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OSCILLATING PDE IN A ROUGH DOMAIN WITH A CURVED INTERFACE: HOMOGENIZATION OF AN OPTIMAL CONTROL PROBLEM

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Abstract. Homogenization of an elliptic PDE with periodic oscillating coefficients and associated optimal control problems with energy type cost functional is considered. The domain is a 3-dimensional region (method applies to any n dimensional region) with oscillating boundary, where the base of the oscillation is curved and it is given by a Lipschitz function. Further, we consider general elliptic PDE with oscillating coefficients. We also include very general type functional of Dirichlet type given with oscillating coefficients which can be different from the coefficient matrix of the equation. We introduce appropriate unfolding operators and approximate unfolded domain to study the limiting analysis. The present article is new in this generality.

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1. Introduction

In this paper, we consider a general second order elliptic equation in a general oscillatory domain Ω_ε (see Sect. 2 regarding the description of Ω_ε). Normally the high oscillations are posed on a straight boundary, whereas in this paper, we consider very general oscillations on a curved boundary (see, Fig. 1). In this kind of rough domain, we want to study the homogenization of an elliptic PDE with oscillating coefficients. More precisely, we have considered an equation of the form

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_\varepsilon) + u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.1}$$

where $A^\varepsilon(x) = A\left(x, \dfrac{x'}{\varepsilon}\right)$ is elliptic. Thus the problem under consideration is very general than available in the literature and the aim is to study the homogenization of the above problem. We also study the homogenization of an associated optimal control problem. Again instead of the standard L^2-cost functional, we wish to consider

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cost functional of the form

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B^\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \mathrm{d}x + \frac{\beta}{2}\int_{\Omega_\varepsilon} |\theta_\varepsilon|^2 \mathrm{d}x, \tag{1.2}$$

where $B^\varepsilon(x) = B\left(x, \frac{x'}{\varepsilon}\right)$ is also oscillatory which can be different form A^ε. This indeed will cause additional difficulties in the analysis even in the case of fixed domains. Indeed the oscillations will be sitting on the boundary of the domain which is $n-1$ dimensional. Hence A^ε and B^ε will be of the form $A^\varepsilon(x) = A\left(x, \frac{x'}{\varepsilon}\right)$ and $B^\varepsilon(x) = B\left(x, \frac{x'}{\varepsilon}\right)$ respectively, in generality with oscillations in all directions. Here $x' = (x_1, \cdots, x_{n-1})$. One can simplify the case when there is oscillations only in one direction, that is $A^\varepsilon(x) = A\left(x, \frac{x_1}{\varepsilon}\right)$. Similarly $B^\varepsilon(x) = B\left(x, \frac{x_1}{\varepsilon}\right)$. For example, when $n = 3$ the oscillatory domain has a slab-like structure if the oscillation is only in one direction. In fact, we see clearly the laminate like effect in this case as in general homogenization theory. On the other hand, we get pillar type oscillations if we consider oscillations in both directions, that is $A^\varepsilon(x) = A\left(x, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$. Similarly $B^\varepsilon(x) = B\left(x, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$.

In addition to the generality, we have discussed so far, one of our major concerns is the development of the unfolding operator for the general oscillating domain under consideration in this paper. We also introduce an approximate unfolded domain together with a fixed limit unfolded domain which is important for the analysis in our work (see Figs. 5 and 6). The unfolding operator which we develop is quiet new and we derive various properties together with the convergences enjoyed by the newly introduced unfolding operators. In the last 10 years or so the unfolding operators have been used extensively by various authors including the present authors and their collaborators (see [1, 2, 19, 35, 36]). Thus, we deal with, at least, three important aspects in this paper, namely

(i) Consideration of two different oscillating matrices A^ε and B^ε, respectively for the equation and cost functional. The homogenization is quite different even in the usual homogenization.

(ii) Consideration of curved boundary, where the oscillatory part of the domain is placed.

(iii) Development of appropriate unfolding operators for the domain and for the boundary.

This requires the introduction of certain approximate unfolded domains. Thus, the article is new and novel even with fixed matrices A and B in place of A^ε and B^ε, respectively. For the clarity and understanding of the results, we first present the work when the oscillations are only in one direction. Before going to the literature, we briefly discuss the various sections of the article. The domain and the problem under consideration are explained in Section 2. In Section 3, we present the approximate unfolded domain, the unfolding operators both for the domain and boundary, the relevant properties satisfied by them, limit function space *etc.* The homogenization without optimal control is carried out in Section 4, whereas in Section 5, we study the homogenization of optimal control problem. In the remaining sections, we quickly analyze the homogenization of the general case, where we consider the oscillations in multi-directions. To simplify matters, we consider the case when $n = 3$ (practical situation) with oscillations in two directions. We have realized that, without any difficulty, this can be generalized to general dimension with multi-directional oscillations. The domain and problem description is given in Section 6 and, the unfolding operators, related properties and homogenization results are presented in Section 7.

Two different oscillating matrices for the equation and cost functional have been considered earlier in fixed domains (see [25–29]) and this is the first time, we are considering it in oscillatory domain. It is natural to expect a new homogenized matrix $B_\#$, which is not the homogenized limit B^* of B^ε. The matrix $B_\#$ has the contribution from both A and B.

Boundary-value problems in a domain with oscillating boundary arise in many fields of biology, physics and engineering sciences. For instance, to understand the motion of ciliated microorganisms, the flow in a channel with rugose boundary, heat transmission through winglets, propagation of electromagnetic waves in regions with rough boundaries, air flow through compression system in turbo machine such as a jet engine, the vibrations of foundations of buildings (see [13, 18, 30] and references there in). It is often impossible to approach these problems directly with numerical methods, because the rough boundary requires a large number of mesh points in its neighborhood. Thus, the computational cost associated with such a problem grows rapidly when the parameter ε gets smaller. Moreover, it can occur that the required discretization step becomes too small for the machine precision. Then, the goal is to approach the problem on Ω_ε, when the oscillating parameter gets smaller, with a non-oscillating or homogenized problem on Ω which can be numerically solved. Hence the optimal control are also equally important in such domains.

We do not present a detailed survey. However we present some of the papers relevant to the present manuscript. The homogenization in oscillating boundary domain with non flat base and with general elliptic operators is relatively new in the literature and not many articles are available especially in the context of optimal control problems. Further, we also consider optimal control problem with the cost functional involving general oscillating matrix which is different from the coefficient elliptic matrix in the system. One of the main article in this direction is [23], where the homogenization of an elliptic problem with L^1 data posed on a pillar-type domain with non-flat base and nonuniform cross sections has been studied by A. Gaudiello *et al.* Here they consider a fixed coefficient matrix, that is without oscillations. Another significant result is the work of Mossino and Sili [34], where they consider monotone operators with oscillating coefficients in a flat boundary. Needless to say, it is not an optimal control problem and they use the method of two-scale convergence. In our several papers, we used unfolding operators not only to do homogenization, but to characterize the optimal controls as well. Thanks to the work of Mossino and Sili, we also establish certain short propositions in the last section (Props. 7.5 and 7.6) in the context of unfolding operators analogous to the case of two scale convergence as in [34].

Regarding literature on optimal control problems, in [35], authors have considered an interior periodic optimal control problem corresponding to Laplace operator in a pillar-type oscillating domain with L^2-cost functional. The authors have used unfolding operator corresponding to the pillar-type oscillations to characterize the optimal control for the first time. In [1], unfolding operator for general periodic oscillating domain with flat base has been introduced and authors have investigated asymptotic behavior of a semilinear PDE with principle part as the Laplace operator and a corresponding interior optimal control problem have been studied in [2] by the same authors. In [3], unfolding operators for locally periodic oscillating domain has been defined, again the base of the oscillations is flat. In the context of homogenization of non-linear problems in oscillating domains, one can see [22], where authors have analysed asymptotic behavior of a monotone type operator with nonlinear signorini boundary condition. In [32], non-linear parabolic problem using asymptotic expansion has been studied. In all the above cases, the base of the oscillations is flat wherever in our case we consider non-flat base and non-pillar type oscillations. For further reading in this direction we suggest the readers to see [8–10, 19, 36] and references there in.

The literature on oscillating thin domain that is where the amplitude of the oscillations is of $O(\varepsilon)$ and dimension reduction happens in the limit is quite rich. In [4, 5] authors have considered an elliptic PDE to investigate the asymptotic behavior in thin oscillating domain with homogeneous Numann boundary condition. Homogenization of an elliptic PDE in a localy periodic thin domain have been studied in [6], and in [7], authors have developed unfolding operators for locally periodic thin domain. Moreover, they have obtained corrector results in the same article. For further study in this direction, we refer to the readers [11, 14, 20, 21, 33] and references there in. For general reading on homogenization of partial differential equations, readers can look into [12, 15, 37] and references there in.

2. Domain and problem description

2.1. Domain description

For a fixed parameter $\varepsilon = \frac{1}{m}$ with $m \in \mathbb{N}$, we consider the oscillating domain $\Omega_\varepsilon \subset \mathbb{R}^3$. Throughout this article, we denote any element $x \in \mathbb{R}^3$ as $x = (x', x_3)$ where $x' = (x_1, x_2) \in \mathbb{R}^2$. In fact, we can consider any

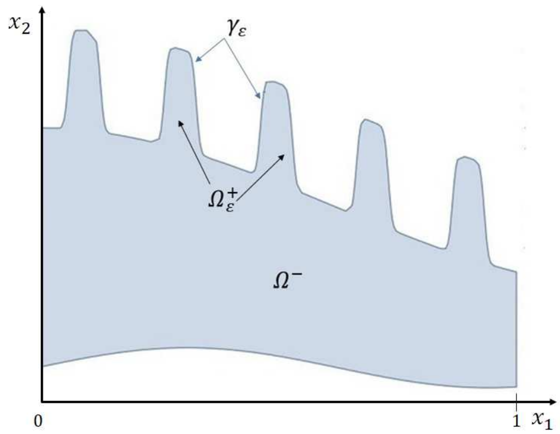

FIGURE 1. Ω_ε.

$\varepsilon > 0$ that converges to zero. Again, the result is true in any $\mathbb{R}^n$. Now, we will give the full description of the oscillating domain Ω_ε and its boundaries below (Fig. 1).

Let $g, h : (0,1)^2 \to \mathbb{R}$ be Lipschitz functions such that $0 < g(x') < h(x')$ for all $x' \in (0,1)^2$ and $\eta : \mathbb{R} \to \mathbb{R}$ be a Lipschitz, 1-periodic real valued function having the following properties:

(1) There exist $0 < a < b < 1$, such that $supp(\eta|_{(0,1)}) = [a, b]$ and $\eta \geqslant 0$.
(2) The supremum $M = \sup\{\eta(x_1) : \ x_1 \in (0,1)\}$ is strictly positive and the Lebesgue measure $|\{x_1 \in (0,1) : \eta(x_1) = M\}| \geq \delta > 0$.
(3) Let $\Omega^+ = \{(x', x_3) : \ x' \in (0,1)^2, \ h(x') < x_3 < h(x') + M\}$. Now for each $(x', x_3) \in \Omega^+$, the set $Y(x', x_3) = \{y_1 \in (0,1) : h(x') + \eta(y_1) > x_3\}$ is connected.

For $x_1 \in (0,1)$, we denote $\eta^\varepsilon(x_1) = \eta\left(\dfrac{x_1}{\varepsilon}\right)$. We define the domain Ω_ε as

$$\Omega_\varepsilon = \left\{(x', x_3) : x' \in (0,1)^2, \ g(x') < x_3 < h(x') + \eta^\varepsilon(x_1)\right\}.$$

We denote the boundary of Ω_ε by $\partial\Omega_\varepsilon$. The oscillating part of the domain is denoted by Ω_ε^+, which is defined by

$$\Omega_\varepsilon^+ = \left\{(x', x_3) : x' \in (0,1)^2, \ h(x') < x_3 < h(x') + \eta^\varepsilon(x_1)\right\}$$

and the lower fixed part Ω^- can be described as

$$\Omega^- = \{(x', x_3) : \ x' \in (0,1)^2, \ g(x') < x_3 < h(x')\}.$$

We denote the common boundary of Ω_ε^+ and Ω^- by γ_c^ε:

$$\gamma_c^\varepsilon = \{(x', x_3) \in \Omega_\varepsilon : x_3 = h(x')\}.$$

The full or limit domain which is the Hausdorff limit of Ω_ε is denoted by Ω (see Fig. 2):

$$\Omega = \{(x', x_3) : x' \in (0,1)^2, g(x') < x_3 < h(x') + M\}.$$

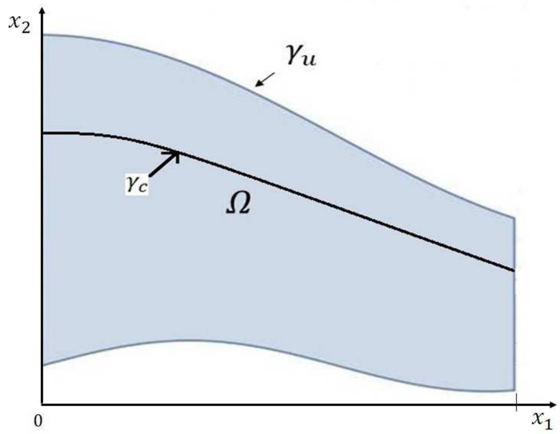

FIGURE 2. Ω.

Now from the definition of Ω^+ and Ω^- we have $\Omega = interior(\overline{\Omega^+ \cup \Omega^-})$. The common boundary of Ω^+ and Ω^- is denoted by γ_c which is defined by

$$\gamma_c = \{(x', x_3) : x' \in (0,1)^2,\ x_3 = h(x')\}$$

We denote the boundary of Ω by $\partial\Omega$.

2.2. Problem description

Let $A(x, y_1) = (a_{i,j}(x, y_1))_{i,j=1}^3$, be a 3×3 matrix where the entries $a_{i,j} : \Omega \times \mathbb{R} \to \mathbb{R}$ are Caratheodory type, $1-$periodic in y_1, and $A(x, y_1)$ is uniformly elliptic, symmetric and bounded in $\Omega \times (0,1)$, that is, there exists $\alpha, \beta \in \mathbb{R}^+$ such that,

$$\langle A(x,y_1)v, v\rangle \geqslant \alpha\|v\|^2, \quad \|A(x,y_1)v\| \leqslant \beta\|v\|, \quad \text{for all } v \in \mathbb{R}^3, \quad x \in \Omega,\ y_1 \in (0,1).$$

Notations: To write in a more conventional way we use the following notations in the sequel: $A = A(x, y_1)$ and $A^\varepsilon = A\left(x, \dfrac{x_1}{\varepsilon}\right)$.

Let us consider the following elliptic PDE

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) + u_\varepsilon = f \quad \text{in} \quad \Omega_\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{2.1}$$

where ν_ε is the outward unit normal vector to $\partial\Omega_\varepsilon$ and $f \in L^2(\Omega)$. The weak formulation to the above PDE is given as follows: find $u_\varepsilon \in H^1(\Omega_\varepsilon)$ such that,

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \cdot \nabla\phi \mathrm{d}x + \int_{\Omega_\varepsilon} u_\varepsilon \phi \mathrm{d}x = \int_{\Omega_\varepsilon} f\phi \mathrm{d}x, \quad \text{for all } \phi \in H^1(\Omega_\varepsilon). \tag{2.2}$$

The Lax-milligram lemma guarantees the existence and uniqueness of u_ε (see [24]) satisfying (2.2). Taking $\phi = u_\varepsilon$ in the weak formulation, we get the uniform bound for $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$, that is, there exists $C \in \mathbb{R}^+$, independent of ε such that $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$. Hereafter C will be denoted as a generic constant independent of ε. Our aim is to analyze the asymptotic behavior of u_ε as $\varepsilon \to 0$. We also consider interior optimal control problems, which we will state in Sections 5 and 7. The existence, uniqueness and the estimate do not use anything special about the oscillations in one direction and it is true with multi-directional oscillations.

To analyze the asymptotic behavior of these problems, a modified unfolding operator will be used which is going to be introduced in the following section.

3. Unfolding operator

The periodic unfolding method is one of the most effective and latest tool in the theory of homogenization which was first introduced in [16]. Here we first recall the definition and properties of periodic unfolding operator for the fixed domain Ω^-.

