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Locally periodic unfolding operator for highly oscillating rough domains

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Abstract
This article aims to understand the locally periodic oscillating domain via unfolding operators. A three-dimensional rough domain Ω_ε, $\varepsilon > 0$ a small parameter, has been considered for the study where the boundary is rapidly oscillating with high amplitude. Though there are some articles with locally periodic boundary oscillations with small amplitude we do not see any literature with high-amplitude $(O(1))$ locally periodic oscillating domains. In this article, we attempt to study a problem in locally periodic rough domains with an eye towards the general oscillating domains without periodicity. With our experience of handling such domains and unfolding operators, we develop locally periodic unfolding operators to study our problems. We consider a nonlinear inhomogeneous Robin boundary value problem posed on this domain to demonstrate the utility of the newly defined operator.

Keywords Asymptotic analysis · Unfolding operator · Locally periodic oscillating boundary domain · Homogenization

Mathematics Subject Classification 80M35 · 80M40 · 35B27

1 Introduction

Study of partial differential equations in domains with rough oscillating boundaries has been of interest for the past several years due to its wide range of applications in several fields. Due to the rapid oscillations of the boundary (the amplitude can be small like $O(\varepsilon^\alpha),\ \alpha > 0$

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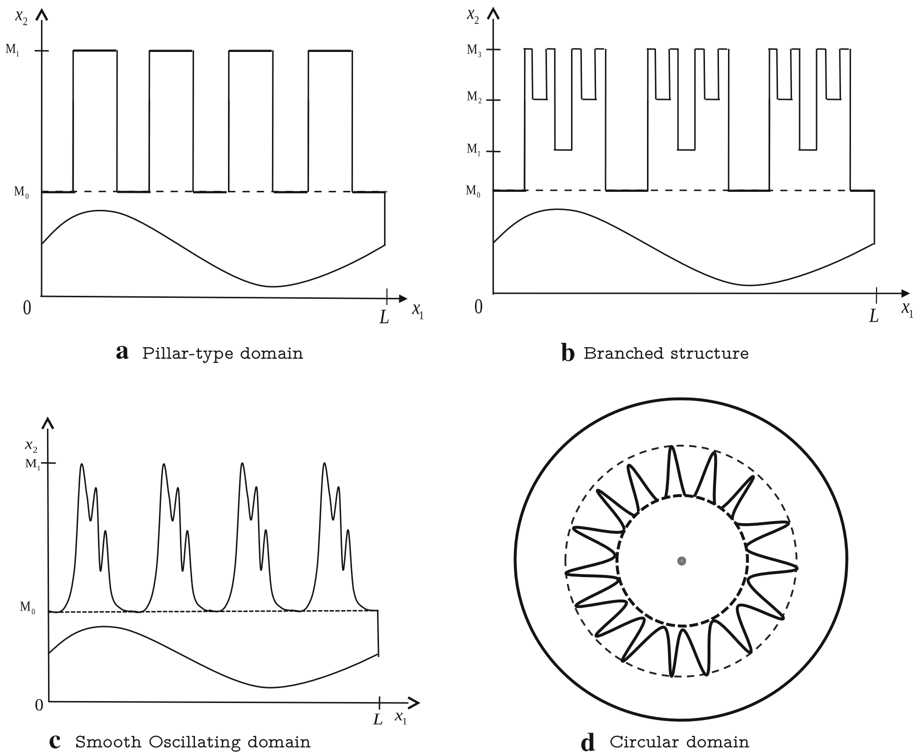
or large like $O(1)$), an asymptotic analysis is called for to obtain the limit equation and it is the topic of homogenization. There are various real-life applications like heat radiators, flows in channels with rough boundaries, propagation of electromagnetic waves in regions having rough interface, absorption–diffusion in biological structures, acoustic vibrations in a medium with narrow channels, etc. For the literature on homogenization of boundary value problems in rough domains, we refer to [2,5,6,10,13,15,19]. Recently, Braides et al. have studied the homogenization of networks on oscillating boundary domain [14]. As mentioned, study on oscillating domains is an active area of research and there is a vast amount of literature and the present authors and collaborators have also published several papers in this direction. See [2,3,21,24–26] and references therein. But, one of the fundamental assumption is the periodicity of oscillating boundaries. We have made substantial progress starting with pillar-type domains (see Fig. 1a); to branched structure domains (see Fig. 1b); to general smooth domains (see Fig. 1c, d). Of course, these domains are motivated from applications like heat radiators, jet engines, etc. (see Fig. 2a, b). Figure 2a leads to pillar-type domains, whereas Fig. 2b models branched structures/circular domains.

In [18], the authors consider an oscillating boundary domain without the periodicity assumption and study the asymptotic behavior of a brush problem with a L^1 source term using oscillating test functions method. But not much literature is available for rugous domain when the periodicity is removed. In this paper, we are making an attempt to consider a domain with oscillating boundaries which are locally periodic (see Fig. 3). We can also consider locally periodic circular domains. This is a precursor to the study of general oscillating domains



a Pillar-type domain

b Branched structure

c Smooth Oscillating domain

d Circular domain

Fig. 1 Oscillating domains

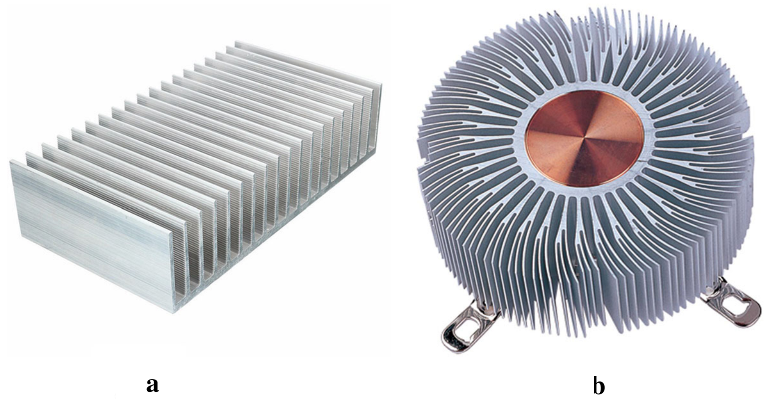


Fig. 2 Heat radiators

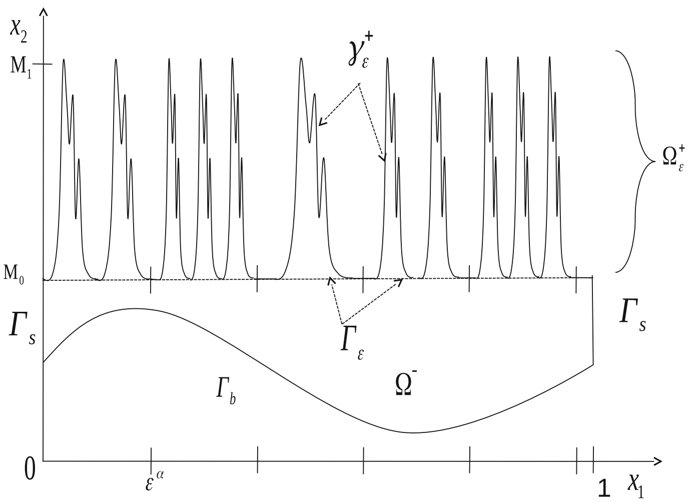


Fig. 3 Locally periodic smooth oscillating domain Ω_ε

without periodicity assumption. Among several methods developed in the last five decades, the periodic unfolding is the latest. We have successfully used it to study the asymptotic analysis of optimal control/controllability problems and characterize optimal control, etc (see [3,25,26]). Further, we have also introduced new unfolding operators as and when it is necessary.

In fact, one of our novel and new approach was to characterize optimal controls using unfolding operators (see [3,25]). Otherwise, we do not see how to characterize optimal controls in oscillating domains. The periodic unfolding was developed by Cioranescu et. al. to study the homogenization of boundary value problems with oscillating coefficients [16]. In [11], Blanchard et al. have modified the unfolding operators to study problems in periodic rectangular oscillating domains (pillar-type domains). The problems with smooth oscillating boundaries were much more difficult. In a novel way, we have developed general unfolding operators for a wide class of oscillating domains (see [2]). For more literature on periodic unfolding operators, we refer to [4,8,12,17,24,26]. For locally periodic oscillating domains, one can look at the work of Arrieta et al. [7,9], where they study the homogenization problems posed on locally periodic oscillating domains with amplitude of order ε. Here, the oscillating part shrinks down to the boundary of non-oscillating part of the domain as ε tends to zero. These types of domains are referred as thin structures in the literature. For further works on homogenization of locally periodic setup, we refer to [27,28].

In this article, we consider domains with locally periodic oscillations of amplitude $O(1)$ and there is no literature in this direction. We develop suitable unfolding operators for these domains, and we use this successfully to study the homogenization of an elliptic problem with nonlinear boundary condition. In [29], Ptashnyk has developed unfolding operators for problems with locally periodic oscillating coefficients. The development of unfolding operators by Ptashnyk for problems with locally periodic highly oscillating coefficients motivated us to develop unfolding operators for oscillating domains.

In the first part, we introduce and explain full geometry of the locally periodic oscillating domain $\Omega_\varepsilon \subset \mathbb{R}^3$, rigorously in Sect. 2. A two-dimensional schematic representation of Ω_ε is given in Fig. 3. After defining Ω_ε, we introduce the unfolding operator T^ε in Sect. 3 and study its properties. To demonstrate the applications of the new unfolding operator, we consider the following linear elliptic equation in the domain Ω_ε with a nonlinear, non-homogeneous boundary condition on the oscillating domain. This type of model problem has been considered by Mel'nyk on a periodic thick junction of type 3:2:1 (see [22]). All the notations which are used in the following are given in Sect. 2.

$$\begin{cases} -\Delta y_\varepsilon + y_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ \partial_\nu y_\varepsilon + \varepsilon^{\alpha_1}\mu(x_3, y_\varepsilon) = u_\varepsilon^\varepsilon \text{ on } \gamma_\varepsilon^+, \\ y_\varepsilon = 0 \text{ on } \Gamma_b, \\ y_\varepsilon \; is \; \Gamma_s - periodic. \end{cases} \tag{1.1}$$

Here, the boundary data $u_\varepsilon^\varepsilon$ is given by

$$u_\varepsilon^\varepsilon(x) = u_\varepsilon^\varepsilon(x_1, x_2, x_3) =: \begin{cases} \varepsilon^{\alpha_2} u^\varepsilon(x) & if \;\; x_3 \in (M_0, M_1) \\ u^\varepsilon(x) & if \;\; x_3 = M_0 \; or \; M_1. \end{cases} \tag{1.2}$$

where u^ε is a locally periodic data, defined on the oscillating boundary γ_ε^+, arising from a fixed data $u \in L^2_\#(\gamma)$. The detailed definition is given in Sect. 4. In future projects, it is possible to use these data as control function. Further, f is a given function in $L^2_\#(\Omega)$; $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$ are fixed constants; $\mu : [M_0, M_1] \times \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth function such that $\mu(\cdot, 0) = 0$ and

$$|\partial_{x_3}\mu(x_3, s)| \leq C_0, \;\; C_1 \leq \partial_s \mu(x_3, s) \leq C_2. \tag{1.3}$$

The well-posedness of the problem is given in Sect. 4.

Our aim is to study the asymptotic behavior of the solution y_ε of problem (1.1) as ε tends to zero. That is, when the number of pillars in the domain Ω_ε grows to infinity and the diameter of each pillar boils down to zero. Generally, in homogenization of oscillating domains, people construct an extension operator (which will be problem dependent) to a fixed domain and using that extension operator, one derives the limit equation. Sometimes, this becomes difficult for certain problems and varies from problems to problems. Here, we use the unfolding operator that we have developed to study the homogenization of the problem. Note that this method just depends on the domain not on the problem unlike extension operators. Now, we present the main homogenization result in the following theorem.

