ that is, } \varepsilon T^\varepsilon \frac{\partial \overline{y}_\varepsilon^+}{\partial \theta} \rightharpoonup \frac{\partial y_0^+}{\partial \tau} \quad \text{weakly in } L^2(\mathcal{O}_u). \tag{5.10}$$

From the Proposition 3.5, we have

$$\left\| T^{\varepsilon} \frac{\partial \bar{y}_{\varepsilon}^{+}}{\partial \theta} \right\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi} \left\| \frac{\partial \bar{y}_{\varepsilon}^{+}}{\partial \theta} \right\|_{L^2(\mathcal{O}_{\varepsilon}^{+})} \leqslant \sqrt{2\pi} \|\bar{y}_{\varepsilon}\|_{H^1(\mathcal{O}_{\varepsilon})}.$$

The above estimate implies the boundedness of the sequence $T^{\varepsilon} \frac{\partial \bar{y}_{\varepsilon}^{+}}{\partial \theta}$ in the space $L^2(\mathcal{O}_u)$. From convergence (5.10), it follows that $\frac{\partial y_0^+}{\partial \tau} = 0$. Hence, we conclude that

$$\widetilde{\bar{y}_{\varepsilon}^{+}} \rightharpoonup \frac{1}{2\pi} \int_{Y(r)} y_0^+ d\tau \;\text{ and }\; \widetilde{\frac{\partial \bar{y}_{\varepsilon}^{+}}{\partial r}} \rightharpoonup \frac{1}{2\pi} \int_{Y(r)} \frac{\partial y_0^+}{\partial r} d\tau \text{ weakly in } L^2(\mathcal{O}^+) \tag{5.11}$$

with the help of Proposition 3.8. Since y_0^+ is independent of τ, we write

$$\int_{Y(r)} y_0^+ \, d\tau = h(r) y_0^+ \text{ and } \int_{Y(r)} \frac{\partial y_0^+}{\partial r} \, d\tau = h(r) \frac{\partial y_0^+}{\partial r}. \tag{5.12}$$

Thus, (5.11) becomes

$$\widetilde{\bar{y}_{\varepsilon}^{+}} \rightharpoonup \frac{h(r)}{2\pi} y_0^+ \;\text{ and }\; \widetilde{\frac{\partial \bar{y}_{\varepsilon}^{+}}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi} \frac{\partial y_0^+}{\partial r} \text{ weakly in } L^2(\mathcal{O}^+). \tag{5.13}$$

Recall that $T^{\varepsilon} \frac{\partial \bar{y}_{\varepsilon}^{+}}{\partial \theta}$ is bounded in $L^2(\mathcal{O}_u)$. Hence, by the weak compactness, there is an element $P \in L^2(\mathcal{O}_u)$ such that up to a subsequence (still denoted by ε),

$$T^{\varepsilon} \frac{\partial \bar{y}_{\varepsilon}^{+}}{\partial \theta} \rightharpoonup P \quad \text{weakly in } L^2(\mathcal{O}_u). \tag{5.14}$$

Using the estimate for $\|\bar{y}_{\varepsilon}\|_{H^1(\mathcal{O}_{\varepsilon})}$, we have the boundedness of $\bar{y}_{\varepsilon}^{-}$ in the space $H^1(\mathcal{O}^-)$. Thus, up to a subsequence (still denoted by ε)

$$\bar{y}_{\varepsilon}^{-} \rightharpoonup y_0^- \text{ weakly in } H^1(\mathcal{O}^-). \tag{5.15}$$

Define, y_0 as

$$y_0(x) = \begin{cases} y_0^+ \text{ if } x \in \mathcal{O}^+, \\ y_0^- \text{ if } x \in \mathcal{O}^-. \end{cases} \tag{5.16}$$

Claim: $y_0 \in W(\mathcal{O})$.
We know that $y_0 \in L^2(\mathcal{O})$ and $\frac{\partial y_0}{\partial r} \in L^2(\mathcal{O}^-)$. To prove $y_0 \in W(\mathcal{O})$, we need to show $\frac{\partial y_0}{\partial r} \in L^2(\mathcal{O})$. Note that y_0^+ is independent of τ and so is $\frac{\partial y_0^+}{\partial r}$. Hence, we have $\frac{\partial y_0^+}{\partial r} \in L^2(\mathcal{O}^+)$ and also $\frac{\partial y_0}{\partial r} \in$

$L^2(\mathcal{O}^-)$. Thus, to show that $\dfrac{\partial y_0}{\partial r} \in L^2(\mathcal{O})$, it is enough to prove that the trace of y_0^+ and y_0^- are equal on Γ_0. Since $\bar{y}_\varepsilon^+|_{\Gamma_\varepsilon} = \bar{y}_\varepsilon^-|_{\Gamma_\varepsilon}$ implies the equality of the traces for the boundary unfolding operator. More precisely, we have $T^\varepsilon_{r_1}(\bar{y}_\varepsilon^+|_{\Gamma_\varepsilon}) = T^\varepsilon_{r_1}(\bar{y}_\varepsilon^-|_{\Gamma_\varepsilon})$ i.e.

$$\left(T^\varepsilon(\bar{y}_\varepsilon^+)\right)|_{r=r_1} = T^\varepsilon_{r_1}\left(\bar{y}_\varepsilon^-|_{\Gamma_\varepsilon}\right). \tag{5.17}$$

From the weak continuity of the trace operator, we can write

$$\left(T^\varepsilon(\bar{y}_\varepsilon^+)\right)|_{r=r_1} \rightharpoonup y_0^+|_{r=r_1} \text{ weakly in } L^2((0,2\pi)\times Y(r_0))$$

and from (5.15), we get

$$\bar{y}_\varepsilon{}^-|_{r=r_1} \to y_0^-|_{r=r_1} \text{ strongly in } L^2(0,2\pi).$$

This implies

$$T^\varepsilon_{r_1}\left(\bar{y}_\varepsilon{}^-|_{r=r_1}\right) \to y_0^-|_{r=r_1} \text{ in } L^2((0,2\pi)\times Y(r_1)).$$

Passing to the limit in (5.17) as $\varepsilon \to 0$ we get

$$y_0^+|_{r=r_1} = y_0^-|_{r=r_1} \text{ in } L^2(0,2\pi),$$

since y_0^+ and y_0^- are independent of τ.

Identification of the limit P in (5.14): Finally, we identify P which is identically zero.