3.1. Periodic unfolding method

Let for any $x \in \Omega^-$, $[x] = \big([x_1],[x_2],[x_3]\big)$, where $[x_i]$ denote the greatest integer part x_i and $\{x\} = \big(\{x_1\},\{x_2\},\{x_3\}\big)$, where $\{x_i\} = x_i - [x_i]$ is the fractional part of x_i. To define unfolding operator in Ω^-, we use the following notations

$$\begin{aligned} &E_\varepsilon = \{\xi \in \Omega^- :\ \varepsilon(\xi + (0,1)^3) \subset \Omega^-\} \\ &\widehat{\Omega}_\varepsilon^- = interior \left\{ \bigcup_{\xi \in E_\varepsilon} (\varepsilon(\xi + (0,1)^3)) \right\} \text{ and } \Lambda_\varepsilon = \Omega^- \backslash \widehat{\Omega}_\varepsilon^-. \end{aligned} \tag{3.1}$$

Definition 3.1. For a Lebesgue-measurable function ϕ on Ω^-, the unfolding operator T_ε is defined as follows:

$$T_\varepsilon(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) & \text{for a.e. } (x,y) \in \widehat{\Omega}_\varepsilon^- \times (0,1)^3 \\ 0 & \text{for a.e. } (x,y) \in \Lambda_\varepsilon \times (0,1)^3 \end{cases}.$$

The following is a compactness theorem:

Theorem 3.2. *[16] Let $\{\psi_\epsilon\}$ be a sequence in $H^1(\Omega^-)$ such that $\psi_\epsilon \rightharpoonup \psi$ weakly in $H^1(\Omega^-)$, then there exists a sub-sequence of $\{\psi_\varepsilon\}$ still denote by $\{\psi_\varepsilon\}$ and $\psi_1 \in L^2(\Omega, H^1_{per}((0,1)^3))$, such that*

$$T_\varepsilon(\nabla\psi_\epsilon) \rightharpoonup \nabla\psi(x) + \nabla_y\psi_1(x,y) \ in\ L^2(\Omega^- \times (0,1)^3).$$

A modified definition of Definition 3.1 was used in [19] to do homogenization in pillar type oscillating domain. Later in [1], authors defined an unfolding operator for general periodic oscillating domains. For further study on unfolding operators see [17] and references therein. In all the above articles, the base of the oscillation is a horizontal plane. In this article, we are allowing the base of the periodic oscillation to be a graph of a Lipschitz function, that is it can be a curved surface. The unfolding operators can be defined with oscillations in any number of directions which is discussed in Section 7. Now, we define the modified unfolding operator for our analysis in one directional oscillating domain. For every $\varepsilon > 0$, we define the approximate unfolded domain corresponding to Ω_ε^+ as:

$$\Omega_\varepsilon^u = \left\{ (x', x_3, y_1) :\ (x', y_1) \in (0,1)^3,\ h\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right) < x_3 < h\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right) + \eta(y_1) \right\}.$$

Definition 3.3. The unfolding operator $\mathcal{T}^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{\mathcal{T}^\varepsilon u : \Omega_\varepsilon^u \to \mathbb{R}\}$ is defined by

$$\mathcal{T}^\varepsilon u = \chi_{\Omega_\varepsilon^u} T^\varepsilon u,$$

where T^ε is the unfolding operator given by

$$T^\varepsilon v(x_1,x_2,x_3,y_1) = v\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2, x_3\right). \tag{3.2}$$

If $\mathcal{O} \subset \mathbb{R}^3$ containing Ω_ε^+ and v is a real valued function on $\mathcal{O}$, then

$$\mathcal{T}^\varepsilon v = \mathcal{T}^\varepsilon \left(v|_{\Omega_\varepsilon^+}\right).$$

Like other unfolding operators it also enjoys the following integral equality:

Lemma 3.4. *Let $u \in L^2(\Omega_\varepsilon^+)$. Then*

$$\int_{\Omega_\varepsilon^u} \mathcal{T}^\varepsilon u \mathrm{d}x' \mathrm{d}x_3 \mathrm{d}y_1 = \int_{\Omega_\varepsilon^+} u \mathrm{d}x' \mathrm{d}x_3.$$

Proof. From the definition of unfolding,

$$\begin{aligned}
&\int_{\Omega_\varepsilon^u} \mathcal{T}^\varepsilon u(x',x_3,y_1)\mathrm{d}x'\mathrm{d}x_3\mathrm{d}y_1 \\
&= \int_{x'\in(0,1)^2}\int_{y_1\in(0,1)}\int_{h\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon y_1,x_2\right)}^{h\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon y_1,x_2\right)+\eta(y_1)} u\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon y_1, x_3\right)\mathrm{d}x'\mathrm{d}x_3\mathrm{d}y_1 \\
&= \sum_{i=0}^{m-1}\int_{x_1\in\varepsilon i+\varepsilon(0,1)}\int_{x_2\in(0,1)}\int_{y_1\in(0,1)}\int_{h(\varepsilon(i+\varepsilon y_1,x_2,x_3)}^{h(\varepsilon i+\varepsilon y_1,x_2)+\eta(y_1)} u\left(\varepsilon i+\varepsilon y_1,x_2,x_3\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1 \\
&= \sum_{i=0}^{m-1}\int_{y_1\in(0,1)}\int_{x_2\in(0,1)}\int_{h(\varepsilon i+\varepsilon y_1,x_2)}^{h(\varepsilon i+\varepsilon y_1,x_2)+\eta(y_1)} \varepsilon u\left(\varepsilon i+\varepsilon y_1,x_2\right)\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}x_3.
\end{aligned}$$

By making the change of variable, $x_1 = \varepsilon i + \varepsilon y_1$, we get,

$$\int_{\Omega_\varepsilon^u} \mathcal{T}^\varepsilon u(x',x_3,y_1)\mathrm{d}x'\mathrm{d}x_3\mathrm{d}y_1 = \sum_{i=0}^{m-1}\int_{\varepsilon i+\varepsilon(0,1)}\int_{(0,1)}\int_{h(x')}^{h(x')+\eta^\varepsilon(x_1)} u(x',x_3)\mathrm{d}x'\mathrm{d}x_3.$$

Hence the claim is proved. □

Notice that the characteristic function, $\chi_{\Omega_\varepsilon^u} \to \chi_{\Omega^u}$ pointwise in $\mathbb{R}^2$, where Ω^u is the fixed unfolded domain defined as

$$\Omega^u = \left\{(x',x_3,y_1):\ x'\in(0,1)^2, y_1\in(0,1),\ h(x') < x_3 < h(x')+\eta(y_1)\right\}.$$

To get a transparent vision on the unfolded domain, we give 2-dimensional figures of $h, \eta, \Omega_\varepsilon^u, \Omega^u$, see Figures 3, 4, 5, 6. Also, since h, η are Lipschitz functions, we have that

$$|\Omega_\varepsilon^u \Delta \Omega^u| = O(\varepsilon) \tag{3.3}$$

where Δ denote the symmetric difference between Ω_ε^+, Ω^u that is $\Omega_\varepsilon^u\Delta\Omega^u = (\Omega_\varepsilon^u\backslash\Omega^u)\bigcup(\Omega^u\backslash\Omega_\varepsilon^u)$ and $|\cdot|$ denotes the usual Lebesgue measure. Because of the above relation (3.3), we have the following equality,

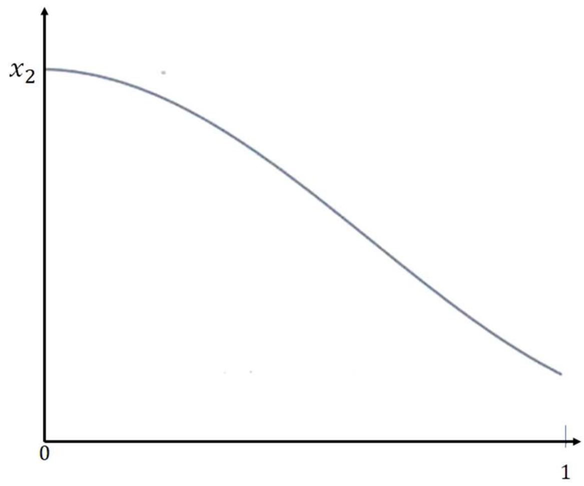

FIGURE 3. h.

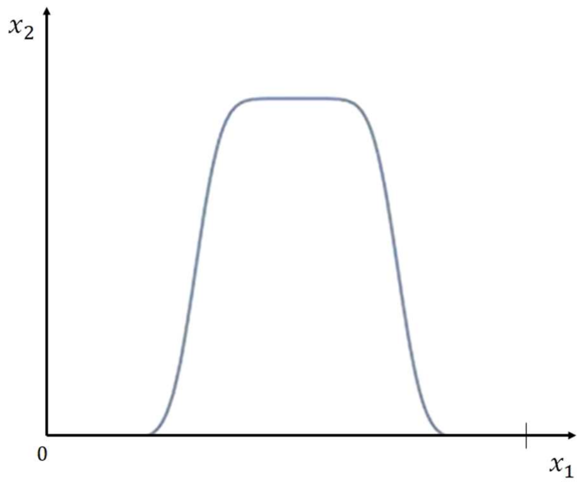

FIGURE 4. η.

for any $v \in L^2_{loc}(\mathbb{R}^3)$:

$$\int_{\Omega^u_\varepsilon} \mathcal{T}^\varepsilon v \mathrm{d}x\mathrm{d}y_1 = \int_{\Omega^u} \mathcal{T}^\varepsilon v \mathrm{d}x\mathrm{d}y_1 + O(\varepsilon). \tag{3.4}$$

Since our aim is to analyze the asymptotic behavior of the sequence u_ε the solution of (2.2), whenever we apply unfolding, we will make use of the relation (3.4).

Note: Throughout this article, we will use $\mathcal{T}^\varepsilon$ and T_ε as unfolding operators for the upper oscillating part and the fixed lower part respectively.

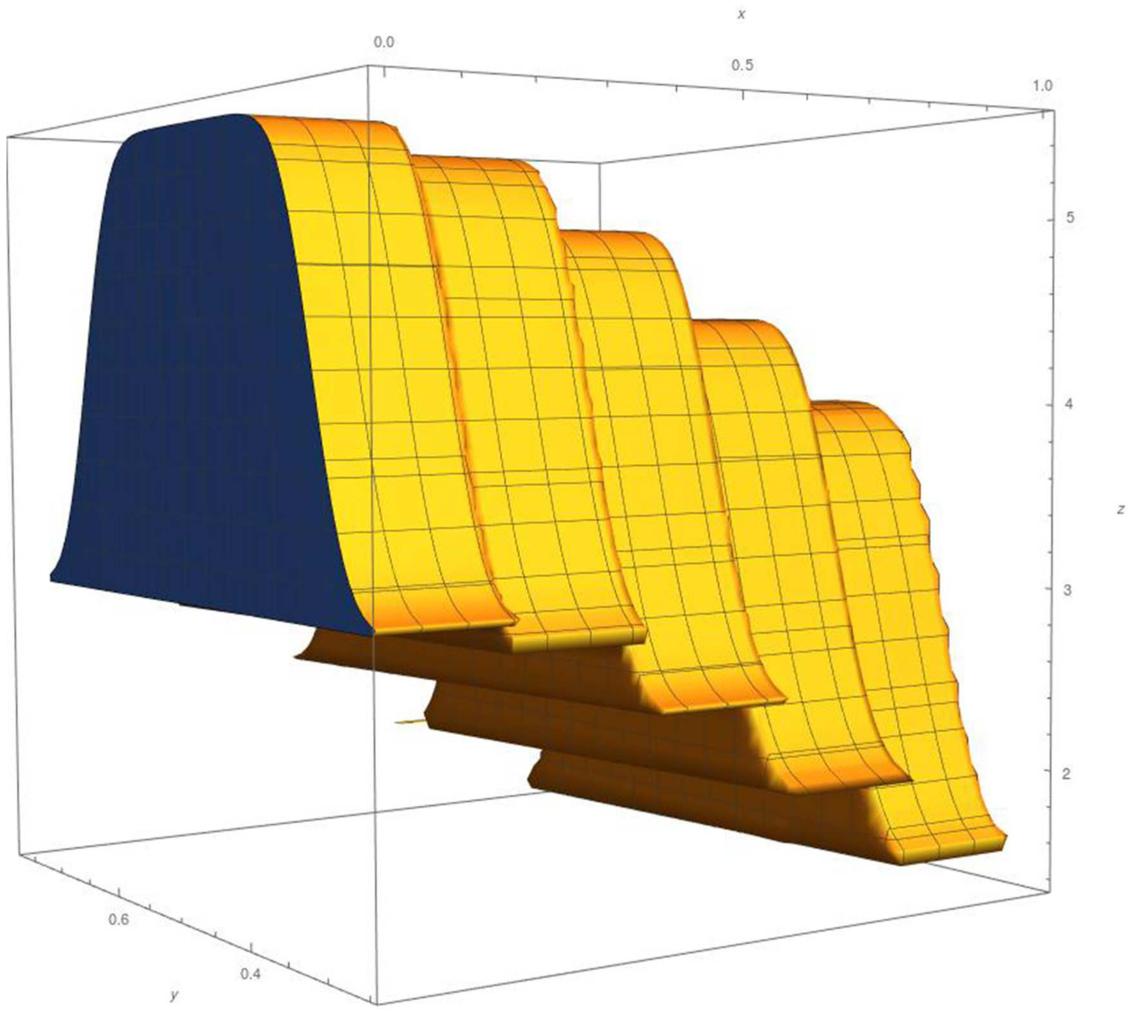

FIGURE 5. Ω_ε^u for $\varepsilon = \frac{1}{5}$.

3.2. Boundary unfolding operator

In order to get the interface conditions, we now introduce the following boundary unfolding operator $\mathcal{T}_b^\varepsilon$ on γ_c^ε. For every $\varepsilon > 0$, let us denote the unfolded boundary of γ_c^ε by γ_c^u, defined by

$$\gamma_c^u = \{(x', h(x'), y_1) : \ x' \in (0,1)^2, y_1 \in (a,b)\}.$$

Define $\phi_{\gamma_c}^\varepsilon : \gamma_c^u \to \gamma_c^\varepsilon$ as

$$\phi_{\gamma_c}^\varepsilon(x', h(x'), y_1) = \left(\varepsilon \left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2, h\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right)\right).$$

Define boundary unfolding operator $\mathcal{T}_b^\varepsilon : \{u : \gamma_c^\varepsilon \to \mathbb{R}\} \to \{\mathcal{T}_b^\varepsilon : \gamma_c^u \to \mathbb{R}\}$ as $\mathcal{T}_b^\varepsilon u = u \circ \phi_{\gamma_c}^\varepsilon$.

Proposition 3.5. *The boundary unfolding operator $\mathcal{T}_b^\varepsilon$ enjoys the following properties:*

(i) $\mathcal{T}_b^\varepsilon$ is linear. Further, if $u,\ v : \gamma_c^\varepsilon \to \mathbb{R}$, then, $\mathcal{T}_b^\varepsilon(uv) = \mathcal{T}_b^\varepsilon(u)\mathcal{T}_b^\varepsilon(v)$,

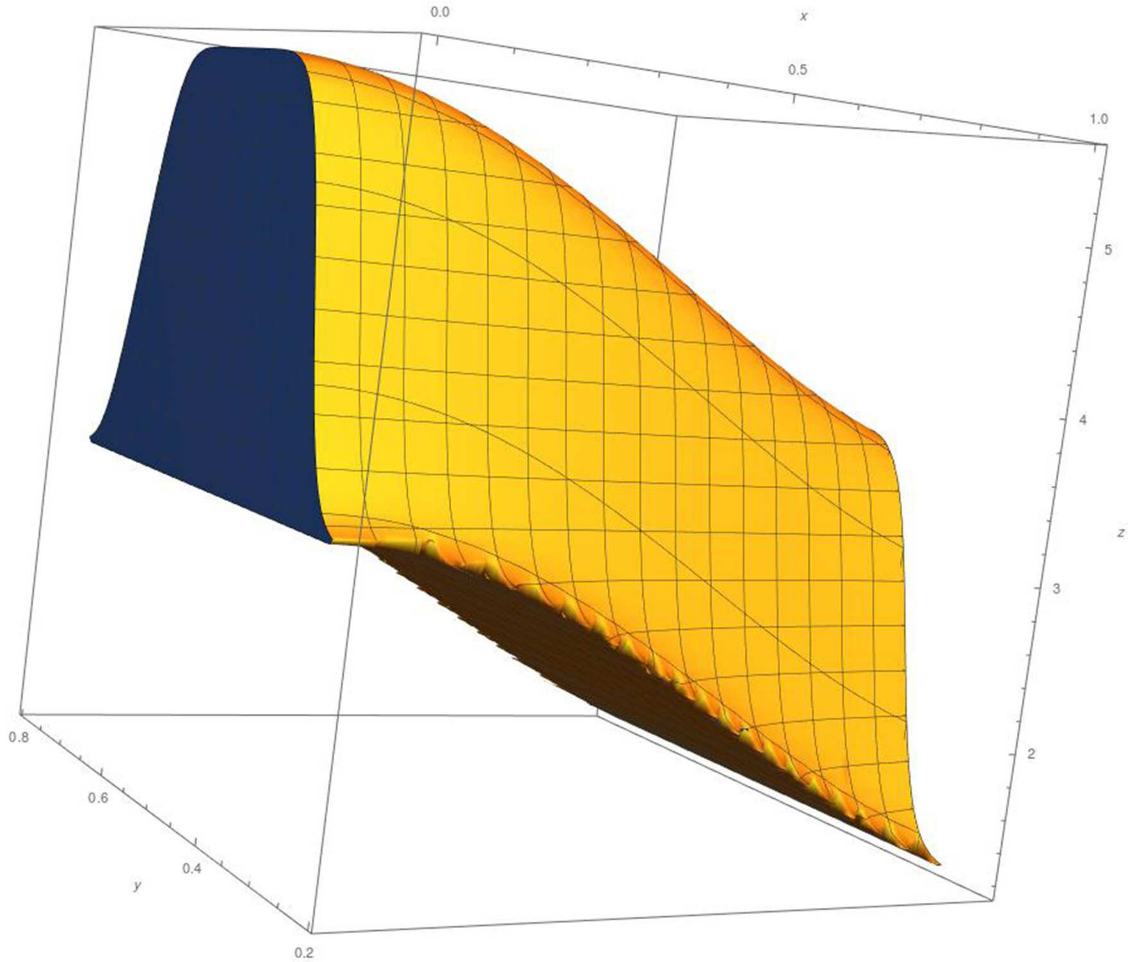

FIGURE 6. Ω^u.