Theorem 1.1 (Main theorem). *Let* $y_\varepsilon = y_\varepsilon^+ \chi_{\Omega_\varepsilon^+} + y_\varepsilon^- \chi_{\Omega^-}$ *be the weak solution of* (1.1). *Then, we have*

$$y_\varepsilon^- \rightharpoonup y^- \ \text{ weakly in } \ H^1\left(\Omega^-\right) \tag{1.4}$$

$$\widetilde{y_\varepsilon^+} \rightharpoonup h(x_3)\, y^+ \ \text{ weakly in } \ L^2(I; H^1(M_0, M_1)), \tag{1.5}$$

where $y = y^+\chi_{\Omega^+} + y^-\chi_{\Omega^-}$ *is the weak solution of*

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial y^+}{\partial x_3}\right) + h(x_3)y^+ + \delta_{1\alpha_1}\dfrac{l}{d}\mu(x_3, y^+) = h(x_3)f^+ + \delta_{1\alpha_2}\dfrac{1}{d}\theta & in\ \Omega^+, \\ -\Delta y^- + y^- = f^- & in\ \Omega^-, \\ \dfrac{\partial y^+}{\partial \nu} = \rho_1 & on\ \Gamma_u, \\ y^- = 0 & on\ \Gamma_b, \\ y^+ = y^-, \quad h(M_0)\dfrac{\partial y^+}{\partial x_3} + \rho_0 = \dfrac{\partial y^-}{\partial x_3} & on\ \Gamma_0, \\ y\ \ is\ \Gamma_s - periodic. \end{cases} \tag{1.6}$$

Here, $f = f_0^+\chi_{\Omega^+} + f^-\chi_{\Omega^-}$*;* $l(x_3) = |\partial Y(x_3)|$*-the perimeter of the boundary of the reference set* $Y(x_3)$*;* $h(x_3) = |Y(x_3)|$*-the Lebesgue measure of the set* $Y(x_3)$*;* $d(x_1, x_2)$ *is the Lipschitz function which decides the local periodicity as defined in Sect.* 2 *and*

$$\theta(x_3) = \int_{z' \in \partial Y(x_3)} u(z', x_3)\mathrm{d}\gamma_{x_3}(z').$$

The constants ρ_0 *and* ρ_1 *are given by* $\rho_0 = \int_{z' \in Y'(M_0)} u\left(z', M_0\right)\mathrm{d}z'$, $\rho_1 = \int_{z' \in Y(M_1)} u\left(z', M_1\right)$ $\mathrm{d}z'$*, respectively and* δ_{ab} *is the usual Kronecker delta function. Here, and in the sequel* $\widetilde{\psi}$ *represents the zero extension of* ψ *to* Ω^+*.* □

The layout of the paper is as follows. In Sect. 2, the locally periodic domain is described. The unfolding operator for this domain is defined, and its properties are studied in Sect. 3. The description of model problem and the uniform estimate of the solution (independent of the parameter ε) are given in Sect. 4. The limit problem and the limit space are also presented in this section. The proof of the main theorem is given in Sect. 5.

2 Domain description

The oscillating domain Ω_ε consists of two parts, namely the oscillating upper part Ω_ε^+ and a fixed lower part Ω^-. First, we will describe the upper part Ω_ε^+. Let $\eta : \overline{I} =: [0, 1] \times [0, 1] \to [M_0, M_1]$ be a smooth function such that $supp(\eta - M_0) \subset I =: (0, 1) \times (0, 1)$. Then, it is extended to $\mathbb{R}^2$, (1, 1)-periodically. Here, M_0 and M_1 are the minimum and maximum of the smooth function η on $\overline{I}$. Now we define the reference cell Λ as $\Lambda = \{(x', x_3) : x' = (x_1, x_2) \in \overline{I}, 0 < x_3 < \eta(x')\}$ and $\Lambda^+ = \{(x', x_3) : x' \in \overline{I}, M_0 < x_3 < \eta(x')\}$. We define γ, the top upper surface of Λ, as $\gamma = \{(x', \eta(x')) : x' \in \overline{I}\}$. The surface γ is divided into two parts, namely the union of flat surfaces F and the lateral/non-flat surface S. That is, F is defined as $F =: \{(x', \eta(x')) \in \gamma : \nabla\eta(x') = 0\}$ and it can be thought of union flat surfaces at different heights as $F = \bigcup_{l=0}^{m} F_l$ with the flat surface at $x_3 = t_l$ for $M_0 \le t_l \le M_1$ is given by $F_l = \{(x', x_3) \in F : x_3 = t_l\}$ and $S = (\partial\Lambda)^+ \backslash \overline{F}$. For $a \in [M_0, M_1)$, define the reference set

$$Y(a) = \{y' \in I : \eta(y') > a\}.$$

and $Y(M_1) = \{y' \in I : \eta(y') = M_1\}$. We remark that $Y(a)$ plays a major role in defining the unfolding operator. This is the novel approach in the new definition of our unfolding operators in locally periodic domain. We hope this study will be a precursor to the analysis of general non-periodic case in future. Another reference set at $x_3 = M_0$ which will be used later is $Y'(M_0)$, which is defined as $Y'(M_0) = \overline{I} \backslash Y(M_0)$. We choose η such that $h(x_3) =: |Y(x_3)| > \kappa$ for all $x_3 \in [M_0, M_1]$ and for some $\kappa > 0$, where $h(x_3)$ is the two-dimensional Lebesgue measure of the reference set $Y(x_3)$. Let α be an arbitrary constant with $0 < \alpha < 1$ and N_ε be defined as the smallest integer such that $[\varepsilon^\alpha N_\varepsilon] = 1$. We assume $\varepsilon > 0$ represents a family of real numbers converging to zero. The mesocell, Ω^ε_K for $K = (k_1, k_2) \in \mathcal{K} = \{(n_1, n_2) \in \mathbb{Z}^2 : 0 \leq n_1, n_2 \leq N_\varepsilon - 1\}$, is defined as

$$\Omega^\varepsilon_K = (k_1\varepsilon^\alpha, (k_1 + 1)\varepsilon^\alpha) \times (k_2\varepsilon^\alpha, (k_2 + 1)\varepsilon^\alpha).$$

Note that the unit cell $\overline{I}$ is covered by the closure of the union of the mesocells Ω^ε_K. The Lipschitz function $d : \overline{I} \to [d_1, d_2]$ decides the local periodicity where d_1 and d_2 are the minimum and maximum of the Lipschitz function d on $\overline{I}$ with $0 < d_1 < d_2$. Let $x^\varepsilon_K = (x^\varepsilon_{k_1}, x^\varepsilon_{k_2})$ be an arbitrarily point chosen in Ω^ε_K. The translated unit cell I_Z, where $Z = (i, j) \in \mathbb{Z}^2$, is defined by $I_Z = (i, i + 1) \times (j, j + 1)$. Now, define the counting set E^ε_K as

$$E^\varepsilon_K = \{Z = (i, j) \in \mathbb{Z}^2 : \varepsilon d_{x^\varepsilon_K} I_Z \subset \Omega^\varepsilon_K \cap I\}.$$

Here, $d_{x^\varepsilon_K}$ denotes the value of d at x^ε_K that is $d_{x^\varepsilon_K} = d(x^\varepsilon_K)$. Now define $\hat{\Omega}^\varepsilon_K$ as

$$\hat{\Omega}^\varepsilon_K = \bigcup_{Z \in E^\varepsilon_K} \varepsilon d_{x^\varepsilon_K} I_Z$$

and the microcells $\Omega^\varepsilon_{K,Z}$ are given by

$$\Omega^\varepsilon_{K,Z} = \varepsilon d_{x^\varepsilon_K} I_Z.$$

To reduce the complexity in the notation, we represent any $x \in \mathbb{R}^3$ as $x = (x', x_3)$ where $x' = (x_1, x_2) \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$. The oscillating function $\eta_\varepsilon : \overline{I} \to [M_0, M_1]$ is defined as

$$\eta_\varepsilon(x') = \sum_{K \in \mathcal{K}} \eta\left(d^{-1}_{x^\varepsilon_K} \frac{x'}{\varepsilon}\right) \chi_{\hat{\Omega}^\varepsilon_K}(x') + M_0\, \chi_{\left\{\overline{I} \backslash \cup_K \hat{\Omega}^\varepsilon_K\right\}}(x').$$

Here, $\dfrac{x'}{\varepsilon} = \left(\dfrac{x_1}{\varepsilon}, \dfrac{x_2}{\varepsilon}\right)$. Now, define the oscillating upper part of the domain Ω_ε to be

$$\Omega^+_\varepsilon = \{(x', x_3) :\ x' \in I,\ M_0 < x_3 < \eta_\varepsilon(x')\}.$$

The fixed lower part Ω^- is defined as $\Omega^- = \{(x', x_3) : x' \in I, g(x') < x_3 < M_0\}$, where $g : \mathbb{R}^2 \to (0, M')$ is a smooth and periodic function with period I. Here, g is chosen so that $M' < M_0$.

Now, the domain under consideration is defined by $\Omega_\varepsilon = \text{Interior}\ \left(\overline{\Omega^+_\varepsilon \cup \Omega^-}\right)$. The oscillating boundary γ^+_ε is given by $\gamma^+_\varepsilon = \{(x', x_3) :\ x' \in \overline{I},\ x_3 = \eta_\varepsilon(x')\}$. The bottom boundary Γ_b of Ω_ε is defined by $\Gamma_b = \{(x', x_3) : \ x_3 = g(x'),\ x' \in \overline{I}\}$. The lateral and top boundaries of Ω^- denoted by Γ_s and Γ_0 are defined as

$$\begin{aligned}\Gamma_s = \{&(i, x_2, x_3) : \ g(i, x_2) \leq x_3 \leq M_0, i = 0, 1, \}\\ &\cup\{(x_1, i, x_3) : \ g(x_1, i) \leq x_3 \leq M_0, i = 0, 1\}\end{aligned}$$

and $\Gamma_0 = \{(x_1, x_2, M_0) : \ (x_1, x_2) \in \overline{I}\}$, respectively. The common boundary Γ_ε is the lower boundary of Ω_ε^+ given as $\Gamma_\varepsilon = \{(x', x_3) \in \Omega_\varepsilon : x_3 = M_0\}$. The full domain or the limit domain Ω is defined as $\Omega = \{(x', x_3) : x' \in I,\ \ g(x') < x_3 < M_1\}$, and the upper part Ω^+ of the limit domain Ω is then defined as $\Omega^+ = \{(x', x_3) : x' \in I,\ \ M_0 < x_3 < M_1\}$. The lateral boundaries $\Gamma_{s'}$ and top boundary Γ_u of the full domain Ω are defined as

$$\begin{aligned}\Gamma_{s'} = \{&(i, x_2, x_3) : \ g(i, x_2) \leq x_3 \leq M_1, i = 0, 1\}\\ &\cup\{(x_1, i, x_3) : \ g(x_1, i) \leq x_3 \leq M_1, i = 0, 1\}\end{aligned}$$

and $\Gamma_u = \{(x', M_1) : \ x' \in \overline{I}\}$. The periodic Sobolev space $H^1_\#(\Omega_\varepsilon)$ is defined as

$$H^1_\#(\Omega_\varepsilon) = \{f|_{\Omega_\varepsilon} : f \in H^1_{loc}(\mathbb{R}^3),\ f \text{ is } \Gamma_s - \text{periodic}\}.$$

We call a function Γ_s-periodic if it takes the same value on the opposite lateral sides of the domain Ω^- in the sense of trace.

3 Unfolding operator

We now present the unfolding operator for the locally periodic oscillating domain Ω_ε. For $x' \in \mathbb{R}^2$, we denote by $[x'] = ([x_1], [x_2])$ and $\{x'\} = (\{x_1\}, \{x_2\})$, where $[t]$ represents the integral part of t and $\{t\} = t - [t]$.

The unfolded domain Ω_U is defined as

$$\Omega_U =: \{(x', x_3, z')|\ x' \in I, M_0 < x_3 < M_1,\ \ z' \in Y(x_3)\}.$$

In other words, $\Omega_U = I \times \mathcal{G}$, where $\mathcal{G} = \{(x_3, z')|M_0 < x_3 < M_1,\ z' \in Y(x_3)\}$.

Definition 3.1 (The unfolding operator) Let Ω_ε^+ and Ω_U be the oscillating domain and the unfolded domain, respectively. The operator which maps every function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator which is denoted by T^ε. That is,

$$T^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{v : \Omega_U \to \mathbb{R}\}$$

defined by

$$T^\varepsilon \phi(x', x_3, z') = \sum_{K \in \mathcal{K}} \phi\left(\varepsilon d_{x_K^\varepsilon}\left[d^{-1}_{x_K^\varepsilon}\frac{x'}{\varepsilon}\right] + \varepsilon d_{x_K^\varepsilon} z', x_3\right) \chi_{\hat{\Omega}_K^\varepsilon}(x') \ \text{ for } \ (x', x_3, z') \in \Omega_U.$$

Some of the important properties of the unfolding operators are listed below.

Proposition 3.2 *For each fixed $\varepsilon > 0$, T^ε is linear. Further, if $u, v : \Omega_\varepsilon^+ \to \mathbb{R}$, then, $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.*

The proof follows directly from the definition.

