Communications in Contemporary Mathematics
(2019) 1950029 (26 pages)

DOI: 10.1142/S0219199719500299

Semi-linear optimal control problem on a smooth oscillating domain

S. Aiyappan[*,‡], A. K. Nandakumaran[*,§] and Ravi Prakash[†,¶]

[*]*Department of Mathematics*
Indian Institute of Science, Bangalore, India

[†]*Department of Mathematics*
Faculty of Physical Sciences and Mathematics
University of Concepcion, Concepcion, Chile
[‡]*aiyappans@iisc.ac.in*
[§]*nands@iisc.ac.in*
[¶]*rprakash@udec.cl*

Received 8 June 2018
Revised 27 January 2019
Accepted 6 February 2019
Published 1 April 2019

We demonstrate the asymptotic analysis of a semi-linear optimal control problem posed on a smooth oscillating boundary domain in the present paper. We have considered a more general oscillating domain than the usual "pillar-type" domains. Consideration of such general domains will be useful in more realistic applications like circular domain with rugose boundary. We study the asymptotic behavior of the problem under consideration using a new generalized periodic unfolding operator. Further, we are studying the homogenization of a non-linear optimal control problem and such non-linear problems are limited in the literature despite the fact that they have enormous real-life applications. Among several other technical difficulties, the absence of a sufficient criteria for the optimal control is one of the most attention-grabbing issues in the current setting. We also obtain corrector results in this paper.

Keywords: Optimal control; homogenization; asymptotic analysis; oscillating boundary; unfolding operator.

Mathematics Subject Classification 2010: 49J20, 80M35, 80M40, 35B27

1. Introduction

In this paper, we wish to study a semi-linear optimal control problem whose state is governed by the following equation:

$$\begin{cases} -\Delta u_\varepsilon + k(u_\varepsilon) + u_\varepsilon = f_0 + \chi_\omega \theta \quad \text{in } \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \quad u_\varepsilon \text{ is } \Gamma_s\text{-periodic} \end{cases}$$

[§]Corresponding author.

with appropriate cost functional (see Sec. 4). Here, Ω_ε is a two-dimensional domain whose boundary is rapidly oscillating with high amplitude of $O(1)$. There are various reasons for studying the homogenization of such a non-linear problem. First of all, the literature of optimal control problems with semi-linear state equation is very limited. Second, we consider oscillatory domain whose oscillations are smooth even though it is periodic. Normally, in the literature, oscillations are of pillar-type of height $O(1)$ and width ε. But extensions to general periodic oscillations need more delicate analysis. It is true that there is a large amount of literature as far as oscillatory domains are concerned, but most of them (except few) consider non-smooth periodic oscillations, like a large number of pillars attached to a fixed domain. Now, we are in a process to consider a smooth arbitrary periodic oscillating boundary domain. We hope this will be a step towards general oscillations without periodicity. The study will also have high impact on fluid flow problems with rugose boundary. In fact, fluid flow problems on such domains are the main motivating factors for us to consider the problems in oscillatory domains. Our major tool is the method of unfolding which is a success in dealing with homogenization, but for the first time, we have used unfolding operators to characterize optimal controls. Standard unfolding operators can directly be applied to non-smooth domain (pillar-type), but it is a challenge to define unfolding for smooth domains. In this paper, we have successfully used (generalized) unfolding operators which we have developed in [1] to study the non-linear optimal control problem. Here, we have considered more general set of domains in the sense that the top boundary of the branches is not necessarily to be flat. The homogenization on these types of domains is investigated in very few papers.

There has been a lot of literature in homogenization of problems posed on rough domains using various techniques. We omit the earlier literature and cite some of the recent references. In 2004, Amirat *et al.* [3] used oscillating test functions to understand the asymptotic behavior of Laplace equation with zero Dirichlet boundary condition on the oscillating part Gaudiello and Sili [21] studied the homogenization of highly oscillating boundaries with strongly contrasting diffusivity. Homogenization of an elliptic problem with L^1 data posed on a pillar-type domain with non-flat base has been studied in [18]. In [20], the authors study the asymptotic analysis of a monotone problem with non-linear Signorini boundary condition on a rough domain. See [5, 8, 13, 23] and the references therein for more literature on homogenization of oscillating boundary.

In 2002, the periodic unfolding operator was introduced by Cioranescu *et al.* (see [11]). Blanchard *et al.* [6, 7] modified the definition of unfolding operator to study the homogenization on oscillating boundary domain Damlamian and Pettersson [12] used the modified operator for their study. Recently, Nandakumaran *et al.* [24] used the unfolding operator for characterizing the optimal control and also in getting the homogenized control problem posed on oscillating domains. See [2, 25] for the successful application of the unfolding operator for the characterization of the optimal control in oscillating domain. For more literature on homogenization of optimal control problems posed on oscillating domains, see [15, 16].

For homogenization of non-linear problems in oscillating domain, one can look into the work of Esposito *et al.* [17], where p-laplacian has been studied using Gamma convergence technique. Mel'nyk [23] has studied semi-linear parabolic problem using asymptotic expansion method. See [4, 5, 16, 19, 22] and the references therein for asymptotic analysis of non-linear problem in oscillating domain. For homogenization of optimal control problems, one can see [2, 14–16, 24–28].

The layout of this paper is as follows. In Sec. 2, we introduce the oscillating domain Ω_ε. In Sec. 3, we introduce the general unfolding operators for the domain under consideration. In Sec. 4, we describe the optimal control problem (P_ε) on the oscillating domain Ω_ε. We obtain a new limit problem which is described in Sec. 5. Since, it is a new optimal control problem, we prove the existence result as well in this paper. The homogenization and convergence analysis are given in Sec. 6. It is interesting to remark that we prove certain strong convergence in the oscillating part which is completely new to the best of our knowledge. Further, we also prove certain corrector results.

2. Oscillating Boundary Domain

For $L > 0$, consider a small parameter ε with $\varepsilon = \frac{L}{N}$, $N \in \mathbb{Z}^+$. We now describe the domain $\Omega_\varepsilon \subset \mathbb{R}^2$ (see Fig. 1) and its boundaries. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth and periodic function with period L. Let η be a smooth real-valued function defined on $[0, L]$ such that $\eta(0) = M_0 = \eta(L)$, where $M_0 =: \min_{x_1 \in (0,L)} \eta(x_1)$.

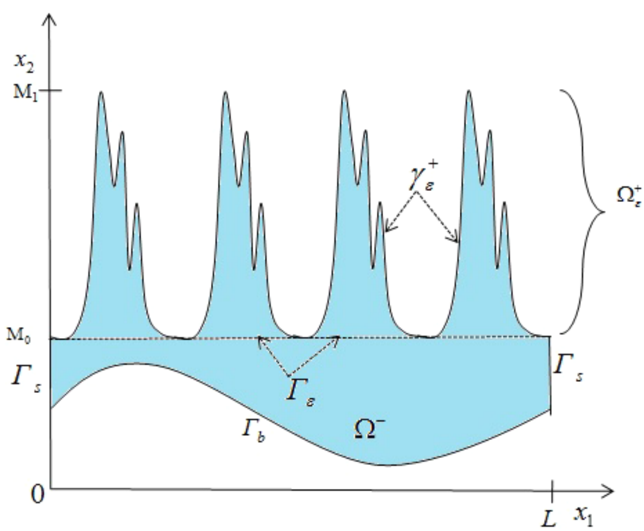

Fig. 1. Oscillating domain Ω_ε.

Also, assume that the function $\eta - M_0$ is compactly supported in $(0, L)$. Now, extend η to the whole of $\mathbb{R}$ by L-periodically. Let $M_1 =: \max_{x_1 \in (0,L)} \eta(x_1)$ and $\mathfrak{m} > \max_{x_1 \in (0,L)} g(x_1)$ with $M_1 > M_0 > \mathfrak{m}$. We define the domain Ω_ε as

$$\Omega_\varepsilon = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (0, L), g(x_1) < x_2 < \eta_\varepsilon(x_1) = \eta\left(\frac{x_1}{\varepsilon}\right)\right\}.$$

The top boundary of Ω_ε is denoted by γ_ε^+ and is defined by

$$\gamma_\varepsilon^+ = \{(x_1, x_2) : x_1 \in [0, L], x_2 = \eta_\varepsilon(x_1)\}.$$

The bottom boundary Γ_b of Ω_ε is defined by

$$\Gamma_b = \{(x_1, x_2) : x_2 = g(x_1), x_1 \in [0, L]\}.$$

Let Ω_ε^+ be the top part of the domain Ω_ε, which is defined by

$$\Omega_\varepsilon^+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (0, L), M_0 < x_2 < \eta_\varepsilon(x_1)\}.$$

The reference set $Y(a)$, for $a \in (M_0, M_1)$, is defined as

$$Y(a) = \{y \in (0, L) : \eta(y) > a\}.$$

Note that $Y(a)$ is Lebesgue measurable as η is assumed to be a smooth function. The varying reference set is crucially used in the definition of new unfolding operators.

Denote Ω^-, the fixed part of the domain Ω_ε, which is described by

$$\Omega^- = \{(x_1, x_2) : x_1 \in (0, L), g(x_1) < x_2 < M_0\}.$$

The lateral and top boundaries of Ω^- denoted by Γ_s and Γ_0 are defined as

$$\Gamma_s = \{(x_1, x_2) : g(x_1) \leq x_2 \leq M_0, x_1 = 0 \text{ or } x_1 = L\} \quad \text{and}$$

$$\Gamma_0 = \{(x_1, M_0) : x_1 \in [0, L]\},$$

respectively. The common boundary Γ_ε is the lower boundary of Ω_ε^+ which is defined as

$$\Gamma_\varepsilon = \{(x_1, x_2) \in \Omega_\varepsilon : x_2 = M_0\}.$$

We can also write Ω_ε as

$$\Omega_\varepsilon = \mathrm{Int}(\overline{\Omega_\varepsilon^+ \cup \Omega^-}).$$

Our full domain or the limiting domain Ω (see Fig. 3) is described by

$$\Omega = \{(x_1, x_2) : x_1 \in (0, L), g(x_1) < x_2 < M_1\}.$$

The upper part of the limit domain Ω^+ is defined by

$$\Omega^+ = \{(x_1, x_2) : x_1 \in (0, L), M_0 < x_2 < M_1\}.$$

The lower boundary of Ω is same as that of Ω_ε, namely Γ_b. The upper boundary Γ_u and the lateral boundaries $\Gamma_{s'}$ are defined as follows:

$$\Gamma_u = \{(x_1, M_1) : x_1 \in [0, L]\} \quad \text{and}$$

$$\Gamma_{s'} = \{(x_1, x_2) : g(x_1) \leq x_2 \leq M_1, x_1 = 0 \text{ or } x_1 = L\}.$$

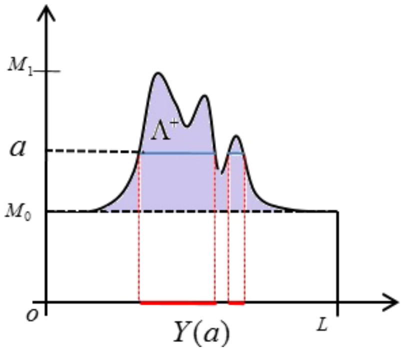

Fig. 2. Reference domain.