(ii) *for any* $\phi \in L^2(\gamma_c)$, $\|\mathcal{T}_b^\varepsilon \phi - \phi\|_{L^2(\gamma_c^u)} \to 0$ *as* $\varepsilon \to 0$,
(iii) *let* $\{\phi_\varepsilon\}$ *is a sequence in* $L^2(\gamma_c)$ *such that* $\|\phi_\varepsilon - \phi\|_{L^2(\gamma_c)} \to 0$, *then* $\|\mathcal{T}_b^\varepsilon \phi_\varepsilon - \phi\|_{L^2(\gamma_c^u)} \to 0$,
(iv) *for* $u \in L^2(\gamma_\varepsilon)$, *then* $\|\mathcal{T}_b^\varepsilon u\|_{L^2(\gamma_c^u)} \simeq \|u\|_{L^2(\gamma_c^\varepsilon)}$ *when* ε *is small.*

Proof. Proof of (i) is obvious, so we will give proof of $(ii), (iii)$ and (iv).

Proof of (ii) and (iii): Let $\phi \in C_c^\infty(\gamma_c)$, hence ϕ is Lipschitz, say with lipschitz constant L_1. Also let the Lipschitz constant for h is L_2. Then

$$\left|\phi\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2, h\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right)\right) - \phi(x', h(x'))\right| \leqslant (L_1 + L_1 L_2)\varepsilon.$$

This implies $\mathcal{T}_b^\varepsilon \phi \to \phi$ pointwise in γ_c^u, hence in $L^2(\gamma_c^u)$. The density of $C_c^\infty(\gamma_c)$ in $L^2(\gamma_c)$ completes the proof of (ii). To prove (iii), use triangle in inequality to get

$$\|\mathcal{T}_b^\varepsilon \phi_\varepsilon - \phi\|_{L^2(\gamma_c^u)} \leqslant \|\mathcal{T}_b^\varepsilon \phi_\varepsilon - \phi_\varepsilon\|_{L^2(\gamma_c^u)} + \|\phi_\varepsilon - \phi\|_{L^2(\gamma_c^u)}.$$

Now, using (ii) and the convergence of $\phi_\varepsilon \to \phi$ in $L^2(\gamma_c)$, we get (iii).

Proof of (iv): We have

$$
\begin{aligned}
&\|u\|^2_{L^2(\gamma_c^\varepsilon)}\\
&=\sum_{i=0}^{m-1}\int_{\varepsilon i+\varepsilon(a,b)}\int_{(0,1)} u^2(x',h(x'))\sqrt{1+|\nabla h(x')|^2}\mathrm{d}x'\\
&=\sum_{i=0}^{m-1}\int_{\varepsilon i+\varepsilon(0,1)}\int_{(0,1)}\int_{(a,b)} u^2\left(\varepsilon i+\varepsilon y_1,x_2,h\left(\varepsilon i+\varepsilon y_1,x_2\right)\right)\sqrt{1+|\nabla h|^2\left(\varepsilon i+\varepsilon y_1,x_2\right)}\mathrm{d}x'\mathrm{d}y_1\\
&=\int_{(0,1)^2}\int_{(a,b)}(\mathcal{T}_b^\varepsilon u)^2(x',h(x'),y_1)\frac{\sqrt{1+|\nabla h|^2\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon y_1,x_2\right)}}{\sqrt{1+|\nabla h(x')|^2}}\sqrt{1+|\nabla h(x')|^2}\mathrm{d}x'\mathrm{d}y_1.
\end{aligned}
$$

Since $\dfrac{\sqrt{1+|\nabla h|^2\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]+\varepsilon y_1,x_2\right)}}{\sqrt{1+|\nabla h|^2(x')^2}}\to 1$ as $\varepsilon\to 0$, the result follows. □

3.3. Limit function space

Let us introduce the function space in the limit domain Ω, where the solution corresponding to homogenized system belongs to. Let $\omega(x)=|Y(x',x_3)|$, the Lebesgue measure of $Y(x',x_3)$, where $Y(x',x_3)=\{y_1\in(0,1):\ h(x')+\eta(y_1)>x_3\}$. For any $\psi:\Omega\to\mathbb{R}$, denote $\psi^+=\psi\chi_{\Omega^+}$ and $\psi^-=\psi\chi_{\Omega^-}$. Define

$$\widehat{H}(\Omega,\omega)=\left\{\psi\in C^\infty(\Omega):\psi\in L^2(\Omega),\frac{\partial\psi}{\partial x_i}\in L^2(\Omega)\text{ for }i=2,3,\ \nabla\psi^-\in H^1(\Omega^-)\right\}$$

with the following inner product for $\psi,\phi\in\widehat{H}(\Omega,\omega)$

$$\langle\psi,\phi\rangle_{H(\Omega,\omega)}=\int_{\Omega^+}\omega\left(\psi\phi+\sum_{i=2}^{3}\frac{\partial\psi}{\partial x_i}\frac{\partial\phi}{\partial x_i}\right)\mathrm{d}x+\int_{\Omega^-}(\psi\phi+\nabla\psi\cdot\nabla\phi)\mathrm{d}x. \tag{3.5}$$

Now define $H(\Omega,\omega)$ to be the completion of $\widehat{H}(\Omega,\omega)$ with respect to the norm defined by the inner product (3.5). We can characterize the space $H(\Omega,\omega)$ as

$$H(\Omega,\omega)=\left\{\psi\in L^2(\Omega):\frac{\partial\psi}{\partial x_i}\in L^2(\Omega)\text{ for }i=2,3,\ \nabla\psi^-\in H^1(\Omega^-)\right\}.$$

4. Homogenization

In this section, we obtain the limit problem corresponding to (2.1) by passing to the limit $\varepsilon\to 0$ in (2.2). We also prove the existence and uniqueness of the solution of the limit problem. From the definition of unfolding operator, we have

$$
\begin{aligned}
&\|\mathcal{T}^\varepsilon(u_\varepsilon)\|^2_{L^2(\Omega^u)}+\|\mathcal{T}^\varepsilon\nabla u_\varepsilon\|^2_{L^2(\Omega^u)}\leqslant\|\mathcal{T}^\varepsilon u_\varepsilon\|^2_{L^2(\Omega^u_\varepsilon)}+\|\mathcal{T}^\varepsilon(\nabla u_\varepsilon)\|^2_{L^2(\Omega^u_\varepsilon)}+o(1)\\
&=\int_{\Omega^u_\varepsilon}\left(|\mathcal{T}^\varepsilon u|^2+|T^\varepsilon\nabla u_\varepsilon|^2\right)\mathrm{d}x\mathrm{d}y_1+o(1)=\int_{\Omega^+_\varepsilon}\left(|u_\varepsilon|^2+|\nabla u_\varepsilon|^2\right)\mathrm{d}x+o(1)\leqslant C.
\end{aligned}
$$

The last inequality follows from the boundedness of $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$. Thus, we have the following proposition:

Proposition 4.1. *The sequence of solutions u_ε to (2.1) satisfies the estimates:*

$$\|\mathcal{T}^\varepsilon u_\varepsilon\|_{L^2(\Omega^u)} \leqslant C, \qquad \|\mathcal{T}^\varepsilon \nabla u_\varepsilon\|_{L^2(\Omega^u)} \leqslant C.$$

Analogous to the Propositions 7 and 8 in [34], we prove two propositions in the context of unfolded operators. Let for each $\varepsilon > 0$, u_ε be the unique solution of (2.1). Let us define another sequence using the unfolded sequence $\mathcal{T}^\varepsilon u_\varepsilon$, say $\mathcal{T}^\varepsilon U_\varepsilon$ as

$$\mathcal{T}^\varepsilon U_\varepsilon(x', y_1, x_3) = \frac{1}{\varepsilon}\left(\mathcal{T}^\varepsilon u_\varepsilon - \frac{1}{|Y(x', x_3|)} \int_{Y(x',x_3)} \mathcal{T}^\varepsilon u_\varepsilon(x_1, y_1, x_2, x_3)\mathrm{d}y_1\right). \tag{4.1}$$

Proposition 4.2. *The sequence defined in (4.1) is uniformly bounded in $L^2(\Omega^u)$.*

Proof. Using Poincare-Writinger inequality in (4.1), we get

$$\begin{aligned}\int_{Y(x',x_3)} |\mathcal{T}^\varepsilon U_\varepsilon|^2 \mathrm{d}y &\leqslant C \int_{Y(x',x_3)} \left|\frac{\partial}{\partial y_1}\left(\frac{1}{\varepsilon}\left(\mathcal{T}^\varepsilon u_\varepsilon - \frac{1}{|Y(x', x_3|)}\int_{Y(x',x_3)} \mathcal{T}^\varepsilon u_\varepsilon(x_1, y_1, x_2, x_3)\mathrm{d}y_1\right)\right)\right|^2 \mathrm{d}y_1 \\ &\leqslant C \int_{Y(x',x_3)} \left|\frac{1}{\varepsilon}\frac{\partial \mathcal{T}^\varepsilon u_\varepsilon}{\partial y_1}\right|^2 = C \int_{Y(x',x_3)} \left|\mathcal{T}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_1}\right|^2 \mathrm{d}y_1.\end{aligned}$$

Now integrating both side with respect to x' and x_3 we get,

$$\int_{\Omega^u} |\mathcal{T}^\varepsilon U_\varepsilon|^2 \leqslant C \int_{\Omega^u} \left|\mathcal{T}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_1}\right|^2 = C \int_{\Omega^+_\varepsilon} \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 \leqslant C,$$

where C is a generic constant independent of ε. □

As $\|\mathcal{T}^\varepsilon U_\varepsilon\| \leqslant C$, by compactness there exist $U_1 \in L^2(\Omega^u)$ such that $\mathcal{T}^\varepsilon U_\varepsilon \rightharpoonup U_1$ in $L^2(\Omega^u)$. Then the following proposition hold.

Proposition 4.3. *Let for each $\varepsilon > 0$, u_ε be the unique solution of (2.1). Then*

$$\mathcal{T}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_1} \rightharpoonup \frac{\partial U_1}{\partial y_1} \text{ weakly in } L^2(\Omega^u).$$

Proof. Let $\psi(x, y_1) \in C_c^\infty(\Omega^u)$. Let $\mathcal{T}^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_1}\right) \rightharpoonup P_1$ in $L^2(\Omega^u)$. A simple integration by parts gives us the following,

$$\int_{\Omega^u} \mathcal{T}^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)\psi(x, y_1) = \int_{\Omega^u} \frac{1}{\varepsilon}\frac{\partial}{\partial y_1}(\mathcal{T}^\varepsilon u_\varepsilon)\psi(x, y_1) = -\int_{\Omega^u} \frac{1}{\varepsilon}\mathcal{T}^\varepsilon u_\varepsilon \frac{\partial \psi}{\partial y_1} = -\int_{\Omega^u} \mathcal{T}^\varepsilon U_\varepsilon \frac{\partial \psi}{\partial x_1}.$$

Now letting $\varepsilon \to 0$ in the above equation on both side, we get

$$\int_{\Omega^u} P_1 \psi = -\int_{\Omega^u} U_1 \frac{\partial \psi}{\partial y_1}.$$

Since ψ is arbitrary, implies $P_1 = \dfrac{\partial U_1}{\partial y_1}$ a.e. in Ω^u. $\square$

In the main convergence theorem we will explicitly calculate P_1 in terms of $a_{i,j}$, x_2 and x_3 directional derivatives of the limit solution.

Limit Problem: We denote the limit matrices A_0^+ and A_0^- corresponding to the upper and lower parts. The matrix A_0^+ is defined as

$$A_0^+ = A_0^+(x) = \int_{Y(x',x_3)} \frac{1}{a_{11}} \begin{bmatrix} a_{11}a_{22} - a_{12}^2 & a_{23}a_{11} - a_{12}a_{13} \\ a_{11}a_{23} - a_{13}a_{12} & a_{11}a_{33} - a_{13}^2 \end{bmatrix} \mathrm{d}y_1, \text{ for } x \in \Omega^+. \tag{4.2}$$

To introduce the matrix A_0^- we need the following standard cell problems: For $i = 1, 2, 3$, let $\chi_i = \chi_i(y)$ solves:

$$\begin{cases} -\mathrm{div}_y(A(x, y_1)\nabla\chi_i) = \mathrm{div}_y(A(x, y_1)e_i), \quad \text{in } (0, 1)^3 \\ \chi_i \text{ is } (0, 1)^3\text{-periodic}, \\ M(\chi_i) = 0, \end{cases} \tag{4.3}$$

where $\{e_i : i = 1, 2, 3\}$ is the standard basis of $\mathbb{R}^3$ and $M(\phi) = \displaystyle\int_{(0,1)^3} \phi(y)\mathrm{d}y$. Let $\chi = [\chi_1\ \chi_2\ \chi_3]$. Now, A_0^- is defined by

$$A_0^-(x) = \int_{(0,1)^3} A(x, y_1)\big(I + \nabla_y\chi\big)\mathrm{d}y, \text{ for } x \in \Omega^-, \tag{4.4}$$

where I is the 3×3 identity matrix and $\nabla_y\chi = [\nabla_y\chi_1\ \nabla_y\chi_2\ \nabla_y\chi_3]$ is a 3×3 matrix. The homogenized equation is given by,

$$\begin{cases} -\mathrm{div}_{2,3}\left(A_0^+\nabla_{2,3}u^+\right) + \omega(x)u^+ = \omega(x)f \quad \text{in } \ \Omega^+, \\ -\mathrm{div}(A_0^-\nabla u^-) + u^- = f \quad \text{in } \ \Omega^-, \\ A_0^+\nabla_{2,3}u^+ \cdot (\nu_2, \nu_3) = 0 \quad \text{on } \ \gamma_u, \\ u^+ = u^-, A_0^+\nabla_{2,3}u^+ \cdot (\sigma_2, \sigma_3) - A_0^-\nabla u^- \cdot \sigma = 0 \quad \text{on } \ \gamma_c, \\ A_0^-\nabla u \cdot \nu = 0 \quad \text{on } \ \partial\Omega^-\backslash\gamma_c, \end{cases} \tag{4.5}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the outward unit normal vector on $\partial\Omega$ and

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) = \left(\sqrt{1 + |\nabla h(x')|^2}\right)^{-1}\left(\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, -1\right)$$

is the downward unit normal on γ_c and

$$\nabla_{2,3} = \begin{bmatrix} \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{bmatrix}, \ \mathrm{div}_{2,3} = \left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

The limit system (4.5) is quite interesting and it shows the feature of laminate structure. Recall that we have oscillations only in one direction, namely x_1. Hence the oscillatory part is like laminates and it averages out in that direction like usual homogenization theory and we only have the presence of x_2 and x_3 variable in the upper part of the domain. Indeed the oscillatory part is like laminates. The variational form for the above system is given by: find $u = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-} \in H(\Omega,\omega)$ such that

$$\int_{\Omega^+} \left(A_0^+\nabla_{2,3}u^+\nabla_{2,3}\phi^+ + \omega(x)u^+\phi^+\right) + \int_{\Omega^-} (A_0^-\nabla u^-\nabla\phi^- + u^-\phi^-) = \int_{\Omega^+} \omega(x)f\phi^+\mathrm{d}x + \int_{\Omega^-} f\phi^-,$$

for all $\phi = \phi^+\chi_{\Omega^+} + \phi^-\chi_{\Omega^-} \in H(\Omega,\omega)$. We write $u_\varepsilon^+ = u_\varepsilon|_{\Omega_\varepsilon^+}$ and $u_\varepsilon^- = u_\varepsilon|_{\Omega^-}$

Now we state the main theorem of this section:

Theorem 4.4. *(Homogenization) Let u_ε be the sequence of solution to (2.1). Then the following convergences hold:*

(i) $\mathcal{T}^\varepsilon u_\varepsilon^+ \rightharpoonup u^+$ and $T^\varepsilon\left(\frac{\partial u_\varepsilon^+}{\partial x_i}\right) \rightharpoonup \frac{\partial u^+}{\partial x_i}$ weakly in $L^2(\Omega^u)$ for $i = 2,3$,

(ii) $u_\varepsilon^- \rightharpoonup u^-$ weakly in $H^1(\Omega^-)$.