For $\phi \in \mathcal{D}(\mathcal{O}^+)$ and $\zeta(z) \in \mathcal{D}(0,2\pi)$, choose $\psi \in \mathcal{D}(0,2\pi)$ such that $\psi'(z) = \zeta(z)$. Now choose a test function

$$\phi^\varepsilon(r,\theta) = \varepsilon\phi(r,\theta)\psi\left(\left\{\tfrac{\theta}{\varepsilon}\right\}\right),$$

in such a way that ϕ^ε is continuous on $\mathcal{O}_\varepsilon^+$. From the definition of the ε-unfolding of ϕ^ε and by Proposition 3.5, we get

$$\begin{aligned}
&T^\varepsilon\phi^\varepsilon = \varepsilon\phi\left(r, \varepsilon\left[\tfrac{\theta}{\varepsilon}\right] + \varepsilon\tau\right)\psi(\tau),\\
&T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial\theta} = \frac{1}{\varepsilon}\frac{\partial}{\partial\tau}T^\varepsilon\phi^\varepsilon = \varepsilon\frac{\partial\phi}{\partial\theta}\left(r,\varepsilon\left[\frac{\theta}{\varepsilon}\right]+\varepsilon\tau\right)\psi(\tau) + \phi\left(r,\varepsilon\left[\frac{\theta}{\varepsilon}\right]+\varepsilon\tau\right)\psi'(\tau) \text{ and}\\
&T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial r} = \varepsilon\frac{\partial\phi}{\partial r}\left(r,\varepsilon\left[\frac{\theta}{\varepsilon}\right]+\varepsilon\tau\right)\psi(\tau).
\end{aligned}$$

From the above equations, we derive the following convergences as $\varepsilon \to 0$:

$$T^\varepsilon\phi^\varepsilon \to 0, \quad T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial\theta} \to \phi(r,\theta)\psi'(\tau), \quad \text{and} \quad T^\varepsilon\frac{\partial\phi^\varepsilon}{\partial r} \to 0 \quad \text{in } L^2(\mathcal{O}_u).$$

Let us recall the variational formulation of (4.1) with the test function ϕ^ε given by

$$\int_{\mathcal{O}_\varepsilon}\left(\frac{\partial \overline{y}_\varepsilon}{\partial r}\frac{\partial \phi^\varepsilon}{\partial r}+\frac{1}{r^2}\frac{\partial \overline{y}_\varepsilon}{\partial \theta}\frac{\partial \phi^\varepsilon}{\partial \theta}+\overline{y}_\varepsilon\phi^\varepsilon\right) rdrd\theta = \int_{\mathcal{O}_\varepsilon}(f_0+\chi_{\mathcal{O}_\varepsilon^+}\overline{Q}_\varepsilon)\phi^\varepsilon\, rdrd\theta \tag{5.18}$$

Notice that

$$\begin{aligned}\int_{\mathcal{O}_\varepsilon}\left(\frac{\partial \overline{y}_\varepsilon}{\partial r}\frac{\partial \phi^\varepsilon}{\partial r}+\frac{1}{r^2}\frac{\partial \overline{y}_\varepsilon}{\partial \theta}\frac{\partial \phi^\varepsilon}{\partial \theta}\right) rdrd\theta &= \int_{\mathcal{O}_\varepsilon^+}\left(\frac{\partial \overline{y}_\varepsilon^+}{\partial r}\frac{\partial \phi^\varepsilon}{\partial r}+\frac{1}{r^2}\frac{\partial \overline{y}_\varepsilon^+}{\partial \theta}\frac{\partial \phi^\varepsilon}{\partial \theta}\right) rdrd\theta\\ &= \frac{1}{2\pi}\int_{\mathcal{O}_u}\left(T^\varepsilon\frac{\partial \overline{y}_\varepsilon^+}{\partial r}\,T^\varepsilon\frac{\partial \phi^\varepsilon}{\partial r}+\frac{1}{r^2}T^\varepsilon\frac{\partial \overline{y}_\varepsilon^+}{\partial \theta}\,T^\varepsilon\frac{\partial \phi^\varepsilon}{\partial \theta}\right) rdrd\theta d\tau\\ &\to \frac{1}{2\pi}\int_{\mathcal{O}_u} P\,\phi(r,\theta)\psi'(\tau)\, rdrd\theta d\tau \quad \text{as } \varepsilon\to 0;\end{aligned} \tag{5.19}$$

$$\int_{\mathcal{O}_\varepsilon}\overline{y}_\varepsilon\phi^\varepsilon\, rdrd\theta = \int_{\mathcal{O}_\varepsilon^+}\overline{y}_\varepsilon^+\phi^\varepsilon\, rdrd\theta = \frac{1}{2\pi}\int_{\mathcal{O}_u}T^\varepsilon\overline{y}_\varepsilon^+\,T^\varepsilon\phi^\varepsilon\, rdrd\theta d\tau \to 0 \quad \text{as } \varepsilon\to 0 \tag{5.20}$$

and

$$\begin{aligned}\int_{\mathcal{O}_\varepsilon}(f_0+\chi_{\mathcal{O}_\varepsilon^+}\overline{Q}_\varepsilon)\phi^\varepsilon\, rdrd\theta &= \int_{\mathcal{O}_\varepsilon^+}(f_0^+ +\overline{Q}_\varepsilon)\phi^\varepsilon\, rdrd\theta\\ &= \frac{1}{2\pi}\int_{\mathcal{O}_u}T^\varepsilon\left(f_0^+ +\overline{Q}_\varepsilon\right)\,T^\varepsilon\phi^\varepsilon\, rdrd\theta d\tau\\ &= \frac{1}{2\pi}\int_{\mathcal{O}_u}\left(T^\varepsilon f_0^+ +\overline{q}_\varepsilon\right)\,T^\varepsilon\phi^\varepsilon rdrd\theta d\tau \;\to 0 \quad \text{as } \varepsilon\to 0.\end{aligned} \tag{5.21}$$

Combining (5.19), (5.20) and (5.21), we get

$$\int_{\mathcal{O}_u} P\,\phi(r,\theta)\psi'(\tau) = 0,$$

which implies

$$\int_{\mathcal{O}^+}\left(\int_{Y(r)} P\psi'(\tau)\, d\tau\right)\phi(r,\theta)\, rdrd\theta = 0\;\forall\,\phi\in\mathcal{D}(\mathcal{O}^+).$$

That is,

$$\int_{Y(r)} P(r,\theta,\tau)\psi'(\tau)d\tau = 0.$$

Hence,

$$\int_{Y(r)} P(r,\theta,\tau)\zeta(\tau)\,d\tau = 0 \quad \text{a.e. in } \mathcal{O}^+,\ \forall \zeta \in \mathcal{D}(0,2\pi).$$

Thus, $P \equiv 0$ a.e. on $\mathcal{O}^+$.

Claim: y_0 satisfies the limit equation: Choose a test function $\psi \in C^\infty(\overline{\mathcal{O}})$ in the variational formulation of the polar form of (5.1), which is given by

$$\begin{aligned}\int_{\mathcal{O}_\varepsilon}\left(\frac{\partial \overline{y}_\varepsilon}{\partial r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial \overline{y}_\varepsilon}{\partial \theta}\frac{\partial \psi}{\partial \theta}+\overline{y}_\varepsilon\psi\right) rdrd\theta &:= \int_{\mathcal{O}_\varepsilon^+}\left(\frac{\partial \overline{y}_\varepsilon^+}{\partial r}\frac{\partial \psi^+}{\partial r}+\frac{1}{r^2}\frac{\partial \overline{y}_\varepsilon^+}{\partial \theta}\frac{\partial \psi^+}{\partial \theta}+\overline{y}_\varepsilon^+\psi^+\right) rdrd\theta \\ &\quad + \int_{\mathcal{O}^-}\left(\frac{\partial \overline{y}_\varepsilon^-}{\partial r}\frac{\partial \psi^-}{\partial r}+\frac{1}{r^2}\frac{\partial \overline{y}_\varepsilon^-}{\partial \theta}\frac{\partial \psi^-}{\partial \theta}+\overline{y}_\varepsilon^-\psi^-\right) rdrd\theta \\ &= \int_{\mathcal{O}_\varepsilon}(f_0+\chi_{\mathcal{O}_\varepsilon^+}\overline{Q}_\varepsilon)\psi \; rdrd\theta\end{aligned}$$