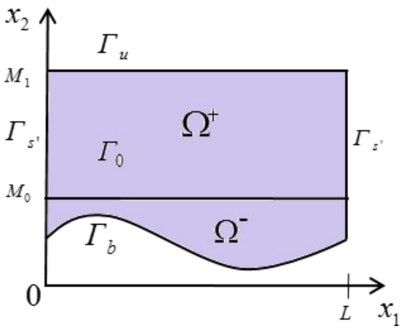

Fig. 3. Full domain.

Define a set $\mathbb{E}_\varepsilon = \{k \in \mathbb{Z} : \varepsilon kL + \varepsilon(0, L) \in (0, L)\}$ and the reference cell Λ^+ is defined as (see Fig. 2)

$$\Lambda^+ =: \{(y_1, y_2) : y_1 \in (0, L), M_0 < y_2 < \eta(y_1)\}.$$

The space $H^1_\#(\Omega_\varepsilon)$ is defined as $H^1_\#(\Omega_\varepsilon) = \{f|_{\Omega_\varepsilon} : f \in H^1_{\text{loc}}(\mathbb{R}^2), f \text{ is } \Gamma_s\text{-periodic}\}$. We call a function Γ_s-periodic if it takes the same value on the opposite lateral sides of the domain Ω^- in the sense of trace. And $C^\infty_\#(\overline{\Omega})$ is the set of all $C^\infty(\overline{\Omega})$ functions which are Γ_s-periodic.

Remark 2.1. In this paper, we have only considered the problem in two-dimensional domain. It is also possible to extend the results to three-dimensional domains. However, extending the results to dimension greater than 3 requires higher regularity on the data. Then, one can derive analogous results.

3. Unfolding Operator and its Properties

In this section, we recall the new periodic unfolding operator (T^ε) which is suitable to study the asymptotic behavior of the solution of control problem posed on a domain with highly oscillating smooth boundary (see [1]). Let us define the unfolded (fixed) domain Ω_u, where the unfolded functions are defined as follows.

Let $\mathcal{G} = \{(x_2, y) : x_2 \in (M_0,\ M_1), y \in Y(x_2)\}$, then Ω_u is defined as $\Omega_u = (0, L) \times \mathcal{G}$, which can be written as

$$\Omega_u =: \{(x_1, x_2, y) \,|\, x_1 \in (0, L), M_0 < x_2 < M_1, y \in Y(x_2)\}.$$

Now, we present the unfolding operators that we have developed in [1] and their properties. For $x_1 \in \mathbb{R}$, we write $[x_1]_L$ as the integer part of x_1 with respect to L, that is, $[x_1]_L = kL$, where k is the largest integer such that $kL \le x_1$ and the fractional part of x_1 with respect to L, namely $x_1 - [x_1]$ will be denoted by $\{x_1\}_L$.

Definition 3.1 (The unfolding operator). Let $\phi^\varepsilon : \Omega_u \to \Omega_\varepsilon^+$ be defined by $(x_1, x_2, y) \to (\varepsilon[\frac{x_1}{\varepsilon}]_L + \varepsilon y, x_2)$, that is, $\phi^\varepsilon(x_1, x_2, y) = (\varepsilon[\frac{x_1}{\varepsilon}]_L + \varepsilon y, x_2)$. The ε-unfolding of a function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \Omega_u \to \mathbb{R}$. The operator which maps every function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. Let the unfolding operator is denoted by T^ε, that is,

$$T^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{v : \Omega_u \to \mathbb{R}\}$$

defined by

$$T^\varepsilon u(x_1, x_2, y) = u \circ \phi^\varepsilon(x_1, x_2, y) = u\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]_L + \varepsilon y, x_2\Big).$$

If U is an open subset of $\mathbb{R}^2$ containing Ω_ε^+ and u is a real-valued function on U, $T^\varepsilon u$ will mean T^ε acting on the restriction of u to Ω_ε^+.

One can easily observe that we are calling the above-defined operator as a "new" periodic unfolding operator not because of its definition but Ω_u which appeared because of the nature of the oscillations in the boundary of the domain Ω_ε^+. Some of the important properties of T^ε are given as follows. We refer [1] for the proofs.

Proposition 3.2. (a) *For each fixed $\varepsilon > 0$, T^ε is linear. Further, if $u, v : \Omega_\varepsilon^+ \to \mathbb{R}$, then $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.*

(b) *Let $u \in L^1(\Omega_\varepsilon^+)$. Then, $\int_{\Omega_u} T^\varepsilon u dx dy = L \int_{\Omega_\varepsilon^+} u dx$.*

(c) *Let $u \in L^2(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2(\Omega_u)$ and $\|T^\varepsilon u\|_{L^2(\Omega_u)} = \sqrt{L}\|u\|_{L^2(\Omega_\varepsilon^+)}$.*

(d) *Let $u \in H^1(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2((0, L); H^1(\mathcal{G}))$. Moreover, $\frac{\partial}{\partial x_2} T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial x_2}$ and $\frac{\partial}{\partial y} T^\varepsilon u = \varepsilon T^\varepsilon \frac{\partial u}{\partial x_1}$.*

(e) *Let $u \in L^2(\Omega^+)$. Then, $T^\varepsilon u \to u$ in $L^2(\Omega_u)$. More generally, let $u_\varepsilon \to u$ in $L^2(\Omega^+)$. Then, $T^\varepsilon u_\varepsilon \to u$ in $L^2(\Omega_u)$.*

(f) *Let, for every ε, $u_\varepsilon \in L^2(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega_u)$. Then,*

$$\widetilde{u}_\varepsilon \rightharpoonup \frac{1}{L} \int_{y \in Y(x_2)} u(x_1, x_2, y) dy$$

weakly in $L^2(\Omega^+)$. Here, $\widetilde{u}_\varepsilon$ is the zero extension of u_ε to Ω^+.

(g) *Let $u_\varepsilon \in H^1(\Omega_\varepsilon^+)$ for every $\varepsilon > 0$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0, L); H^1(\mathcal{G}))$. Then, $\widetilde{u}_\varepsilon \rightharpoonup \frac{1}{L} \int_{Y(x_2)} u dy$ and $\widetilde{\frac{\partial u_\varepsilon}{\partial x_2}} \rightharpoonup \frac{1}{L} \int_{Y(x_2)} \frac{\partial u}{\partial x_2} dy$ weakly in $L^2(\Omega^+)$.*

4. Optimal Control

This section is devoted to study a semi-linear optimal control problem posed on an oscillating boundary domain which has been described in the previous section. We apply control in the fixed interior part. In our earlier works (see [2, 24, 25]), we have considered control in the oscillating part, where we have characterized the optimal control using unfolding operators. In this non-linear problem, we have considered control only on the non-oscillating part. Applying control on the oscillating part is highly challenging in non-linear problems, and is the topic of our future work.

Consider the following optimal control problem: Find $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon) \in H^1_\#(\Omega_\varepsilon) \times L^2(\omega)$ such that

$$(P_\varepsilon) \quad J_\varepsilon(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta) \,|\, (u_\varepsilon, \theta) \in H^1_\#(\Omega_\varepsilon) \times L^2(\omega), (u_\varepsilon, \theta) \text{ satisfies (4.2)}\}.$$

The cost functional $J_\varepsilon(u_\varepsilon, \theta)$ is given by

$$J_\varepsilon(u_\varepsilon, \theta) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_{\Omega^-} \chi_\omega |\theta|^2, \tag{4.1}$$

where ω is an open subset of Ω^-, $u_d \in L^2(\Omega)$ is the desired state and β is a regularization parameter. Given $\theta \in L^2(\omega)$, the state u_ε satisfies the following semi-linear state equation:

$$\begin{cases} -\Delta u_\varepsilon + k(u_\varepsilon) + u_\varepsilon = f_0 + \chi_\omega \theta & \text{in } \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \quad u_\varepsilon \text{ is } \Gamma_s\text{-periodic}. \end{cases} \tag{4.2}$$

Here, f_0 is a given function in $L^2(\Omega)$; ∂_ν is the outward normal derivative; $k : \mathbb{R} \to \mathbb{R}$ is a smooth real-valued function, such that

$$0 < C_1 \le k'(t) \le C_2, \quad k(0) = 0 \quad \text{and} \quad k'' \text{ is bounded.} \tag{4.3}$$

By applying monotone operator theory (see [29, 30]), it has been shown that if $f_0 \in L^2(\Omega_\varepsilon)$ and $\theta \in L^2(\omega)$, then, Eq. (4.2) admits a unique weak solution u_ε in $H^1_\#(\Omega_\varepsilon)$. Moreover, the solution satisfies the following *a priori* estimate:

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C(\|f_0\|_{L^2(\Omega_\varepsilon)} + \|\theta\|_{L^2(\omega)}), \tag{4.4}$$

where $C > 0$ is independent of ε which can be easily verified. Let us recall the following well-known result on non-linear optimal control problems (see [9, 30]).

Theorem 4.1. *For each $\varepsilon > 0$, the minimization problem (P_ε) admits at least one solution. Further, let $(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon) \in H^1_\#(\Omega_\varepsilon) \times L^2(\omega)$ be an optimal solution to (P_ε). Then, the optimal control satisfies*

$$\overline{\theta}_\varepsilon = \frac{-1}{\beta}\overline{p}_\varepsilon \chi_\omega, \tag{4.5}$$

where the state $\overline{u}_\varepsilon$ *and the adjoint state* $\overline{p}_\varepsilon$ *satisfy*

$$\begin{cases} -\Delta\overline{u}_\varepsilon + k(\overline{u}_\varepsilon) + \overline{u}_\varepsilon = f_0 + \chi_\omega\overline{\theta}_\varepsilon & in\ \Omega_\varepsilon, \\ -\Delta\overline{p}_\varepsilon + k'(\overline{u}_\varepsilon)\overline{p}_\varepsilon + \overline{p}_\varepsilon = \overline{u}_\varepsilon - u_d & in\ \Omega_\varepsilon, \\ \partial_\nu\overline{u}_\varepsilon = 0, \quad \partial_\nu\overline{p}_\varepsilon = 0 \quad on\ \Gamma_\varepsilon, \quad \overline{u}_\varepsilon, \overline{p}_\varepsilon\ are\ \Gamma_s\text{-}periodic. \end{cases} \tag{4.6}$$

Remark 4.2. It is interesting to observe that the adjoint state $\overline{p}_\varepsilon$ is defined via a linear boundary value problem.