Further u^+ is independent of y_1, $u = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-} \in H(\Omega,\omega)$ and u is the unique solution of the homogenized system (4.5).

Proof. The proof will be accomplished in several steps.

Step 1: Taking $\phi \in C_c^\infty(\Omega^+)$ as a test function in (2.2), we get

$$\int_{\Omega_\varepsilon^+} \left(A^\varepsilon\nabla u_\varepsilon^+ \cdot \nabla\phi + u_\varepsilon^+\phi\right)\mathrm{d}x = \int_{\Omega_\varepsilon^+} f\phi\mathrm{d}x.$$

Applying unfolding operator on both side of the above equation, by (3.4), we have

$$\int_{\Omega^u} \left(\mathcal{T}^\varepsilon\left(A^\varepsilon\right)\mathcal{T}^\varepsilon(\nabla u_\varepsilon^+)\cdot\mathcal{T}^\varepsilon(\nabla\phi) + \mathcal{T}^\varepsilon(u_\varepsilon^+)\mathcal{T}^\varepsilon\phi\right)\mathrm{d}x\mathrm{d}y_1 = \int_{\Omega^u} \mathcal{T}^\varepsilon(f)\mathcal{T}^\varepsilon(\phi)\mathrm{d}x\mathrm{d}y_1 + o(1). \tag{4.6}$$

Thus,

$$\lim_{\varepsilon\to 0}\int_{\Omega^u} \left(T^\varepsilon\left(A^\varepsilon\right)\mathcal{T}^\varepsilon(\nabla u_\varepsilon^+)\cdot\mathcal{T}^\varepsilon(\nabla\phi) + \mathcal{T}^\varepsilon(u_\varepsilon^+)\mathcal{T}^\varepsilon(\phi)\right)\mathrm{d}x\mathrm{d}y_1 = \lim_{\varepsilon\to 0}\int_{\Omega^u} \mathcal{T}^\varepsilon(f)\mathcal{T}^\varepsilon(\phi)\mathrm{d}x\mathrm{d}y_1.$$

Now we try to pass to the limit as $\varepsilon \to 0$ in the above identity. By Proposition 4.1, we have $\|\mathcal{T}^\varepsilon u_\varepsilon^+\|_{L^2(\Omega^u)} \leqslant C$, $\left\|\mathcal{T}^\varepsilon\left(\dfrac{\partial u_\varepsilon^+}{\partial x_i}\right)\right\|_{L^2(\Omega^u)} \leqslant C$ for $i = 1,2,3$. Hence, there exist $u^+, P_1, P_2, P_3 \in L^2(\Omega^u)$ such that

$$\begin{aligned}
&\mathcal{T}^\varepsilon u_\varepsilon^+ \rightharpoonup u^+ \text{ weakly in } L^2(\Omega^u),\\
&\mathcal{T}^\varepsilon\left(\frac{\partial u_\varepsilon^+}{\partial x_i}\right) \rightharpoonup P_i \text{ weakly in } L^2(\Omega^u), \text{ for } i = 1,2,3.
\end{aligned}$$

Main issue is to compute P_i and connect it to u^+. This is easy for P_2 and P_3 as the domain has oscillations in x_1 direction. We have

$$\frac{\partial}{\partial x_i}(\mathcal{T}^\varepsilon) = \mathcal{T}^\varepsilon\left(\frac{\partial}{\partial x_i}\right) \text{ for } \ i = 2,3.$$

But this is not the case for $i = 1$ and has to be dealt separately. First, we find out P_2 and P_3.

Claim: For $i = 2, 3$, $P_i = \dfrac{\partial u^+}{\partial x_i}$.

Proof: We have $\mathcal{T}^\varepsilon\left(\dfrac{\partial u_\varepsilon^+}{\partial x_i}\right) \rightharpoonup P_i$ in $L^2(\Omega^u)$. Let $\phi \in C_c^\infty(\Omega^u)$, so after sufficiently small $\varepsilon > 0$, $supp(\phi) \subset \Omega_\varepsilon^u \cap \Omega^u$. Now, consider the following,

$$\int_{\Omega^u} \mathcal{T}^\varepsilon\left(\frac{\partial u_\varepsilon^+}{\partial x_i}\right)\phi \mathrm{d}x\mathrm{d}y_1 = \int_{\Omega^u} \frac{\partial}{\partial x_i}(\mathcal{T}^\varepsilon u_\varepsilon^+)\phi \mathrm{d}x\mathrm{d}y_1 = -\int_{\Omega^u} \mathcal{T}^\varepsilon u_\varepsilon^+ \frac{\partial \phi}{\partial x_i}\mathrm{d}x\mathrm{d}y_1.$$

Letting $\varepsilon \to 0$ in the above equality, we obtain,

$$\int_{\Omega^u} P_i \phi \mathrm{d}x\mathrm{d}y_1 = -\int_{\Omega^u} u^+ \frac{\partial \phi}{\partial x_i}\mathrm{d}x\mathrm{d}y_1.$$

Since, $\phi \in C_c^\infty(\Omega^u)$ is arbitrary, we get, $P_i = \dfrac{\partial u^+}{\partial x_i}$ for $i = 2, 3$.

In order to find P_1, we will make use of the fact that u^+ is independent of y_1.

Claim: The limit solution on upper part, that is u^+, is independent of y_1.

Proof: From the definition of unfolding operator, we have $\chi_{\Omega_\varepsilon^u} \dfrac{\partial}{\partial y_1}(T^\varepsilon u_\varepsilon^+) = \varepsilon \chi_{\Omega_\varepsilon^u} T^\varepsilon \left(\dfrac{\partial u_\varepsilon^+}{\partial x_1}\right)$. Hence $\left\|\dfrac{\partial}{\partial y_1}(T^\varepsilon u_\varepsilon^+)\right\|_{L^2(\Omega^u)} < C\varepsilon$. From the weak convergence of $\mathcal{T}^\varepsilon(u_\varepsilon^+)$ in $L^2(\Omega^u)$, and boundedness of $\left\|\dfrac{\partial}{\partial y_1}(T^\varepsilon u_\varepsilon^+)\right\|$, we get

$$\frac{\partial}{\partial y_1}\mathcal{T}^\varepsilon u_\varepsilon^+ \rightharpoonup \frac{\partial u^+}{\partial y_1} \quad \text{weakly in } L^2(\Omega^u). \tag{4.7}$$

But, we have, $\left\|\dfrac{\partial}{\partial y_1}\mathcal{T}^\varepsilon u_\varepsilon^+\right\|_{L^2(\Omega^u)} = \left\|\varepsilon \mathcal{T}^\varepsilon\left(\dfrac{\partial u_\varepsilon^+}{\partial x_1}\right)\right\|_{L^2(\Omega^u)} \to 0$ as $\varepsilon \to 0$. This implies $\dfrac{\partial u^+}{\partial y_1} = 0$. Hence this proves the claim, since $Y(x', x_3)$ is connected for each $(x', x_3) \in \Omega^+$.

Now we will find P_1. For that we use the oscillating test function $\phi^\varepsilon(x) = \varepsilon\phi(x)\psi\left(\left\{\frac{x_1}{\varepsilon}\right\}\right)$, where $\phi \in C_c^\infty(\Omega^+)$ and $\psi \in C_{per}^\infty((0,1))$, where $C_{per}^\infty((0,1))$ denotes the space of smooth 1- periodic real valued functions. This test function has the following properties,

$$\begin{aligned}
&\mathcal{T}^\varepsilon(\phi^\varepsilon) \to 0 \ \text{ strongly in } L^2(\Omega^u),\\
&\mathcal{T}^\varepsilon\left(\frac{\partial \phi^\varepsilon}{\partial x_i}\right) \to 0 \ \text{ strongly in } L^2(\Omega^u), \ \text{for } i = 2, 3,\\
&\mathcal{T}^\varepsilon\left(\frac{\partial \phi^\varepsilon}{\partial x_1}\right) = \varepsilon \mathcal{T}^\varepsilon\left(\frac{\partial \phi}{\partial y_1}\right)\psi(y_1) + \mathcal{T}^\varepsilon(\phi)\frac{\mathrm{d}\psi}{\mathrm{d}y_1}(y_1).
\end{aligned}$$

Hence, $\mathcal{T}^\varepsilon\left(\dfrac{\partial \phi^\varepsilon}{\partial x_1}\right) \to \phi \dfrac{\mathrm{d}\psi}{\mathrm{d}y_1}$ strongly in $L^2(\Omega^u)$. Also from the assumptions on the coefficient matrix A, we have $\mathcal{T}^\varepsilon(A^\varepsilon) \to A$ in $L^2(\Omega^u)$. Using ϕ^ε as a test function in the weak formulation (4.6) and passing to the limit as $\varepsilon \to 0$, we get

$$\int_{\Omega^u} A \begin{bmatrix} P_1 \\ \\ \nabla_{2,3} u^+ \end{bmatrix} \cdot \begin{bmatrix} \phi\psi' \\ 0 \\ 0 \end{bmatrix} \mathrm{d}x\mathrm{d}y_1 = 0.$$

By simple matrix multiplication, we obtain

$$\int_{\Omega^u} \left(a_{11} P_1 + a_{12} \frac{\partial u^+}{\partial x_2} + a_{13} \frac{\partial u^+}{\partial x_3} \right) \phi \frac{\mathrm{d}\psi}{\mathrm{d}y_1} = 0.$$

Above equality holds for all $\phi \in C_c^\infty(\Omega^+)$. Hence, we get

$$P_1 = -\frac{a_{12}}{a_{11}} \frac{\partial u^+}{\partial x_2} - \frac{a_{13}}{a_{11}} \frac{\partial u^+}{\partial x_3} \qquad \text{a.e in } \Omega^u. \tag{4.8}$$

Since, $\psi \in C^\infty_{per}((0,1))$ and A is uniformly elliptic, implies $a_{11} > \alpha$ (elliptic constant). Now, taking $\phi \in C_c^\infty(\Omega^+)$ as a test function in the weak formulation (2.2) and passing to the limit $\varepsilon \to 0$, we obtain

$$\begin{cases} \displaystyle\int_{\Omega^u} \left(a_{11} P_1 + a_{12} \frac{\partial u^+}{\partial x_2} + a_{13} \frac{\partial u^+}{\partial x_3} \right) \frac{\partial \phi}{\partial x_1} + \int_{\Omega^u} \left(a_{12} P_1 + a_{22} \frac{\partial u^+}{\partial x_2} + a_{23} \frac{\partial u^+}{\partial x_3} \right) \frac{\partial \phi}{\partial x_2} \\ \displaystyle\quad + \int_{\Omega^u} \left(a_{13} P_1 + a_{23} \frac{\partial u^+}{\partial x_2} + a_{33} \frac{\partial u^+}{\partial x_3} \right) \frac{\partial \phi}{\partial x_3} + \int_{\Omega^u} u^+ \phi = \int_{\Omega^u} f\phi. \end{cases} \tag{4.9}$$

The first integral in (4.9) is 0 by (4.8). Now, substituting the value of P_1 given in (4.8) in the above expression, it reduces to,

$$\int_{\Omega^u} \left(\frac{1}{a_{11}} \begin{bmatrix} a_{11}a_{22} - a_{12}^2 & a_{23}a_{11} - a_{12}a_{13} \\ a_{11}a_{23} - a_{13}a_{12} & a_{11}a_{33} - a_{13}^2 \end{bmatrix} \nabla_{2,3} u^+ \cdot \nabla_{2,3}\phi + u^+\phi \right) \mathrm{d}x\mathrm{d}y_1 = \int_{\Omega^u} f\phi \mathrm{d}x\mathrm{d}y_1.$$

Since u^+ and ϕ are independent of y_1, using the definition of Ω^u, we get

$$\int_{\Omega^+} A_0^+ \nabla_{2,3} u^+ \cdot \nabla_{2,3}\phi \mathrm{d}x + \int_{\Omega^+} \omega(x) u^+ \phi \mathrm{d}x = \int_{\Omega^+} \omega(x) f\phi \mathrm{d}x.$$

Step 2: Now, we look into to the lower fixed part. In the weak formulation (2.2), take $\phi \in C_c^\infty(\Omega^-)$, we get

$$\int_{\Omega^-} (A^\varepsilon \nabla u_\varepsilon \cdot \nabla\phi + u_\varepsilon \phi) \,\mathrm{d}x = \int_{\Omega^-} f\phi \mathrm{d}x. \tag{4.10}$$

Here we will use the unfolding operator for fixed domain. Since we already have $\|u_\varepsilon\|_{H^1(\Omega^-)} \leqslant C$, then by compactness Theorem 3.2, there exist, $u^- \in H^1(\Omega^-)$ and $u_1 \in L^2(\Omega; H^1_{per}((0,1)^3))$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u^- \ \text{ weakly in } \ H^1(\Omega^-), \\ T_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u^-(x) + \nabla_y u_1(x,y) \text{ weakly in } L^2(\Omega \times (0,1)^3). \end{cases} \tag{4.11}$$

Now consider $\phi^\varepsilon = \phi_0(x) + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)$, where $\phi_0 \in C_c^\infty(\Omega^-)$ and $\phi_1 \in C_c^\infty(\Omega; C_{per}^\infty((0,1)^3))$, as an oscillating test function in the weak formulation (4.10), we obtain

$$\int_{\Omega^-} (A^\varepsilon \nabla u_\varepsilon \cdot \nabla \phi^\varepsilon + u_\varepsilon \phi^\varepsilon)\mathrm{d}x = \int_{\Omega^-} f\phi^\varepsilon \mathrm{d}x. \tag{4.12}$$

Applying unfolding operator on both sides, using the properties of unfolding operator ([16], Prop. 2.5) and convergence (4.11), by letting $\varepsilon \to 0$ in (4.12), we obtain,

$$\int_{\Omega^- \times (0,1)^3} A(\nabla u + \nabla_y u_1) \cdot (\nabla \phi_0 + \nabla_y \phi_1)\mathrm{d}x\mathrm{d}y + \int_{\Omega^-} u\phi_0 \mathrm{d}x = \int_{\Omega^-} f\phi_0 \mathrm{d}x. \tag{4.13}$$

Put, $\phi_0 = 0$ in (4.13), we have

$$\int_{\Omega^- \times (0,1)^3} A(\nabla u + \nabla_y u_1) \cdot \nabla_y \phi_1 \mathrm{d}x\mathrm{d}y = 0.$$

This can be written as

$$\int_{\Omega^- \times (0,1)^3} A\nabla_y u_1 \cdot \nabla_y \phi_1 \mathrm{d}x\mathrm{d}y = -\int_{\Omega^- \times (0,1)^3} A\nabla u^- \cdot \nabla_y \phi_1 \mathrm{d}x\mathrm{d}y.$$

Since $\phi_1 \in C_c^\infty(\Omega; C_{per}^\infty((0,1)^3))$ was arbitrary, $u_1(x,y)$ satisfy the following PDE in y variable,

$$\begin{cases} -\mathrm{div}_y\left(A(x,y_1)\nabla_y u_1(x,y)\right) = \mathrm{div}_y(A(x,y_1)\nabla u^-(x)) & \text{in } (0,1)^3 \\ u_1(x,y) \text{ is } (0,1)^3-\text{periodic.} \end{cases} \tag{4.14}$$

Using the solution of the cell problems (4.3), we can write $u_1(x,y)$ in $L^2\left(\Omega; H_{per}^1((0,1)^3)/\mathbb{R}\right)$ as $u_1(x,y) = \displaystyle\sum_{i=1}^3 \frac{\partial u^-}{\partial x_i}\chi_i$. Now, take any $\phi \in C_c^\infty(\Omega^-)$ as a test function in (4.10) and pass to the limit as $\varepsilon \to 0$ to get,

$$\int_{\Omega^- \times (0,1)^3} A(x,y_1)\left(\nabla u^- + \sum_{i=1}^3 \frac{\partial u^-}{\partial x_i}\nabla_y \chi_i\right) \cdot \nabla\phi \mathrm{d}x\mathrm{d}y_1 + \int_{\Omega^-} u^-\phi \mathrm{d}x = \int_{\Omega^-} f\phi \mathrm{d}x.$$

Using the definition of χ, above equality becomes

$$\int_{\Omega^-}\left(\int_{(0,1)^3} A(x,y_1)(I + \nabla_y \chi)\mathrm{d}y\right)\nabla u^- \cdot \nabla\phi \mathrm{d}x + \int_{\Omega^-} u^-\phi \mathrm{d}x = \int_{\Omega^-} f\phi \mathrm{d}x.$$

Now applying the definition of A_0^- given in (4.4) in the above expression, we get

$$\int_{\Omega^-} A_0^- \nabla u^- \cdot \nabla\phi \mathrm{d}x + \int_{\Omega^-} u^-\phi \mathrm{d}x = \int_{\Omega^-} f\phi \mathrm{d}x.$$

Step 3: Now, we will show $u^+ = u^-$ on γ_c. We have $u_\varepsilon^+ = u_\varepsilon^-$ on γ_c^ε. Let $\phi \in C^\infty(\Omega^u)$ such that $\phi = 0$ on $\partial\Omega^u \backslash \gamma_c^u$. A simple integration by parts gives the following

$$\int_{\Omega^u} \mathcal{T}^\varepsilon \left(\frac{\partial u_\varepsilon^+}{\partial x_3} \right) \phi \mathrm{d}x \mathrm{d}y_1 = -\int_{\Omega^u} \mathcal{T}^\varepsilon u_\varepsilon^+ \frac{\partial \phi}{\partial x_3} \mathrm{d}x \mathrm{d}y_1 + \int_{\gamma_c^u} \mathcal{T}_b^\varepsilon (u_\varepsilon^-) \phi \sigma_3 + o(1),$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3, 0)$ is the downward unit normal on γ_c^u. By passing to limit $\varepsilon \to 0$ in the above identity using (3.3) and Lemma 3.5 to get,

$$\int_{\Omega^u} \frac{\partial u^+}{\partial x_3} \phi \mathrm{d}x \mathrm{d}y_1 + \int_{\Omega^u} u^+ \frac{\partial \phi}{\partial x_3} \mathrm{d}x \mathrm{d}y_1 = \int_{\gamma_c^u} u^- \sigma_3 \phi.$$

Combining the first two integrals and applying integration by parts, we have

$$\int_{\gamma_c^u} u^+ \sigma_3 \phi = \int_{\gamma_c^u} u^- \sigma_3 \phi, \quad \text{for all} \quad \phi \in C_c^\infty(\Omega) \text{ with } \phi = 0 \ on \ \partial\Omega^u \backslash \gamma_c^u.$$

Hence, we have

$$u^+ = u^- \text{ on } \gamma_c^u.$$

Since u^+ are u^- are independent of y_1, we have $u^+ = u^-$ on γ_c.