First, let us look at the integral on $\mathcal{O}_\varepsilon^+$. As $\varepsilon \to 0$, we get

$$\begin{aligned}&\int_{\mathcal{O}_\varepsilon^+}\left(\frac{\partial \overline{y}_\varepsilon^+}{\partial r}\frac{\partial \psi^+}{\partial r}+\frac{1}{r^2}\frac{\partial \overline{y}_\varepsilon^+}{\partial \theta}\frac{\partial \psi^+}{\partial \theta}+\overline{y}_\varepsilon^+\psi^+\right) rdrd\theta \\ &\quad = \frac{1}{2\pi}\int_{\mathcal{O}_u}\left(T^\varepsilon\frac{\partial \overline{y}_\varepsilon^+}{\partial r}\,T^\varepsilon\frac{\partial \psi^+}{\partial r}+\frac{1}{r^2}T^\varepsilon\frac{\partial \overline{y}_\varepsilon^+}{\partial \theta}\,T^\varepsilon\frac{\partial \psi^+}{\partial \theta}+T^\varepsilon\overline{y}_\varepsilon^+T^\varepsilon\psi^+\right) rdrd\theta d\tau \\ &\quad \to \frac{1}{2\pi}\int_{\mathcal{O}_u}\left(\frac{\partial y_0^+}{\partial r}\,\frac{\partial \psi^+}{\partial r}+y_0^+\psi^+\right)\; rdrd\theta d\tau.\end{aligned}$$

The terms on $\mathcal{O}^-$ become

$$\int_{\mathcal{O}^-}\left(\frac{\partial \overline{y}_\varepsilon^-}{\partial r}\frac{\partial \psi^-}{\partial r}+\frac{1}{r^2}\frac{\partial \overline{y}_\varepsilon^-}{\partial \theta}\frac{\partial \psi^-}{\partial \theta}+\overline{y}_\varepsilon^-\psi^-\right) rdrd\theta \to \int_{\mathcal{O}^-}\left(\frac{\partial y_0^-}{\partial r}\frac{\partial \psi^-}{\partial r}+\frac{1}{r^2}\frac{\partial y_0^-}{\partial \theta}\frac{\partial \psi^-}{\partial \theta}+y_0^-\psi^-\right) rdrd\theta$$

and on $\mathcal{O}_\varepsilon$ the terms become

$$\begin{aligned}\int_{\mathcal{O}_\varepsilon}(f_0+\chi_{\mathcal{O}_\varepsilon^+}\overline{Q}_\varepsilon)\psi\; rdrd\theta &= \int_{\mathcal{O}_\varepsilon^+}(f_0^+ + \overline{Q}_\varepsilon)\psi^+\; rdrd\theta + \int_{\mathcal{O}^-} f_0^-\psi^-\; rdrd\theta \\ &= \frac{1}{2\pi}\int_{\mathcal{O}_u}\left(T^\varepsilon f_0^+ + T^\varepsilon\overline{Q}_\varepsilon\right)\, T^\varepsilon\psi^+ rdrd\theta d\tau + \int_{\mathcal{O}^-} f_0^-\psi^- rdrd\theta \\ &= \frac{1}{2\pi}\int_{\mathcal{O}_u}\left(T^\varepsilon f_0^+ + \overline{q}_\varepsilon\right)\, T^\varepsilon\psi^+ rdrd\theta d\tau + \int_{\mathcal{O}^-} f_0^-\psi^- rdrd\theta \\ &\to \frac{1}{2\pi}\int_{\mathcal{O}_u}(f_0^+ + q_0)\psi^+ rdrd\theta d\tau + \int_{\mathcal{O}^-} f_0^-\psi^- rdrd\theta.\end{aligned}$$

Hence, as $\varepsilon \to 0$, the limit variational formulation becomes

$$\frac{1}{2\pi}\int_{\mathcal{O}_u}\left(\frac{\partial y_0^+}{\partial r}\frac{\partial \psi^+}{\partial r}+y_0^+\psi^+\right) rdrd\theta d\tau+\int_{\mathcal{O}^-}\left(\frac{\partial y_0^-}{\partial r}\frac{\partial \psi^-}{\partial r}+\frac{1}{r^2}\frac{\partial y_0^-}{\partial \theta}\frac{\partial \psi^-}{\partial \theta}+y_0^-\psi^-\right) rdrd\theta$$
$$=\frac{1}{2\pi}\int_{\mathcal{O}_u}(f_0^+ + q_0)\psi^+\, rdrd\theta d\tau+\int_{\mathcal{O}^-} f_0^-\psi^-\, rdrd\theta.$$

This shows that y_0 satisfies the equation

$$\int_{\mathcal{O}^+}\frac{h(r)}{2\pi}\left(\frac{\partial y_0^+}{\partial r}\frac{\partial \psi^+}{\partial r}+y_0^+\psi^+\right) rdrd\theta+\int_{\mathcal{O}^-}\left(\frac{\partial y_0^-}{\partial r}\frac{\partial \psi^-}{\partial r}+\frac{1}{r^2}\frac{\partial y_0^-}{\partial \theta}\frac{\partial \psi^-}{\partial \theta}+y_0^-\psi^-\right) rdrd\theta$$
$$=\int_{\mathcal{O}^+}\frac{h(r)}{2\pi}(f_0^+ + q_0)\psi^+\, rdrd\theta+\int_{\mathcal{O}^-} f_0^-\psi^-\, rdrd\theta$$

for all $\psi \in C^\infty(\overline{\mathcal{O}})$. As we know that $C^\infty(\overline{\mathcal{O}})$ is dense in $W(\mathcal{O})$, the above equation is true for all ψ in $W(\mathcal{O})$. Therefore, y_0 satisfies (5.1). Hence, we proved the convergences in (5.13).

Similarly, we can prove the following convergences.

$$\widetilde{\overline{p}_\varepsilon^+} \rightharpoonup \frac{h(r)}{2\pi}\, p_0^+\,,\quad \widetilde{\frac{\partial \overline{p}_\varepsilon^+}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi}\frac{\partial p_0^+}{\partial r} \text{ weakly in } L^2(\mathcal{O}^+), \text{ and}$$
$$\overline{y}_\varepsilon^- \rightharpoonup y_0^-,\ \overline{p}_\varepsilon^- \rightharpoonup p_0^- \text{ weakly in } H^1(\mathcal{O}^-),$$

where y_0 satisfies (5.1) with $q = q_0$ and p_0 satisfies (5.3) with $\overline{y} = y_0$. To prove the convergence of the optimality system, it is enough to prove $\overline{q} = q_0$. Recall the optimality condition (4.4):

$$\overline{q}_\varepsilon(r,\tau)|_D = \frac{-1}{2\pi\beta}\int_0^{2\pi}(T^\varepsilon \overline{p}_\varepsilon^+)(r,\theta,\tau)\, d\theta. \tag{5.22}$$

By the convergences of $\overline{q}_\varepsilon$ and $T^\varepsilon \overline{p}_\varepsilon^+$ as $\varepsilon \to 0$, the equation (5.22) becomes

$$q_0(r,\tau) = \frac{-1}{2\pi\beta}\int_0^{2\pi} p_0^+(r,\theta,\tau)\, d\theta.$$