5. Limit Optimal Control Problem

In this section, our aim is to introduce the limit optimal control problem and then we prove the existence result. We begin by introducing certain function spaces required for our analysis. Let $h(x_2) = |Y(x_2)|$, where $|Y(x_2)|$ is the Lebesgue measure of the set $Y(x_2)$ at $x_2 \in (M_0, M_1)$ and η is chosen such that $h(x_2) > 0$ in (M_0, M_1). For any measurable function ϕ defined on Ω, we denote $\phi^+ = \phi|_{\Omega^+}$ and $\phi^- = \phi|_{\Omega^-}$. Define the limit space $W(\Omega)$ as

$$W(\Omega) =: \{\phi : h^{1/2}\phi^+ \in L^2(\Omega^+), h^{1/2}\partial_{x_2}\phi^+ \in L^2(\Omega^+),$$
$$\phi^- \in H^1(\Omega^-) \text{ and } \phi^+ = \phi^- \text{ on } \Gamma_0\}.$$

Note that $W(\Omega)$ is a Hilbert space (see [10]) with the inner product

$$\langle u, v\rangle_W = \langle h^{1/2}u, h^{1/2}v\rangle_{L^2(\Omega^+)} + \langle h^{1/2}\partial_{x_2}u, h^{1/2}\partial_{x_2}v\rangle_{L^2(\Omega^+)} + \langle u, v\rangle_{H^1(\Omega^-)}. \tag{5.1}$$

The limit problem is defined as: Find $u_0 \in W(\Omega)$ such that

$$\begin{aligned} &\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial u_0^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + (k(u_0^+) + u_0^+)\psi\right) + \int_{\Omega^-} \nabla u_0^- \cdot \nabla\psi + (k(u_0^-) + u_0^-)\psi \\ &\quad = \frac{1}{L}\int_{\Omega^+} hf_0^+\psi + \int_{\Omega^-} f_0^-\psi, \quad \forall\,\psi \in W(\Omega), \end{aligned} \tag{5.2}$$

where $u_0 = u_0^+\chi_{\Omega^+} + u_0^-\chi_{\Omega^-}$. Using the monotonicity of the operator k and though the problem is new, one can adopt the techniques of the proof in [29, 30] to get the well-posedness result of the above problem. We state the result as follows.

Theorem 5.1. *Let* $f_0 \in L^2(\Omega)$ *and* k *be as in* (4.3). *Then, the problem* (5.3) *has a unique weak solution* u_0 *in* $W(\Omega)$. *Moreover, the solution satisfies the following a priori estimate*:

$$\|u_0\|_{W(\Omega)} \leq C\|f_0\|_{L^2(\Omega)}.$$

Remark 5.2. If the function h is positive in $[M_0, M_1]$, then u_0 solves the following strong form for the limit problem:

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(h\dfrac{\partial u_0^+}{\partial x_2}\right) + hk(u_0^+) + hu_0^+ = hf_0^+ & \text{in } \Omega^+, \\ -\Delta u_0^- + k(u_0^-) + u_0^- = f_0^- & \text{in } \Omega^- \end{cases} \tag{5.3}$$

with the boundary and interface conditions

$$(BIFC) \quad \begin{cases} \dfrac{\partial u_0^+}{\partial \nu} = 0 \quad \text{on } \Gamma_b \cup \Gamma_u, \\ \dfrac{h(M_0)}{L}\dfrac{\partial u_0^+}{\partial x_2} = \dfrac{\partial u_0^-}{\partial x_2} \quad \text{on } \Gamma_0, \\ u_0 \quad \text{is } \Gamma_{s'}\text{-periodic}, \end{cases} \tag{5.4}$$

where $u_0 = u_0^+\chi_{\Omega^+} + u_0^-\chi_{\Omega^-}$.

Since we are considering more general domains, namely we allow h to be zero at M_1, we look for solutions in more general space unlike $L^2(\Omega)$.

We, now describe the limit system which we will show in Theorem 5.6 that it is indeed an optimality system of a control problem (P).

Find $(u_0, p_0) \in W(\Omega) \times W(\Omega)$ such that

$$\begin{cases} \displaystyle\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial u_0^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + (u_0^+ + k(u_0^+))\psi\right) + \int_{\Omega^-} \nabla u_0^- \cdot \nabla\psi + (u_0^+ + k(u_0^-))\psi \\ \displaystyle\qquad = \int_{\Omega^-} (f_0 + \theta_0\chi_\omega)\psi + \int_{\Omega^+} \frac{h}{L} f_0\psi \\ \displaystyle\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial p_0^+}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + (p_0^+ + k(p_0^+))\varphi\right) + \int_{\Omega^-} \nabla p_0^- \cdot \nabla\psi + (p_0^+ + k(p_0^-))\varphi \\ \displaystyle\qquad = \int_{\Omega^-} (u_0^- - u_d)\varphi + \int_{\Omega^+} \frac{h}{L}(u_0^+ - u_d)\varphi \end{cases} \tag{5.5}$$

for all $(\psi, \varphi) \in W(\Omega) \times W(\Omega)$ with

$$\theta_0 = \frac{-1}{\beta} p_0\chi_\omega.$$

Using the similar techniques as in Theorem 5.1, it is easy to see that the above coupled system has a unique solution $(u_0, p_0) \in W(\Omega) \times W(\Omega)$, for given $f_0 \in L^2(\Omega)$ and $\theta_0 \in L^2(\omega)$.

We will now propose a limit optimal control problem. Find $(\overline{u}, \overline{\theta}) \in W(\Omega) \times L^2(\omega)$ such that

$$(P_0) \quad J_0(\overline{u}, \overline{\theta}) = \inf\{J_0(u, \theta) \,|\, (u, \theta) \in W(\Omega) \times L^2(\omega) \text{ satisfies (5.7)}\}.$$

The cost functional $J_0(u,\theta)$ is given by

$$J_0(u,\theta) = \frac{1}{2}\int_{\Omega^-} |u^- - u_d|^2 + \frac{1}{2L}\int_{\Omega^+} h|u^+ - u_d|^2 + \frac{\beta}{2}\int_{\Omega} \chi_\omega |\theta|^2. \tag{5.6}$$

Given $\theta \in L^2(\omega)$, the state $u \in W(\Omega)$ satisfies the following state equation: Find $u \in W(\Omega)$ such that

$$\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial u^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + (u^+ + k(u^+))\psi\right) + \int_{\Omega^-} \nabla u^- \cdot \nabla \psi + (u^+ + k(u^-))\psi$$
$$= \int_{\Omega^-} (f_0 + \theta\chi_\omega)\psi + \int_{\Omega^+} \frac{h}{L} f_0 \psi \tag{5.7}$$

for all $\psi \in W(\Omega)$. Since, it is a new optimal control problem, we will present a proof for the existence of optimal control and optimal state. This is an important component of the present paper.

Theorem 5.3. *The optimal control problem* (P_0) *admits at least one solution.*

Proof. Note that $F(\theta) = J_0(u(\theta),\theta) \geq 0$ for all $\theta \in L^2(\omega)$. Hence, F is bounded below which guarantees the infimum, say m^*. Let θ_n be a minimizing sequence such that $F(\theta_n) \to m^*$. From this, we infer that $\{\theta_n\}$ is bounded in $L^2(\omega)$. By using Theorem 5.1, we derive that $\|u_n(\theta_n)\|_{W(\Omega)} \leq C$ as θ_n is bounded. Here, u_n solves the state equation (5.7) with $\theta = \theta_n$ and C is a constant independent of n. As $\{\theta_n\}$ and $\{u_n\}$ are bounded, we can extract a subsequence such that

$$\theta_n \rightharpoonup \overline{\theta} \quad \text{weakly in } L^2(\omega),$$
$$u_n \rightharpoonup u_0 \quad \text{weakly in } W(\Omega)$$

for some $\overline{\theta}$ and u_0 in the corresponding spaces. We want to show that $u_0 = u(\overline{\theta})$, that is u_0 solves (5.7) with $\theta = \overline{\theta}$, which will conclude that $\overline{\theta}$ is an optimal control as norm is weakly lower semi-continuous. Also note that $\|h^{1/2}k(u_n^+)\|_{L^2(\Omega^+)} \leq C$ and $\|k(u_n^-)\|_{L^2(\Omega^-)} \leq C$ as u_n is bounded in W and k satisfies (4.3). This implies that there exist $h^{1/2}\xi_1$ in $L^2(\Omega^+)$ and ξ_2 in $L^2(\Omega^-)$ such that

$$h^{1/2}k(u_n^+) \rightharpoonup h^{1/2}\xi_1 \quad \text{weakly in } L^2(\Omega^+),$$
$$k(u_n^-) \rightharpoonup \xi_2 \quad \text{weakly in } L^2(\Omega^-).$$

Recall the state equation (5.7) satisfied by u_n and θ_n:

$$\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial u_n^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + (u_n^+ + k(u_n^+))\psi\right) + \int_{\Omega^-} \nabla u_n^- \cdot \nabla \psi + (u_n^- + k(u_n^-))\psi$$
$$= \frac{1}{L}\int_{\Omega^+} f_0^+ \psi + \int_{\Omega^-} (f_0^- + \chi_\omega \theta_n)\psi, \quad \forall\, \psi \in W(\Omega). \tag{5.8}$$

Using the above convergence, we get as $n \to \infty$,

$$\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial u_0^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + (u_0^+ + \xi_1)\psi\right) + \int_{\Omega^-} \nabla u_0^- \cdot \nabla \psi + (u_0^- + \xi_2)\psi$$
$$= \frac{1}{L}\int_{\Omega^+} f_0^+ \psi + \int_{\Omega^-} (f_0^- + \chi_\omega \overline{\theta})\psi, \quad \forall\, \psi \in W(\Omega). \tag{5.9}$$

One of the major step in the proof is the following claim, namely, the identification of ξ_1 and ξ_2.

Claim ($\boldsymbol{\xi_1 = k(u_0^+)}$ and $\boldsymbol{\xi_2 = k(u_0^-)}$). *We prove the claim using Browder–Minty-type argument. Consider the inequality with $\phi \in W(\Omega)$:*

$$\int_{\Omega^+} \frac{h}{L}|\partial_{x_2} u_n^+ - \partial_{x_2}\phi|^2 + \int_{\Omega^-} |\nabla u_n^- - \nabla \phi|^2 + \int_{\Omega^+} \frac{h}{L}|u_n^+ - \phi|^2$$
$$+ \int_{\Omega^-} |u_n^- - \phi|^2 + \int_{\Omega^+} \frac{h}{L}(k(u_n^+) - k(\phi))(u_n^+ - \phi)$$
$$+ \int_{\Omega^-} (k(u_n^-) - k(\phi))(u_n^- - \phi) \geq 0.$$

Upon expanding the above inequality, we obtain

$$\int_{\Omega^+} \frac{h}{L}(|\partial_{x_2} u_n^+|^2 + k(u_n^+)u_n^+ + |u_n^+|^2) + \int_{\Omega^-} (|\nabla u_n^-|^2 + k(u_n^-)u_n^- + |u_n^-|^2)$$
$$+ \int_{\Omega^+} \frac{h}{L}(|\partial_{x_2}\phi|^2 - 2\partial_{x_2} u_n^+ \partial_{x_2}\phi + k(\phi)\phi - k(u_n^+)\phi - k(\phi)u_n^+ + |\phi|^2 - 2u_n^+\phi)$$
$$+ \int_{\Omega^-} (|\nabla\phi|^2 - 2\nabla u_n^- \nabla\phi + k(\phi)\phi - k(u_n^-)\phi - k(\phi)u_n^- + |\phi|^2 - 2u_n^-\phi) \geq 0. \tag{5.10}$$