Step 4: Taking $\phi \in C^\infty(\Omega)$ as a test function in the weak formulation (2.2), we get,

$$\int_{\Omega_\varepsilon^+} \left(A^\varepsilon \nabla u_\varepsilon^+ \cdot \nabla \phi + u_\varepsilon^+ \phi \right) + \int_{\Omega^-} \left(A^\varepsilon \nabla u_\varepsilon^- \cdot \nabla \phi + u_\varepsilon^- \phi \right) = \int_{\Omega_\varepsilon^+} f \phi + \int_{\Omega^-} f \phi. \tag{4.15}$$

We have,

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^+} \left(A^\varepsilon \nabla u_\varepsilon^+ \cdot \nabla \phi + u_\varepsilon^+ \phi \right) \mathrm{d}x = \int_{\Omega^+} \left(A_0^+ \nabla_{2,3} u^+ \cdot \nabla_{2,3} \phi + \omega(x) u^+ \phi \right) \mathrm{d}x, \tag{4.16}$$

and

$$\lim_{\varepsilon \to 0} \int_{\Omega^-} A^\varepsilon \nabla u_\varepsilon^- \cdot \nabla \phi \mathrm{d}x + u_\varepsilon^- \phi \mathrm{d}x = \int_{\Omega^-} (A_0^- \nabla u^- \cdot \nabla \phi + u^- \phi) \mathrm{d}x. \tag{4.17}$$

Hence, if we pass to the limit $\varepsilon \to 0$ in (4.15), using the above convergences (4.16) and (4.17), we get,

$$\begin{aligned} \int_{\Omega^+} \left(A_0^+ \nabla_{2,3} u^+ \cdot \nabla_{2,3} \phi + \omega(x) u^+ \phi \right) \mathrm{d}x + \int_{\Omega^-} (A_0^- \nabla u^- \cdot \nabla \phi + u^- \phi) \mathrm{d}x \\ = \int_{\Omega^+} \omega(x) f \phi \mathrm{d}x + \int_{\Omega^-} f \phi \mathrm{d}x. \end{aligned} \tag{4.18}$$

Since $\phi \in C^\infty(\Omega)$ is arbitrary, hence by density, the above equality holds for all $\phi \in H(\Omega, \omega)$, and this is the weak formulation corresponding to (4.5). Hence u is a weak solution to (4.5). To get the uniqueness of $u \in H(\Omega, \omega)$, we will use Lax Miligram lemma. For that one has to verify the following bilinear form corresponding to the left hand side of the variational equality (4.18), $F : H(\Omega,\omega)\times H(\Omega,\omega)\to\mathbb{R}$ defined by

$$F(\phi,\psi)=\int_{\Omega^+}\left(A_0^+\nabla_{2,3}\phi^+\cdot\nabla_{2.3}\psi^+ +\omega(x)\phi^+\psi^+\right)+\int_{\Omega^-}\left(A_0^-\nabla\phi^-\cdot\nabla\psi^- +\phi^-\psi^-\right)$$

is continuous and elliptic. This will follow from the fact that, A_0^+ and A_0^- are uniformly elliptic and bounded. Ellipticity of A_0^- is a classical result. Let us prove the ellipticity of A_0^+: for $(\lambda_2,\lambda_3)\in\mathbb{R}^2$, we have

$$\begin{aligned}\alpha\left\|\left(-\frac{a_{12}}{a_{11}}\lambda_2-\frac{a_{13}}{a_{11}}\lambda_3,\lambda_2,\lambda_3\right)\right\|^2 &\leqslant A\begin{bmatrix}-\frac{a_{12}}{a_{11}}\lambda_2-\frac{a_{13}}{a_{11}}\lambda_3\\ \lambda_2\\ \lambda_3\end{bmatrix}\cdot\begin{bmatrix}-\frac{a_{12}}{a_{11}}\lambda_2-\frac{a_{13}}{a_{11}}\lambda_3\\ \lambda_2\\ \lambda_3\end{bmatrix}\\ &=\frac{1}{a_{11}}\begin{bmatrix}a_{11}a_{22}-a_{12}^2 & a_{23}a_{11}-a_{12}a_{13}\\ a_{11}a_{23}-a_{13}a_{12} & a_{11}a_{33}-a_{13}^2\end{bmatrix}\begin{bmatrix}\lambda_2\\ \lambda_3\end{bmatrix}\cdot\begin{bmatrix}\lambda_2\\ \lambda_3\end{bmatrix}.\end{aligned}\tag{4.19}$$

Now integrate both sides over the set $Y(x',x_3)$ with respect to y_1, we obtain

$$\alpha\omega(x)\|(\lambda_2,\lambda_3)\|^2\leqslant A_0^+\begin{bmatrix}\lambda_2\\ \lambda_3\end{bmatrix}\cdot\begin{bmatrix}\lambda_2\\ \lambda_3\end{bmatrix}.$$

Since $\omega(x)\geqslant\delta>0$, implies A_0^+ elliptic. Hence, this also shows the convergence of the full sequence u_ε. This completes the proof. $\square$

Remark 4.5. (1) Instead of taking the source term $f\in L^2(\Omega)$, one can take $f_\varepsilon\in L^2(\Omega_\varepsilon)$ whose zero extension $\tilde{f}_\varepsilon$ to the whole domain Ω, weakly converges to f in $L^2(\Omega)$.
(2) Note that in the above analysis, we could compute P_2 and P_3 directly as there were no oscillations in x_2 and x_3 directions. Then, we computed P_1 in terms of the computed values of P_2 and P_3 . If we have oscillations in more than one directions, say along x_1 and x_2 directions, then we will compute P_1 and P_2 in terms of the computed value of P_3. This result has shown in Section 7.

5. Homogenization of an optimal control problem

In this section, we consider an interior optimal control problem with general energy type cost functional in Ω_ε. Here we restrict ourselves to 3-dimensional case to make simpler presentation. But the results can be extend to any finite dimensional domain Ω_ε. Let $B^\varepsilon(x)=B\left(x,\frac{x_1}{\varepsilon}\right)=\left(b_{i,j}\left(x,\frac{x_1}{\varepsilon}\right)\right)_{i,j=1}^3$, be a family of 3×3 matrices and have the same properties as A^ε defined in Section 2.2 namely the uniform ellipticity and boundedness.

Notations: To write in a more conventional way, we use the following notations for this section: $B=B(x,y_1)$ and $B^\varepsilon=B\left(x,\dfrac{x_1}{\varepsilon}\right)$.
We also assume that A^ε and B^ε are symmetric. Let us consider the following cost functional

$$J_\varepsilon=J_\varepsilon(\theta_\varepsilon)=\frac{1}{2}\int_{\Omega_\varepsilon}B^\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon\mathrm{d}x+\frac{\beta}{2}\int_{\Omega_\varepsilon}|\theta_\varepsilon|^2\mathrm{d}x,\tag{5.1}$$

where $\beta>0$ is a fixed constant and given $\theta_\varepsilon\in L^2(\Omega_\varepsilon)$, the function $u_\varepsilon\in H^1(\Omega_\varepsilon)$ satisfies the following PDE,

$$\begin{cases}-\mathrm{div}(A^\varepsilon\nabla u_\varepsilon)+u_\varepsilon=f+\theta_\varepsilon & \text{in } \Omega_\varepsilon,\\ A^\varepsilon\nabla u_\varepsilon\cdot\nu_\varepsilon=0 & \text{on } \partial\Omega_\varepsilon.\end{cases}\tag{5.2}$$

Here $f \in L^2(\Omega_\varepsilon)$ is the source term and θ_ε is the control. The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon) : \ (u_\varepsilon, \theta_\varepsilon) \text{ obeys } (5.2)\}. \tag{5.3}$$

This type of problem was first considered in [29] in a fixed domain, in the case when the coefficient matrices A^ε and B^ε are periodic. In [27], in fixed domain, authors have considered the case when coefficient matrices A^ε and B^ε are H-convergent. In [28], this control problem with fixed A and B have been considered in perforated domain. For further study in this direction we suggest the reader to see [25, 26] and references therein. Here we consider the above optimal control problem (5.3) in an oscillating domain having curved interface. It is known that the equation (5.2) admits a unique solution $u_\varepsilon \in H^1(\Omega_\varepsilon)$ and also satisfies the following estimate,

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C(\|f\|_{L^2(\Omega_\varepsilon)} + \|\theta_\varepsilon\|_{L^2(\Omega_\varepsilon)}), \tag{5.4}$$

where C is a generic constant independent of ε. We also have the existence and uniqueness result for the optimal control problem (5.3) for each fixed $\varepsilon > 0$ (see [31]).

Theorem 5.1. *For each fixed $\varepsilon > 0$, the minimization problem (5.3) admits a unique solution $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$.*

The topic of discussion for this section is to analyze the asymptotic behavior of $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon))$ as the oscillating parameter $\varepsilon \to 0$. To achieve our goal, we will make use of the following characterization theorem.

Theorem 5.2. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (5.3), then the optimal control is characterized by*

$$\bar{\theta}_\varepsilon = -\frac{1}{\beta}\bar{v}_\varepsilon,$$

where the adjoint state $\bar{v}_\varepsilon$, satisfies the following adjoint PDE

$$\begin{cases} -div(A^\varepsilon \nabla \bar{v}_\varepsilon) + \bar{v}_\varepsilon = -div(B^\varepsilon \nabla \bar{u}_\varepsilon) \quad in \ \ \Omega_\varepsilon, \\ (A^\varepsilon \nabla \bar{v}_\varepsilon - B^\varepsilon \nabla \bar{u}_\varepsilon) \cdot \nu_\varepsilon = 0 \quad on \ \ \partial\Omega_\varepsilon. \end{cases} \tag{5.5}$$

Conversely, let $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ satisfies the following system

$$\begin{cases} -div(A^\varepsilon \nabla \bar{u}_\varepsilon) + \bar{u}_\varepsilon = f + \bar{\theta}_\varepsilon \quad in \ \ \Omega_\varepsilon, \\ -div(A^\varepsilon \nabla \bar{v}_\varepsilon) + \bar{v}_\varepsilon = -div(B^\varepsilon \nabla \bar{u}_\varepsilon) \quad in \ \ \Omega_\varepsilon, \\ A^\varepsilon \nabla \bar{u}_\varepsilon \cdot \nu_\varepsilon = 0, \ (A^\varepsilon \nabla \bar{v}_\varepsilon - B^\varepsilon \nabla \bar{u}_\varepsilon) \cdot \nu_\varepsilon = 0 \quad on \ \ \partial\Omega_\varepsilon. \\ \bar{\theta}_\varepsilon = -\dfrac{1}{\beta}\bar{v}_\varepsilon. \end{cases} \tag{5.6}$$

Then $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ is the optimal solution to the optimal control problem (5.3).

Proof. We present a quick proof for the sake of completion. Given $\theta_\varepsilon \in L^2(\Omega_\varepsilon)$, denote $F_\varepsilon(\theta_\varepsilon) = J_\varepsilon(u_\varepsilon(\theta_\varepsilon), \theta_\varepsilon)$ where $u_\varepsilon(\theta_\varepsilon)$ is the solution to (5.2). Evaluating the limit of

$$\frac{1}{\lambda}(F_\varepsilon(\bar{\theta}_\varepsilon + \lambda\theta_\varepsilon) - F_\varepsilon(\bar{\theta}_\varepsilon))$$

as $\lambda \to 0$ and denoting the limit by $F'(\bar{\theta}_\varepsilon)\theta_\varepsilon$, we get

$$F'_\varepsilon(\bar{\theta}_\varepsilon)\theta_\varepsilon = \int_{\Omega_\varepsilon} A^\varepsilon \nabla \bar{u}_\varepsilon \cdot \nabla w_{\theta_\varepsilon} \mathrm{d}x + \beta \int_{\Omega_\varepsilon} \bar{\theta}_\varepsilon \theta_\varepsilon \mathrm{d}x,$$

where w_{θ_ε} is the solution to the following PDE,

$$\begin{cases} -\mathrm{div}\,(A^\varepsilon \nabla w_{\theta_\varepsilon}) + w_{\theta_\varepsilon} = \theta_\varepsilon \ \text{ in } \ \Omega_\varepsilon, \\ A^\varepsilon \nabla w_{\theta_\varepsilon} \cdot \nu_\varepsilon = 0 \ \text{ on } \ \partial\Omega_\varepsilon. \end{cases} \tag{5.7}$$

As $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ is the optimal solution, we have

$$F'_\varepsilon(\bar{\theta}_\varepsilon)\theta_\varepsilon = 0, \quad \text{for all} \ \ \theta_\varepsilon \in L^2(\Omega_\varepsilon).$$

Hence, we get,

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla \bar{u}_\varepsilon \cdot \nabla w_{\theta_\varepsilon} \mathrm{d}x = -\beta \int_{\Omega_\varepsilon} \bar{\theta}_\varepsilon \theta_\varepsilon \mathrm{d}x. \tag{5.8}$$

Let $\bar{v}_\varepsilon$ satisfies (5.5). Using w_{θ_ε} as a test function in (5.5) and $\bar{v}_\varepsilon$ in (5.7), we obtain

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla \bar{u}_\varepsilon \cdot \nabla w_{\theta_\varepsilon} \mathrm{d}x = \int_{\Omega_\varepsilon} \bar{v}_\varepsilon \theta_\varepsilon \mathrm{d}x. \tag{5.9}$$

Hence from (5.8) and (5.9), we have

$$\bar{\theta}_\varepsilon = -\frac{1}{\beta}\bar{v}_\varepsilon.$$

This completes the proof of forward part.