By noting the fact that p_0 is independent of τ, we conclude that q_0 is also independent of the variable τ. Hence, $q_0 = \frac{-1}{2\pi\beta}\int_0^{2\pi} p_0^+ d\theta$. Therefore, we get the optimality system corresponding to the minimization problem (P). Then the Theorem 5.2 says that, the optimal solution is (y_0, q_0). Hence by the uniqueness, we have $\overline{y} = y_0$, $\overline{p} = p_0$ and $\overline{q} = q_0$ which completes the proof. □

6. Dirichlet Cost Functional

In this section, we study the homogenization of the control problem with a Dirichlet cost functional G_ε defined by

$$G_\varepsilon(z_\varepsilon,\varrho)=\frac{1}{2}\int_{\mathcal{O}_\varepsilon}|\nabla z_\varepsilon-\nabla z_d|^2 rdrd\theta+\frac{\beta}{2}\int_{\mathcal{O}_\varepsilon^+}|\varrho^\varepsilon|^2 rdrd\theta \tag{6.1}$$

with $\varrho^\varepsilon(r,\theta)=\varrho(r,\frac{\theta}{\varepsilon})$ and the desired state $z_d\in H^1(\mathcal{O})$. Given $\varrho\in L^2(D)$ and the source term $f_0\in L^2(\mathcal{O})$, the function z_ε satisfies the state equation:

$$\begin{cases}-\Delta z_\varepsilon+z_\varepsilon=f_0+\chi_{\mathcal{O}_\varepsilon^+}\varrho^\varepsilon & \text{in } \mathcal{O}_\varepsilon,\\ \partial_\nu z_\varepsilon=0 & \text{on } \partial\mathcal{O}_\varepsilon.\end{cases} \tag{6.2}$$

The optimal control problem with Dirichlet cost functional is described as follows.

Find $(\bar{z}_\varepsilon,\bar{\varrho}_\varepsilon)\in H^1(\mathcal{O}_\varepsilon)\times L^2(D)$ such that

$$(E_\varepsilon)\quad G_\varepsilon(\bar{z}_\varepsilon,\bar{\varrho}_\varepsilon)=\inf\{G_\varepsilon(z_\varepsilon,\varrho)|(z_\varepsilon,\varrho)\in H^1(\mathcal{O}_\varepsilon)\times L^2(D),\ (z_\varepsilon,\varrho)\text{ satisfies (6.2)}\}$$

6.1. Optimality System

We state the necessary and sufficient conditions for the optimality and also characterize the optimal control via unfolding operator in the following theorem. The proofs of the results are either skipped or sketched as they can be proved following the similar arguments as in Section 5.

Theorem 6.1. *For each $\varepsilon>0$, the minimization problem (E_ε) admits a unique solution. Let $(\bar{z}_\varepsilon,\bar{\varrho}_\varepsilon)\in H^1(\mathcal{O}_\varepsilon)\times L^2(D)$ be the optimal solution to (E_ε), then the optimal control is characterized by*

$$\bar{\varrho}_\varepsilon(r,\tau)|_D=\frac{-1}{2\pi\beta}\int_0^{2\pi}(T^\varepsilon\bar{v}_\varepsilon^+)(r,\theta,\tau)\,d\theta \tag{6.3}$$

where $\bar{z}_\varepsilon$ satisfies (6.2) with $\varrho^\varepsilon=\overline{\mathcal{P}}_\varepsilon$, that is $\varrho^\varepsilon(r,\theta)=\overline{\mathcal{P}}_\varepsilon(r,\theta)=\bar{\varrho}_\varepsilon(r,\frac{\theta}{\varepsilon})$ and the adjoint state $\bar{v}_\varepsilon$ satisfies the problem

$$\begin{cases}-\Delta\bar{v}_\varepsilon+\bar{v}_\varepsilon=-\Delta(\bar{z}_\varepsilon-z_d) & \text{in } \mathcal{O}_\varepsilon,\\ \partial_\nu\bar{v}_\varepsilon=(\nabla\bar{z}_\varepsilon-\nabla z_d)\cdot\nu & \text{on } \partial\mathcal{O}_\varepsilon.\end{cases} \tag{6.4}$$

Conversely, if a pair $(\hat{z}_\varepsilon,\hat{v}_\varepsilon)$ satisfies the following system

$$\begin{cases}-\Delta\hat{z}_\varepsilon+\hat{z}_\varepsilon=f_0+\chi_{\mathcal{O}_\varepsilon^+}\hat{\mathcal{P}}_\varepsilon;\ -\Delta\hat{v}_\varepsilon+\hat{v}_\varepsilon=-\Delta(\hat{z}_\varepsilon-z_d) & \text{in } \mathcal{O}_\varepsilon,\\ \partial_\nu\hat{z}_\varepsilon=0;\ \partial_\nu\hat{v}_\varepsilon=(\nabla\hat{z}_\varepsilon-\nabla z_d)\cdot\nu & \text{on } \partial\mathcal{O}_\varepsilon,\\ \hat{\varrho}_\varepsilon(r,\tau)=\dfrac{-1}{2\pi\beta}\displaystyle\int_0^{2\pi}T^\varepsilon\hat{v}_\varepsilon^+\,d\theta & \text{on } D,\end{cases} \tag{6.5}$$

then the pair $(\hat{z}_\varepsilon,\hat{\varrho}_\varepsilon)$ is the optimal solution to (E_ε). Here $\hat{\mathcal{P}}_\varepsilon(r,\theta)=\hat{\varrho}_\varepsilon(r,\frac{\theta}{\varepsilon})$.

Limit Problem: Now, we will describe the limit problem. Given $f_0 \in L^2(\mathcal{O})$ and $\varrho \in L^2(r_0,r_1)$, consider the limit state equation:

$$\begin{cases} -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial z^+}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial z^+}{\partial r} + h(r)z^+ = h(r)(f_0^+ + \varrho\chi_{\mathcal{O}^+}) & \text{in } \mathcal{O}^+, \\ -\Delta z^- + z^- = f_0^- & \text{in } \mathcal{O}^-, \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial\mathcal{O}, \\ z^+ = z^-, \quad \dfrac{h(r_1)}{2\pi}\dfrac{\partial z^+}{\partial r} = \dfrac{\partial z^-}{\partial r} & \text{on } \Gamma_0, \end{cases} \tag{6.6}$$

where

$$z = \begin{cases} z^+ & \text{if } x \in \mathcal{O}^+ \\ z^- & \text{if } x \in \mathcal{O}^-. \end{cases}$$

The limit optimal control problem is given below.

Find $(\bar{z},\bar{\varrho}) \in W \times L^2(r_0,r_1)$ such that

$$(E) \quad G(\bar{z},\bar{\varrho}) = \inf\{J(z,\varrho) | (z,\varrho) \in W \times L^2(r_0,r_1),\ (z,\varrho) \text{ satisfies } (6.6)\}.$$

where the cost functional G is defined as

$$G(z,\varrho) = \frac{1}{2}\int_{\mathcal{O}^-} |\nabla z^- - \nabla z_d|^2 r dr d\theta + \frac{1}{4\pi}\int_{\mathcal{O}^+} h(r)|\partial_r z^+ - \partial_r z_d|^2 r dr d\theta + \frac{\beta}{2}\int_{r_0}^{r_1} h(r)|\varrho|^2 r dr.$$

Now, we will establish the optimality system for this limit problem. The adjoint state $\bar{v} \in W(\mathcal{O})$ solves

$$\begin{cases} -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial \bar{v}^+}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial \bar{v}^+}{\partial r} + h(r)\bar{v}^+ = -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial(\bar{z}^+ - z_d)}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial(\bar{z}^+ - z_d)}{\partial r} & \text{in } \mathcal{O}^+, \\ -\Delta\bar{v}^- + \bar{v}^- = -\Delta(\bar{z}^- - z_d) & \text{in } \mathcal{O}^-, \\ \dfrac{\partial \bar{v}^+}{\partial \nu} = \dfrac{\partial}{\partial r}(\bar{z}^+ - z_d) & \text{on } \Gamma_b, \\ \dfrac{\partial \bar{v}^-}{\partial \nu} = \dfrac{\partial}{\partial \nu}(\bar{z}^- - z_d) & \text{on } \Gamma_u, \\ \bar{v}^+ = \bar{v}^-, \quad \dfrac{h(r_1)}{2\pi}\dfrac{\partial \bar{v}^+}{\partial r} = \dfrac{\partial \bar{v}^-}{\partial r} & \text{on } \Gamma_0. \end{cases} \tag{6.7}$$

The following result can be easily verified as in the previous section.