Let us look at the first line of the inequality (5.10). *From* (5.8), *it follows that*

$$\lim_{n\to\infty}\left[\int_{\Omega^+} \frac{h}{L}(|\partial_{x_2} u_n^+|^2 + k(u_n^+)u_n^+ + |u_n^+|^2) + \int_{\Omega^-} (|\nabla u_n^-|^2 + k(u_n^-)u_n^- + |u_n^-|^2)\right]$$
$$= \lim_{n\to\infty}\left[\frac{1}{L}\int_{\Omega^+} f_0^+ u_n^+ + \int_{\Omega^-} (f_0^- + \chi_\omega \theta_n)u_n^-\right]$$
$$= \frac{1}{L}\int_{\Omega^+} f_0^+ u_0^+ + \int_{\Omega^-} (f_0^- + \chi_\omega \overline{\theta})u_0^-$$
$$= \int_{\Omega^+} \frac{h}{L}(|\partial_{x_2} u_0^+|^2 + \xi_1 u_0^+ + |u_0^+|^2) + \int_{\Omega^-} |\nabla u_0^-|^2 + \xi_2 u_0^- + |u_0^-|^2.$$

The last equality is due to Eq. (5.9) *with $\psi = u_0$. Using the weak convergences of u_n, $h^{1/2}k(u_n^+)$ and $k(u_n^-)$, we can easily pass to the limits in the other two lines of* *the inequality* (5.10). *Hence, the inequality* (5.10) *becomes, as* $n \to \infty$,

$$\int_{\Omega^+} \frac{h}{L}(|\partial_{x_2} u_0^+|^2 + \xi_1 u_0^+ + |u_0^+|^2) + \int_{\Omega^-} |\nabla u_0^-|^2 + \xi_2 u_0^- + |u_0^-|^2$$
$$+ \int_{\Omega^+} \frac{h}{L}(|\partial_{x_2}\phi|^2 - 2\partial_{x_2} u_0^+ \partial_{x_2}\phi + k(\phi)\phi - \xi_1\phi - k(\phi)u_0^+ + |\phi|^2 - 2u_0^+\phi)$$
$$+ \int_{\Omega^-} (|\nabla\phi|^2 - 2\nabla u_0^-\nabla\phi + k(\phi)\phi - \xi_2\phi - k(\phi)u_0^- + |\phi|^2 - 2u_0^-\phi) \geq 0.$$

By rewriting, we get

$$\int_{\Omega^+} \frac{h}{L}|\partial_{x_2} u_0^+ - \partial_{x_2}\phi|^2 + \int_{\Omega^-} |\nabla u_0^- - \nabla\phi|^2 + \int_{\Omega^+} \frac{h}{L}|u_0^+ - \phi|^2$$
$$+ \int_{\Omega^-} |u_0^- - \phi|^2 + \int_{\Omega^+} \frac{h}{L}(\xi_1 - k(\phi))(u_0^+ - \phi) + \int_{\Omega^-} (\xi_2 - k(\phi))(u_0^- - \phi) \geq 0.$$

Now, for $\psi \in C^1(\overline{\Omega})$, *choose* $\phi = u_0 - \lambda\psi$, $\lambda > 0$, *to get*:

$$\int_{\Omega^+} \frac{\lambda h}{L}|\partial_{x_2}\psi|^2 + \lambda\int_{\Omega^-} |\nabla\psi|^2 + \int_{\Omega^+} \frac{\lambda h}{L}|\psi|^2 + \lambda\int_{\Omega^-} |\psi|^2$$
$$+ \int_{\Omega^+} \frac{h}{L}(\xi_1 - k(u_0^+ - \lambda\psi))\psi + \int_{\Omega^-} (\xi_2 - k(u_0^- - \lambda\psi))\psi \geq 0, \quad \forall\, \psi \in C^1(\overline{\Omega}).$$

As $\lambda \to 0$, *we get the following inequality*:

$$\int_{\Omega^+} \frac{h}{L}(\xi_1 - k(u_0^+))\psi + \int_{\Omega^-} (\xi_2 - k(u_0^-))\psi \geq 0, \quad \forall\, \psi \in C^1(\overline{\Omega}).$$

By choosing $\psi \in \mathcal{D}(\Omega^+)$ and $\mathcal{D}(\Omega^-)$, we derive that $\xi_1 = k(u_0^+)$ and $\xi_2 = k(u_0^-)$. This shows that $u(\overline{\theta}) = u_0$, which shows that $\overline{\theta}$ is an optimal control. □

To prove the necessary optimality conditions, we recall a theorem from [30]. We assume an additional condition that the state solution u of (5.7) belongs to $W(\Omega) \cap C(\overline{\Omega})$. With this assumption, we state the following theorem whose proof we omit as it will be along the same lines of proof as in [30]. Let $\mathbb{G}(\theta) = u$ be the control to solution operator, where u is the solution of (5.7) corresponding to θ.

Theorem 5.4. *Let* $\mathbb{G}$ *be the control to state operator as above and* k *satisfies the condition* (4.3). *Then,* $\mathbb{G}$ *is Lipschitz continuous mapping from* $L^2(\omega)$ *to* $W(\Omega) \cap C(\overline{\Omega})$, *that is, there exists a constant* $\mathcal{L} > 0$ *such that*

$$\|u_1 - u_2\|_{W(\Omega)} + \|u_1 - u_2\|_{C(\overline{\Omega})} \leq \mathcal{L}\|\theta_1 - \theta_2\|_{L^2(\omega)}$$

whenever $\theta_i \in L^2(\omega)$ *and* $u_i = \mathbb{G}(\theta_i)$.

The following lemma is crucial in getting the necessary optimality condition. We will give a proof for this lemma as it is in the new set up. Let $\overline{u} = \mathbb{G}(\overline{\theta})$ and w

be the solution of the linearized problem: Find $w \in W(\Omega)$ such that

$$\int_{\Omega^+} h\left(\frac{\partial w^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + (k'(\overline{u}^+)w^+ + w^+)\psi\right) + \int_{\Omega^-} \nabla w^- \cdot \nabla \psi + (k'(\overline{u}^-)w^- + w^-)\psi$$
$$= \int_{\Omega^-} \theta \chi_\omega \psi \tag{5.11}$$

for all $\psi \in W(\Omega)$.

Lemma 5.5. *Let us assume that the state $w \in L^\infty(\Omega)$ for every $\theta \in L^2(\omega)$. Then, the Nemytskii operator $\mathcal{K}$ defined by $\mathcal{K}(u)(x) = k(u(x))$ a.e. $x \in \Omega$ is Fréchet differentiable on $L^\infty(\Omega)$, where k is a smooth function satisfying the condition* (4.3). *Moreover, the control to state operator $\mathbb{G}$ defined above is Fréchet differentiable from $L^2(\omega)$ to $W(\Omega) \cap L^\infty(\Omega)$. Its derivative at $\overline{\theta} \in L^2(\omega)$ in the direction θ is given by*

$$\mathbb{G}'(\overline{\theta})\theta = w,$$

where w as in (5.11).

Proof. Let $u, v \in L^\infty(\Omega)$, choose M such that $\|u(x)\| \leq M$ and $\|v(x)\| \leq M$ a.e. $x \in \Omega$. Then,

$$k(u(x) + v(x)) - k(u(x)) = k'(u(x))v(x) + r(u,v)(x)$$

with the remainder

$$r(u,v)(x) = v(x)\int_0^1 [k'(u(x) + sv(x)) - k'(u(x))]ds.$$

Now,

$$|r(u,v)(x)| \leq C|v(x)|\int_0^1 s|v(x)|ds$$
$$\leq C_1|v(x)|^2 \leq C\|v\|^2_{L^\infty(\Omega)}.$$

Therefore,

$$\|r(u,v)\|_{L^\infty(\Omega)} \leq C\|v\|^2_{L^\infty(\Omega)}.$$

This implies, as $\|v\|_{L^\infty(\Omega)} \to 0$,

$$\frac{\|r(u,v)\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\Omega)}} \leq C\|v\|_{L^\infty(\Omega)} \to 0.$$

This proves that the Nemytskii operator is Fréchet differentiable. For the second part, we need to show

$$\mathbb{G}(\overline{\theta} + \theta) - \mathbb{G}(\overline{\theta}) = D\theta + r(\overline{\theta}, \theta)$$

with a bounded linear operator $D : L^2(\omega) \to W(\Omega) \cap L^\infty(\Omega)$ and mapping r such that

$$\frac{\|r(\overline{\theta}, \theta)\|_{W(\Omega)\cap L^\infty(\Omega)}}{\|\theta\|_{L^2(\omega)}} \to 0 \quad \text{as } \|\theta\|_{L^2(\omega)} \to 0.$$

Here, $\|r\|_{W(\Omega)\cap L^\infty(\Omega)} = \|r\|_{W(\Omega)} + \|r\|_{L^\infty(\Omega)}$. Now, denote $\overline{u}$ and $\hat{u}$ as the weak solutions of the state equation (5.7) with θ replaced by $\overline{\theta}$ and $\overline{\theta}+\theta$, respectively. Subtracting them gives

$$\begin{aligned}\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial}{\partial x_2}(\hat{u}^+ - \overline{u}^+)\frac{\partial\psi}{\partial x_2} + (k(\hat{u}^+) - k(\overline{u}^+))\psi + (\hat{u}^+ - \overline{u}^+)\psi\right)\\ + \int_{\Omega^-} \nabla(\hat{u}^- - \overline{u}^-)\cdot\nabla\psi + ((\hat{u}^- - \overline{u}^-) + (k(\hat{u}^-) - k(\overline{u}^-)))\psi\\ = \int_{\Omega^-} \theta\chi_\omega\psi\end{aligned}$$

for all $\psi \in W(\Omega)$. As the Nemytskii operator is differentiable in $L^\infty(\Omega)$, we have

$$\mathcal{K}(\hat{u}) - \mathcal{K}(\overline{u}) = k'(\overline{u})(\hat{u} - \overline{u}) + r_d$$

with the remainder r_d such that

$$\frac{\|r_d\|_{L^\infty(\Omega)}}{\|\hat{u} - \overline{u}\|_{L^\infty(\Omega)}} \to 0 \quad \text{as } \|\hat{u} - \overline{u}\|_{L^\infty(\Omega)} \to 0.$$

Now, we can write $\hat{u} - \overline{u} = w + u_\rho$, where w solves (5.11) and u_ρ is the solution of

$$\begin{aligned}\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial u_\rho^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + (k'(\overline{u}^+)u_\rho^+ + u_\rho^+)\psi\right) + \int_{\Omega^-} \nabla u_\rho^- \cdot\nabla\psi + (k'(\overline{u}^-)u_\rho^- + u_\rho^-)\psi\\ = -\int_{\Omega^+} \frac{h}{L} r_d\psi - \int_{\Omega^-} r_d\psi \qquad (5.12)\end{aligned}$$

for all $\psi \in W(\Omega)$.