Conversely, from the identity (5.9), Gateaux derivative of F_ε at $\theta_\varepsilon = -\frac{1}{\beta}\bar{v}_\varepsilon$ vanishes in all the direction. Since J is strictly convex, hence F_ε is strictly convex which implies that the pair $\left(\bar{u}_\varepsilon, -\frac{1}{\beta}\bar{v}_\varepsilon\right)$ is the optimal solution to the considered optimal control problem (5.3). □

5.1. Homogenized optimal control problem

To introduce the homogenized or limit problem, we need the following cell problems. For each $x \in \Omega^-$ and $i = 1, 2, 3$, let $\chi_i = \chi_i(y)$ and $N_i = N_i(y)$ solve the following system

$$\begin{cases} -\mathrm{div}_y(A(x, y_1)\nabla\chi_i) = \mathrm{div}_y(A(x, y_1)e_i) \ \text{ in } \ (0,1)^3, \\ -\mathrm{div}_y(A(x, y_1)\nabla N_i) = \mathrm{div}_y(B(x, y_1)\,(I + \nabla_y\chi)\,e_i) \ \text{ in } \ (0,1)^3, \\ \chi_i, \eta_i \text{ are } (0,1)^2 - \text{periodic } in \ y \text{ and } M(\chi_i), \ M(N_i) = 0. \end{cases} \tag{5.10}$$

Now define the matrices $B^\pm_\#$, for $x \in \Omega^\pm$:

$$B^-_\#(x) = \int_{(0,1)^3} (B(x, y_1)(I + \nabla_y\chi) + A(x, y_1)\nabla_y N)\mathrm{d}y, \qquad \text{for } x \in \Omega^-,$$

$$B^+_\#(x) = \int_{(0,1)} \mathcal{B}^+(x,y_1)\mathrm{d}y_1, \quad \text{for } x \in \Omega^-,$$

where

$$\mathcal{B}^+(x,y_1) = \frac{1}{a_{11}} \begin{bmatrix} b_{22}a_{11}^2 - 2a_{11}b_{12}a_{12} + a_{12}^2 b_{11} & b_{23}a_{11}^2 + a_{12}^2 b_{11} - b_{13}a_{12}a_{11} - a_{11}b_{12}a_{13}, \\ b_{23}a_{11}^2 + a_{12}^2 b_{11} - b_{13}a_{12}a_{11} - a_{11}b_{12}a_{13} & b_{33}a_{11}^2 - 2a_{11}b_{13}a_{13} + a_{13}^2 b_{11} \end{bmatrix},$$

$N = [N_1 \ N_2 \ N_3]$ and $\chi = [\chi_1 \ \chi_2 \ \chi_3]$. The state equation for the control $\theta \in L^2(\Omega)$ is given by

$$\begin{cases} -\mathrm{div}_{2,3}\left(A_0^+ \nabla_{2,3} u^+\right) + \omega(x)u^+ = \omega(x)f + \omega(x)\theta \ \text{ in } \ \Omega^+, \\ -\mathrm{div}(A_0^- \nabla u^-) + u^- = f + \theta \ \text{ in } \ \Omega^-, \\ \quad A_0^+ \nabla_{2,3} u^+ \cdot (\nu_2, \nu_3) = 0 \ \text{ on } \ \gamma_u, \\ \quad u^+ = u^-, \qquad A_0^+ \nabla_{2,3} u^+ \cdot (\sigma_2, \sigma_3) - A_0^- \nabla u^- \cdot \sigma = 0 \ \text{ on } \ \gamma_c, \\ A_0^- \nabla u^- \cdot \nu = 0 \ \text{ on } \ \partial\Omega^- \backslash \gamma_c, \end{cases} \tag{5.11}$$

where $A_0^\pm$, $\nabla_{2,3}u^+$ and $\mathrm{div}_{2,3}$ are defined as in Section 4. The limit optimal control problem is defined as follows: find $(\bar{u}, \bar{\theta}) \in H(\Omega, \omega) \times L^2(\Omega)$ such that

$$J(\bar{u}, \bar{\theta}) = \inf\{J(u,\theta) : \ (u,\theta) \in H(\Omega,\omega) \times L^2(\Omega) \text{ satisfies } (5.11)\}, \tag{5.12}$$

where the cost functional J is given by

$$J(u,\theta) = \frac{1}{2}\int_{\Omega^+} B^+_\# \nabla_{2,3}u^+ \cdot \nabla_{2,3}u^+ \mathrm{d}x + \frac{1}{2}\int_{\Omega^-} B^-_\# \nabla u^- \nabla u^- \mathrm{d}x + \frac{\beta}{2}\int_\Omega |\theta|^2 \mathrm{d}x.$$

Remark 5.3. It is quite intuitive that the limit problem will be in the full domain Ω which is the Hausdorff limit of Ω_ε^+. Indeed the limit problem is independent of ε defined in the full domain Ω. We expect that in the limit there will be interactions between A^ε and B^ε, namely the coefficient matrix of the equation and matrix in the cost functional and that is what we get as $B_\#$. In other words, the matrix $B_\#$ has the contribution both from the state and cost, whereas the coefficient matrix of the state equation should not be affected by coefficient matrix of the cost functional. This indeed is demonstrated by our limit problem and we get the limit coefficient matrix as A_0^+.

We have the symmetricity and the ellipticity of $B^+_\#$ and $B^-_\#$ (will prove it in the sequel of proof of Thm. 5.7). This gives the existence and uniqueness of the optimal solution to the optimal control problem (5.12). The characterization and optimality system is given by the following theorem:

Theorem 5.4. *(Existence and characterization) Let $(\bar{u}, \bar{\theta})$ be the optimal solution to the optimal control problem* (5.12)*, then the optimal control is given by*

$$\bar{\theta} = -\frac{1}{\beta}\bar{v},$$

where the adjoint state $\bar{v}$ satisfies the following adjoint PDE

$$\begin{cases} -div_{2,3}\left(A_0^+\nabla_{2,3}\bar{v}^+\right)+\omega(x)\bar{v}^+ = -div_{2,3}\left(B_{\#}^+\nabla_{2,3}\bar{u}^+\right) \quad in \quad \Omega^+, \\ -div(A_0^-\nabla\bar{v}^-)+\bar{v}^- = -div(B_{\#}^-\nabla\bar{u}^-) \quad in \quad \Omega^-, \\ \quad \left(A_0^+\nabla_{2,3}\bar{v}^+ - B_{\#}^+\nabla_{2,3}\bar{u}^+\right)\cdot(\nu_2,\nu_3)=0 \quad on \quad \gamma_u, \\ \quad \bar{v}^+=\bar{v}^-, \qquad \left(A_0^+\nabla_{2,3}\bar{v}^+ - B_{\#}^+\nabla_{2,3}\bar{u}^+\right)\cdot(\sigma_2,\sigma_3) - A_0^-\nabla\bar{v}^-\cdot\sigma = 0 \quad on \quad \gamma_c, \\ A_0^-\nabla\bar{v}\cdot\nu = 0 \quad on \quad \partial\Omega^-\backslash\gamma_c. \end{cases} \tag{5.13}$$

Conversely, assume that $(u,v)\in H(\Omega,\omega)\times H(\Omega,\omega)$ satisfies the following system,

$$\begin{cases} -div_{2,3}(A_0^+\nabla_{2,3}u^+)+\omega(x)u^+ = \omega(x)f^+ - \dfrac{1}{\beta}\omega(x',x_3)v^+ \quad in \quad \Omega^+, \\ -div_{2,3}(A_0^+\nabla_{2,3}v^+)+\omega(x)v^+ = -div_{2,3}(B_{\#}^+\nabla_{2,3}u^+) \quad in \quad \Omega^+, \\ -div(A_0^-\nabla u^-)+u^- = f^- - \dfrac{1}{\beta}v^- \quad in \quad \Omega^-, \\ -div(A_0^-\nabla v^-)+v^- = -div\left(B_{\#}^-\nabla u^-\right) \quad in \quad \Omega^-, \end{cases} \tag{5.14}$$

together with the boundary conditions

$$\begin{cases} \quad A_0^+\nabla_{2,3}u^+\cdot(\nu_2,\nu_3)=0, \qquad \left(A_0^+\nabla_{2,3}v^+ - B_{\#}^+\nabla_{2,3}u^+\right)\cdot(\nu_2,\nu_3)=0 \quad on \quad \gamma_u, \\ A_0^-\nabla u^-\cdot\nu=0, \ (A_0^-\nabla v^-\cdot\nu - B_{\#}^-\nabla u^-\cdot\nu)=0 \quad on \quad \partial\Omega^-\backslash\gamma_c, \end{cases}$$

and the interface conditions

$$\begin{cases} u^+=u^-, \ A_0^+\nabla_{2,3}u^+\cdot(\sigma_2,\sigma_3) - A_0^-\nabla u^-\cdot\sigma = 0 \quad on \quad \gamma_c, \\ v^+=v^-, \ \left(A_0^+\nabla_{2,3}v^+ - B_{\#}^+\nabla_{2,3}u^+\right)\cdot(\sigma_2,\sigma_3) - (A_0^-\nabla v^- - B_{\#}^-\nabla u^-)\cdot\sigma = 0 \quad on \quad \gamma_c. \end{cases}$$

Then the pair $\left(u,-\dfrac{1}{\beta}v\right)$ is the optimal solution to the optimal control problem (5.12).

We now present the convergence analysis.

5.2. Convergence analysis

Let $(\bar{u}_\varepsilon,\bar{\theta}_\varepsilon)$ be the optimal solution to the problem (5.3). Let $u_\varepsilon(0)$ be the solution to the problem (5.2) corresponding to $\theta_\varepsilon=0$. Then, from (5.4) we get, $\|u_\varepsilon(0)\|_{H^1(\Omega_\varepsilon)}\leqslant C$. Now using the optimality of the solution $(\bar{u}_\varepsilon,\bar{\theta}_\varepsilon)$, we have

$$J_\varepsilon(\bar{u}_\varepsilon,\bar{\theta}_\varepsilon)\leqslant J_\varepsilon(u_\varepsilon(0),0).$$

This implies the following inequality

$$\frac{1}{2}\int_{\Omega_\varepsilon}B^\varepsilon\nabla\bar{u}_\varepsilon\cdot\nabla\bar{u}_\varepsilon\mathrm{d}x+\frac{\beta}{2}\int_{\Omega_\varepsilon}|\bar{\theta}_\varepsilon|^2\mathrm{d}x\leqslant C.$$

Since B is uniformly elliptic, it follows that

$$\|\bar{\theta}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant C \text{ and } \|\nabla \bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant C. \tag{5.15}$$

The uniform bound on $\|\bar{\theta}_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ gives the uniform bound on $\|\bar{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ as $\bar{u}_\varepsilon$ satisfies (5.2) and uniform bound of $\|\bar{v}_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ follows from (5.5).

We also have the similar type of convergence as Propositions (4.2) and (4.3).

Let for each $\varepsilon > 0$, $\bar{u}_\varepsilon, \bar{v}_\varepsilon$ be optimal state and corresponding adjoint state respectively. Let us define two sequences using the unfolded sequence $\mathcal{T}^\varepsilon \bar{U}_\varepsilon$ and $\mathcal{T}^\varepsilon \bar{V}_\varepsilon$ as

$$\begin{aligned} \mathcal{T}^\varepsilon \bar{U}_\varepsilon(x', y_1, x_3) &= \frac{1}{\varepsilon}\left(\mathcal{T}^\varepsilon \bar{u}_\varepsilon - \frac{1}{|Y(x', x_3|)} \int_{Y(x',x_3)} \mathcal{T}^\varepsilon \bar{u}_\varepsilon(x_1, y_1, x_2, x_3) \mathrm{d}y_1\right) \\ \mathcal{T}^\varepsilon \bar{V}_\varepsilon(x', y_1, x_3) &= \frac{1}{\varepsilon}\left(\mathcal{T}^\varepsilon \bar{v}_\varepsilon - \frac{1}{|Y(x', x_3|)} \int_{Y(x',x_3)} \mathcal{T}^\varepsilon \bar{v}_\varepsilon(x_1, y_1, x_2, x_3) \mathrm{d}y_1\right). \end{aligned} \tag{5.16}$$

Proposition 5.5. *The sequences defined in (5.16) is uniformly bounded in $L^2(\Omega^u)$.*

As $\|\mathcal{T}^\varepsilon \bar{U}_\varepsilon\|_{L^2(\Omega^u)}, \|\mathcal{T}^\varepsilon \bar{U}_\varepsilon\|_{L^2(\Omega^u)} \leqslant C$, by compactness there exist $\bar{U}_1, \bar{V}_1 \in L^2(\Omega^u)$ such that

$$\mathcal{T}^\varepsilon \bar{U}_\varepsilon \rightharpoonup \bar{U}_1, \text{ and } \mathcal{T}^\varepsilon \bar{V}_\varepsilon \rightharpoonup \bar{V}_1 \text{ in } L^2(\Omega^u).$$

Then the following proposition holds.

Proposition 5.6. *Let for each $\varepsilon > 0$, $\bar{u}_\varepsilon$ and $\bar{v}_\varepsilon$ are as described earlier. Let*

$$\mathcal{T}^\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial x_1} \rightharpoonup p_1, \text{ and } \mathcal{T}^\varepsilon \frac{\partial \bar{v}_\varepsilon}{\partial x_1} \rightharpoonup q_1 \text{ weakly in } L^2(\Omega^u).$$

Then $p_1 = \dfrac{\partial \bar{U}_1}{\partial y_1}$ and $q_1 = \dfrac{\partial \bar{V}_1}{\partial y_1}$.

In the proof of Theorem 5.7 to be given below. We will write explicitly p_1, q_1 in terms of $a_{i,j}, b_{i,j}$, x_2 and x_3 directional derivative of limit optimal state and adjoint state.

The following theorem gives us the full convergence of optimal state, control and adjoint state corresponding to the optimal control problem (5.3). We recall that the notation $\sim$ is used to denote the extended function from Ω_ε^+ to Ω^+ by 0. Recall that $\omega(x)$ is defined on the upper part Ω^+ as in subsection 3.3 and let $\omega(x) = 1$ in Ω^-.

Theorem 5.7. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ and $(\bar{u}, \bar{\theta})$ be the optimal solution to (5.3) and (5.12) respectively. Also, let $\bar{v}_\varepsilon$ and $\bar{v}$ are the adjoint states corresponding to the optimal control problem (5.3) and (5.12) respectively. Then*

$$\begin{aligned} &\widetilde{u_\varepsilon^+} \rightharpoonup \omega(x)\bar{u}^+, \ \widetilde{\frac{\partial u_\varepsilon^+}{\partial x_i}} \rightharpoonup \omega(x)\frac{\partial \bar{u}^+}{\partial x_i} \quad \text{weakly in} \quad L^2(\Omega^+), \text{ for } i = 2,3 \\ &\widetilde{v_\varepsilon^+} \rightharpoonup \omega(x)\bar{v}^+, \ \widetilde{\frac{\partial v_\varepsilon^+}{\partial x_i}} \rightharpoonup \omega(x)\frac{\partial \bar{v}^+}{\partial x_i} \quad \text{weakly in} \quad L^2(\Omega^+), \text{ for } i = 2,3 \\ &\bar{u}_\varepsilon^- \rightharpoonup \bar{u}^-, \ \bar{v}_\varepsilon^- \rightharpoonup \bar{v}^- \text{ weakly in } H^1(\Omega^-), \\ &\widetilde{\bar{\theta}}_\varepsilon \rightharpoonup -\omega(x)\frac{1}{\beta}\bar{v}, \quad \text{weakly in} \quad L^2(\Omega). \end{aligned} \tag{5.17}$$

Proof. The proof will be accomplished in several steps by homogenizing the system (5.6). We will show that the homogenized system corresponding to (5.6) is the system (5.14). To do this, the system (5.6) will be divided into two parts, namely Ω_ε^+ and Ω^-. Then analyze the asymptotic behavior in these two parts separately. These are connected *via* interface conditions.

Step 1: Using the uniform bound on $\|\bar{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ and $\|\bar{v}_\varepsilon\|_{H^1(\Omega_\varepsilon)}$, by Proposition 4.1, we have

$$\begin{aligned} &(i)\ \|\mathcal{T}^\varepsilon \bar{u}_\varepsilon^+\|_{L^2(\Omega^u)} \leqslant C, \quad \|\mathcal{T}^\varepsilon \nabla \bar{u}_\varepsilon^+\|_{L^2(\Omega^u)} \leqslant C \\ &(ii)\ \|\mathcal{T}^\varepsilon \bar{v}_\varepsilon^+\|_{L^2(\Omega^u)} \leqslant C, \quad \|\mathcal{T}^\varepsilon \nabla \bar{v}_\varepsilon^+\|_{L^2(\Omega^u)} \leqslant C. \end{aligned} \tag{5.18}$$

As in the proof of the Theorem 4.4, we get $u^+, p_1 \in L^2(\Omega^u)$ such that

$$\begin{aligned} &(i)\ \mathcal{T}^\varepsilon \bar{u}_\varepsilon^+ \rightharpoonup u^+ \text{ weakly in } L^2(\Omega^u) \\ &(ii)\ \mathcal{T}^\varepsilon \left(\frac{\partial \bar{u}_\varepsilon^+}{\partial x_1}\right) \rightharpoonup p_1 \text{ weakly in } L^2(\Omega^u) \\ &(iii)\ \mathcal{T}^\varepsilon \left(\frac{\partial \bar{u}_\varepsilon^+}{\partial x_i}\right) \rightharpoonup \frac{\partial u^+}{\partial x_i} \text{ weakly in } L^2(\Omega^u),\ \text{for } i = 2,3, \\ &(iv)\ p_1 = -\frac{a_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2} - \frac{a_{13}}{a_{11}}\frac{\partial u^+}{\partial x_3}. \end{aligned} \tag{5.19}$$

Further, we see that u^+ is independent of y_1. Convergence takes place along a subsequence. Similarly, there exist $v^+, q_1 \in L^2(\Omega^u)$ such that

$$\begin{aligned} &(i)\ \mathcal{T}^\varepsilon \bar{v}_\varepsilon^+ \rightharpoonup v^+ \text{ weakly in } L^2(\Omega^u) \\ &(ii)\ \mathcal{T}^\varepsilon \left(\frac{\partial \bar{v}_\varepsilon^+}{\partial x_1}\right) \rightharpoonup q_1 \text{ weakly in } L^2(\Omega^u) \\ &(iii)\ \mathcal{T}^\varepsilon \left(\frac{\partial \bar{v}_\varepsilon^+}{\partial x_i}\right) \rightharpoonup \frac{\partial v^+}{\partial x_i} \text{ weakly in } L^2(\Omega^u),\ \text{for } i = 2,3, \end{aligned} \tag{5.20}$$

and v^+ is independent of y_1.