Proposition 3.3 *Let $u \in L^2(\Omega_\varepsilon^+)$. Then,*

$$\int_{\Omega_U} T^\varepsilon u \,\mathrm{d}x\mathrm{d}z' = \int_{\Omega_\varepsilon^+} u \,\mathrm{d}x \text{ and } \|T^\varepsilon u\|_{L^2(\Omega_u)} \le \|u\|_{L^2(\Omega_\varepsilon^+)}.$$

Proof

$$\begin{aligned}
\int_{\Omega_U} T^\varepsilon u \,\mathrm{d}x\mathrm{d}z' &= \int_{x_3=M_0}^{M_1} \int_{z' \in Y(x_3)} \sum_{K\in\mathcal{K}} \int_{x'\in\hat{\Omega}_K^\varepsilon} u\left(\varepsilon d_{x_K^\varepsilon}\left[d_{x_K^\varepsilon}^{-1}\frac{x'}{\varepsilon}\right] + \varepsilon d_{x_K^\varepsilon} z', x_3\right) \mathrm{d}x'\mathrm{d}z'\mathrm{d}x_3 \\
&= \int_{M_0}^{M_1} \int_{z'\in Y(x_3)} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{x'\in\Omega_{K,Z}^\varepsilon} u(\varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} z', x_3) \,\mathrm{d}x'\mathrm{d}z'\mathrm{d}x_3 \\
&= \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \varepsilon^2 d_{x_K^\varepsilon}^2 \int_{M_0}^{M_1} \int_{\zeta'\in\varepsilon d_{x_K^\varepsilon}(Z+Y(x_3))} u(\zeta', x_3)\, \varepsilon^{-2} d_{x_K^\varepsilon}^{-2} \mathrm{d}\zeta'\mathrm{d}x_3 \\
&= \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{M_0}^{M_1} \int_{x'\in\varepsilon d_{x_K^\varepsilon}(Z+Y(x_3))} u(x', x_3)\,\mathrm{d}x'\mathrm{d}x_3 \\
&= \int_{\Omega_\varepsilon^+} u(x)\,\mathrm{d}x.
\end{aligned}$$

This proves the first part. The second part follows from the first part by taking $u = |u|^2$ and applying Proposition 3.2. $\square$

Proposition 3.4 *Let $u \in H^1(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2(I; H^1(\mathcal{G}))$ and $\|T^\varepsilon u\|_{L^2(I;H^1(\mathcal{G}))} \le \|u\|_{H^1(\Omega_\varepsilon^+)}$. Moreover, $\dfrac{\partial}{\partial x_3} T^\varepsilon u = T^\varepsilon \dfrac{\partial u}{\partial x_3}$ and $\dfrac{\partial}{\partial z_j} T^\varepsilon u = \varepsilon \sum_{K\in\mathcal{K}} d_{x_K^\varepsilon} T^\varepsilon \dfrac{\partial u}{\partial x_j} \chi_{\hat{\Omega}_K^\varepsilon} = \varepsilon d_\varepsilon T^\varepsilon \dfrac{\partial u}{\partial x_j}$, for $j = 1, 2$. Here, d_ε is a step function defined as $d_\varepsilon(x') =: \sum_{K\in\mathcal{K}} d_{x_K^\varepsilon} \chi_{\hat{\Omega}_K^\varepsilon}(x')$.*

Remark 3.5 It is easy see that the step function d_ε converges to d pointwise in I as $\varepsilon \to 0$. $\square$

Proof Since there is no oscillation in x_3 direction (and hence no unfolding), we obtain that $\dfrac{\partial}{\partial x_3} T^\varepsilon u = T^\varepsilon \dfrac{\partial u}{\partial x_3}$. Now look at the derivative of unfolding with respect to z_j for $j = 1, 2$.

$$\begin{aligned}
\partial_{z_j}(T^\varepsilon u) &= \sum_{K\in\mathcal{K}} \partial_{z_j}\left(u\left(\varepsilon d_{x_K^\varepsilon}\left[d_{x_K^\varepsilon}^{-1}\frac{x'}{\varepsilon}\right] + \varepsilon d_{x_K^\varepsilon} z', x_3\right)\right) \chi_{\hat{\Omega}_K^\varepsilon}(x') \\
&= \sum_{K\in\mathcal{K}} \partial_{x_j} u\left(\varepsilon d_{x_K^\varepsilon}\left[d_{x_K^\varepsilon}^{-1}\frac{x'}{\varepsilon}\right] + \varepsilon d_{x_K^\varepsilon} z', x_3\right) \varepsilon d_{x_K^\varepsilon} \chi_{\hat{\Omega}_K^\varepsilon}(x') \\
&= \sum_{K\in\mathcal{K}} T^\varepsilon \partial_{x_j} u(x', x_3, z') \varepsilon d_{x_K^\varepsilon} \chi_{\hat{\Omega}_K^\varepsilon}(x') = \varepsilon d_\varepsilon T^\varepsilon \left(\partial_{x_i} u\right)
\end{aligned}$$

Now, we have

$$\begin{aligned}
\|T^\varepsilon u\|^2_{L^2(I;H^1(\mathcal{G}))} &= \int\limits_I \|T^\varepsilon u\|^2_{H^1(\mathcal{G})}\,\mathrm{d}x' \\
&= \int\limits_{\Omega_U} \left(\sum_{j=1}^{2} \varepsilon^2 d_\varepsilon^2 T^\varepsilon \left|\frac{\partial u}{\partial x_j}\right|^2 + T^\varepsilon \left|\frac{\partial u}{\partial x_3}\right|^2 + T^\varepsilon |u|^2\right)\,\mathrm{d}x'\mathrm{d}x_3\mathrm{d}z' \\
&\le \int\limits_{\Omega_U} T^\varepsilon \left(\sum_{j=1}^{2} \varepsilon^2 d_2^2 \left|\frac{\partial u}{\partial x_i}\right|^2 + \left|\frac{\partial u}{\partial x_3}\right|^2 + |u|^2\right)\,\mathrm{d}x\mathrm{d}z' \\
&= \int\limits_{\Omega_\varepsilon^+} \left(\sum_{j=1}^{2} \varepsilon^2 \left|\frac{\partial u}{\partial x_j}\right|^2 + \left|\frac{\partial u}{\partial x_3}\right|^2 + |u|^2\right)\,\mathrm{d}x \le \|u\|^2_{H^1(\Omega_\varepsilon^+)} < \infty.
\end{aligned}$$

□

Proposition 3.6 *Let $u \in L^2(\Omega^+)$. Then, $T^\varepsilon u \to u$ strongly in $L^2(\Omega_u)$.*

Proof Consider $\phi \in \mathcal{D}(\Omega^+)$. Write $\|T^\varepsilon\phi - \phi\|^2_{L^2(\Omega_U)} = I_1 + I_2$ where I_1 and I_2 are integrals taken over $\cup_K \hat{\Omega}_K^\varepsilon$ and $\cup_K \Omega_K^\varepsilon \backslash \hat{\Omega}_K^\varepsilon$, respectively. We estimate them separately.

$$\begin{aligned}
I_1 &= \sum_{K\in\mathcal{K}} \int\limits_{x'\in\hat{\Omega}_K^\varepsilon} \int\limits_{x_3=M_0}^{M_1} \int\limits_{z'\in Y(x_3)} \left|\phi\left(\varepsilon d_{x_K^\varepsilon}\left[d_{x_K^\varepsilon}^{-1}\frac{x'}{\varepsilon}\right] + \varepsilon d_{x_K^\varepsilon} z', x_3\right) - \phi\left(x', x_3\right)\right|^2 \\
&\le C \sum_{K\in\mathcal{K}} \int\limits_{x'\in\hat{\Omega}_K^\varepsilon} \int\limits_{M_0}^{M_1} \int\limits_{z'\in Y(x_3)} \left|\left(\varepsilon d_{x_K^\varepsilon}\left[d_{x_K^\varepsilon}^{-1}\frac{x'}{\varepsilon}\right] + \varepsilon d_{x_K^\varepsilon} z', x_3\right) - \left(x', x_3\right)\right|^2 \\
&\le C \sum_{K\in\mathcal{K}} \int\limits_{x'\in\hat{\Omega}_K^\varepsilon} \int\limits_{M_0}^{M_1} \int\limits_{z'\in Y(x_3)} \left|\left(\varepsilon d_{x_K^\varepsilon}\left[d_{x_K^\varepsilon}^{-1}\frac{x'}{\varepsilon}\right] + \varepsilon d_{x_K^\varepsilon} z'\right) - x'\right|^2 \\
&\le C \sum_{K\in\mathcal{K}} \int\limits_{x'\in\hat{\Omega}_K^\varepsilon} 2 d_{x_K^\varepsilon}^2 \varepsilon^2 \\
&\le 2C d_2^2 \varepsilon^2
\end{aligned}$$

Recall that here d_2 is the maximum of the function d in $\overline{I}$. In the second line above, we have used the fact that ϕ is Lipschitz as $\phi \in \mathcal{D}(\Omega^+)$ and C is the Lipschitz constant. Let C_1 be the maximum value of $|\phi|^2$ in Ω^+ and C_2 be the volume between the surfaces $x_3 = \eta$ and $x_3 = M_0$. Then, the integral I_2 becomes

$$\begin{aligned}
I_2 &= \sum_{K\in\mathcal{K}} \int\limits_{x'\in\{\Omega_K^\varepsilon\backslash\hat{\Omega}_K^\varepsilon\}} \int\limits_{x_3=M_0}^{M_1} \int\limits_{z'\in Y(x_3)} \left|\phi\left(x', x_3\right)\right|^2 \\
&\le C_1 C_2 \sum_{K\in\mathcal{K}} |\Omega_K^\varepsilon \backslash \hat{\Omega}_K^\varepsilon| \le C \sum_{K\in\mathcal{K}} 2 d_{x_K^\varepsilon}^2 \varepsilon^{(1+\alpha)} \le C d_2^2 \varepsilon^{(1+\alpha)} N_\varepsilon^2 \le C\varepsilon^{(1-\alpha)}
\end{aligned}$$

where $C = C_1C_2$ is a constant independent of ε. Hence, $\|T^\varepsilon\phi - \phi\|_{L^2(\Omega_U)} \to 0$ as $\varepsilon \to 0$. Thus,

$$T^\varepsilon\phi \to \phi \; in \; L^2(\Omega_U) \quad \forall\, \phi \in \mathcal{D}(\Omega^+).$$

The density of $\mathcal{D}(\Omega^+)$ in $L^2(\Omega^+)$ completes the proof. □

Using the above proposition, one can prove the following result.

Proposition 3.7 *If $u_\varepsilon \to u$ in $L^2(\Omega^+)$, then, $T^\varepsilon u_\varepsilon \to u$ strongly in $L^2(\Omega_u)$.*

Proposition 3.8 *Let $u_\varepsilon \in L^2(\Omega_\varepsilon^+)$ and if $T^\varepsilon u_\varepsilon \rightharpoonup \hat{u}$ weakly in $L^2(\Omega_u)$. Then,*

$$\widetilde{u}_\varepsilon \rightharpoonup \int\limits_{Y(x_3)} \hat{u}\mathrm{d}z' \; weakly \; in \; L^2(\Omega^+).$$

Here and in the sequel $\widetilde{\psi}$ represents the zero extension of ψ to Ω^+.