As $k' \geq C_1 > 0$, the above system has a unique solution. Recall that $\mathbb{G}$ is Lipschitz continuous from $L^2(\omega)$ to $W(\Omega)\cap L^\infty(\Omega)$. Hence,

$$\|\hat{u} - \overline{u}\|_{L^\infty(\Omega)} + \|\hat{u} - \overline{u}\|_{W(\Omega)} \leq \mathcal{L}\|\theta\|_{L^2(\omega)}.$$

Here, $\mathcal{L}$ is the Lipschitz constant. Note that

$$\frac{\|r_d\|_{L^\infty(\Omega)}}{\|\theta\|_{L^2(\omega)}} = \frac{\|r_d\|_{L^\infty(\Omega)}}{\|\hat{u} - \overline{u}\|_{L^\infty(\Omega)}}\frac{\|\hat{u} - \overline{u}\|_{L^\infty(\Omega)}}{\|\theta\|_{L^2(\omega)}} \leq \mathcal{L}\frac{\|r_d\|_{L^\infty(\Omega)}}{\|\hat{u} - \overline{u}\|_{L^\infty(\Omega)}}$$

and thus $\|r_d\|_{L^\infty(\Omega)} = o(\|\theta\|_{L^2(\omega)})$. Also from (5.12), we have

$$\|u_\rho\|_{L^\infty(\Omega)} + \|u_\rho\|_{W(\Omega)} = o(\|\theta\|_{L^2(\omega)}).$$

Denoting the linear map $\theta \mapsto w$ by D, we conclude that

$$\mathbb{G}(\overline{\theta} + \theta) - \mathbb{G}(\overline{\theta}) = \hat{u} - \overline{u} = D\theta + u_\rho = D\theta + r(\overline{\theta}, \theta),$$

where $r(\overline{\theta}, \theta) = u_\rho$ has the required properties. This proves the lemma. ∎

5.1. *Optimality condition*

Let us look at the necessary optimality condition for the optimal control problem (P_0). The cost functional J_0 can be written as a function of θ, that is, $F(\theta) = J_0(\mathbb{G}(\theta), \theta)$, where

$$\begin{aligned} J_0(\mathbb{G}(\theta), \theta) &= J_0(u, \theta) \\ &= \frac{1}{2}\int_{\Omega^-} |\mathbb{G}(\theta)^- - u_d|^2 + \frac{1}{2L}\int_{\Omega^+} h|\mathbb{G}(\theta)^+ - u_d|^2 + \frac{\beta}{2}\int_\Omega \chi_\omega |\theta|^2. \end{aligned}$$

The Fréchet differentiability of $\mathbb{G}$ implies that of F. Since $\overline{\theta}$ is optimal, we have $F'(\overline{\theta})\theta = 0$ for all $\theta \in L^2(\omega)$. That is,

$$F'(\overline{\theta})\theta = \int_{\Omega^-} (\overline{u}^- - u_d)w^+ + \frac{1}{L}\int_{\Omega^+} h(\overline{u}^+ - u_d)w^- + \beta\int_\Omega \chi_\omega \overline{\theta}\theta = 0 \tag{5.13}$$

$\forall\, \theta \in L^2(\omega)$, where w is the unique solution of (5.11). Now, let us define the adjoint equation

$$\begin{aligned} &\int_{\Omega^+} \frac{h}{L}\left(\frac{\partial p^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + (k'(\overline{u}^+)p^+ + p^+)\psi\right) + \int_{\Omega^-} \nabla p^- \cdot \nabla\psi + (k'(\overline{u}^-)p^- + p^-)\psi \\ &\quad = \int_{\Omega^+} \frac{h}{L}(\overline{u}^+ - u_d)\psi + \int_{\Omega^-} (\overline{u}^- - u_d)\psi \end{aligned} \tag{5.14}$$

for all $\psi \in W(\Omega)$.

Choosing p and w as the test functions in (5.11) and (5.14), respectively, we get

$$\int_{\Omega^-} (\overline{u}^- - u_d)w^- + \frac{1}{L}\int_{\Omega^+} h(\overline{u}^+ - u_d)w^+ = \int_\Omega \chi_\omega \theta p^-. \tag{5.15}$$

Now comparing (5.13) and (5.15), we get

$$\overline{\theta} = \frac{-1}{\beta} p\chi_\omega.$$

Thus, we have proved the following theorem under the assumption that the state $u_0 \in L^\infty(\Omega)$.

Theorem 5.6. *Let $(\overline{u}, \overline{\theta})$ be an optimal control for (P_0) and assume that the state solution belongs to $W(\Omega) \cap C(\overline{\Omega})$, then we have the following optimality condition*:

$$\overline{\theta} = \frac{-1}{\beta} p\chi_\omega,$$

where p is the adjoint state solving (5.14).

6. Homogenization

6.1. *Convergence analysis*

In this section, we will prove the homogenization results, as $\varepsilon \to 0$, of the optimal control problem (P_ε). In this direction, let us state the well-known lemma (see [12]) which will assist us in proving the convergence of optimality system.

Lemma 6.1. *Let m be a fixed integer, α_ε^j, $j = 1, \ldots, m$ be m bounded sequences of real numbers and α^j, $j = 1, \ldots, m$ be m real numbers. Suppose that $\sum_{j=1}^m \alpha_\varepsilon^j \to \sum_{j=1}^m \alpha^j$ and for every $j = 1, \ldots, m$, $\liminf \alpha_\varepsilon^j \geq \alpha^j$. Then, $\lim_{\varepsilon\to 0} \alpha_\varepsilon^j = \alpha^j$ for every $j = 1, \ldots, m$.*

We, now state the homogenization theorem for the optimality system.

Theorem 6.2 (Main theorem). *If $\overline{u}_\varepsilon, \overline{p}_\varepsilon$ and $\overline{\theta}_\varepsilon$ satisfy the optimality system (4.5)–(4.6) and $s \mapsto k(s)s$ is convex, then,*

$$\widetilde{\overline{u}_\varepsilon^+} \rightharpoonup \frac{h}{L} u_0^+; \quad \widetilde{\overline{p}_\varepsilon^+} \rightharpoonup \frac{h}{L} p_0^+ \quad \textit{weakly in } L^2(\Omega^+),$$

$$\widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_1}} \rightharpoonup 0; \quad \widetilde{\frac{\partial \overline{p}_\varepsilon^+}{\partial x_1}} \rightharpoonup 0 \quad \textit{weakly in } L^2(\Omega^+),$$

$$\widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_2}} \rightharpoonup \frac{h}{L}\frac{\partial u_0^+}{\partial x_2}; \quad \widetilde{\frac{\partial \overline{p}_\varepsilon^+}{\partial x_2}} \rightharpoonup \frac{h}{L}\frac{\partial p_0^+}{\partial x_2} \quad \textit{weakly in } L^2(\Omega^+),$$

$$\overline{u}_\varepsilon^- \to u_0^-; \quad \overline{p}_\varepsilon^- \to p_0^- \quad \textit{strongly in } H^1(\Omega^-),$$

$$\overline{\theta}_\varepsilon \to \theta_0 = \frac{-1}{\beta} p_0 \chi_\omega \quad \textit{strongly in } H^1(\omega),$$

where u_0, p_0 and θ_0 satisfy the system (5.5). Here, $\overline{u}_\varepsilon^+$ and $\overline{u}_\varepsilon^-$ are the restrictions of u_ε to Ω_ε^+ and Ω^-, respectively, and $\tilde{}$ represents the trivial extension by zero.

Remark 6.3. It is important to note that the optimal triplet $(\overline{u}_\varepsilon, \overline{p}_\varepsilon, \overline{\theta}_\varepsilon)$ converges to the triplet (u_0, p_0, θ_0) in the appropriate sense described in Theorem 6.2 but it is just a candidate to be an optimal one but it is not necessarily to be the one. The obvious reason is the absence of a sufficient criteria of optimality because of the non-linearity present in the model (semi-linear partial differential equation).

Proof. The proof consists of three steps. In the first step, we will prove the weak convergence of the optimal state $\overline{u}_\varepsilon$ as given in the theorem. By using Lemma 6.1, we will prove the strong convergence of $T^\varepsilon u_\varepsilon^+$ and u_ε^- in the appropriate spaces in Step II. In Step III, using these strong convergences, we will prove the convergence of the adjoint state and the control.

Step I.

The continuity of the solution operator gives the following estimate:

$$\|\overline{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C(\|f_0\|_{L^2(\Omega_\varepsilon)} + \|\overline{\theta}_\varepsilon\|_{L^2(\omega)}), \tag{6.1}$$

where $C > 0$ is independent of ε. As $\overline{\theta}_\varepsilon$ is an optimal control, we have $J_\varepsilon(\overline{\theta}_\varepsilon) \leq J_\varepsilon(0)$. That is

$$J_\varepsilon(\overline{u}_\varepsilon, \overline{\theta}_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} |\overline{u}_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_\Omega \chi_\omega |\overline{\theta}_\varepsilon|^2 \leq \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon(0) - u_d|^2,$$

where $u_\varepsilon(0)$ is the solution of the state equation (4.2) with $\theta = 0$. Hence, $\|u_\varepsilon(0)\|_{H^1(\Omega_\varepsilon)} \leq C$, where C is independent of ε. Thus, we have $\|\overline{\theta}_\varepsilon\|_{L^2(\omega)} \leq C$

and hence,

$$\|\overline{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C. \tag{6.2}$$

Let us estimate $T^\varepsilon \overline{u}_\varepsilon^+$ in the space $L^2(0, L; H^1(\mathcal{G}))$ using the properties of the unfolding operator, which are given in Proposition 3.2.

The following estimate is derived by applying Proposition 3.2(c):

$$\|T^\varepsilon \overline{u}_\varepsilon^+\|^2_{L^2(0,L;H^1(\mathcal{G}))} \leq L\|\overline{u}_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}. \tag{6.3}$$

The boundedness of the sequence $T^\varepsilon \overline{u}_\varepsilon^+$ in $L^2(0, L; H^1(\mathcal{G}))$ follows from the estimate (6.2). By weak compactness, there exists a subsequence (still denoted by ε) such that

$$T^\varepsilon \overline{u}_\varepsilon^+ \rightharpoonup u_0^+ \quad \text{weakly in } L^2(0, L; H^1(\mathcal{G})), \tag{6.4}$$

which implies

$$T^\varepsilon \overline{u}_\varepsilon^+ \rightharpoonup u_0^+ \quad \text{weakly in } L^2(\Omega_u), \tag{6.5}$$

$$\frac{\partial}{\partial x_2} T^\varepsilon \overline{u}_\varepsilon^+ \rightharpoonup \frac{\partial u_0^+}{\partial x_2} \quad \text{and hence} \quad T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_2} \rightharpoonup \frac{\partial u_0^+}{\partial x_2} \quad \text{weakly in } L^2(\Omega_u) \tag{6.6}$$

and

$$\frac{\partial}{\partial y} T^\varepsilon \overline{u}_\varepsilon^+ \rightharpoonup \frac{\partial u_0^+}{\partial y} \quad \text{and thus} \quad \varepsilon T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1} \rightharpoonup \frac{\partial u_0^+}{\partial y} \quad \text{weakly in } L^2(\Omega_u). \tag{6.7}$$

Observe that $k(\overline{u}_\varepsilon^+)$ is bounded in $L^2(\Omega_\varepsilon^+)$ as $\overline{u}_\varepsilon^+$ is bounded in the same space and k satisfies (4.3). Hence, there exists $\zeta \in L^2(\Omega_u)$ such that

$$T^\varepsilon k(\overline{u}_\varepsilon^+) \rightharpoonup \zeta \quad \text{weakly in } L^2(\Omega_u). \tag{6.8}$$

Now, Proposition 3.2(c) allows us to get

$$\left\| T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1} \right\|_{L^2(\Omega_u)} = \sqrt{L} \left\| \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1} \right\|_{L^2(\Omega_\varepsilon^+)} \leq \sqrt{L} \|\overline{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)}.$$