Step 2: We now compute q_1. To do so, we use oscillating test functions of the form $\phi^\varepsilon(x) = \varepsilon\phi(x)\psi\left(\left\{\frac{x_1}{\varepsilon}\right\}\right)$, where $\phi \in C_c^\infty(\Omega^+)$ and $\psi \in C_{per}^\infty((0,1))$ in the weak formulation of (5.5), to obtain

$$\int_{\Omega_\varepsilon^+} (A^\varepsilon \nabla \bar{v}_\varepsilon \nabla \phi^\varepsilon + \bar{v}_\varepsilon \phi^\varepsilon)\mathrm{d}x = \int_{\Omega_\varepsilon^+} B^\varepsilon \nabla \bar{u}_\varepsilon \nabla \phi^\varepsilon \mathrm{d}x. \tag{5.21}$$

As in the proof of Theorem 4.4, applying unfolding operator for the upper part on both sides and letting $\varepsilon \to 0$ to obtain

$$\int_{\Omega^u} A \begin{bmatrix} q_1 \\ \\ \nabla_{2,3} v^+ \end{bmatrix} \cdot \begin{bmatrix} \phi\psi'(y_1) \\ 0 \\ 0 \end{bmatrix} \mathrm{d}x\mathrm{d}y_1 = \int_{\Omega^u} B \begin{bmatrix} p_1 \\ \\ \nabla_{2,3} u^+ \end{bmatrix} \cdot \begin{bmatrix} \phi\psi'(y_1) \\ 0 \\ 0 \end{bmatrix} \mathrm{d}x\mathrm{d}y_1.$$

By simple matrix multiplication on the above equality gives,

$$\int_{\Omega^u} \left(a_{11}q_1 + a_{12}\frac{\partial v^+}{\partial x_2} + a_{13}\frac{\partial v^+}{\partial x_3}\right)\phi\frac{\mathrm{d}\psi}{\mathrm{d}y_1}\mathrm{d}x\mathrm{d}y_1 = \int_{\Omega^u} \left(b_{11}p_1 + b_{12}\frac{\partial u^+}{\partial x_2} + b_{13}\frac{\partial u^+}{\partial x_3}\right)\phi\frac{\mathrm{d}\psi}{\mathrm{d}y_1}\mathrm{d}x\mathrm{d}y_1.$$

Since ϕ and ψ are arbitrary, the above equality implies

$$a_{11}q_1 + a_{12}\frac{\partial v^+}{\partial x_2} + a_{13}\frac{\partial v^+}{\partial x_3} = b_{11}p_1 + b_{12}\frac{\partial u^+}{\partial x_2} + b_{13}\frac{\partial u^+}{\partial x_3} \qquad a.e \ in \ \Omega^u.$$

The uniform ellipticity of A, implies $a_{11} > \alpha$ (elliptic constant). Hence, we have

$$q_1 = -\frac{a_{12}}{a_{11}}\frac{\partial v^+}{\partial x_2} - \frac{a_{13}}{a_{11}}\frac{\partial v^+}{\partial x_3} + \frac{b_{11}}{a_{11}}p_1 + \frac{b_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2} + \frac{b_{13}}{a_{11}}\frac{\partial u^+}{\partial x_3} \qquad a.e \ in \ \Omega^u. \tag{5.22}$$

Let us denote

$$\tilde{p}_1 = \frac{b_{11}}{a_{11}}p_1 + \frac{b_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2} + \frac{b_{13}}{a_{11}}\frac{\partial u^+}{\partial x_3}. \tag{5.23}$$

Then $q_1 = -\frac{a_{12}}{a_{11}}\frac{\partial v^+}{\partial x_2} - \frac{a_{13}}{a_{11}}\frac{\partial v^+}{\partial x_3} + \tilde{p}_1$.
Take $\phi^+ \in C_c^\infty(\Omega^+)$ as a test function in (5.5), and applying unfolding operator on both sides and let $\varepsilon \to 0$. As in the proof of Theorem 4.4, we obtain

$$\int_{\Omega^u}\left(A\begin{bmatrix} q_1 \\ \\ \nabla_{2,3}v^+\end{bmatrix}\cdot\nabla\phi^+ + v^+\phi^+\right)\mathrm{d}x\mathrm{d}y_1 = \int_{\Omega^u} B\begin{bmatrix} p_1 \\ \\ \nabla_{2,3}u^+\end{bmatrix}\cdot\nabla\phi^+\mathrm{d}x\mathrm{d}y_1. \tag{5.24}$$

Use (5.19)(iv) in the above equation to get

$$\begin{aligned}
&\int_{\Omega^u}\Bigg[\left(-a_{12}\frac{\partial v^+}{\partial x_2} - a_{13}\frac{\partial v^+}{\partial x_3} + a_{11}\tilde{p}_1 + a_{12}\frac{\partial v^+}{\partial x_2} + a_{13}\frac{\partial v^+}{\partial x_3}\right)\frac{\partial \phi^+}{\partial x_1} \\
&\quad + \left(-\frac{a_{12}^2}{a_{11}}\frac{\partial v^+}{\partial x_2} - \frac{a_{12}a_{13}}{a_{11}}\frac{\partial v^+}{\partial x_3} + a_{12}\tilde{p}_1 + a_{22}\frac{\partial v^+}{\partial x_2} + a_{23}\frac{\partial v^+}{\partial x_3}\right)\frac{\partial \phi^+}{\partial x_2} \\
&\quad + \left(-\frac{a_{12}a_{13}}{a_{11}}\frac{\partial v^+}{\partial x_2} - \frac{a_{13}^2}{a_{11}}\frac{\partial v^+}{\partial x_3} + a_{13}\tilde{p}_1 + a_{23}\frac{\partial v^+}{\partial x_2} + a_{33}\frac{\partial v^+}{\partial x_3}\right)\frac{\partial \phi^+}{\partial x_3} + v^+\phi^+\Bigg]\mathrm{d}x\mathrm{d}y_1 \\
&= \int_{\Omega^u}\Bigg[\left(b_{11}p_1 + b_{12}\frac{\partial u^+}{\partial x_2} + b_{13}\frac{\partial u^+}{\partial x_3}\right)\frac{\partial \phi^+}{\partial x_1} + \left(b_{12}p_1 + b_{22}\frac{\partial u^+}{\partial x_2} + b_{23}\frac{\partial u^+}{\partial x_2}\right)\frac{\partial \phi^+}{\partial x_2} \\
&\quad + \left(b_{13}p_1 + b_{23}\frac{\partial u^+}{\partial x_2} + b_{33}\frac{\partial u^+}{\partial x_3}\right)\frac{\partial \phi^+}{\partial x_3}\Bigg]\mathrm{d}x\mathrm{d}y_1.
\end{aligned} \tag{5.25}$$

The first term of the left hand side integral in (5.25) vanishes due to (5.22). Thus, we have the following identity

$$\begin{aligned}
&\int_{\Omega^u}\Bigg[\left(-\frac{a_{12}^2}{a_{11}}\frac{\partial v^+}{\partial x_2} - \frac{a_{12}a_{13}}{a_{11}}\frac{\partial v^+}{\partial x_2} + a_{22}\frac{\partial v^+}{\partial x_2} + a_{23}\frac{\partial v^+}{\partial x_3}\right)\frac{\partial \phi^+}{\partial x_2} \\
&\quad + \left(-\frac{a_{12}a_{13}}{a_{11}}\frac{\partial v^+}{\partial x_2} - \frac{a_{13}^2}{a_{11}}\frac{\partial v^+}{\partial x_2} + a_{23}\frac{\partial v^+}{\partial x_2} + a_{33}\frac{\partial v^+}{\partial x_3}\right)\frac{\partial \phi^+}{\partial x_3} + v^+\phi^+\Bigg]\mathrm{d}x\mathrm{d}y_1 \\
&= \int_{\Omega^u}\Bigg[\left(b_{12}p_1 + b_{22}\frac{\partial u^+}{\partial x_2} + b_{23}\frac{\partial u^+}{\partial x_2}\right)\frac{\partial \phi^+}{\partial x_2} + \left(b_{13}p_1 + b_{23}\frac{\partial u^+}{\partial x_2} + b_{33}\frac{\partial u^+}{\partial x_3}\right)\frac{\partial \phi^+}{\partial x_3} \\
&\quad - a_{12}\tilde{p}_1\frac{\partial \phi^+}{\partial x_2} - a_{13}\tilde{p}_1\frac{\partial \phi^+}{\partial x_3}\Bigg]\mathrm{d}x\mathrm{d}y_1.
\end{aligned}$$

Substituting the expression for p_1 given in (5.19) and $\tilde{p_1}$ given in (5.23) in the above equality, we get the following

$$\int_{\Omega+} \left(A_0^+ \nabla_{2,3} v^+ \cdot \nabla_{2,3} \phi^+ + \omega(x) v^+ \phi^+\right) \mathrm{d}x = \int_{\Omega+} B_\#^+ \nabla_{2,3} u^+ \cdot \nabla_{2,3} \phi^+ \mathrm{d}x. \tag{5.26}$$

Now, for the state equation, choosing $\psi^+ \in C_c^\infty(\Omega^+)$, we can pass to the limit in the following equation

$$\int_{\Omega_\varepsilon^+} (A^\varepsilon \nabla \bar{u}_\varepsilon^+ \cdot \nabla \psi^+ + \bar{u}_\varepsilon^+ \psi^+) \mathrm{d}x = \int_{\Omega_\varepsilon^+} (f + \bar{\theta}_\varepsilon) \psi^+ \mathrm{d}x.$$

By the characterization of the optimal control, above equality can be written as

$$\int_{\Omega_\varepsilon^+} (A^\varepsilon \nabla \bar{u}_\varepsilon^+ \cdot \nabla \psi^+ + \bar{u}_\varepsilon^+ \psi^+) \mathrm{d}x = \int_{\Omega_\varepsilon^+} (f - \frac{1}{\beta} \bar{v}_\varepsilon) \psi^+ \mathrm{d}x. \tag{5.27}$$

As in the proof of Theorem 4.4, applying unfolding operator for the upper part on both sides and letting $\varepsilon \to 0$, we obtain

$$\int_{\Omega+} \left(A_0^+ \nabla_{2,3} v^+ \cdot \nabla_{2,3} \psi^+ + \omega(x) v^+ \psi^+\right) \mathrm{d}x = \int_{\Omega+} \omega(x) \left(f - \frac{1}{\beta} v^+\right) \psi^+ \mathrm{d}x. \tag{5.28}$$

Now we consider the state equation in the lower part: for any $\psi^- \in C_c^\infty(\Omega^-)$, we have

$$\int_{\Omega^-} (A^\varepsilon \nabla \bar{u}_\varepsilon^- \cdot \nabla \psi^- + \bar{u}_\varepsilon^- \psi^-) \mathrm{d}x = \int_{\Omega^-} (f + \bar{\theta}_\varepsilon) \psi^- \mathrm{d}x.$$

By the characterization of the optimal control, above equation can be written as

$$\int_{\Omega^-} (A^\varepsilon \nabla \bar{u}_\varepsilon^- \cdot \nabla \psi^- + \bar{u}_\varepsilon^- \psi^-) \mathrm{d}x = \int_{\Omega^-} \left(f - \frac{1}{\beta} \bar{v}_\varepsilon^-\right) \psi^- \mathrm{d}x. \tag{5.29}$$

Since $\|\bar{u}_\varepsilon^-\|_{H^1(\Omega^-)} \leqslant C$ and $\|\bar{v}_\varepsilon^-\|_{H^1(\Omega^-)} \leqslant C$, there exist $u^-, v^- \in H^1(\Omega^-)$ such that

$$T_\varepsilon \bar{u}_\varepsilon^- \to u^-, \text{ and } T_\varepsilon \bar{v}_\varepsilon^- \to v^-, \text{ strongly in } L^2(\Omega^- \times (0,1)^3). \tag{5.30}$$

Moreover, by compactness Theorem 3.2, there exist $u_1, v_1 \in L^2(\Omega; H^1_{per}((0,1)^3))$ such that

$$T_\varepsilon(\nabla \bar{u}_\varepsilon^-) \rightharpoonup \nabla u^- + \nabla_y u_1, \text{ and } T_\varepsilon(\nabla \bar{v}_\varepsilon) \rightharpoonup \nabla v^- + \nabla_y v_1, \text{ weakly in } L^2(\Omega \times (0,1)^3). \tag{5.31}$$

Similar to the proof of Theorem 4.4, applying unfolding for the fixed lower part on both sides of (5.29) and letting $\varepsilon \to 0$, we obtain the following

$$\int_{\Omega^-} (A_0^- \nabla u^- \cdot \nabla \psi^- + u^- \psi^-) \mathrm{d}x = \int_{\Omega^-} \left(f - \frac{1}{\beta} v^-\right) \psi^- \mathrm{d}x. \tag{5.32}$$

Let us consider the adjoint equation in the lower part: For any $\phi^- \in C_c^\infty(\Omega^-)$ we have

$$\int_{\Omega^-} (A^\varepsilon \nabla \bar{v}_\varepsilon^- \cdot \nabla \phi^- + v_\varepsilon^- \phi^-) \mathrm{d}x = \int_{\Omega^-} B^\varepsilon \nabla \bar{u}_\varepsilon^- \cdot \nabla \phi^- \mathrm{d}x. \tag{5.33}$$

Take $\phi^\varepsilon = \phi_0(x) + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)$, where $\phi_0 \in C_c^\infty(\Omega^-)$ and $\phi_1 \in C_c^\infty(\Omega; C_{per}^\infty((0,1)^3))$, as a test function in (5.33), we get

$$\int_{\Omega^-} (A^\varepsilon \nabla \bar{v}_\varepsilon^- \cdot \nabla\phi^\varepsilon + v_\varepsilon^- \phi^\varepsilon)\mathrm{d}x = \int_{\Omega^-} B^\varepsilon \nabla \bar{u}_\varepsilon^- \cdot \nabla \phi^\varepsilon \mathrm{d}x. \tag{5.34}$$

Apply unfolding operator on both sides of (5.34) and letting $\varepsilon \to 0$, by (5.31)(ii), we obtain

$$\begin{aligned}\int_{\Omega^- \times (0,1)^3} &(A(\nabla v^- + \nabla v_1^-) \cdot (\nabla\phi_0 + \nabla_y \phi_1) + v^-\phi_0)\mathrm{d}x\mathrm{d}y \\ &= \int_{\Omega^- \times (0,1)^3} B(x, y_1)(\nabla u^- + \nabla_y u_1) \cdot (\nabla \phi_0 + \nabla_y \phi_1)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{5.35}$$

Put, $\phi_0 = 0$ in (5.35) to get

$$\int_{\Omega^- \times (0,1)^3} A(\nabla v^- + \nabla v_1^-) \cdot \nabla_y \phi_1 \mathrm{d}x\mathrm{d}y = \int_{\Omega^- \times (0,1)^3} B(x, y_1)(\nabla u^- + \nabla_y u_1) \cdot \nabla_y \phi_1 \mathrm{d}x\mathrm{d}y.$$

In the same way as in the proof of Theorem 4.4, we have $u_1(x, y) = \sum_{i=1}^{3} \frac{\partial u^-}{\partial x_i}\chi_i$. Hence, above equality reduces to

$$\int_{\Omega^- \times (0,1)^3} A\nabla v_1^- \cdot \nabla_y\phi_1 \mathrm{d}x\mathrm{d}y = \int_{\Omega^- \times (0,1)^3} (B(x, y_1)(I + \nabla_y\chi)\nabla u^- \cdot \nabla_y \phi_1 - A\nabla v^- \cdot \nabla_y\phi_1)\mathrm{d}x\mathrm{d}y_1.$$

Since $\phi_1 \in C_c^\infty(\Omega; C_{per}^\infty((0,1)^3))$ is arbitrary, for each $x \in \Omega^-$, $v_1(x, y)$ satisfies the following PDE in y variable,

$$\begin{cases} -\mathrm{div}_y\,(A(x, y_1)\nabla_y v_1(x, y)) = -\mathrm{div}_y(B(x, y_1)(I + \nabla_y\chi)\nabla u^- - A\nabla v^-) \text{ in } (0,1)^3 \\ v_1(x, y) \text{ is } (0,1)^3 - \text{periodic.} \end{cases} \tag{5.36}$$

Using the cell problem (5.10), we can write $v_1(x, y) = \sum_{i=1}^{3}\left(\frac{\partial v^-}{\partial x_i}\chi_i - \frac{\partial u^-}{\partial x_i}N_i\right)$.