Proof Let $\psi \in \mathcal{D}(\Omega^+)$, then,

$$\begin{aligned}
\int\limits_{\Omega^+} \widetilde{u}_\varepsilon \psi &= \int\limits_{\Omega_U} T^\varepsilon u_\varepsilon T^\varepsilon \psi \\
&\to \int\limits_{\Omega_U} \hat{u}\psi \text{ as } \varepsilon \to 0, \text{ by Proposition 3.6} \\
&= \int\limits_{x' \in I} \int\limits_{x_3 = M_0}^{M_1} \left(\int\limits_{z' \in Y(x_3)} \hat{u}\,\mathrm{d}z' \right) \psi \,\mathrm{d}x_3 \mathrm{d}x' \\
&= \int\limits_{\Omega^+} \left(\int\limits_{z' \in Y(x_3)} \hat{u}\,\mathrm{d}z' \right) \psi \,\mathrm{d}x \quad \forall \psi \in \mathcal{D}(\Omega^+)
\end{aligned}$$

This completes the proof as $\mathcal{D}(\Omega^+)$ is dense in $L^2(\Omega^+)$. □

Proposition 3.9 *Let $u_\varepsilon \in H^1(\Omega_\varepsilon^+)$ for every $\varepsilon > 0$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(I; H^1(\mathcal{G}))$. Then, $\widetilde{u}_\varepsilon \rightharpoonup \int\limits_{Y(x_3)} u \,\mathrm{d}z'$ and $\widetilde{\dfrac{\partial u_\varepsilon}{\partial x_3}} \rightharpoonup \int\limits_{Y(x_3)} \dfrac{\partial u}{\partial x_3}\,\mathrm{d}z'$ weakly in $L^2(\Omega^+)$.*

Proof Given that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(I; H^1(\mathcal{G}))$, which implies

$$T^\varepsilon u_\varepsilon \rightharpoonup u \text{ and } \frac{\partial}{\partial x_3} T^\varepsilon u_\varepsilon \rightharpoonup \frac{\partial u}{\partial x_3} \text{ weakly in } L^2(\Omega_U).$$

That is,

$$T^\varepsilon u_\varepsilon \rightharpoonup u \text{ and } T^\varepsilon \frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3} \text{ weakly in } L^2(\Omega_U).$$

Using Proposition 3.8, we get $\widetilde{u}_\varepsilon \rightharpoonup \int\limits_{Y(x_3)} u \;\mathrm{d}z'$ in $L^2(\Omega^+)$ and $\widetilde{\dfrac{\partial u_\varepsilon}{\partial x_3}} \rightharpoonup \int\limits_{Y(x_3)} \dfrac{\partial u}{\partial x_3}\,\mathrm{d}z'$ in $L^2(\Omega^+)$. This proves the proposition. □

Remark 3.10 It is interesting to observe that the locally periodic unfolding and the periodic unfolding operators are close in the following sense. Let T_p^ε be the periodic unfolding that we have developed for smooth oscillating domain in [2]. Then, we have

$$\|T^\varepsilon\phi - T_p^\varepsilon\phi\|_{L^2(\Omega_U)} \le \|T^\varepsilon\phi - \phi\|_{L^2(\Omega_U)} + \|T_p^\varepsilon\phi - \phi\|_{L^2(\Omega_U)} \to 0$$

as $\varepsilon \to 0$. □

4 Homogenization

In this section, we will establish the well-posedness of the problem (1.1). Define the space $\mathcal{V}_\varepsilon$ to be the set all functions $\phi \in H^1_\#(\Omega_\varepsilon)$ with $\phi|_{\Gamma_b} = 0$. The weak formulation of (1.1) is: find $y_\varepsilon \in \mathcal{V}_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} (\nabla y_\varepsilon \nabla\phi + y_\varepsilon\phi) + \varepsilon^{\alpha_1}\int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon)\phi = \int_{\Omega_\varepsilon} f\phi + \int_{\gamma_\varepsilon^+} u_\varepsilon^\varepsilon\phi, \quad \forall\phi \in \mathcal{V}_\varepsilon. \tag{4.1}$$

Recall here that the boundary data is given by

$$u_\varepsilon^\varepsilon = \begin{cases} \varepsilon^{\alpha_2} u^\varepsilon & if \;\; x_3 \in (M_0, M_1) \\ u^\varepsilon & if \;\; x_3 = M_0 \; or \; M_1, \end{cases}$$

where u^ε is defined by $u^\varepsilon(x_1, x_2, x_3) =: \sum_{K\in\mathcal{K}} u\left(\frac{x_1}{\varepsilon d_{x_K^\varepsilon}}, \frac{x_2}{\varepsilon d_{x_K^\varepsilon}}, x_3\right)\chi_{\hat{\Omega}_K^\varepsilon}(x_1, x_2)$ for $u \in L^2_\#(\gamma)$. The second condition on μ in (1.3) makes μ monotonic. Thus, using the monotone operator theory (see, for example, Lemma 2.1 and Corollary 2.2, Chapter 2 of [30]), we can prove the existence and uniqueness of a weak solution of (4.1). We will derive the *a priori* estimate in following subsection.

4.1 A priori estimate

The *a priori* estimate will be established once we estimate the surface integral of $u_\varepsilon^\varepsilon y_\varepsilon$ over γ_ε^+. The upper boundary γ_ε^+ can be divided into two surfaces, namely F_ε and S_ε, which are defined as $F_\varepsilon = \{(x_1, x_2, x_3) \in \gamma_\varepsilon^+ : \nabla\eta(x_1, x_2) = 0\}$ and $S_\varepsilon = \gamma_\varepsilon^+\backslash F_\varepsilon$, respectively. We derive the estimates in the following propositions.

Proposition 4.1 *Let $y_\varepsilon \in \mathcal{V}_\varepsilon$ be the solution of the problem (4.1) and $u_\varepsilon^\varepsilon$ be as in (1.2). Then, there is a constant $C > 0$ such that*

$$\int_{S_\varepsilon} u_\varepsilon^\varepsilon y_\varepsilon \le C\varepsilon^{\alpha_2 - 1}\|T^\varepsilon y_\varepsilon\|_{L^2(I;H^1(\mathcal{G}))} \le C\epsilon^{\alpha_2-1}\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \tag{4.2}$$

Proof The main ingredients of the proof are the slicing lemma for Hausdorff measures (see Lemma 7.6.1 in [20] or Theorem 7.2 in [21]) and the properties of the unfolding operators. We define the subset $B \subset I$ to be such that $S = \{(x', \eta(x')) : x' \in B\}$.

$$\int_{S_\varepsilon} \varepsilon^{\alpha_2} u^\varepsilon y_\varepsilon \, \mathrm{d}s_\varepsilon$$

$$= \varepsilon^{\alpha_2} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{\varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} B} u\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}, \eta_\varepsilon(x')\right) y_\varepsilon(x', \eta_\varepsilon(x'))$$

$$\sqrt{1+\frac{1}{(\varepsilon d_{x_K^\varepsilon})^2}\left|\nabla_{z'}\eta\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}\right)\right|^2}\,\mathrm{d}x'$$

$$= \varepsilon^{\alpha_2-1}\int_I d_\varepsilon^{-1}\int_S u(z',\eta(z'))\,T^\varepsilon y_\varepsilon(x',\eta(z'),z')\frac{\sqrt{(\varepsilon d_\varepsilon)^2+\left|\nabla_{z'}\eta(z')\right|^2}}{\sqrt{1+\left|\nabla_{z'}\eta(z')\right|^2}}\mathrm{d}s\mathrm{d}x'$$

$$\leq \varepsilon^{\alpha_2-1}\int_I d_1^{-1}\int_S \left|u(z',\eta(z'))\,T^\varepsilon y_\varepsilon(x',\eta(z'),z')\right|\mathrm{d}s\mathrm{d}x'$$

$$\leq \varepsilon^{\alpha_2-1}d_1^{-1}\|u\|_{L^2(\gamma)}\|T^\epsilon y_\varepsilon\|_{L^2(I;L^2(\partial\mathcal{G}))} \leq C\varepsilon^{\alpha_2-1}\|T^\varepsilon y_\varepsilon\|_{L^2(I;H^1(\mathcal{G}))}$$

$$\leq\; C\varepsilon^{\alpha_2-1}\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \tag{4.3}$$

In the last line of the estimate, we have used the continuity of the trace map. For the explicit calculation of the term (4.3), see Lemma 5.3 in Sect. 5. ◻

We derive the estimate on the flat boundaries in the proposition below.

Proposition 4.2 *Let $y_\varepsilon \in \mathcal{V}_\varepsilon$ be a solution of the problem (4.1) and $u_\varepsilon^\varepsilon$ be as in (1.2). Then, there is a constant $C > 0$ such that*

$$\int_{F_\varepsilon^t} u_\varepsilon^\varepsilon y_\varepsilon \leq C\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}\;\; for\ t = M_0 = t_0, t_1, t_2, \dots t_m = M_1.$$

Here, F_ε^t are the flat surfaces in $\partial\Omega^\varepsilon$ such that $F_\varepsilon = \bigcup_{l=0}^m F_\varepsilon^{t_l}$.

Proof Let us first look at the integration on $F_\varepsilon^{M_0}$. Define the set $Y'(M_0) = \overline{I}\backslash Y(M_0)$.

$$\int_{F_\varepsilon^{M_0}} u^\varepsilon y_\varepsilon ds = \sum_{K\in\mathcal{K}}\int_{\hat{\Omega}_K^\varepsilon} u\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}, M_0\right) y_\varepsilon(x', M_0)\,\mathrm{d}x'$$

$$= \sum_{K\in\mathcal{K}}\sum_{Z\in E_K^\varepsilon}\int_{x'\in\varepsilon d_{x_K^\varepsilon}(Z+Y'_{M_0})} u\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}, M_0\right) y_\varepsilon(x', M_0)\,\mathrm{d}x'$$

$$= \sum_{K\in\mathcal{K}}\sum_{Z\in E_K^\varepsilon}\int_{z'\in Y'_{M_0}} u\left(z', M_0\right) y_\varepsilon(\varepsilon d_{x_K^\varepsilon}Z+\varepsilon d_{x_K^\varepsilon}z', M_0)\,\varepsilon^2\, d_{x_K^\varepsilon}^2\,\mathrm{d}z'$$

$$\leq \sum_{K\in\mathcal{K}}\sum_{Z\in E_K^\varepsilon}\varepsilon^2\, d_{x_K^\varepsilon}^2\left(\int_{z'\in Y'_{M_0}}|u(z',M_0)|^2\mathrm{d}z'\right)^{1/2}\left(\int_{z'\in Y'_{M_0}}|y_\varepsilon(\varepsilon d_{x_K^\varepsilon}Z+\varepsilon d_{x_K^\varepsilon}z', M_0)|^2\,\mathrm{d}z'\right)^{1/2}$$

$$\leq \left(\int_\gamma |u|^2 d\sigma\right)^{1/2}\sum_{K\in\mathcal{K}}\sum_{Z\in E_K^\varepsilon}\varepsilon^2\, d_{x_K^\varepsilon}^2\left(\int_{z'\in Y'_{M_0}}|y_\varepsilon(\varepsilon d_{x_K^\varepsilon}Z+\varepsilon d_{x_K^\varepsilon}z', M_0)|^2\,\mathrm{d}z'\right)^{1/2}$$

$$\leq \left(\int_\gamma |u|^2 d\sigma\right)^{1/2}\sum_{K\in\mathcal{K}}\sum_{Z\in E_K^\varepsilon}\varepsilon^2 d_{x_K^\varepsilon}^2\left(\int_I |y_\varepsilon(z', M_0)|^2\,\mathrm{d}z'\right)^{1/2}$$

$$\leq C \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \varepsilon^2 \, d_{x_K^\varepsilon}^2 \|y_\varepsilon\|_{H^1(\Omega^-)} \;\leq\; C\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}.$$

Note that in the penultimate estimate, we have used the facts that $u \in L^2(\gamma)$ and the trace estimate for y_ε. Now we will estimate the integral on $F_\varepsilon^{t_l}$ for $l = 1, 2, \ldots m$.

$$\begin{aligned}
\int_{F_\varepsilon^{t_l}} u^\varepsilon y_\varepsilon ds &= \sum_{K\in\mathcal{K}} \int_{\hat{\Omega}_K^\varepsilon} u\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}, t_l\right) y_\varepsilon(x', t_l)\,\mathrm{d}x' \\
&= \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{x'\in\varepsilon d_{x_K^\varepsilon}(Z+Y(t_l))} u\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}, t_l\right) y_\varepsilon(x', t_l)\,\mathrm{d}x' \\
&= \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{z'\in Y(t_l)} u\left(z', t_l\right) y_\varepsilon(\varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} z', t_l)\,\varepsilon^2\, d_{x_K^\varepsilon}^2\,\mathrm{d}z' \\
&= \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{x'\in\varepsilon d_{x_K^\varepsilon} I_Z} \int_{z'\in Y(t_l)} u\left(z', t_l\right) y_\varepsilon(\varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} z', t_l)\,\mathrm{d}z'\mathrm{d}x' \\
&= \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{x'\in\hat{\Omega}_K^\varepsilon} \int_{z'\in Y(t_l)} u\left(z', t_l\right) T^\varepsilon y_\varepsilon(x', t_l, z')\,\mathrm{d}z'\mathrm{d}x' \\
&= \int_{x'\in\overline{I}} \int_{z'\in Y(t_l)} u\left(z', t_l\right) T^\varepsilon y_\varepsilon(x', t_l, z')\,\mathrm{d}z'\mathrm{d}x' \\
&\leq \left(\int_{z'\in Y(t_l)} |u(z', t_l)|^2 \mathrm{d}z'\right)^{1/2} \left(\int_{x'\in\overline{I}} \int_{z'\in Y(t_l)} |T^\varepsilon y_\varepsilon(x', t_l, z')|^2\,\mathrm{d}z'\mathrm{d}x'\right)^{1/2} \\
&\leq C\|T^\varepsilon y_\varepsilon\|_{L^2(I;L^2(Y(t_l)))} \leq C\|T^\varepsilon y_\varepsilon\|_{L^2(I;H^1(\mathcal{G}))} \leq \; C\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