Theorem 6.2. *The optimal control problem (E) has a unique solution. If $(\bar{z},\bar{\varrho})$ is an optimal solution to (E), then*

$$\bar{\varrho} = \frac{-1}{2\pi\beta}\int_0^{2\pi} \bar{v}^+ d\theta \tag{6.8}$$

where $\overline{v} \in W(\mathcal{O})$ is the solution to the adjoint problem (6.7). The converse is also true (similar to Theorem 5.2).

Theorem 6.3. *Let $(\overline{z}_\varepsilon, \overline{\varrho}_\varepsilon)$ and $(\overline{z}, \overline{\varrho})$ be the optimal solutions to (E_ε) and (E), respectively. Then*

$$\begin{aligned}
&\widetilde{\overline{z}_\varepsilon^+} \rightharpoonup \frac{h(r)}{2\pi}\,\overline{z}^+, \quad \widetilde{\frac{\partial \overline{z}_\varepsilon^+}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi}\frac{\partial \overline{z}^+}{\partial r} \quad \text{weakly in } L^2(\mathcal{O}^+),\\
&\widetilde{\overline{v}_\varepsilon^+} \rightharpoonup \frac{h(r)}{2\pi}\,\overline{v}^+, \quad \widetilde{\frac{\partial \overline{v}_\varepsilon^+}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi}\frac{\partial \overline{v}^+}{\partial r} \quad \text{weakly in } L^2(\mathcal{O}^+), \quad \text{and}\\
&\overline{z}_\varepsilon^- \rightharpoonup \overline{z}^-, \ \ \overline{v}_\varepsilon^- \rightharpoonup \overline{v}^- \quad \text{weakly in } H^1(\mathcal{O}^-),
\end{aligned} \tag{6.9}$$

where $\overline{\varrho} = \dfrac{-1}{2\pi\beta}\displaystyle\int_0^{2\pi} \overline{v}^+ d\theta$ and $\overline{v}_\varepsilon$, $\overline{v}$ are the solutions of (6.4) and (6.7), respectively.

Proof. As in the proof of Theorem 5.4, we get

$$\begin{aligned}
&\widetilde{\overline{z}_\varepsilon^+} \rightharpoonup \frac{h(r)}{2\pi}\,z_0^+, \quad \widetilde{\frac{\partial \overline{z}_\varepsilon^+}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi}\frac{\partial z_0^+}{\partial r} \quad \text{weakly in } L^2(\mathcal{O}^+), \quad \text{and}\\
&\overline{z}_\varepsilon^- \rightharpoonup z_0^- \quad \text{weakly in } H^1(\mathcal{O}^-).
\end{aligned}$$

Now, we look at the convergence of the adjoint state $\overline{v}_\varepsilon$. As the state $\overline{z}_\varepsilon$ is uniformly bounded, we get

$$\|\overline{v}_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} \leqslant C\left(\overline{z}_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)} + \|z_d\|_{H^1(\mathcal{O})}\right) \leqslant C, \tag{6.10}$$

where $C > 0$ is independent of ε. Using this estimate and following the similar arguments of Theorem 5.4, we get the following convergences:

$$\widetilde{\overline{v}_\varepsilon^+} \rightharpoonup \frac{h(r)}{2\pi} v_0^+ \ \text{ and } \ \widetilde{\frac{\partial \overline{v}_\varepsilon^+}{\partial r}} \rightharpoonup \frac{h(r)}{2\pi}\frac{\partial v_0^+}{\partial r} \quad \text{weakly in } L^2(\mathcal{O}^+). \tag{6.11}$$

Also, we know that $T^\varepsilon \dfrac{\partial \overline{v}_\varepsilon^+}{\partial \theta}$ is bounded in $L^2(\mathcal{O}_u)$. Hence, by the weak compactness, there is an element $R \in L^2(\mathcal{O}_u)$ such that up to subsequence (still denoted by ε),

$$T^\varepsilon \frac{\partial \overline{v}_\varepsilon^+}{\partial \theta} \rightharpoonup R \quad \text{weakly in } L^2(\mathcal{O}_u).$$

Using the estimate for $\|\overline{v}_\varepsilon\|_{H^1(\mathcal{O}_\varepsilon)}$, we have the boundedness of $\overline{v}_\varepsilon^-$ in the space $H^1(\mathcal{O}^-)$. Thus, up to a subsequence (still denoted by ε)

$$\overline{v}_\varepsilon^- \rightharpoonup v_0^- \quad \text{weakly in } H^1(\mathcal{O}^-). \tag{6.12}$$

Define v_0 as

$$v_0(x) = \begin{cases} v_0^+ \text{ if } x \in \mathcal{O}^+, \\ v_0^- \text{ if } x \in \mathcal{O}^-. \end{cases} \tag{6.13}$$

Identification of the limit R:

We choose the same test function ϕ^ε as in Theorem 5.4 which satisfies, as $\varepsilon \to 0$,

$$T^\varepsilon \phi^\varepsilon \to 0, \quad T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial \theta} \to \phi(r,\theta)\psi'(\tau), \quad \text{and} \quad T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial r} \to 0 \text{ in } L^2(\mathcal{O}_u).$$

Let us recall the variational formulation of (6.4) with the test function ϕ^ε given by

$$\int_{\mathcal{O}_\varepsilon} \left(\frac{\partial \overline{v}_\varepsilon}{\partial r} \frac{\partial \phi^\varepsilon}{\partial r} + \frac{1}{r^2} \frac{\partial \overline{v}_\varepsilon}{\partial \theta} \frac{\partial \phi^\varepsilon}{\partial \theta} + \overline{v}_\varepsilon \phi^\varepsilon \right) r dr d\theta = \int_{\mathcal{O}_\varepsilon} \left(\frac{\partial (\overline{z}_\varepsilon - z_d)}{\partial r} \frac{\partial \phi^\varepsilon}{\partial r} + \frac{1}{r^2} \frac{\partial (\overline{z}_\varepsilon - z_d)}{\partial \theta} \frac{\partial \phi^\varepsilon}{\partial \theta} \right) r dr d\theta,$$