By estimate (6.2), the sequence $T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1}$ is uniformly bousnded in the space $L^2(\Omega_u)$. This implies $\frac{\partial u_0^+}{\partial y} = 0$ by (6.7). Thus, u_0^+ is independent of y. Further, we have

$$\widetilde{\overline{u}_\varepsilon^+} \rightharpoonup \frac{1}{L} \int_{Y(x_2)} u_0^+ dy \quad \text{and} \quad \widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_2}} \rightharpoonup \frac{1}{L} \int_{Y(x_2)} \frac{\partial u_0^+}{\partial x_2} dy \quad \text{weakly in } L^2(\Omega^+), \tag{6.9}$$

with the help of Proposition 3.2(f). Since u_0^+ is independent of y variable, we obtain

$$\int_{Y(x_2)} u_0^+ dy = h(x_2) u_0^+ \quad \text{and} \quad \int_{Y(x_2)} \frac{\partial u_0^+}{\partial x_2} dy = h(x_2) \frac{\partial u_0^+}{\partial x_2}. \tag{6.10}$$

Thus, (6.9) becomes

$$\widetilde{\overline{u}_\varepsilon^+} \rightharpoonup \frac{h(x_2)}{L} u_0^+ \quad \text{and} \quad \widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_2}} \rightharpoonup \frac{h(x_2)}{L} \frac{\partial u_0^+}{\partial x_2} \quad \text{weakly in } L^2(\Omega^+). \tag{6.11}$$

As $T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1}$ is bounded in $L^2(\Omega_u)$, by weak compactness, there is an element $P \in L^2(\Omega_u)$ such that, up to a subsequence (still denoted by ε),

$$T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1} \rightharpoonup P \quad \text{weakly in } L^2(\Omega_u). \tag{6.12}$$

Similar to the techniques used in [2, 24, 25, 28], one can show $P \equiv 0$. This implies

$$\widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_1}} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega^+).$$

Now, using the estimate of $\|\overline{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)}$, we have the boundedness of $\overline{u}_\varepsilon^-$ in the space $H^1(\Omega^-)$. Thus, up to a subsequence (still denoted by ε),

$$\overline{u}_\varepsilon^- \rightharpoonup u_0^- \quad \text{weakly in } H^1(\Omega^-). \tag{6.13}$$

Define u_0 as $u_0 = u_0^+ \chi_{\Omega^+} + u_0^- \chi_{\Omega^-}$.

Claim. *We claim that u_0 defined as above is in the space $W(\Omega)$ and satisfies the limit problem* (5.5)*. This can be shown by using the standard techniques of unfolding operators as in* [2, 24].

The next major difficulty is the passage to the limit in the non-linear terms. Now, note that $k(u_\varepsilon^-) \to k(u_0^-)$ in $L^2(\Omega^-)$ as u_ε^- converges strongly to u_0^- in $L^2(\Omega^-)$ and also $u_\varepsilon^- \in C(\Omega^-)$. But, calculating the limit of $k(u_\varepsilon^+)$ is a bit non-trivial task as we have only the weak convergence of u_ε^+ in the upper part. In this direction, we have the following result.

Claim. $T^\varepsilon k(\overline{u}_\varepsilon^+) \rightharpoonup \zeta = k(u_0^+)$ *in* $L^2(\Omega_u)$: *We now proceed to prove the claim.*

Let $\phi \in C^1(\overline{\Omega})$. Then, the monotonicity of k gives

$$\begin{aligned}
&\int_{\Omega^-} |\nabla \overline{u}_\varepsilon^- - \nabla\phi|^2 + \int_{\Omega_\varepsilon^+} |\partial_{x_2}\overline{u}_\varepsilon^+ - \partial_{x_2}\phi|^2 + \int_{\Omega_\varepsilon^+} |\partial_{x_1}\overline{u}_\varepsilon^+|^2 + \int_{\Omega^-} |\overline{u}_\varepsilon^- - \phi|^2 \\
&\quad + \int_{\Omega_\varepsilon^+} |\overline{u}_\varepsilon^+ - \phi|^2 + \int_{\Omega^-} (k(\overline{u}_\varepsilon^-) - k(\phi))(\overline{u}_\varepsilon^- - \phi) \\
&\quad + \int_{\Omega_\varepsilon^+} (k(\overline{u}_\varepsilon^+) - k(\phi))(\overline{u}_\varepsilon^+ - \phi) \geq 0.
\end{aligned}$$

By applying unfolding, we get

$$\begin{aligned}
&\int_{\Omega^-} |\nabla \overline{u}_\varepsilon^- - \nabla\phi|^2 + \frac{1}{L}\int_{\Omega_u} |T^\varepsilon\partial_{x_2}\overline{u}_\varepsilon^+ - T^\varepsilon\partial_{x_2}\phi|^2 + |T^\varepsilon\partial_{x_1}\overline{u}_\varepsilon^+|^2 + \int_{\Omega^-} |\overline{u}_\varepsilon^- - \phi|^2 \\
&\quad + \frac{1}{L}\int_{\Omega_u} |T^\varepsilon\overline{u}_\varepsilon^+ - T^\varepsilon\phi|^2 + \int_{\Omega^-} (k(\overline{u}_\varepsilon^-) - k(\phi))(\overline{u}_\varepsilon^- - \phi) \\
&\quad + \frac{1}{L}\int_{\Omega_u} (T^\varepsilon k(\overline{u}_\varepsilon^+) - T^\varepsilon k(\phi))(T^\varepsilon\overline{u}_\varepsilon^+ - T^\varepsilon\phi) \geq 0.
\end{aligned}$$

Upon expanding the above inequality, we obtain

$$\begin{aligned}
&\int_{\Omega^-}(|\nabla\overline{u}_\varepsilon^-|^2+k(\overline{u}_\varepsilon^-)\overline{u}_\varepsilon^-+|\overline{u}_\varepsilon^-|^2)+\frac{1}{L}\int_{\Omega_u}(|T^\varepsilon\nabla\overline{u}_\varepsilon^+|^2+T^\varepsilon k(\overline{u}_\varepsilon^+)T^\varepsilon\overline{u}_\varepsilon^++|T^\varepsilon\overline{u}_\varepsilon^+|^2)\\
&\quad+\int_{\Omega^-}(|\nabla\phi|^2-2\nabla\overline{u}_\varepsilon^-\nabla\phi+k(\phi)\phi-k(\overline{u}_\varepsilon^-)\phi-k(\phi)\overline{u}_\varepsilon^-+|\phi|^2-2\overline{u}_\varepsilon^-\phi)\\
&\quad+\frac{1}{L}\int_{\Omega_u}(|T^\varepsilon\partial_{x_2}\phi|^2-2T^\varepsilon\partial_{x_2}\overline{u}_\varepsilon^+T^\varepsilon\partial_{x_2}\phi+T^\varepsilon k(\phi)T^\varepsilon\phi-T^\varepsilon k(\overline{u}_\varepsilon^+)T^\varepsilon\phi)\\
&\quad+\frac{1}{L}\int_{\Omega_u}(-T^\varepsilon k(\phi)T^\varepsilon\overline{u}_\varepsilon^++|T^\varepsilon\phi|^2-2T^\varepsilon\overline{u}_\varepsilon^+T^\varepsilon\phi)\geq 0. \qquad (6.14)
\end{aligned}$$

Now, let us recall the variational formulation of the state equation:

$$\begin{aligned}
&\int_{\Omega^-}\nabla\overline{u}_\varepsilon^-\nabla\phi+(k(\overline{u}_\varepsilon^-)+\overline{u}_\varepsilon^-)\phi+\int_{\Omega_\varepsilon^+}\nabla\overline{u}_\varepsilon^+\nabla\phi+(k(\overline{u}_\varepsilon^+)+\overline{u}_\varepsilon^+)\phi\\
&\qquad=\int_{\Omega^-}(f_0+\overline{\theta}_\varepsilon\chi_\omega)\phi+\int_{\Omega_\varepsilon^+}f_0\phi.
\end{aligned}$$

On applying unfolding, we get

$$\begin{aligned}
&\int_{\Omega^-}\nabla\overline{u}_\varepsilon^-\nabla\phi+(k(\overline{u}_\varepsilon^-)+\overline{u}_\varepsilon^-)\phi+\frac{1}{L}\int_{\Omega_u}T^\varepsilon\nabla\overline{u}_\varepsilon^+T^\varepsilon\nabla\phi+(T^\varepsilon k(\overline{u}_\varepsilon^+)+T^\varepsilon\overline{u}_\varepsilon^+)T^\varepsilon\phi\\
&\qquad=\int_{\Omega^-}(f_0+\overline{\theta}_\varepsilon\chi_\omega)\phi+\frac{1}{L}\int_{\Omega_u}T^\varepsilon f_0T^\varepsilon\phi.
\end{aligned}$$

Using the convergence of $T^\varepsilon\overline{u}_\varepsilon$, we can pass to the limit in the above equation and obtain

$$\begin{aligned}
&\int_{\Omega^-}\nabla u_0^-\nabla\phi+(k(u_0^-)+u_0^-)\phi+\frac{1}{L}\int_{\Omega_u}\partial_{x_2}u_0^+\partial_{x_2}\phi+\frac{1}{L}\int_{\Omega_u}(\zeta+u_0^+)\phi\\
&\qquad=\int_{\Omega^-}(f_0+\theta_0\chi_\omega)\phi+\frac{1}{L}\int_{\Omega_u}f_0\phi,\quad\forall\,\phi\in C^1(\overline{\Omega}).
\end{aligned}$$

Here, θ_0 is the weak L^2-limit of $\overline{\theta}_\varepsilon$. As $C^1(\overline{\Omega})$ is dense in $W(\Omega)$, we have (by choosing $\phi=u_0$)

$$\begin{aligned}
&\int_{\Omega^-}|\nabla u_0^-|^2+k(u_0^-)u_0^-+|u_0^-|^2+\frac{1}{L}\int_{\Omega_u}(|\partial_{x_2}u_0^+|^2+\zeta u_0^++|u_0^+|^2)\\
&\qquad=\int_{\Omega^-}(f_0+\theta_0\chi_\omega)u_0^-+\frac{1}{L}\int_{\Omega_u}f_0u_0^+. \qquad (6.15)
\end{aligned}$$

Note that, by (6.15),

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega^-}|\nabla\overline{u}_\varepsilon^-|^2+k(\overline{u}_\varepsilon^-)\overline{u}_\varepsilon+|\overline{u}_\varepsilon^-|^2+\frac{1}{L}\int_{\Omega_u}|T^\varepsilon\nabla\overline{u}_\varepsilon^+|^2+T^\varepsilon k(\overline{u}_\varepsilon^+)T^\varepsilon\overline{u}_\varepsilon^++|T^\varepsilon\overline{u}_\varepsilon^+|^2\\
&\quad=\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon}(f_0+\overline{\theta}_\varepsilon\chi_\omega)\overline{u}_\varepsilon\\
&\quad=\int_{\Omega^-}(f_0+\theta_0\chi_\omega)u_0^-+\frac{1}{L}\int_{\Omega_u}f_0u_0^+\\
&\quad=\int_{\Omega^-}|\nabla u_0^-|^2+k(u_0^-)u_0^-+|u_0^-|^2+\frac{1}{L}\int_{\Omega_u}|\partial_{x_2}u_0^+|^2+\zeta u_0^++|u_0^+|^2. \qquad (6.16)
\end{aligned}$$