□

Proposition 4.3 *Let $y_\varepsilon \in \mathcal{V}_\varepsilon$ be a solution of the problem* (4.1) *and $u_\varepsilon^\varepsilon$ be as in* (1.2). *Then, there is a constant $C > 0$ such that*

$$\int_{\gamma_\varepsilon^+} u_\varepsilon^\varepsilon y_\varepsilon \leq C.$$

Proof Using Propositions 4.1 and 4.2, we get

$$\int_{\gamma_\varepsilon^+} u_\varepsilon^\varepsilon y_\varepsilon \leq C\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \tag{4.4}$$

for a constant $C > 0$ independent of ε. The second condition (see (1.3)) on μ allows us to get some positive constants C_1, C_2 such that

$$C_1 s^2 \leq \mu(x_3, s)s \leq C_2 s^2 \quad \forall\, s \in \mathbb{R}. \tag{4.5}$$

By taking $\phi = y_\varepsilon$ in the weak formulation (4.1) and using the estimates (4.4) and (4.5), we observe that the weak solution y_ε of Eq. (1.1) satisfies

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \tag{4.6}$$

where $C > 0$ is a constant independent of ε. Hence, the result follows. □

4.2 Limit problem

Now, we present the existence and uniqueness of the limit problem (1.6). Recall that $h(x_3) = |Y(x_3)|$, where $|Y(x_3)|$ is the Lebesgue measure of the set $Y(x_3)$ at $x_3 \in [M_0, M_1]$. Note that the reference function η is chosen such that h is a strictly positive function in $[M_0, M_1]$. For any function $\psi \in L^2(\Omega)$, we denote $\psi^+ = \psi|_{\Omega^+}$ and $\psi^- = \psi|_{\Omega^-}$. Now, consider the space $W(\Omega)$

$$W(\Omega) = \{\psi \in L^2(\Omega) : \frac{\partial \psi}{\partial x_3} \in L^2(\Omega),\ \ \psi^- \in H^1(\Omega^-)\}.$$

Note that $W(\Omega)$ is a Hilbert space with the inner product given by

$$\langle u, v\rangle_W = \langle hu, v\rangle_{L^2(\Omega^+)} + \langle h\partial_{x_3} u, \partial_{x_3} v\rangle_{L^2(\Omega^+)} + \langle u, v\rangle_{H^1(\Omega^-)}. \tag{4.7}$$

The weak formulation of the limit equation (1.6) is: find $y \in W(\Omega)$ such that

$$\begin{aligned} &\int_{\Omega^+} h\left(\partial_{x_3} y^+ \partial_{x_3}\phi + y^+\phi\right) + \int_{\Omega^-}\left(\nabla y^- \nabla\phi + y^-\phi\right) + \int_{\Omega^+} \delta_{1\alpha_1}\frac{l}{d}\mu(x_3, y^+)\phi \\ &= \int_{\Omega^+} hf^+\phi + \int_{\Omega^-} f^-\phi + \int_{\Omega^+}\delta_{1\alpha_2}\frac{1}{d}\theta\phi + \sum_{l=0}^{1}\int_{\bar{I}}\rho_l\phi(x', M_l)\mathrm{d}x' \end{aligned} \tag{4.8}$$

for all $\psi \in W(\Omega)$. Once again the monotonicity of μ will help us get the well-posedness of the limit problem in the space $W(\Omega)$ as in the beginning of Sect. 4.

5 Convergence result

We will provide the proof of the main homogenization result in this section. First, we will prove some preliminary results before dissolving the theorem. Recall the a priori estimate (4.6): There exists a constant $C > 0$ independent of ε such that

$$\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C. \tag{5.1}$$

By using Proposition 3.4 and the estimate (5.1), we get

$$\left\|T^\varepsilon y_\varepsilon^+\right\|_{L^2(I;H^1(\mathcal{G}))} \le \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C. \tag{5.2}$$

As $T^\varepsilon u_\varepsilon^+$ is uniformly bounded in $L^2(I; H^1(\mathcal{G}))$, by the weak compactness, there exists a subsequence (still denoted by ε) such that

$$T^\varepsilon y_\varepsilon^+ \rightharpoonup y^+ \text{ weakly in } L^2(I; H^1(\mathcal{G})). \tag{5.3}$$

This implies

$$\begin{aligned} &T^\varepsilon y_\varepsilon^+ \rightharpoonup y^+ \text{ weakly in } L^2(\Omega_U), \\ &T^\varepsilon \frac{\partial y_\varepsilon^+}{\partial x_3} = \frac{\partial}{\partial x_3} T^\varepsilon y_\varepsilon^+ \rightharpoonup \frac{\partial y^+}{\partial x_3} \text{ weakly in } L^2(\Omega_U) \end{aligned}$$

and for $j = 1, 2$

$$\varepsilon T^\varepsilon \frac{\partial y_\varepsilon^+}{\partial x_j} = \frac{\partial}{\partial z_j} T^\varepsilon y_\varepsilon^+ \rightharpoonup \frac{\partial y^+}{\partial z_j} \text{ weakly in } L^2(\Omega_U). \tag{5.4}$$

Proposition 3.3 lends us a hand to get

$$\left\| T^{\varepsilon} \frac{\partial y_{\varepsilon}^{+}}{\partial x_j} \right\|_{L^2(\Omega_U)} \leq \left\| \frac{\partial y_{\varepsilon}^{+}}{\partial x_j} \right\|_{L^2(\Omega_{\varepsilon}^{+})} \leq \| y_{\varepsilon} \|_{H^1(\Omega_{\varepsilon})} \leq C. \tag{5.5}$$

Here, the last inequality is derived by using the estimate (5.1). Now that $T^{\varepsilon} \dfrac{\partial y_{\varepsilon}^{+}}{\partial x_j}$ is uniformly bounded in the space $L^2(\Omega_U)$ for $i = 1, 2$. Hence, from (5.4), we conclude that $\dfrac{\partial y^{+}}{\partial z_j} = 0$, for $1 \leq j \leq 2$. This implies y^{+} is independent of z'. Now, we have

$$\widetilde{y_{\varepsilon}^{+}} \rightharpoonup \int_{Y(x_3)} y^{+} \mathrm{d}z' \ \text{ and } \ \widetilde{\frac{\partial y_{\varepsilon}^{+}}{\partial x_3}} \rightharpoonup \int_{Y(x_3)} \frac{\partial y^{+}}{\partial x_3} \mathrm{d}z' \text{ weakly in } L^2(\Omega^{+}) \tag{5.6}$$

with the assistance of Propositions 3.8 and 3.9. This shows that

$$\widetilde{y_{\varepsilon}^{+}} \rightharpoonup h(x_3) y^{+} \ \text{ and } \ \widetilde{\frac{\partial y_{\varepsilon}^{+}}{\partial x_3}} \rightharpoonup h(x_3) \frac{\partial y^{+}}{\partial x_3} \text{ weakly in } L^2(\Omega^{+}) \ \text{ as } y^{+} \text{ is independent of } z'. \tag{5.7}$$

Also we get the weak convergence of y_{ε}^{-} as $\|y_{\varepsilon}^{-}\|_{H^1(\Omega^{-})} \leq \|y_{\varepsilon}^{-}\|_{H^1(\Omega_{\varepsilon})} \leq C$. Hence, we have proved the following proposition.

Proposition 5.1 *Let y_{ε} be the weak solution of* (1.1). *Then, there exist $y^{+} \in L^2(I; H^1(M_0, M_1))$ and $y^{-} \in H^1(\Omega^{-})$ such that*

$$\widetilde{y_{\varepsilon}^{+}} \rightharpoonup h(x_3) y^{+}; \quad \widetilde{\frac{\partial y_{\varepsilon}^{+}}{\partial x_3}} \rightharpoonup h(x_3) \frac{\partial y^{+}}{\partial x_3} \textit{ weakly in } L^2(\Omega^{+}) \textit{ and}$$
$$y_{\varepsilon}^{-} \rightharpoonup y^{-} \textit{ weakly in } H^1(\Omega^{-}).$$

□

By using the estimate (5.5), we can find $P_j \in L^2(\Omega_U)$ for $j = 1, 2$ such that for a subsequence

$$T^{\varepsilon} \frac{\partial y_{\varepsilon}}{\partial x_j} \rightharpoonup P_j \text{ weakly in } L^2(\Omega_U). \tag{5.8}$$

The following proposition will throw more lights on P_j's.

Proposition 5.2 *For $j = 1, 2$, we have*

$$\int_{Y(x_3)} P_j(x', x_3, z') \, \mathrm{d}z' = 0 \ \textit{ on } \Omega^{+} \textit{ where } P_j \textit{ is defined as in } (5.8).$$

Proof For $\phi \in \mathcal{D}(\Omega^{+})$, choose a test function

$$\phi_j^{\varepsilon}(x', x_3) = \varepsilon \phi(x', x_3) \sum_{K \in \mathcal{K}} d_{x_K^{\varepsilon}} \chi_{\hat{\Omega}_{\varepsilon}^{k}} \left\{ \frac{x_j}{\varepsilon d_{x_K^{\varepsilon}}} \right\},$$

for $j = 1, 2$ in such a way that ϕ_j^{ε} are continuous on Ω_{ε}^{+}. Recall the step function $d_{\varepsilon}(x') =: \sum_{K \in \mathcal{K}} d_{x_K^{\varepsilon}} \chi_{\hat{\Omega}_K^{\varepsilon}}(x')$. By applying unfolding operator on ϕ_j^{ε} and by Proposition 3.4, we get

$$T^{\varepsilon} \phi_j^{\varepsilon} = \sum_{K \in \mathcal{K}} \varepsilon d_{x_K^{\varepsilon}} z_j T^{\varepsilon} \phi \chi_{\hat{\Omega}_K^{\varepsilon}} = \varepsilon d_{\varepsilon} z_j T^{\varepsilon} \phi,$$

$$\begin{aligned}
T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_j} &= \sum_{K\in\mathcal{K}} \frac{1}{\varepsilon d_{x_K^\varepsilon}} \frac{\partial}{\partial z_j} T^\varepsilon \phi_j^\varepsilon \chi_{\hat{\Omega}_K^\varepsilon} \\
&= \sum_{K\in\mathcal{K}} \left(\varepsilon d_{x_K^\varepsilon} z_j T^\varepsilon \frac{\partial \phi}{\partial x_j} + T^\varepsilon \phi \right) \chi_{\hat{\Omega}_K^\varepsilon} = \varepsilon d_\varepsilon z_j T^\varepsilon \frac{\partial \phi}{\partial x_j} + T^\varepsilon \phi, \\
T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_i} &= \varepsilon d_\varepsilon z_j T^\varepsilon \frac{\partial \phi}{\partial x_i}, i \neq j,\ i \in \{1,2\}, \\
T^\varepsilon \frac{\partial \phi_j^\varepsilon}{\partial x_3} &= \varepsilon d_\varepsilon z_j T^\varepsilon \frac{\partial \phi}{\partial x_3}.
\end{aligned}$$

Let us recall the variational formulation of (4.1) with the test function $\widetilde{\phi_j^\varepsilon}$.