for all $\varphi \in H^1(\mathcal{O}_\varepsilon)$. Now notice that

$$\begin{aligned} \int_{\mathcal{O}_\varepsilon} \left(\frac{\partial \overline{v}_\varepsilon}{\partial r} \frac{\partial \phi^\varepsilon}{\partial r} + \frac{1}{r^2} \frac{\partial \overline{v}_\varepsilon}{\partial \theta} \frac{\partial \phi^\varepsilon}{\partial \theta} \right) r dr d\theta &= \int_{\mathcal{O}_\varepsilon^+} \left(\frac{\partial \overline{v}_\varepsilon^+}{\partial r} \frac{\partial \phi^\varepsilon}{\partial r} + \frac{1}{r^2} \frac{\partial \overline{v}_\varepsilon^+}{\partial \theta} \frac{\partial \phi^\varepsilon}{\partial \theta} \right) r dr d\theta \\ &= \frac{1}{2\pi} \int_{\mathcal{O}_u} \left(T^\varepsilon \frac{\partial \overline{v}_\varepsilon^+}{\partial r} T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial r} + \frac{1}{r^2} T^\varepsilon \frac{\partial \overline{v}_\varepsilon^+}{\partial \theta} T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial \theta} \right) r dr d\theta d\tau \\ &\to \frac{1}{2\pi} \int_{\mathcal{O}_u} \frac{1}{r^2} R\, \phi(r,\theta)\psi'(\tau)\ r dr d\theta d\tau \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{6.14}$$

$$\int_{\mathcal{O}_\varepsilon} \overline{v}_\varepsilon \phi^\varepsilon\ r dr d\theta = \int_{\mathcal{O}_\varepsilon^+} \overline{v}_\varepsilon^+ \phi^\varepsilon\ r dr d\theta = \frac{1}{2\pi} \int_{\mathcal{O}_u} T^\varepsilon \overline{v}_\varepsilon^+ T^\varepsilon \phi^\varepsilon\ r dr d\theta d\tau \to 0 \quad \text{as } \varepsilon \to 0, \tag{6.15}$$

and

$$\begin{aligned} &\int_{\mathcal{O}_\varepsilon} \left(\frac{\partial (\overline{z}_\varepsilon - z_d)}{\partial r} \frac{\partial \phi^\varepsilon}{\partial r} + \frac{1}{r^2} \frac{\partial (\overline{z}_\varepsilon - z_d)}{\partial \theta} \frac{\partial \phi^\varepsilon}{\partial \theta} \right) r dr d\theta \\ &= \frac{1}{2\pi} \int_{\mathcal{O}_u} \left(T^\varepsilon \frac{\partial (\overline{z}_\varepsilon^+ - z_d)}{\partial r} T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial r} + \frac{1}{r^2} T^\varepsilon \frac{\partial (\overline{z}_\varepsilon^+ - z_d)}{\partial \theta} T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial \theta} \right) r dr d\theta d\tau \\ &\to \frac{1}{2\pi} \int_{\mathcal{O}_u} \frac{1}{r^2} \frac{\partial z_d}{\partial \theta} \phi(r,\theta)\psi'(\tau)\ r dr d\theta d\tau \quad \text{as } \varepsilon \to 0; \end{aligned} \tag{6.16}$$

Combining the above equations, we get

$$\int_{\mathcal{O}_u} \frac{1}{r^2} R\, \phi(r,\theta)\psi'(\tau)\ r dr d\theta d\tau = \int_{\mathcal{O}_u} \frac{1}{r^2} \frac{\partial z_d}{\partial \theta} \phi(r,\theta)\psi'(\tau)\ r dr d\theta d\tau,$$

which implies

$$\int_{\mathcal{O}^+}\left(\int_{Y(r)} R\psi'(\tau)\,d\tau\right)\phi(r,\theta)\,rdrd\theta = 0\ \forall\,\phi\in\mathcal{D}(\mathcal{O}^+).$$

That is (we skip the details here)

$$R = \frac{\partial z_d}{\partial\theta}.$$

Claim: v_0 satisfies the limit equation. Choose a test function $\psi\in C^\infty(\overline{\mathcal{O}})$ in the variational formulation of the polar form of (6.7), that is,

$$\begin{aligned}\int_{\mathcal{O}_\varepsilon}\left(\frac{\partial\overline{v}_\varepsilon}{\partial r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial\overline{v}_\varepsilon}{\partial\theta}\frac{\partial\psi}{\partial\theta}+\overline{v}_\varepsilon\psi\right)rdrd\theta &:= \int_{\mathcal{O}_\varepsilon^+}\left(\frac{\partial\overline{v}_\varepsilon^+}{\partial r}\frac{\partial\psi^+}{\partial r}+\frac{1}{r^2}\frac{\partial\overline{v}_\varepsilon^+}{\partial\theta}\frac{\partial\psi^+}{\partial\theta}+\overline{v}_\varepsilon^+\psi^+\right)rdrd\theta\\ &\quad+\int_{\mathcal{O}^-}\left(\frac{\partial\overline{v}_\varepsilon^-}{\partial r}\frac{\partial\psi^-}{\partial r}+\frac{1}{r^2}\frac{\partial\overline{v}_\varepsilon^-}{\partial\theta}\frac{\partial\psi^-}{\partial\theta}+\overline{v}_\varepsilon^-\psi^-\right)rdrd\theta\\ &=\int_{\mathcal{O}_\varepsilon}\left(\frac{\partial(\overline{z}_\varepsilon-z_d)}{\partial r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial(\overline{z}_\varepsilon-z_d)}{\partial\theta}\frac{\partial\psi}{\partial\theta}\right)rdrd\theta.\end{aligned}$$

Using the value of R and the convergence of $\overline{v}_\varepsilon$, we show that v_0 satisfies the equation

$$\begin{aligned}&\int_{\mathcal{O}^+}\frac{h(r)}{2\pi}\left(\frac{\partial v_0^+}{\partial r}\frac{\partial\psi^+}{\partial r}+v_0^+\psi^+\right)rdrd\theta+\int_{\mathcal{O}^-}\left(\frac{\partial v_0^-}{\partial r}\frac{\partial\psi^-}{\partial r}+\frac{1}{r^2}\frac{\partial v_0^-}{\partial\theta}\frac{\partial\psi^-}{\partial\theta}+v_0^-\psi^-\right)rdrd\theta\\ &=\int_{\mathcal{O}^+}\frac{h(r)}{2\pi}\left(\frac{\partial(z_0^+-z_d)}{\partial r}\frac{\partial\psi^+}{\partial r}\right)rdrd\theta+\int_{\mathcal{O}^-}\left(\frac{\partial(z_0^--z_d)}{\partial r}\frac{\partial\psi^-}{\partial r}+\frac{1}{r^2}\frac{\partial(z_0^--z_d)}{\partial\theta}\frac{\partial\psi^-}{\partial\theta}\right)rdrd\theta\end{aligned}$$

for all $\psi\in C^\infty(\overline{\mathcal{O}})$. As we know that $C^\infty(\overline{\mathcal{O}})$ is dense in $W(\mathcal{O})$, the above equation is true for all ψ in $W(\mathcal{O})$. Therefore, v_0 satisfies the weak formulation of the adjoint problem (6.7). Hence, we have the following convergences:

$$\begin{aligned}&\widetilde{\overline{v}_\varepsilon^+}\rightharpoonup\frac{h(r)}{2\pi}v_0^+,\ \widetilde{\frac{\partial\overline{v}_\varepsilon^+}{\partial r}}\rightharpoonup\frac{h(r)}{2\pi}\frac{\partial v_0^+}{\partial r}\quad\text{weakly in } L^2(\mathcal{O}^+),\ \text{and}\\ &\overline{v}_\varepsilon^-\rightharpoonup v_0^-\quad\text{weakly in } H^1(\mathcal{O}^-).\end{aligned}$$