Now, we utilize (6.16) *in passing to the limit in the first line of the inequality* (6.14). *In the other components, we just apply weak convergence of* $T^\varepsilon\overline{u}_\varepsilon^+$ *and* $\overline{u}_\varepsilon^-$, *in the respective spaces. Thus, we get*

$$\begin{aligned}
&\int_{\Omega^-}(|\nabla u_0^-|^2+k(u_0^-)u_0^-+|u_0^-|^2)+\frac{1}{L}\int_{\Omega_u}(|\partial_{x_2}u_0^+|^2+\zeta u_0^++|u_0^+|^2)\\
&\quad+\int_{\Omega^-}(|\nabla\phi|^2-2\nabla u_0^-\nabla\phi+k(\phi)\phi-k(u_0^-)\phi-k(\phi)u_0^-+|\phi|^2-2u_0^-\phi)\\
&\quad+\frac{1}{L}\int_{\Omega_u}(|\partial_{x_2}\phi|^2-2\partial_{x_2}u_0^+\partial_{x_2}\phi+k(\phi)\phi-\zeta\phi-k(\phi)u_0^++|\phi|^2-2u_0^+\phi)\geq 0.
\end{aligned}$$

That is,

$$\begin{aligned}
&\int_{\Omega^-}|\nabla u_0^--\nabla\phi|^2+\int_{\Omega^-}(k(u_0^-)-k(\phi))(u_0^--\phi)+\int_{\Omega^-}|\overline{u}_0^--\phi|^2\\
&\quad+\frac{1}{L}\int_{\Omega_u}|u_0^+-\phi|^2+\frac{1}{L}\int_{\Omega_u}|\partial_{x_2}u_0^+-\partial_{x_2}\phi|^2+\frac{1}{L}\int_{\Omega_u}(\zeta-k(\phi))(u_0^+-\phi)\geq 0.
\end{aligned}$$

Now, for $\psi\in C^1(\overline{\Omega})$, *choose* $\phi=u_0-\lambda\psi$, $\lambda>0$, *to get*

$$\begin{aligned}
&\lambda\int_{\Omega^-}|\nabla\psi|^2+\int_{\Omega^-}(k(u_0^-)-k(u_0^--\lambda\psi))\psi+\lambda\int_{\Omega^-}|\psi|^2+\frac{\lambda}{L}\int_{\Omega_u}|\psi|^2\\
&\quad+\frac{\lambda}{L}\int_{\Omega_u}|\partial_{x_2}\psi|^2+\frac{1}{L}\int_{\Omega_u}(\zeta-k(u_0^+-\lambda\psi))\psi\geq 0,\quad\forall\,\psi\in C^1(\overline{\Omega}).
\end{aligned}$$

As $\lambda\to 0$, *we get*

$$\int_{\Omega^+}(\zeta-k(u_0^+))\psi\geq 0,\quad\forall\,\psi\in C^1(\overline{\Omega}).$$

Thus, we conclude the claim that $\zeta=k(u_0^+)$. *Finally, in Step I, we prove the following claim.*

Claim ($\boldsymbol{u_0}$ satisfies the limit equation). *Choose a test function $\psi \in C^\infty(\overline{\Omega})$ in the variational formulation of the state equation in* (5.5).

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla\psi + \int_{\Omega_\varepsilon} k(u_\varepsilon)\psi + \int_{\Omega_\varepsilon} u_\varepsilon \psi \\
&\quad = \int_{\Omega_\varepsilon^+} \nabla u_\varepsilon^+ \cdot \nabla\psi + \int_{\Omega_\varepsilon^+} k(u_\varepsilon^+)\psi + \int_{\Omega_\varepsilon^+} u_\varepsilon^+\psi + \int_{\Omega^-} (\nabla u_\varepsilon^- \cdot \nabla\psi + k(u^-)\psi + u^-\psi) \\
&\quad = \frac{1}{L}\int_{\Omega_u}\left(T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_1} T^\varepsilon \frac{\partial\psi}{\partial x_1} + T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_2} T^\varepsilon \frac{\partial\psi}{\partial x_2} + T^\varepsilon k(\overline{u}_\varepsilon^+) T^\varepsilon\psi + T^\varepsilon u_\varepsilon^+ T^\varepsilon \psi\right) \\
&\qquad + \int_{\Omega^-} (\nabla u_\varepsilon^- \cdot \nabla\psi + k(\overline{u}_\varepsilon^-)\psi + \overline{u}_\varepsilon^-\psi) \\
&\quad \to \frac{1}{L}\int_{\Omega_u}\left(\frac{\partial u_0^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + k(u_0^+)\psi + u_0^+\psi\right) + \int_{\Omega^-} (\nabla u_0{}^- \cdot \nabla\psi + k(u_0^-)\psi + u_0^-\psi)
\end{aligned}$$

and

$$\int_{\Omega_\varepsilon} (f_0 + \overline{\theta}_\varepsilon\chi_\omega)\psi = \int_{\Omega^-} (f_0 + \overline{\theta}_\varepsilon\chi_\omega)\psi + \int_{\Omega_\varepsilon^+} f_0\psi \to \int_{\Omega^-} (f_0 + \theta_0\chi_\omega)\psi + \frac{1}{L}\int_{\Omega_u} f_0\psi.$$

Hence,

$$\begin{aligned}
&\frac{1}{L}\int_{\Omega_u} \frac{\partial u_0^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + u_0^+\psi + k(u_0^+)\psi + \int_{\Omega^-} \nabla u_0^- \cdot \nabla\psi + u_0^+\psi + k(u_0^-)\psi \\
&\qquad = \int_{\Omega^-} (f_0 + \theta_0\chi_\omega)\psi + \frac{1}{L}\int_{\Omega_u} f_0\psi \\
&\qquad \Rightarrow \int_{\Omega^+} \frac{h}{L}\left(\frac{\partial u_0^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + (u_0^+ + k(u_0^+))\psi\right) + \int_{\Omega^-} \nabla u_0^- \cdot \nabla\psi + (u_0^+ + k(u_0^-))\psi \\
&\qquad = \int_{\Omega^-} (f_0 + \theta_0\chi_\omega)\psi + \int_{\Omega^+} \frac{h}{L} f_0\psi,
\end{aligned}$$

$\forall\, \psi \in C^\infty(\overline{\Omega})$. *Since $C^\infty(\overline{\Omega})$ is dense in $W(\Omega)$, the above identity is true for all ψ in $W(\Omega)$. Therefore, u_0 satisfies the state equation in* (5.5).

Step II.

Here, we will prove the following strong convergences:

$$T^\varepsilon \overline{u}_\varepsilon^+ \to u_0^+ \quad \text{strongly in } L^2(0, L; H^1(\mathcal{G})), \tag{6.17}$$

$$T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1} \to 0 \quad \text{strongly in } L^2(\Omega_u), \tag{6.18}$$

$$\overline{u}_\varepsilon^- \to u_0^- \quad \text{strongly in } H^1(\Omega^-) \tag{6.19}$$

with the assistance of Lemma 6.1. Let us put

$$\alpha_\varepsilon^1 = \frac{1}{L}\|T^\varepsilon \overline{u}_\varepsilon^+\|^2_{L^2(\Omega_u)}, \quad \alpha_\varepsilon^2 = \frac{1}{L}\left\|T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1}\right\|^2_{L^2(\Omega_u)}, \quad \alpha_\varepsilon^3 = \frac{1}{L}\left\|T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_2}\right\|^2_{L^2(\Omega_u)},$$

$$\alpha_\varepsilon^4 = \frac{1}{L}\int_{\Omega_u} k(T^\varepsilon \overline{u}_\varepsilon^+)T^\varepsilon \overline{u}_\varepsilon^+ dxdy, \quad \alpha_\varepsilon^5 = \|\overline{u}_\varepsilon^-\|^2_{L^2(\Omega^-)}, \quad \alpha_\varepsilon^6 = \left\|\frac{\partial \overline{u}_\varepsilon^-}{\partial x_1}\right\|^2_{L^2(\Omega^-)},$$

$$\alpha_\varepsilon^7 = \left\|\frac{\partial \overline{u}_\varepsilon^-}{\partial x_2}\right\|^2_{L^2(\Omega^-)}, \quad \alpha_\varepsilon^8 = \int_{\Omega^-} k(\overline{u}_\varepsilon^-)\overline{u}_\varepsilon^- dx,$$

$$\alpha^1 = \frac{1}{L}\|u_0^+\|^2_{L^2(\Omega_u)}, \quad \alpha^2 = 0, \quad \alpha^3 = \frac{1}{L}\left\|\frac{\partial u_0^+}{\partial x_2}\right\|^2_{L^2(\Omega_u)},$$

$$\alpha^4 = \frac{1}{L}\int_{\Omega_u} k(u_0)u_0 dxdy, \quad \alpha^5 = \|u_0^+\|^2_{L^2(\Omega^-)}, \quad \alpha^6 = \left\|\frac{\partial u_0^-}{\partial x_1}\right\|^2_{L^2(\Omega^-)},$$

$$\alpha^7 = \left\|\frac{\partial u_0^-}{\partial x_2}\right\|^2_{L^2(\Omega^-)}, \quad \alpha^8 = \int_{\Omega^-} k(u_0^-)u_0^- dx.$$

From Eq. (6.16) and since $\zeta = k(u_0^+)$, we have the following energy convergence:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega^-} &|\nabla \overline{u}_\varepsilon|^2 + k(\overline{u}_\varepsilon)\overline{u}_\varepsilon + |\overline{u}_\varepsilon|^2 + \frac{1}{L}\int_{\Omega_u} |T^\varepsilon \nabla \overline{u}_\varepsilon|^2 + T^\varepsilon k(\overline{u}_\varepsilon)T^\varepsilon \overline{u}_\varepsilon + |T^\varepsilon \overline{u}_\varepsilon|^2\\ &= \int_{\Omega^-} |\nabla u_0^-|^2 + k(u_0^-)u_0^- + |u_0^-|^2 + \frac{1}{L}\int_{\Omega_u} |\partial_{x_2} u_0^+|^2 + k(u_0^+)u_0^+ + |u_0^+|^2.\end{aligned}$$

That is, $\sum_{j=1}^m \alpha_\varepsilon^j \to \sum_{j=1}^m \alpha^j$ as $\varepsilon \to 0$. Also, we have $\liminf \alpha_\varepsilon^j \geq \alpha^j$ for $j = 1,\ldots,8$. Here, we have used the fact that norm is weakly lower semi-continuous and the function $u \mapsto k(u)u$ is weakly lower semi-continuous in the appropriate spaces as $sk(s)$ is a convex real-valued function. Thus, Lemma 6.1 guarantees the norm convergence of $T^\varepsilon \overline{u}_\varepsilon^+$, $T^\varepsilon \frac{\partial \overline{u}_\varepsilon^+}{\partial x_1}$ and u_ε^- in $L^2(0,L;H^1(\mathcal{G}))$, $L^2(\Omega_u)$ and $H^1(\Omega^-)$, respectively. The weak convergence and norm convergence together gives us the strong convergence, which we are interested in. This completes the Step II.