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \left(\nabla y_\varepsilon \nabla \phi_j^\varepsilon + y_\varepsilon \phi_j^\varepsilon \right) + \varepsilon^{\alpha_1} \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon) \phi_j^\varepsilon \\
&\quad = \int_{\Omega_\varepsilon} f \phi_j^\varepsilon + \int_{\gamma_\varepsilon^+} \varepsilon^{\alpha_2} u^\varepsilon \phi_j^\varepsilon, \quad \forall \phi \in \mathcal{D}(\Omega^+). \qquad (5.9)
\end{aligned}$$

As $\phi_j^\varepsilon \in \mathcal{D}(\Omega^+)$, we have

$$\begin{aligned}
&\int_{\Omega_\varepsilon^+} \left(\nabla y_\varepsilon \nabla \phi_j^\varepsilon + y_\varepsilon \phi_j^\varepsilon \right) + \varepsilon^{\alpha_1} \int_{\gamma_\varepsilon^+ \setminus \Gamma_0 \cup \Gamma_u} \mu(x_3, y_\varepsilon) \phi_j^\varepsilon \\
&\quad = \int_{\Omega_\varepsilon^+} f \phi_j^\varepsilon + \int_{\gamma_\varepsilon^+ \setminus \Gamma_0 \cup \Gamma_u} \varepsilon^{\alpha_2} u^\varepsilon \phi_j^\varepsilon. \qquad (5.10)
\end{aligned}$$

Now we will concentrate on the second integral of the above expression. Let $S_1^\varepsilon = \gamma_\varepsilon^+ \setminus \Gamma_0 \cup \Gamma_u$, $B_1 \subset I$ be such that $S_1^\varepsilon = \{(x', \eta_\varepsilon(x')) : x' \in B_1\}$ and $S_{K,Z}^\varepsilon = \{(x', x_3) \in S_1^\varepsilon : x' \in \hat{\Omega}_{K,Z}^\varepsilon\}$.

$$\begin{aligned}
I_1 =: \varepsilon^{\alpha_1} \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon) \phi_j^\varepsilon \mathrm{d}s_\varepsilon &= \varepsilon^{\alpha_1} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{S_{K,Z}^\varepsilon} \mu(x_3, y_\varepsilon(x', x_3)) \phi_j^\varepsilon \,\mathrm{d}s_\varepsilon \\
= \varepsilon^{\alpha_1} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{\varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} B_1} &\mu(\eta_\varepsilon(x'), y_\varepsilon(x', \eta_\varepsilon(x'))) \phi_j^\varepsilon \\
&\sqrt{1 + \frac{1}{(\varepsilon d_{x_K^\varepsilon})^2} \left| \nabla_{z'} \eta \left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}} \right) \right|^2} \,\mathrm{d}x'
\end{aligned}$$

Now change $x' = \varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} z'$ and use the periodicity of the reference function η.

$$\begin{aligned}
I_1 &= \varepsilon^{\alpha_1} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{B_1} (\eta(z'), y_\varepsilon(\varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} z', \eta(z'))) \varepsilon d_{x_K^\varepsilon} z_j \phi \\
&\qquad \sqrt{(\varepsilon d_{x_K^\varepsilon})^2 + \left| \nabla_{z'} \eta(z') \right|^2} \,\varepsilon d_{x_K^\varepsilon} \mathrm{d}z' \\
&= \varepsilon^{\alpha_1} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{\varepsilon d_{x_K^\varepsilon} I_Z} \int_{B_1} z_j \mu(\eta(z'), T^\varepsilon y_\varepsilon(x', \eta(z'), z')) T^\varepsilon \phi \\
&\qquad \sqrt{(\varepsilon d_{x_K^\varepsilon})^2 + \left| \nabla_{z'} \eta(z') \right|^2} \mathrm{d}z' \mathrm{d}x'
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
|I_1| &\leq \varepsilon^{\alpha_1}\int_I\int_{B_1}|z_j\mu(\eta(z'),T^\varepsilon y_\varepsilon(x',\eta(z'),z'))T^\varepsilon\phi|\sqrt{1+\left|\nabla_{z'}\eta(z')\right|^2}\mathrm{d}z'\mathrm{d}x'\\
&=\varepsilon^{\alpha_1}\int_I\int_{\gamma_1}|z_j\mu(\eta(z'),T^\varepsilon y_\varepsilon(x',\eta(z'),z'))T^\varepsilon\phi|\mathrm{d}s\mathrm{d}x'\\
&\leq C\varepsilon^{\alpha_1}\|\mu(x_3,T^\varepsilon y_\varepsilon)\|_{L^2(I;H^1(\mathcal{G}))}\|z_jT^\varepsilon\phi\|_{L^2(I;L^2(\gamma))}\leq C\varepsilon^{\alpha_1}\|T^\varepsilon y_\varepsilon\|_{L^2(I;H^1(\mathcal{G}))}.
\end{aligned}
$$

Here, $\gamma_1=\gamma\backslash\Gamma_0\cup\Gamma_u$. In the above estimate, we have used the continuity of trace on the space $L^2(I;H^1(\mathcal{G}))$. As both the norms are bounded, we have $I_1\to 0$ as $\varepsilon\to 0$. Similarly, the last integration in (5.10) can also be proved to be zero as $\varepsilon\to 0$. Now notice, for j fixed

$$
\begin{aligned}
\int\limits_{\Omega_\varepsilon^+}\nabla y_\varepsilon^+\cdot\nabla\phi_j^\varepsilon &=\int\limits_{\Omega_U}\left(T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_i}T^\varepsilon\frac{\partial\phi_j^\varepsilon}{\partial x_i}\right)+T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_j}T^\varepsilon\frac{\partial\phi_j^\varepsilon}{\partial x_j}+T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_3}T^\varepsilon\frac{\partial\phi_j^\varepsilon}{\partial x_3},\quad i\neq j\\
&=\int\limits_{\Omega_U}T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_i}\varepsilon d_\varepsilon z_jT^\varepsilon\frac{\partial\phi}{\partial x_i}+T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_j}\left(\varepsilon d_\varepsilon z_jT^\varepsilon\frac{\partial\phi}{\partial x_j}+T^\varepsilon\phi\right)+T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_3}\varepsilon d_\varepsilon z_jT^\varepsilon\frac{\partial\phi}{\partial x_3}.
\end{aligned}
$$

Equation (5.10) gives,

$$
\begin{aligned}
\int\limits_{\Omega_U}T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_j}T^\varepsilon\phi=-\int\limits_{\Omega_U}\varepsilon d_\varepsilon z_j&\\
&\left(T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_i}T^\varepsilon\frac{\partial\phi}{\partial x_i}+T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_j}T^\varepsilon\frac{\partial\phi}{\partial x_j}+T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_3}T^\varepsilon\frac{\partial\phi}{\partial x_3}+T^\varepsilon y_\varepsilon T^\varepsilon\phi\right)\\
&-\varepsilon^{\alpha_1}\int_{\gamma_\varepsilon^+\backslash\Gamma_0\cup\Gamma_u}\mu(x_3,y_\varepsilon)\phi_j^\varepsilon+\int_{\Omega_U}\varepsilon d_\varepsilon z_jT^\varepsilon fT^\varepsilon\phi\\
&+\int_{\gamma_\varepsilon^+\backslash\Gamma_0\cup\Gamma_u}\varepsilon^{\alpha_2}u^\varepsilon\phi_j^\varepsilon.
\end{aligned}
$$

This implies,

$$
\left|\int\limits_{\Omega_U}T^\varepsilon\frac{\partial u_\varepsilon^+}{\partial x_j}T^\varepsilon\phi\right|\leq\varepsilon^{\min\{1,\alpha_1,\alpha_2\}}C\|T^\varepsilon\phi\|_{L^2(I;H^1(\mathcal{G}))}.
$$

Hence,

$$
\lim_{\varepsilon\to 0}\int_{\Omega_U}T^\varepsilon\frac{\partial y_\varepsilon^+}{\partial x_j}T^\varepsilon\phi=\int_{\Omega_U}P_j\phi=0\;\forall\phi\in\mathcal{D}(\Omega^+).
$$

Thus, we conclude that $\displaystyle\int\limits_{Y(x_3)}P_j\mathrm{d}z'\equiv 0$ a.e Ω^+ for $j=1,2$. □

Proposition 5.3 *Let $u_\varepsilon^\varepsilon$ be the boundary data defined as in (1.1). Then,*

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon^+}u_\varepsilon^\varepsilon\phi\,\mathrm{d}s_\varepsilon=&\int_{\Omega^+}\frac{1}{d}\left(\int_{\partial Y(x_3)}u(z',x_3)\mathrm{d}\gamma_t\right)\phi\,\mathrm{d}x\\
&+\sum_{l=0}^m\int_{\overline{I}}\left(\int_{z'\in Y(t_l)}u\left(z',t_l\right)\,\mathrm{d}z'\right)\phi(x',t_l)\mathrm{d}x'
\end{aligned}
$$

for all $\phi\in C^\infty(\overline{\Omega})$.

Proof First let us look at the integration on the lateral surface of γ_ε^+.

$$\int_{S_\varepsilon} \varepsilon^{\alpha_2} u^\varepsilon \phi \,\mathrm{d}s$$
$$= \varepsilon^{\alpha_2} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{\varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} B_1} u\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}, \eta_\varepsilon(x')\right) \phi(x', \eta_\varepsilon(x'))$$
$$\sqrt{1 + \frac{1}{(\varepsilon d_{x_K^\varepsilon})^2} \left|\nabla_{z'}\eta\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}\right)\right|^2}\,\mathrm{d}x'$$

Now change $x' = \varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} z'$ and use the periodicity of the reference function η and u. We define the set $B \subset I$ to be such that $S = \{(x', \eta(x')) : x' \in B\}$. Then,

$$\int_{S_\varepsilon} \varepsilon^{\alpha_2} u^\varepsilon \phi \,\mathrm{d}s$$
$$= \varepsilon^{\alpha_2} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_B u(z', \eta(z'))\phi(\varepsilon d_{x_K^\varepsilon} Z + \varepsilon d_{x_K^\varepsilon} z', \eta(z'))$$
$$\sqrt{(\varepsilon d_{x_K^\varepsilon})^2 + \left|\nabla_{z'}\eta(z')\right|^2}\, \varepsilon d_{x_K^\varepsilon} \mathrm{d}z'$$
$$= \varepsilon^{\alpha_2 - 1} \sum_{K\in\mathcal{K}} \sum_{Z\in E_K^\varepsilon} \int_{\varepsilon d_{x_K^\varepsilon} I_Z} d_{x_K^\varepsilon}^{-1} \int_B u(z', \eta(z'))\, T^\varepsilon \phi \sqrt{(\varepsilon d_{x_K^\varepsilon})^2 + \left|\nabla_{z'}\eta(z')\right|^2} \mathrm{d}z' \mathrm{d}x'$$
$$= \varepsilon^{\alpha_2 - 1} \int_I d_\varepsilon^{-1} \int_B u(z', \eta(z'))\, T^\varepsilon \phi(x', \eta(z'), z') \frac{\sqrt{(\varepsilon d_\varepsilon)^2 + \left|\nabla_{z'}\eta(z')\right|^2}}{\sqrt{1 + \left|\nabla_{z'}\eta(z')\right|^2}}$$
$$\sqrt{(1 + \left|\nabla_{z'}\eta(z')\right|^2} \mathrm{d}z' \mathrm{d}x'$$
$$= \varepsilon^{\alpha_2 - 1} \int_I d_\varepsilon^{-1} \int_S u(z', \eta(z'))\, T^\varepsilon \phi(x', \eta(z'), z') \frac{\sqrt{(\varepsilon d_\varepsilon)^2 + \left|\nabla_{z'}\eta(z')\right|^2}}{\sqrt{1 + \left|\nabla_{z'}\eta(z')\right|^2}} \mathrm{d}s \mathrm{d}x'$$

Taking $\alpha_2 = 1$, we see that as $\varepsilon \to 0$

$$\int_{S_\varepsilon} \varepsilon u^\varepsilon \phi \,\mathrm{d}s \longrightarrow \int_I d^{-1} \int_S u(z', \eta(z'))\, \phi(x', \eta(z')) \frac{\sqrt{\left|\nabla_{z'}\eta(z')\right|^2}}{\sqrt{1 + \left|\nabla_{z'}\eta(z')\right|^2}} \mathrm{d}s \mathrm{d}x'$$
$$= \int_I d^{-1} \int_{M_0}^{M_1} \int_0^1 u(z', t)\, \phi(x', t) \frac{\sqrt{\left|\nabla_{z'}\eta(z')\right|^2}}{\sqrt{1 + \left|\nabla_{z'}\eta(z')\right|^2}} \frac{1}{|\nabla_S \pi|} \mathrm{d}\gamma_t \mathrm{d}t \mathrm{d}x'$$

Here, $|\nabla_S \pi|$ is given by

$$|\nabla_S \pi| = \frac{|\nabla_{z'}\eta|}{\sqrt{1 + |\nabla_{z'}\eta|^2}} \tag{5.11}$$

and $d\gamma_t$ is the line element of the curve $\left\{z' \in \mathbb{R}^2 : \eta(z') = t\right\}$. Here, we have used the slicing lemma for Hausdorff measures (see Lemma 7.6.1 in [20] or Theorem 7.2 in [21]). Thus,

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \varepsilon u^\varepsilon \phi \,\mathrm{d}s_\varepsilon = \int_I d^{-1}\int_{M_0}^{M_1}\int_0^1 u(z',t)\,\phi(x',t)\mathrm{d}\gamma_t \mathrm{d}t\mathrm{d}x'.$$