To prove the convergence of the optimality system, it is enough to prove $\overline{\varrho}=\varrho_0$. Recall the optimality condition (6.3) given by

$$\overline{\varrho}_\varepsilon(r,\tau)|_D=\frac{-1}{2\pi\beta}\int_0^{2\pi}(T^\varepsilon\overline{v}_\varepsilon^+)(r,\theta,\tau)\,d\theta. \tag{6.17}$$

By the convergences of $\overline{\varrho}_\varepsilon$ and $T^\varepsilon \overline{v}_\varepsilon^+$, as $\varepsilon \to 0$, the optimality condition becomes

$$\varrho_0(r,\tau) = \frac{-1}{2\pi\beta} \int_0^{2\pi} v_0^+(r,\theta,\tau)\, d\theta.$$

By the fact that v_0 is independent of τ, we conclude that q_0 also independent of the variable τ. Hence, $\varrho_0 = \dfrac{-1}{2\pi\beta} \displaystyle\int_0^{2\pi} v_0^+ d\theta$. Therefore, we get the optimality system corresponding to the minimization problem (E). Then Theorem 6.1 says that the optimal solution is (v_0, ϱ_0). Hence by the uniqueness, we have $\overline{z} = z_0$, $\overline{v} = v_0$ and $\overline{\varrho} = \varrho_0$ which completes the proof. $\square$

7. Conclusions

We have analyzed the homogenization of an optimal control problem with two different cost functionals posed on a rough circular domain. We used unfolding operators for this study. First, we converted the problem into polar coordinates and then using the unfolding operator, we derived the limit problem. The novelty, in addition to the main result namely the convergence analysis, is the characterization of the optimal control in the non-homogenized level itself using the unfolding operator.

Appendix A

A.1. (Proof of Proposition 3.3)

$$\begin{aligned}
\int_{\mathcal{O}_u} T^\varepsilon u \; r dr d\theta d\tau &= \int_{r_0}^{r_1} \int_{Y(r)} \int_0^{2\pi} u\left(r, \varepsilon \left[\tfrac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right)\; r d\theta d\tau dr \\
&= \int_{r_0}^{r_1} \int_{Y(r)} \sum_{k=0}^{N-1} \int_{2k\varepsilon\pi}^{2(k+1)\varepsilon\pi} u(r, 2k\varepsilon\pi + \varepsilon\tau)\; r d\theta d\tau dr \\
&= \sum_{k=0}^{N-1} \int_{2k\varepsilon\pi}^{2(k+1)\varepsilon\pi} d\theta \int_{r_0}^{r_1} \int_{Y(r)} u(r, 2k\varepsilon\pi + \varepsilon\tau)\; d\tau r dr \\
&= 2\pi\varepsilon \sum_{k=0}^{N-1} \int_{r_0}^{r_1} \int_{Y(r)} u(r, 2k\varepsilon\pi + \varepsilon\tau)\; d\tau r dr \\
&= 2\pi \sum_{k=0}^{N-1} \int_{r_0}^{r_1} \int_{2k\varepsilon\pi + \varepsilon Y(r)} u(r,z)\; r dz dr \\
&= 2\pi \int_{\mathcal{O}_\varepsilon^+} u(r,\theta)\; r dr d\theta.
\end{aligned}$$

A.2. *(Proof of Proposition 3.4)*

For any $u \in L^2(\mathcal{O}_\varepsilon^+)$, we get $|u|^2 \in L^1(\mathcal{O}_\varepsilon^+)$. Hence, Proposition 3.3 implies

$$\int_{\mathcal{O}_u} |T^\varepsilon u|^2 \, rdrd\theta = \int_{\mathcal{O}_u} T^\varepsilon |u|^2 \, rdrd\theta \; = 2\pi \int_{\mathcal{O}_\varepsilon^+} |u|^2 \, rdrd\theta.$$

Hence, we get $\|T^\varepsilon u\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\|u\|_{L^2(\mathcal{O}_\varepsilon^+)}$ and thus we have the result. □

A.3. *(Proof of Proposition 3.5)*

By using the Definition 3.1, we can easily check that

$$\frac{\partial}{\partial r} T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial r} \text{ and } \frac{\partial}{\partial \tau} T^\varepsilon u = \varepsilon T^\varepsilon \frac{\partial u}{\partial \theta}.$$

By the Proposition 3.4, we have

$$\|T^\varepsilon u\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi}\|u\|_{L^2(\mathcal{O}_\varepsilon^+)},$$

$$\left\| T^\varepsilon \frac{\partial u}{\partial r} \right\|_{L^2(\mathcal{O}_u)} = \sqrt{2\pi} \left\| \frac{\partial u}{\partial r} \right\|_{L^2(\mathcal{O}_\varepsilon^+)}$$

and

$$\left\| \frac{\partial}{\partial \tau} T^\varepsilon u \right\|_{L^2(\mathcal{O}_u)} = \varepsilon\sqrt{2\pi} \left\| \frac{\partial u}{\partial \theta} \right\|_{L^2(\mathcal{O}_\varepsilon^+)}.$$

Hence, the result follows from the hypothesis. □

A.4. *(Proof of Proposition 3.6)*

Consider $\phi \in \mathcal{D}(\mathcal{O}^+)$.

$$\begin{aligned}\sup_{(r,\theta,\tau)\in\mathcal{O}_u} |(T^\varepsilon\phi)(r,\theta,\tau) - \phi(r,\theta)| &= \sup_{(r,\theta,\tau)\in\mathcal{O}_u} \left|\phi(r, \varepsilon[\tfrac{\theta}{\varepsilon}]_{2\pi} + \varepsilon\tau) - \phi(r,\theta)\right| \\ &\leqslant m_\phi(2\pi\varepsilon),\end{aligned}$$

where m_ϕ is the modulus of continuity of the function ϕ which is defined as

$$m_\phi(\delta) = \sup_{z_1,z_2\in\mathcal{O}^+} \{|\phi(z_1) - \phi(z_2)| : |z_1 - z_2| < \delta\}.$$

Since ϕ is uniformly continuous in $\mathcal{O}^+$, $m_\phi(2\pi\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence,

$$\sup_{\mathcal{O}_u} |T^\varepsilon \phi - \phi| \to 0 \; as \; \varepsilon \to 0.$$

Thus, $T^\varepsilon \phi \to \phi$ in $L^2(\mathcal{O}_u) \;\; \forall \, \phi \in \mathcal{D}(\mathcal{O}^+)$. The density of $\mathcal{D}(\mathcal{O}^+)$ in $L^2(\mathcal{O}^+)$ completes the first part of the proposition. To see the second part, suppose that $y_\varepsilon \to y$ in $L^2(\mathcal{O}^+)$. Then, Proposition 3.4 implies

$$\begin{aligned}
\|T^\varepsilon y_\varepsilon - y\|_{L^2(\mathcal{O}_u)} &= \|T^\varepsilon y_\varepsilon - T^\varepsilon y + T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&\leqslant \|T^\varepsilon y_\varepsilon - T^\varepsilon y\|_{L^2(\mathcal{O}_u)} + \|T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&= \|T^\varepsilon (y_\varepsilon - y)\|_{L^2(\mathcal{O}_u)} + \|T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&= \sqrt{2\pi}\|y_\varepsilon - y\|_{L^2(\mathcal{O}_\varepsilon^+)} + \|T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&\leqslant \sqrt{2\pi}\|y_\varepsilon - y\|_{L^2(\mathcal{O}^+)} + \|T^\varepsilon y - y\|_{L^2(\mathcal{O}_u)} \\
&\to 0 \;\; as \;\; \varepsilon \to 0.
\end{aligned}$$