Step III.

Now that, we have the strong convergence of $T^\varepsilon \overline{u}_\varepsilon^+$ and $\overline{u}_\varepsilon^-$, we can show that $T^\varepsilon k'(\overline{u}_\varepsilon^+) = k'(T^\varepsilon \overline{u}_\varepsilon^+)$ converges to $k'(u_0^+)$ strongly in $L^2(\Omega_u)$. Using the same procedure as in Step I, in fact, with lesser difficulty as the problem is linear, one can prove the convergence in the adjoint problem. Hence, the theorem is proved. $\square$

6.2. *Correctors*

In this section, we prove certain strong convergences known as corrector results. Of course, we do not shave any error estimates. Recall Eq. (6.17). That is,

$$T^\varepsilon \overline{u}_\varepsilon^+ \to u_0^+ \quad \text{strongly in } L^2(0, L; H^1(\mathcal{G})).$$

In other words, $\|T^\varepsilon \overline{u}_\varepsilon^+ - u_0\|_{L^2(\Omega_u)} \to 0$ as $\varepsilon \to 0$ and also the derivatives with respect to x_2 and y. As

$$\|T^\varepsilon \overline{u}_\varepsilon^+ - T^\varepsilon u_0\|_{L^2(\Omega_u)} \leq \|T^\varepsilon \overline{u}_\varepsilon^+ - u_0\|_{L^2(\Omega_u)} + \|u_0 - T^\varepsilon u_0\|_{L^2(\Omega_u)},$$

and both of the terms in the right-hand side are converging to zero, we get

$$\|T^\varepsilon \overline{u}_\varepsilon^+ - T^\varepsilon u_0\|_{L^2(\Omega_u)} \to 0$$

as $\varepsilon \to 0$. Now,

$$\begin{aligned} \|T^\varepsilon \overline{u}_\varepsilon^+ - T^\varepsilon u_0\|_{L^2(\Omega_u)}^2 &= \int_{\Omega_u} |T^\varepsilon \overline{u}_\varepsilon^+ - T^\varepsilon u_0|^2 = \int_{\Omega_u} T^\varepsilon (\overline{u}_\varepsilon^+ - u_0)^2 \\ &= L \int_{\Omega_\varepsilon^+} (\overline{u}_\varepsilon^+ - u_0)^2. \end{aligned}$$

Thus, we get

$$\widetilde{\overline{u}_\varepsilon^+} - \chi_{\Omega_\varepsilon^+} u_0 \to 0 \quad \text{strongly in } L^2(\Omega^+).$$

Similarly, we can prove the results for $\widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_1}}$ and $\widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_2}}$. On collecting these results with the convergence (6.19), we get the following corrector theorem.

Theorem 6.4. *Let $\overline{u}_\varepsilon$ and u_0 be as in Theorem 6.2. Then,*

$$\begin{aligned} \widetilde{\overline{u}_\varepsilon^+} - \chi_{\Omega_\varepsilon^+} u_0 &\to 0 \quad \textit{strongly in } L^2(\Omega^+), \\ \widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_1}} &\to 0 \quad \textit{strongly in } L^2(\Omega^+), \\ \widetilde{\frac{\partial \overline{u}_\varepsilon^+}{\partial x_2}} - \chi_{\Omega_\varepsilon^+} \frac{\partial u_0}{\partial x_2} &\to 0 \quad \textit{strongly in } L^2(\Omega^+), \\ \overline{u}_\varepsilon^- - u_0^- &\to 0 \quad \textit{strongly in } H^1(\Omega^-). \end{aligned}$$

7. Conclusions

This paper has several novelties as far as oscillatory domains are considered. First of all, we have considered a very general periodic domain. In earlier papers, the oscillations are of pillar-type. The analysis on the general oscillations will allow us to consider more realistic physical domains, for example, circular domains with rugose boundary. This may be a topic of a future paper. In this paper, we have used a general periodic unfolding operator required for the homogenization analysis. The second novelty is the consideration of a semi-linear problem. We have obtained a new limit optimal control problem and established the existence of optimal control. Indeed, the non-linearity causes trouble in our entire analysis. Finally, we have also proved a corrector result. The ideas and analysis would be a stepping stone for studying other non-linear control problems. In this paper, we have only considered controls acting away from the oscillating boundary. Unlike our other works, the non-linearity is not allowing us to consider controls on the oscillating part. We may need new techniques to treat such problems.

Acknowledgments

We thank the referee for his/her meticulous reading of the paper and his/her comments which have improved the paper substantially. The first author would like to acknowledge the support from National Board for Higher Mathematics (NBHM), Department of Atomic Energy, India. The second author would like to thank Department of Science and Technology (DST), Government of India as the work was partially supported by Project No. MTR/2017/000087 dtd 18.12.2017. The third author wish to thank the Facultad de Ciencias Fisicas y Matematicas, Universidad de Concepcion (Chile) for their financial support through PROYECTOS VRID INICIACION NO. 216.013.0.41-1.0IN.

References

[1] S. Aiyappan, A. K. Nandakumaran and R. Prakash, Generalization of unfolding operator for highly oscillating smooth boundary domain and homogenization, *Calc. Var. Partial Differential Equations* **57**(3) (2018) Article 86, 30 pp.; https://doi.org/10.1007/s00526-018-1354-6.

[2] S. Aiyappan and A. K. Nandakumaran, Optimal control problem in a domain with branched structure and homogenization, *Math. Methods Appl. Sci.* **40**(8) (2017) 3173–3189.

[3] Y. Amirat, O. Bodart, U. De Maio and A. Gaudiello, Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary, *SIAM J. Math. Anal.* **35**(6) (2004) 1598–1616.

[4] D. Blanchard, L. Carbone and A. Gaudiello, Homogenization of a monotone problem in a domain with oscillating boundary, *ESAIM Math. Model. Numer. Anal.* **33**(5) (1999) 1057–1070.

[5] D. Blanchard and A. Gaudiello, Homogenization of highly oscillating boundaries and reduction of dimension for a monotone problem, *ESAIM Control Optim. Calc. Var.* **9** (2003) 449–460.

[6] D. Blanchard, A. Gaudiello and G. Griso, Junction of a periodic family of elastic rods with a 3D plate. Part I, *J. Math. Pures Appl.* **88**(1) (2007) 1–33.

[7] D. Blanchard, A. Gaudiello and G. Griso, Junction of a periodic family of elastic rods with a thin plate. Part II, *J. Math. Pures Appl.* **88**(2) (2007) 149–190.

[8] D. Blanchard, A. Gaudiello and T. A. Mel'nyk, Boundary homogenization and reduction of dimension in a Kirchhoff–Love plate, *SIAM J. Math. Anal.* **39**(6) (2008) 1764–1787.

[9] J. F. Bonnans, Second-order analysis for control constrained optimal control problems of semilinear elliptic systems, *Appl. Math. Optim.* **38**(3) (1998) 303–325.

[10] R. Brizzi and J.-P. Chalot, Boundary homogenization and Neumann boundary value problem, *Ricerche Mat.* **46**(2) (1997) 341–388.

[11] D. Cioranescu, A. Damlamian and G. Griso, *The Periodic Unfolding Method: Theory and Applications to Partial Differential Problems*, Series in Contemporary Mathematics, Vol. 3 (Springer, Singapore, 2018).

[12] A. Damlamian and K. Pettersson, Homogenization of oscillating boundaries, *Discrete Contin. Dyn. Syst.* **23**(1–2) (2009) 197–210.

[13] U. De Maio, T. Durante and T. A. Mel'nyk, Asymptotic approximation for the solution to the robin problem in a thick multi-level junction, *Math. Models Methods Appl. Sci.* **15**(12) (2005) 1897–1921.

[14] U. De Maio, A. Gaudiello and C. Lefter, Optimal control for a parabolic problem in a domain with highly oscillating boundary, *Appl. Anal.* **83**(12) (2004) 1245–1264.

[15] T. Durante and T. A. Mel'nyk, Asymptotic analysis of an optimal control problem involving a thick two-level junction with alternate type of controls, *J. Optim. Theory Appl.* **144**(2) (2010) 205–225.

[16] T. Durante and T. A. Mel'nyk, Homogenization of quasilinear optimal control problems involving a thick multilevel junction of type 3:2:1, *ESAIM Control Optim. Calc. Var.* **18**(2) (2012) 583–610.

[17] A. C. Esposito, P. Donato, A. Gaudiello and C. Picard, Homogenization of the p-laplacian in a domain with oscillating boundary, *Comm. Appl. Nonlinear Anal.* **4**(4) (1997) 1–23.

[18] A. Gaudiello, O. Guibé and F. Murat, Homogenization of the brush problem with a source term in L^1, *Arch. Ration. Mech. Anal.* **225**(1) (2017) 1–64.

[19] A. Gaudiello, R. Hadiji and C. Picard, Homogenization of the Ginzburg–Landau equation in a domain with oscillating boundary, *Comm. Appl. Anal.* **7**(2–3) (2003) 209–223.

[20] A. Gaudiello and T. A. Mel'nyk, Homogenization of a nonlinear monotone problem with nonlinear Signorini boundary conditions in a domain with highly rough boundary, *J. Differential Equations* **265**(10) (2018) 5419–5454.

[21] A. Gaudiello and A. Sili, Homogenization of highly oscillating boundaries with strongly contrasting diffusivity, *SIAM J. Math. Anal.* **47**(3) (2015) 1671–1692.

[22] T. A. Mel'nyk, Homogenization of a boundary-value problem with a nonlinear boundary condition in a thick junction of type 3:2:1, *Math. Methods Appl. Sci.* **31**(9) (2008) 1005–1027.

[23] T. A. Mel'nyk, Asymptotic approximation for the solution to a semi-linear parabolic problem in a thick junction with the branched structure, *J. Math. Anal. Appl.* **424**(2) (2015) 1237–1260.

[24] A. K. Nandakumaran, R. Prakash and B. C. Sardar, Homogenization of an optimal control via unfolding method, *SIAM J. Control Optim.* **53**(5) (2015) 3245–3269.

[25] A. K. Nandakumaran, R. Prakash and B. C. Sardar, Asymptotic analysis of Neumann periodic optimal boundary control problem, *Math. Methods Appl. Sci.* **39**(15) (2016) 4354–4374.

[26] A. K. Nandakumaran, R. Prakash and J.-P. Raymond, Asymptotic analysis and error estimates for an optimal control problem with oscillating boundaries, *Ann. Univ. Ferrara* **58**(1) (2012) 143–166.

[27] R. Prakash, Optimal control problem for the time-dependent Kirchhoff–Love plate in a domain with rough boundary, *Asymptot. Anal.* **81**(3–4) (2013) 337–355.

[28] R. Prakash and B. C. Sardar, Homogenization of boundary optimal control problem in a domain with highly oscillating boundary via periodic unfolding method, *Nonlinear Stud.* **22**(2) (2015) 213–240.

[29] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Vol. 49 (American Mathematical Society, Providence, RI, 2013).

[30] F. Tröltzsch, *Optimal Control of Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 112 (American Mathematical Society, Providence, RI, 2010).