Hence, we have

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \varepsilon^{\alpha_2} u^\varepsilon \phi \,\mathrm{d}s_\varepsilon = \begin{cases} \int_{\Omega^+} \frac{1}{d}\left(\int_{\partial Y(x_3)} u(z',x_3)\mathrm{d}\gamma_{x_3}\right)\phi(x)\mathrm{d}x & if \;\; \alpha_2 = 1 \\ 0 & if \;\; \alpha_2 > 1. \end{cases} \tag{5.12}$$

Now we will look at the integration on the non-lateral parts.

$$\begin{aligned}
\int_{F_\varepsilon^{M_0}} u^\varepsilon \phi \, ds_\varepsilon &= \sum_{K\in\mathcal{K}}\int_{\hat{\Omega}_K^\varepsilon} u\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}, M_0\right)\phi(x',M_0)\,\mathrm{d}x' \\
&= \sum_{K\in\mathcal{K}}\sum_{Z\in E_K^\varepsilon}\int_{x'\in\varepsilon d_{x_K^\varepsilon}(Z+Y'_{M_0})} u\left(\frac{x'}{\varepsilon d_{x_K^\varepsilon}}, M_0\right)\phi(x',M_0)\,\mathrm{d}x' \\
&= \sum_{K\in\mathcal{K}}\sum_{Z\in E_K^\varepsilon}\int_{z'\in Y'_{M_0}} u\left(z',M_0\right)\phi(\varepsilon d_{x_K^\varepsilon}Z + \varepsilon d_{x_K^\varepsilon}z', M_0)\,\varepsilon^2\, d^2_{x_K^\varepsilon}\,\mathrm{d}z' \\
&= \int_I\int_{z'\in Y'_{M_0}} u\left(z',M_0\right) T^\varepsilon\phi(x',M_0,z')\,\mathrm{d}z'\,\mathrm{d}x' \\
&\xrightarrow{\varepsilon\to 0} \int_I\left(\int_{z'\in Y'_{M_0}} u\left(z',M_0\right)dz'\right)\phi(x',M_0)\,\mathrm{d}x'.
\end{aligned}$$

Similarly, we have

$$\int_{F_\varepsilon^{M_1}} u^\varepsilon \phi ds_\varepsilon \xrightarrow{\varepsilon\to 0} \int_{x'\in\overline{I}}\left(\int_{z'\in Y(M_1)} u\left(z',M_1\right)\,\mathrm{d}z'\right)\phi(x',M_1)\mathrm{d}x'$$

and for $l = 1, 2, \ldots m-1$, we have

$$\begin{aligned}
\varepsilon^{\alpha_2}\int_{F_\varepsilon^{t_l}} u^\varepsilon\phi\mathrm{d}s_\varepsilon &= \varepsilon^{\alpha_2}\int_{x'\in\overline{I}}\int_{z'\in Y(t_l)} u\left(z',t_l\right)T^\varepsilon\phi(x',t_l,z')\,\mathrm{d}z'\mathrm{d}x' \\
&\xrightarrow{\varepsilon\to 0} 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon^+} u_\varepsilon^\varepsilon\phi\,\mathrm{d}s_\varepsilon &= \int_{\Omega^+}\frac{1}{d}\left(\int_{\partial Y(x_3)} u(z',x_3)\mathrm{d}\gamma_t\right)\,\phi(x',x_3)\mathrm{d}x'\mathrm{d}x_3 + \sum_{l=0}^m\int_{F_\varepsilon^{t_l}} u^\varepsilon\phi\,\mathrm{d}s_\varepsilon \\
&= \int_{\Omega^+}\frac{1}{d}\left(\int_{\partial Y(x_3)} u(z',x_3)\mathrm{d}\gamma_t\right)\phi\,\mathrm{d}x \\
&\quad + \int_{\overline{I}}\left(\int_{z'\in Y'(M_0)} u\left(z',M_0\right)\,\mathrm{d}z'\right)\phi(x',M_0)\mathrm{d}x' \\
&\qquad + \int_{\overline{I}}\left(\int_{z'\in Y(M_1)} u\left(z',M_1\right)\,\mathrm{d}z'\right)\phi(x',M_1)\mathrm{d}x' \\
&= \int_{\Omega^+}\frac{1}{d}\theta(x_3)\phi\,\mathrm{d}x + \int_{\overline{I}}\rho_0\phi(x',M_0)\mathrm{d}x' + \int_{\overline{I}}\rho_1\phi(x',M_1)\mathrm{d}x'.
\end{aligned}$$

□

5.1 Proof of Theorem 1.1

As $\|T^\varepsilon y_\varepsilon^+\|_{L^2(I;H^1(\mathcal{G}))}$ is bounded, we have $\mathfrak{M}_\varepsilon = \mu(x_3, T^\varepsilon y_\varepsilon^+)$ is also bounded in $L^2(I; H^1(\mathcal{G}))$. Hence, there exists $\zeta \in L^2(I; H^1(\mathcal{G}))$ such that for a subsequence

$$\mathfrak{M}^\varepsilon \rightharpoonup \zeta \text{ weakly in } L^2(I; H^1(\mathcal{G})).$$

As in Lemma 5.1, we can prove that ζ is independent of z'. We will prove the result for the case when $\alpha_1 = 1$. Now, let us look at the integral on the lateral part of the surface γ_ε^+.

$$\begin{aligned}
\lim_{\varepsilon\to 0} \varepsilon \int_{S_\varepsilon} \mu(x_3, y_\varepsilon)\phi &= \int_I d_\varepsilon^{-1} \int_{B_1} \mu(\eta(z'), T^\varepsilon y_\varepsilon)\, T^\varepsilon \phi(x', \eta(z'), z') \\
&\qquad \frac{\sqrt{(\varepsilon d_\varepsilon)^2 + \left|\nabla_{z'}\eta(z')\right|^2}}{\sqrt{1 + \left|\nabla_{z'}\eta(z')\right|^2}} \mathrm{d}s\mathrm{d}x' \\
&\xrightarrow{\varepsilon\to 0} \int_I d^{-1} \int_{B_1} \zeta\ \phi \frac{\sqrt{\left|\nabla_{z'}\eta(z')\right|^2}}{\sqrt{1 + \left|\nabla_{z'}\eta(z')\right|^2}} \mathrm{d}s\mathrm{d}x' \\
&= \int_I d^{-1} \int_{M_0}^{M_1} \int_{\gamma_t} \zeta(x', t)\ \phi(x', t) \mathrm{d}\gamma_t \mathrm{d}t \mathrm{d}x' \qquad (5.13)\\
&= \int_{\Omega^+} \frac{1}{d} |\partial Y(x_3)|\ \zeta\phi(x)\mathrm{d}x = \int_{\Omega^+} \frac{l}{d}\ \zeta\phi(x)\mathrm{d}x \qquad (5.14)
\end{aligned}$$

where $l(x_3) = |\partial Y(x_3)|$-the perimeter of the boundary of the reference set $Y(x_3)$. In the line (5.13), we have used the slicing lemma. First note that, by using the properties of μ, the Trace theorem and (4.6), we get

$$\begin{aligned}
\left|\varepsilon \int_{\gamma_\varepsilon^+ \setminus S_\varepsilon} \mu(x_3, y_\varepsilon)\phi\right| &\le C\varepsilon \int_{\gamma_\varepsilon^+ \setminus S_\varepsilon} |y_\varepsilon \phi| \\
&\le C\varepsilon \|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \|\phi\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon.
\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon)\phi = \int_{\Omega^+} \frac{l}{d}\ \zeta\phi(x)\mathrm{d}x \quad \forall \phi \in C^\infty(\overline{\Omega}). \qquad (5.15)$$

Next, we will identify ζ. Here, we will use the Browder–Minty technique to retrieve ζ. Recall the variational formulation (4.1): find $y_\varepsilon \in \mathcal{V}_\varepsilon$ such that

$$\begin{aligned}
\int_{\Omega_\varepsilon} (\nabla y_\varepsilon \nabla\phi + y_\varepsilon\phi) + \varepsilon \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon)\phi = \int_{\Omega_\varepsilon} f\phi \\
+ \int_{\gamma_\varepsilon^+} u_\varepsilon^\varepsilon \phi, \quad \forall\phi \in C^\infty(\overline{\Omega}) \text{ with } \phi_{|\Gamma_b} = 0.
\end{aligned}$$

Using Lemmas 5.1, 5.2 and 5.3 and the convergence (5.15), we get

$$\begin{aligned}
&\int_{\Omega^+} h(x_3) \left(\partial_{x_3} y^+ \partial_{x_3}\phi + y^+\phi\right) + \int_{\Omega^-} \left(\nabla y^- \nabla\phi + y^-\phi\right) + \int_{\Omega^+} \frac{l}{d}\ \zeta\phi = \int_{\Omega^+} h(x_3) f\phi \\
&+ \int_{\Omega^-} f\phi + \int_{\Omega^+} \frac{1}{d} \left(\int_{\partial Y(x_3)} u(z', x_3)\mathrm{d}\gamma_{x_3}\right)\phi + \sum_{i=0}^{1} \int_{\overline{I}} \rho_i \phi(x', M_i)\mathrm{d}x' \qquad (5.16)
\end{aligned}$$

As μ is monotonic and the other terms are nonnegative, we have

$$\begin{aligned}
&\int_{\Omega^-} |\nabla y_\varepsilon - \nabla\phi|^2 + \int_{\Omega^-} |y_\varepsilon - \phi|^2 + \int_{\Omega_\varepsilon^+} \left|\partial_{x_3} y_\varepsilon - \partial_{x_3}\phi\right|^2 \\
&+ \int_{\Omega_\varepsilon^+} |\nabla_{x'} y_\varepsilon|^2 + \int_{\Omega_\varepsilon^+} |y_\varepsilon - \phi|^2 \\
&+ \varepsilon \int_{\gamma_\varepsilon} (\mu(x_3, y_\varepsilon) - \mu(x_3, \phi))(y_\varepsilon - \phi) \geq 0.
\end{aligned}$$

Upon expanding the above inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon^+} \left|\partial_{x_3} y_\varepsilon\right|^2 + \int_{\Omega_\varepsilon^+} |\nabla_{x'} y_\varepsilon|^2 + \int_{\Omega_\varepsilon^+} |y_\varepsilon|^2 + \int_{\Omega^-} |\nabla y_\varepsilon|^2 + \int_{\Omega^-} |y_\varepsilon|^2 + \varepsilon \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon) y_\varepsilon \\
&- 2\int_{\Omega_\varepsilon^+} \partial_{x_3} y_\varepsilon \partial_{x_3}\phi + \int_{\Omega_\varepsilon^+} \left|\partial_{x_3}\phi\right|^2 - 2\int_{\Omega_\varepsilon^+} y_\varepsilon\phi + \int_{\Omega_\varepsilon^+} |\phi|^2 \\
&- 2\int_{\Omega^-} \nabla y_\varepsilon \cdot \nabla\phi + \int_{\Omega^-} |\nabla\phi|^2 + \int_{\Omega^-} |\phi|^2 - 2\int_{\Omega^-} y_\varepsilon\phi \\
&- \varepsilon \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon)\phi - \varepsilon \int_{\gamma_\varepsilon^+} \mu(x_3, \phi) y_\varepsilon + \varepsilon \int_{\gamma_\varepsilon^+} \mu(x_3, \phi)\phi \geq 0.
\end{aligned} \tag{5.17}$$

We can find out the limit of all the terms in the inequality (5.17) except the first line. Thus, we will concentrate on the terms in the first line of (5.17). Now, let us recall the variational formulation with the test function $\phi = y_\varepsilon$:

$$\int_{\Omega_\varepsilon} \left(|\nabla y_\varepsilon|^2 + |y_\varepsilon|^2\right) + \varepsilon \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon) y_\varepsilon = \int_{\Omega_\varepsilon} f y_\varepsilon + \int_{\gamma_\varepsilon^+} u_\varepsilon^\varepsilon y_\varepsilon.$$

Using Lemma 5.1, 5.2 and 5.3 in the RHS of the above equation, we get as $\varepsilon \to 0$

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f y_\varepsilon + \int_{\gamma_\varepsilon^+} u_\varepsilon^\varepsilon y_\varepsilon \\
&= \int_{\Omega^+} h(x_3) f y^+ + \int_{\Omega^-} f y^- + \int_{\Omega^+} \frac{1}{d}\theta(x_3) y^+(x)\, \mathrm{d}x \\
&\quad + \sum_{i=0}^{1} \int_{I} \rho_i y^+(x', M_i) \mathrm{d}x'
\end{aligned} \tag{5.18}$$

Now recall the variational formulation of the limit equation (5.16) with $y = y^+\chi_{\Omega^+} + y^-\chi_{\Omega^-}$ as the test function.