□

A.5. (Proof of Proposition 3.7)

Let $\psi \in \mathcal{D}(\mathcal{O}^+)$. Then, Propositions 3.2, 3.3 and 3.6 imply

$$\begin{aligned}
\int_{\mathcal{O}^+} \widetilde{y}_\varepsilon \psi \, rdrd\theta &= \frac{1}{2\pi}\int_{\mathcal{O}_u} T^\varepsilon y_\varepsilon T^\varepsilon \psi \, rdrd\theta d\tau \\
&\to \frac{1}{2\pi}\int_{\mathcal{O}_u} y\psi \, rdrd\theta d\tau \; as \; \varepsilon \to 0, \\
&= \int_0^{2\pi}\int_{r_0}^{r_1}\left(\frac{1}{2\pi}\int_{Y(r)} y \, d\tau\right)\psi \, rdrd\theta \\
&= \int_{\mathcal{O}^+}\left(\frac{1}{2\pi}\int_{Y(r)} y \, d\tau\right)\psi \, rdrd\theta \;\; \forall \psi \in \mathcal{D}(\mathcal{O}^+).
\end{aligned}$$

This completes the proof as $\mathcal{D}(\mathcal{O}^+)$ is dense in $L^2(\mathcal{O}^+)$. □

A.6. (Proof of Proposition 3.8)

Since

$$T^\varepsilon y_\varepsilon \rightharpoonup y \text{ and } \frac{\partial}{\partial r} T^\varepsilon y_\varepsilon \rightharpoonup \frac{\partial y}{\partial r} \text{ weakly in } L^2(\mathcal{O}_u),$$

which means

$$T^\varepsilon y_\varepsilon \rightharpoonup y \text{ and } T^\varepsilon \frac{\partial y_\varepsilon}{\partial r} \rightharpoonup \frac{\partial y}{\partial r} \text{ weakly in } L^2(\mathcal{O}_u).$$

Using Proposition 3.7, we get $\widetilde{y}_\varepsilon \rightharpoonup \dfrac{1}{2\pi}\displaystyle\int_{Y(r)} y\, d\tau$ in $L^2(\mathcal{O}^+)$ and $\widetilde{\dfrac{\partial y_\varepsilon}{\partial r}} \rightharpoonup \dfrac{1}{2\pi}\displaystyle\int_{Y(r)} \dfrac{\partial y}{\partial r}\, d\tau$ in $L^2(\mathcal{O}^+)$.

□

Appendix B

B.1. (Proof of Theorem 5.2)

Assume $(\overline{y}, \overline{q})$ is an optimal solution to (P) and $y(q)$ is a solution of (5.1) for a fixed arbitrary q. Now, set $F(q) = J(y(q), q)$, then

$$F'(\overline{q})q = \int_{\mathcal{O}^-} (\overline{y}^- - y_d)w^- r dr d\theta + \int_{\mathcal{O}^+} \frac{h(r)}{2\pi}(\overline{y}^+ - y_d)w^+ r dr d\theta + \beta \int_{r_0}^{r_1} h(r)\, \overline{q} q r dr$$

where w is the solution of the equation (B.1) given below.

$$\begin{cases} -\dfrac{\partial}{\partial r}\left(h(r)\dfrac{\partial w^+}{\partial r}\right) - \dfrac{h(r)}{r}\dfrac{\partial w^+}{\partial r} + h(r)w^+ = q & \text{in } \mathcal{O}^+, \\ -\Delta w^- + w^- = 0 & \text{in } \mathcal{O}^-, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\mathcal{O}, \\ w^+ = w^-, \quad \dfrac{h(r_1)}{2\pi}\dfrac{\partial w^+}{\partial r} = \dfrac{\partial w^-}{\partial r} & \text{on } \Gamma_0. \end{cases} \tag{B.1}$$

Since $(\overline{y}, \overline{q})$ is a solution to (P), we have $F'(\overline{q})q = 0$ for all $q \in L^2(r_0, r_1)$. That is

$$\int_{\mathcal{O}^-} (\overline{y}^- - y_d)w^- r dr d\theta + \int_{\mathcal{O}^+} \frac{h(r)}{2\pi}(\overline{y}^+ - y_d)w^+ r dr d\theta = -\beta \int_{r_0}^{r_1} h(r)\, \overline{q} q r dr \tag{B.2}$$

Choosing w and $\overline{p}$ as the test functions in the weak formulation of the equations (5.3) and (B.1) respectively, we get

$$\int_{\mathcal{O}^+} \frac{h(r)}{2\pi}\left(\frac{\partial w^+}{\partial r}\frac{\partial \overline{p}^+}{\partial r} + w^+\overline{p}^+\right) r dr d\theta + \int_{\mathcal{O}^-}\left(\frac{\partial w^-}{\partial r}\frac{\partial \overline{p}^-}{\partial r} + \frac{1}{r^2}\frac{\partial w^-}{\partial \theta}\frac{\partial \overline{p}^-}{\partial \theta} + w^-\overline{p}^-\right) r dr d\theta$$
$$= \frac{1}{2\pi}\int_{\mathcal{O}^+} h(r)\overline{p}^+ q r dr d\theta$$

and

$$\int_{\mathcal{O}^+} \frac{h(r)}{2\pi}\left(\frac{\partial \overline{p}^+}{\partial r}\frac{\partial w^+}{\partial r} + \overline{p}^+ w^+\right) r dr d\theta + \int_{\mathcal{O}^-}\left(\frac{\partial \overline{p}^-}{\partial r}\frac{\partial w^-}{\partial r} + \frac{1}{r^2}\frac{\partial \overline{p}^-}{\partial \theta}\frac{\partial w^-}{\partial \theta} + \overline{p}^- w^-\right) r dr d\theta$$
$$= \int_{\mathcal{O}^+} \frac{h(r)}{2\pi}(\overline{y}^+ - y_d)w^+ r dr d\theta + \int_{\mathcal{O}^-} (\overline{y}^- - y_d)w^- r dr d\theta.$$

This implies

$$\int_{\mathcal{O}^-} (\bar{y}^- - y_d) w^- r dr d\theta + \int_{\mathcal{O}^+} \frac{h(r)}{2\pi} (\bar{y}^+ - y_d) w^+ r dr d\theta = \frac{1}{2\pi} \int_{\mathcal{O}^+} h(r) \overline{p}^+ q r dr d\theta. \tag{B.3}$$

Now, we compare the equations (B.2) and (B.3) to get

$$-\beta \int_{r_0}^{r_1} h(r)\, \overline{q} q r dr = \frac{1}{2\pi} \int_{\mathcal{O}^+} h(r) \overline{p}^+ q r dr d\theta = \int_{r_0}^{r_1} h(r) \left(\int_{\theta=0}^{2\pi} \frac{1}{2\pi} \overline{p}^+ d\theta \right) q r dr.$$

Hence we get $\overline{q} = \dfrac{-1}{2\pi\beta} \displaystyle\int_0^{2\pi} \overline{p}^+ d\theta.$ □

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