$$\begin{aligned}
&\int_{\Omega^-} \left(|\nabla y^-|^2 + |y^-|^2\right) + \int_{\Omega^+} h(x_3) \left|\partial_{x_3} y^+\right|^2 + \int_{\Omega^+} \frac{l}{d}\zeta\, y^+ \mathrm{d}x + \int_{\Omega^+} h(x_3)\, |y^+|^2 \\
&= \int_{\Omega^+} h(x_3) f y^+ + \int_{\Omega^-} f y^- + \int_{\Omega^+} \frac{1}{d}\theta y^+ \mathrm{d}x + \sum_{i=0}^{1} \int_{I} \rho_i y^+(x', M_i) \mathrm{d}x'
\end{aligned} \tag{5.19}$$

Notice that the RHS of (5.18) matches with the RHS of (5.19). Hence,

$$\begin{aligned}
&\int_{\Omega_\varepsilon^+} \left|\partial_{x_3} y_\varepsilon\right|^2 + \int_{\Omega_\varepsilon^+} |\nabla_{x'} y_\varepsilon|^2 + \int_{\Omega_\varepsilon^+} |y_\varepsilon|^2 + \int_{\Omega^-} |\nabla y_\varepsilon|^2 + \int_{\Omega^-} |y_\varepsilon|^2 + \varepsilon \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon) y_\varepsilon \\
&= \int_{\Omega_\varepsilon} f y_\varepsilon + \int_{\gamma_\varepsilon^+} u_\varepsilon^\varepsilon y_\varepsilon
\end{aligned}$$

$$\stackrel{\varepsilon\to 0}{\longrightarrow} \int_{\Omega^+} h(x_3) f y^+ + \int_{\Omega^-} f y^- + \int_{\Omega^+} \frac{1}{d}\theta(x_3) y^+(x)\,\mathrm{d}x + \sum_{i=0}^{1} \int_{\overline{I}} \rho_i y^+(x', M_i)\mathrm{d}x'$$
$$= \int_{\Omega^-} \left(|\nabla y^-|^2 + |y^-|^2\right) + \int_{\Omega^+} h(x_3)\, \left|\partial_{x_3} y^+\right|^2 + \int_{\Omega^+} \frac{l}{d}\zeta\, y^+\mathrm{d}x + \int_{\Omega^+} h(x_3)\, |y^+|^2$$

Using the above limit in the inequality (5.17) and passing to the limit as $\varepsilon \to 0$, we get

$$\int_{\Omega^-} \left|\nabla y^- - \nabla\phi\right|^2 + \int_{\Omega^-} \left|y^- - \phi\right|^2 + \int_{\Omega^+} h(x_3)\left|\partial_{x_3} y^+ - \partial_{x_3}\phi\right|^2$$
$$+ \int_{\Omega^+} \left|y^+ - \phi\right|^2 + \int_{\Omega^+} \frac{l}{d}\left(\zeta - \mu(x_3, \phi)\right)\left(y^+ - \phi\right) \geq 0.$$

By choosing $\phi = y - \lambda\psi$ for $\lambda > 0$ and $\psi \in C_c^1(\Omega)$, we get

$$\lambda\left(\int_{\Omega^-} |\nabla\psi|^2 + \int_{\Omega^-} |\psi|^2 + \left(\int_{\Omega^+} \left|h(x_3)\partial_{x_3}\psi\right|^2 + \int_{\Omega^+} |\psi|^2\right)\right)$$
$$+ \int_{\Omega^+} \frac{l}{d}(\zeta - \mu(x_3, y^+ - \lambda\psi))\psi \geq 0\,.$$

By letting λ to go to 0, using the dominated convergence theorem, we obtain

$$\int_{\Omega^+} \frac{l}{d}(\zeta - \mu(x_3, y^+))\psi \geq 0.$$

Since ψ is an arbitrary element of $C_c^1(\Omega)$ and d and l nonzero functions, we conclude that

$$\zeta(x_1, x_2, x_3) = \mu(x_3, y^+(x_1, x_2, x_3)) \;\text{ a.e. }\; (x_1, x_2, x_3) \in \Omega^+.$$

Hence, Eq. (5.16) becomes

$$\int_{\Omega^+} h(x_3)\left(\partial_{x_3} y^+ \partial_{x_3}\phi + y^+\phi\right) + \int_{\Omega^-} \left(\nabla y^- \nabla\phi + y^-\phi\right) + \int_{\Omega^+} \frac{l}{d}\, \mu(x_3, y^+)\phi$$
$$= \int_{\Omega^+} h(x_3) f\phi + \int_{\Omega^-} f\phi + \int_{\Omega^+} \frac{1}{d}\theta(x_3)\phi + \sum_{i=0}^{1} \int_{\overline{I}} \rho_i\phi(x', M_i)\mathrm{d}x'.$$

Recall that here $\theta(x_3) = \int_{z'\in\partial Y(x_3)} u(z', x_3)\mathrm{d}\gamma_{x_3}(z')$; the constants ρ_0 and ρ_1 are given by $\rho_0 = \int_{z'\in Y'(M_0)} u\left(z', M_0\right)\, \mathrm{d}z'$ and $\rho_1 = \int_{z'\in Y(M_1)} u\left(z', M_1\right)\, \mathrm{d}z'$. Note that when $\alpha_1 > 1$, then (Ref. (5.14))

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha_1} \int_{\gamma_\varepsilon^+} \mu(x_3, y_\varepsilon)\phi = 0. \tag{5.20}$$

Hence, using Eqs. 5.12 and 5.20, we can write the limit for the general case as

$$\int_{\Omega^+} h(x_3)\left(\partial_{x_3} y^+ \partial_{x_3}\phi + y^+\phi\right) + \int_{\Omega^-} \left(\nabla y^- \nabla\phi + y^-\phi\right) + \delta_{1\alpha_1} \int_{\Omega^+} \frac{l}{d}\, \mu(x_3, y^+)\phi$$
$$= \int_{\Omega^+} h(x_3) f\phi + \int_{\Omega^-} f\phi + \delta_{1\alpha_2} \int_{\Omega^+} \frac{1}{d}\theta(x_3)\phi + \sum_{i=0}^{1} \int_{\overline{I}} \rho_i\phi(x', M_i)\mathrm{d}x'.$$

This completes the theorem. □

Remark 5.4 One can also establish the corrector results as in [1,18], but we have chosen to skip the details in this article to keep the presentation simple. In fact, one just need to follow the ideas explained in [1]. It can be observed that the structural contributions, namely the multi-sheeted functions, of the oscillations in the boundary is not specifically visible as appear in the literature like [3,23]. The reason behind this is that we did not split the newly defined unfolding operators as it is done in [3]. We chose not to do so for just to keep the presentation simple as it is already very technical. In other words, the structural contribution of the oscillation in the boundary is hidden in the limit problem and can be extracted using the techniques explained in [3]. □

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References

1. Aiyappan, S., Nandakumaran, A.K., Prakash, R.: Semi-linear optimal control problem on a smooth oscillating domain. Commun. Contemp. Math. (2019). https://doi.org/10.1142/S0219199719500299
2. Aiyappan, S., Nandakumaran, A.K., Prakash, R.: Generalization of unfolding operator for highly oscillating smooth boundary domains and homogenization. Calc. Var. Partial Differ. Equ. **57**(3), 86 (2018)
3. Aiyappan, S., Nandakumaran, A.K.: Optimal control problem in a domain with branched structure and homogenization. Math. Methods Appl. Sci. **40**(8), 3173–3189 (2017)
4. Aiyappan, S.: Sardar, B.C.: Biharmonic equation in a highly oscillating domain and homogenization of an associated control problem. Appl. Anal **23**, 1–19 (2018)
5. Amirat, Y., Bodart, O., De Maio, U., Gaudiello, A.: Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary. SIAM J. Math. Anal. **35**(6), 1598–1616 (2004)
6. Arrieta, J.M., Bruschi, S.M.: Rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a Lipschitz deformation. Math. Models Methods Appl. Sci. **17**(10), 1555–1585 (2007)
7. Arrieta, J.M., Pereira, M.C.: Homogenization in a thin domain with an oscillatory boundary. J. Math. Pures Appl. (9) **96**(1), 29–57 (2011)
8. Arrieta, J.M., Villanueva-Pesqueira, M.: Thin domains with doubly oscillatory boundary. Math. Methods Appl. Sci **37**(2), 158–166 (2014)
9. Arrieta, J.M., Villanueva-Pesqueira, M.: Unfolding operator method for thin domains with a locally periodic highly oscillatory boundary. SIAM J. Math. Anal. **48**(3), 1634–1671 (2016)
10. Blanchard, D., Carbone, L., Gaudiello, A.: Homogenization of a monotone problem in a domain with oscillating boundary. ESAIM Math. Model. Numer. Anal. **33**(5), 1057–1070 (1999)
11. Blanchard, D., Gaudiello, A., Griso, G.: Junction of a periodic family of elastic rods with a 3d plate. Part I. Journal de mathématiques pures et appliquées **88**(1), 1–33 (2007)
12. Blanchard, D., Gaudiello, A., Griso, G.: Junction of a periodic family of elastic rods with a thin plate. Part II. Journal de mathématiques pures et appliquées **88**(2), 149–190 (2007)
13. Bourgeat, A., Paloka, E.M., Mikelić, A.: Effective fluid flow in a porous medium containing a thin fissure. Asymptot. Anal. **11**(3), 241–262 (1995)
14. Braides, A., Piat, C.: Homogenization of networks in domains with oscillating boundaries. Appl. Anal. **98**, 45–63 (2018)
15. Chechkin, G.A., Friedman, A., Piatnitski, A.L.: The boundary-value problem in domains with very rapidly oscillating boundary. J. Math. Anal. Appl. **231**(1), 213–234 (1999)
16. Cioranescu, D., Damlamian, A., Griso, G.: The Periodic Unfolding Method: Theory and Applications to Partial Differential Problems, Series in Contemporary Mathematics 03. Springer, Berlin (2019)
17. Damlamian, A., Pettersson, K.: Homogenization of oscillating boundaries. Discrete Contin. Dyn. Syst. **23**(1–2), 197–210 (2009)
18. Gaudiello, A., Guibé, O., Murat, F.: Homogenization of the brush problem with a source term in L^1. Arch. Ration. Mech. Anal. **225**(1), 1–64 (2017)

19. Gaudiello, A., Mel'nyk, T.A.: Homogenization of a nonlinear monotone problem with nonlinear Signorini boundary conditions in a domain with highly rough boundary. J. Differ. Equ. **265**, 5419–5454 (2018)
20. Krantz, S.G., Parks, H.R.: Geometric Integration Theory, Cornerstones. Birkhäuser Boston Inc., Boston (2008)
21. Mahadevan, R., Nandakumaran, A.K., Prakash, R: Homogenization of an elliptic equation in a domain with oscillating boundary with non-homogeneous non-linear boundary conditions, Appl. Math. Optim. (2018). https://doi.org/10.1007/s00245-018-9499-4
22. Mel'nyk, T.A.: Homogenization of a boundary-value problem with a nonlinear boundary condition in a thick junction of type 3:2:1. Math. Methods Appl. Sci. **31**(9), 1005–1027 (2008)
23. Mel'nyk, T.A.: Asymptotic approximation for the solution to a semi-linear parabolic problem in a thick junction with the branched structure. J. Math. Anal. Appl. **424**(2), 1237–1260 (2015)
24. Nandakumaran, A.K., Prakash, R.: Asymptotic analysis and error estimates for an optimal control problem with oscillating boundaries. Annali dell'Universita di Ferrara **58**(1), 143–166 (2012)
25. Nandakumaran, A.K., Prakash, R., Sardar, B.C.: Homogenization of an optimal control via unfolding method. SIAM J. Control Optim. **53**(5), 3245–3269 (2015)
26. Nandakumaran, A.K., Prakash, R., Sardar, B.C.: Asymptotic analysis of Neumann periodic optimal boundary control problem. Math. Methods Appl. Sci. **39**(15), 4354–4374 (2016)
27. Pankratova, I., Pettersson, K.: Spectral asymptotics for an elliptic operator in a locally periodic perforated domain. Appl. Anal. **94**(6), 1207–1234 (2015)
28. Pettersson, I.: Two-scale convergence in thin domains with locally periodic rapidly oscillating boundary. Differ. Equ. Appl. **9**(3), 393–412 (2017)
29. Ptashnyk, M.: Locally periodic unfolding method and two-scale convergence on surfaces of locally periodic microstructures. Multiscale Model. Simul. **13**(3), 1061–1105 (2015)
30. Showalter, R.E.: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, vol. 49. American Mathematical Society, Providence (2013)