Now put $\phi_1 = 0$ in (5.35), definition of A_0^- and $B_\#^-$ to produce the homogenized equation for v^-

$$\int_{\Omega^-} (A_0^- \nabla v^- \cdot \nabla\phi_0 + v^-\phi_0)\mathrm{d}x = \int_{\Omega^-} B_\#^- \nabla u^- \cdot \nabla\phi_0 \mathrm{d}x. \tag{5.37}$$

Now for $(\psi, \phi) \in C^\infty(\Omega) \times C^\infty(\Omega)$, we can pass to the limit $\varepsilon \to 0$ in the following system

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon} (A^\varepsilon \nabla\bar{u}_\varepsilon \cdot \nabla\psi + \bar{u}_\varepsilon \psi)\mathrm{d}x = \int_{\Omega_\varepsilon} (f + \bar{\theta}_\varepsilon)\psi \mathrm{d}x, \\ \displaystyle\int_{\Omega_\varepsilon} (A^\varepsilon \nabla\bar{v}_\varepsilon \cdot \nabla\phi + \bar{v}_\varepsilon\phi)\mathrm{d}x = \int_{\Omega_\varepsilon} B^\varepsilon \nabla \bar{v}_\varepsilon \cdot \nabla\phi \mathrm{d}x, \end{cases}$$

and the limit is

$$\begin{cases} \int_{\Omega^+} \left(A_0^+\nabla_{2,3}u^+ \cdot \nabla_{2,3}\psi + \omega(x)u^+\psi\right) \mathrm{d}x + \int_{\Omega^-} (A_0^-\nabla v^- \cdot \nabla\psi + u^-\psi)\mathrm{d}x \\ \qquad = \int_{\Omega+} \omega(x)\left(f - \frac{1}{\beta}v^+\right)\psi\mathrm{d}x + \int_{\Omega+}\left(f - \frac{1}{\beta}v^+\right)\psi\mathrm{d}x, \end{cases}$$
$$\begin{cases} \int_{\Omega^+} \left(A_0^+\nabla_{2,3}v^+ \cdot \nabla_{2,3}\phi + \omega(x)v^+\phi\right) \mathrm{d}x + \int_{\Omega^-} (A_0^-\nabla v^- \cdot \nabla\phi + v^-\phi)\mathrm{d}x \\ \qquad = \int_{\Omega+} B_\#^+\nabla_{2,3}u^+ \cdot \nabla_{2,3}\phi\mathrm{d}x + \int_{\Omega^-} B_\#^-\nabla u^- \cdot \nabla\phi\mathrm{d}x. \end{cases} \tag{5.38}$$

Following the same path as in the proof of Theorem 4.4, we can show $u, v \in H(\Omega, \omega)$ by showing $u^+ = u^-$ and $v^+ = v^-$ on γ_c. The system (5.38) holds for all $(\psi, \phi) \in C^\infty(\Omega) \times C^\infty(\Omega)$ and hence it is true for all $(\psi, \phi) \in H(\Omega, \omega) \times H(\Omega, \omega)$. This system is the weak formulation for the limit optimality system (5.14). The system (5.38) has a unique solution if we have $A_0^+, A_0^-, B_\#^+, B_\#^-$ are elliptic. The matrix A_0^- is elliptic and symmetric is a classical result. For ellipticity and symmetricity of $B_\#^-$, we refer to ([27], Thm. 4.3). Ellipticity of A_0^+ has already proved in Section 4. If we replace A by B in (4.19), we get the ellipticity of $B_\#^+$. Hence this shows that (u, v) is the unique weak solution to the limit system (5.14) and hence convergence of the whole sequence follows. Thus, by uniqueness, we have $\bar{u} = u$ and $\bar{v} = v$. □

6. k DIRECTIONAL OSCILLATING DOMIAN

The definition of unfolding operator defined in Section 3 can be generalized to n-dimensional domain with any k ($k < n$) directional oscillating domain. For example, pillar-type oscillations in 2-directions in a 3 dimensional oscillating domain. To present in a simpler way, we consider $n = 3$ and $k = 2$. Here the oscillating domain has 2 directional oscillations and the coefficient matrices under consideration have oscillations in 2−directions (x_1 and x_2). We present the details when $n = 3$ and $k = 2$ and this can be extended without any additional difficulty to any n.

6.1. Domain description

Let $h, g : (0,1)^2 \to \mathbb{R}$ be Lipschitz real valued functions such that $0 < g(x') < h(x')$ for all $x' \in (0,1)^2$ and $\eta : \mathbb{R}^2 \to \mathbb{R}$ be 1-periodic continuous real valued function having the following properties:

(1) there exist $0 < a < b < 1$, *supp* $\left(\eta|_{(0,1)^2}\right) = [a,b]^2$ and $\eta \geqslant 0$,
(2) the supremum $M = \sup\{\eta(x') : x' \in (0,1)^2\}$ is strictly positive and the Lebesgue measure $|\{x' \in (0,1)^2 : \eta(x') = M)\}| \geq \delta > 0$,
(3) for each $(x', x_3) \in \Omega^+ = \{(x', x_3) : x' \in (0,1)^2, h(x') < x_3 < h(x') + M\}$, $Y(x', x_3) = \{y' \in (0,1)^2 : h(x') + \eta(y') > x_3\}$ is connected.

Here 1−periodicity of η means $\eta(x' + (i,j)) = \eta(x')$ for all $x' \in \mathbb{R}^2$ and for all $(i,j) \in \mathbb{Z}^2$. For $x' \in (0,1)^2$, we define $\eta^\varepsilon(x') = \eta\left(\frac{x'}{\varepsilon}\right)$. The oscillating domain Ω_ε is defined as

$$\Omega_\varepsilon = \left\{(x', x_3) : x' \in (0,1)^2,\ g(x') < x_3 < h(x') + \eta^\varepsilon(x')\right\}.$$

We denote the boundary of Ω_ε by $\partial\Omega_\varepsilon$. The oscillating part of the domain is denoted by Ω_ε^+, that is

$$\Omega_\varepsilon^+ = \left\{(x', x_3) : x' \in (0,1)^2,\ h(x') < x_3 < h(x') + \eta^\varepsilon(x')\right\}$$

and the lower fixed part Ω^- is given by

$$\Omega^- = \{(x', x_3) : \ x' \in (0,1)^2, \ g(x') < x_3 < h(x')\}.$$

We denote the common boundary of Ω_ε^+ and Ω^- by γ_c^ε which is defined by

$$\gamma_c^\varepsilon = \{(x', x_3) \in \Omega_\varepsilon : x_3 = h(x')\}.$$

The full or limit domain which is the Hausdorff limit of Ω_ε is denoted by Ω (see Fig. 2), which is defined by

$$\Omega = \{(x', x_3) : x' \in (0,1)^2, \ g(x') < x_3 < h(x') + M\}.$$

The upper and the lower part of the limit domain are Ω^+ and Ω^- and $\Omega = interior(\overline{\Omega^+ \cup \Omega^-})$. The common boundary of Ω^+ and Ω^- is denoted by γ_c which is defined by

$$\gamma_c = \{(x', x_3) : x' \in (0,1)^2, \ x_3 = h(x')\}.$$

We denote the boundary of Ω by $\partial\Omega$.

6.2. Problem description

We consider the following following elliptic PDE in divergence form,

$$\begin{cases} -\mathrm{div}\,(A^\varepsilon \nabla u_\varepsilon) + u_\varepsilon = f \ \text{ in } \ \Omega_\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon = 0 \ \text{ on } \ \partial\Omega_\varepsilon, \end{cases} \tag{6.1}$$

where ν_ε is the outward unit normal vector to $\partial\Omega_\varepsilon$, $A^\varepsilon = A\left(x, \frac{x'}{\varepsilon}\right) = \left(a_{i,j}\left(x, \frac{x'}{\varepsilon}\right)\right)_{i,j=1}^{3}$, where $a_{i,j} : \Omega \times (0,1)^2 \to \mathbb{R}$ with $1-$periodic in y', and $A(x,y')$ is uniformly elliptic and bounded, that is, there exists $\alpha, \beta \in \mathbb{R}^+$ such that,

$$\langle A(x,y')v, v\rangle \geqslant \alpha\|v\|^2, \ \|A(x,y')v\| \leqslant \beta\|v\| \text{ for all } v \in \mathbb{R}^3, \ \ x \in \Omega, \ y' \in (0,1)^2.$$

and $f \in L^2(\Omega)$.

7. Unfolding operator for two-directional oscillating domain and homogenization

For the study of these oscillations we will use periodic unfolding along x'. The periodic unfolding of a function $v : \mathbb{R}^3 \to R$ along x' is defined as

$$T^\varepsilon v(x', x_3, y') = v\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y', x_3\right). \tag{7.1}$$

Now, we define the modified unfolding operator for our analysis. For every $\varepsilon > 0$, the unfolded domain corresponding to Ω_ε^+ denoted by Ω_ε^u, is defied by

$$\Omega_\varepsilon^u = \left\{(x', x_3, y') : \ x', y' \in (0,1)^2, \ h\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y'\right) < x_3 < h\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y'\right) + \eta(y')\right\}.$$

Definition 7.1. The unfolding operator $\mathcal{T}^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{\mathcal{T}^\varepsilon u : \Omega_\varepsilon^u \to \mathbb{R}\}$ is defined by

$$\mathcal{T}^\varepsilon u = \chi_{\Omega_\varepsilon^u} T^\varepsilon u,$$

where T^ε is the unfolding operator given by (7.1).

If $\mathcal{O} \subset \mathbb{R}^3$ containing Ω_ε^+ and v is a real valued function on $\mathcal{O}$, then

$$\mathcal{T}^\varepsilon v = \mathcal{T}^\varepsilon \left(v|_{\Omega_\varepsilon^+} \right).$$

Remark 7.2. Here the base of the oscillation is given by the function h. If we take h as a constant function, then unfolded domain and unfolding operator will be similar to the unfolded domain and unfolding operator defined in [1].

Like other unfolding operators, it also enjoys the following integral equality.

Lemma 7.3. *Let $u \in L^2(\Omega_\varepsilon^+)$. Then*

$$\int_{\Omega_\varepsilon^u} \mathcal{T}^\varepsilon u \mathrm{d}x' \mathrm{d}x_3 \mathrm{d}y' = \int_{\Omega_\varepsilon^+} u \mathrm{d}x' \mathrm{d}x_3.$$

Notice that, $\chi_{\Omega_\varepsilon^u} \to \chi_{\Omega^u}$ pointwise in $\mathbb{R}^2$, where

$$\Omega^u = \left\{ (x', x_3, y) : \ x', y' \in (0,1)^2, \ h(x') < x_3 < h(x') + \eta(y') \right\}.$$

Also, since h, η are Lipschitz functions, we have that

$$|\Omega_\varepsilon^u \Delta \Omega^u| = O(\varepsilon), \tag{7.2}$$

where Δ denote the symmetric difference between Ω_ε^+, Ω^u and $|\cdot|$ denotes the Lebesgue measure. Because of the above relation (7.2), we have the following relation, for any $v \in L^2_{loc}(\mathbb{R}^3)$,

$$\int_{\Omega_\varepsilon^u} \mathcal{T}^\varepsilon v \mathrm{d}x \mathrm{d}y' = \int_{\Omega^u} \mathcal{T}^\varepsilon v \mathrm{d}x \mathrm{d}y' + O(\varepsilon). \tag{7.3}$$

7.1. Boundary unfolding operator

In order to get the interface conditions, we will use the following boundary unfolding which is nothing but the restriction of general unfolding operator on the oscillating common boundary γ_c^ε. For every $\varepsilon > 0$, let us denote the unfolded boundary of γ_c^ε by γ_c^u, which is defined by,

$$\gamma_c^u = \{(x', h(x'), y') : \ x' \in (0,1)^2, y' \in (a,b)^2\}.$$

Define $\phi_{\gamma_c}^\varepsilon : \gamma_c^u \to \gamma_c^\varepsilon$ as

$$\phi_{\gamma_c}^\varepsilon(x', h(x'), y') = \left(\varepsilon \left[\frac{x'}{\varepsilon} \right] + \varepsilon y', h \left(\varepsilon \left[\frac{x'}{\varepsilon} \right] + \varepsilon y' \right) \right).$$

Then define the boundary unfolding operator $\mathcal{T}_b^\varepsilon : \{u : \gamma_c^\varepsilon \to \mathbb{R}\} \to \{\mathcal{T}_b^\varepsilon : \gamma_c^u \to R\}$ as $\mathcal{T}_b^\varepsilon u = u \circ \phi_{\gamma_c}^\varepsilon$

Proposition 7.4. *$\mathcal{T}_b^\varepsilon$ satisfies the following properties:*

(i) $\mathcal{T}_b^\varepsilon$ *is linear. Further, if* $u,\ v : \gamma_c^\varepsilon \to \mathbb{R}$, *then,* $\mathcal{T}_b^\varepsilon(uv) = \mathcal{T}_b^\varepsilon(u)\mathcal{T}_b^\varepsilon(v)$,
(ii) *for any* $\phi \in L^2(\gamma_c)$, $\|\mathcal{T}_b^\varepsilon\phi - \phi\|_{L^2(\gamma_c^u)} \to 0$ *as* $\varepsilon \to 0$,
(iii) *let* $\{\phi_\varepsilon\}$ *is a sequence in* $L^2(\gamma_c)$ *such that* $\|\phi_\varepsilon - \phi\|_{L^2(\gamma_c)} \to 0$, *then* $\|\mathcal{T}_b^\varepsilon\phi_\varepsilon - \phi\|_{L^2(\gamma_c^u)} \to 0$,
(iv) *for* $u \in L^2(\gamma_\varepsilon)$, *we have* $\|\mathcal{T}_b^\varepsilon u\|_{L^2(\gamma_c^u)} \simeq \|u\|_{\gamma_c^\varepsilon}$ *for* ε *small.*

Limit space: As the domain has oscillations in 2-directions, in the limit, on upper part there will be only x_3 directional derivative. We recall that $Y(x', x_3) = \{y' \in (0,1)^2 :\ h(x') + \eta(y') > x_3\}$ and $\omega(x) = |Y(x', x_3)|$. The limit function space is defined as

$$H(\Omega, \omega) = \left\{ \psi \in L^2(\Omega) : \frac{\partial \psi}{\partial x_3} \in L^2(\Omega), \nabla\psi \in L^2(\Omega^-) \right\}.$$

The above function space with the following inner product is a Hilbert space: for $\psi, \phi \in H(\Omega, \omega)$

$$\langle \psi, \phi \rangle_{H(\Omega,\omega)} = \int_{\Omega^+} \omega \left(\psi^+\phi^+ + \frac{\partial \psi^+}{\partial x_3}\frac{\partial \phi^+}{\partial x_3} \right) \mathrm{d}x + \int_{\Omega^-} (\psi^-\phi^- + \nabla\psi^- \cdot \nabla\phi^-)\mathrm{d}x.$$

Let for each $\varepsilon > 0$, u_ε be the unique solution of (6.1). Let us define another sequence using the unfolded sequence $\mathcal{T}^\varepsilon u_\varepsilon$, say $\mathcal{T}^\varepsilon U_\varepsilon$ as

$$\mathcal{T}^\varepsilon U_\varepsilon(x', y', x_3) = \frac{1}{\varepsilon} \left(\mathcal{T}^\varepsilon u_\varepsilon - \frac{1}{|Y(x', x_3|)} \int_{Y(x',x_3)} \mathcal{T}^\varepsilon u_\varepsilon(x', y', x_3)\mathrm{d}y' \right). \tag{7.4}$$

Proposition 7.5. *The sequence defined in* (7.4) *is uniformly bounded in* $L^2(\Omega^u)$.

Proof. Using Poincare-Writinger inequality in (7.4) with respect to y', we get

$$\begin{aligned}\int_{Y(x',x_3)} |\mathcal{T}^\varepsilon U_\varepsilon|^2 \mathrm{d}y &\leqslant C \int_{Y(x',x_3)} \left| \nabla_{y'} \left(\frac{1}{\varepsilon} \left(\mathcal{T}^\varepsilon u_\varepsilon - \frac{1}{|Y(x', x_3|)} \int_{Y(x',x_3)} \mathcal{T}^\varepsilon u_\varepsilon(x', y', x_3)\mathrm{d}y' \right) \right) \right|^2 \mathrm{d}y' \\ &\leqslant C \int_{Y(x',x_3)} \left| \frac{1}{\varepsilon} \nabla_{y'} \mathcal{T}^\varepsilon u_\varepsilon \right|^2 = C \int_{Y(x',x_3)} |\mathcal{T}^\varepsilon \nabla_{x'} u_\varepsilon|^2 \mathrm{d}y'.\end{aligned}$$

Now integrating both sides with respect to x' and x_3 we get,

$$\int_{\Omega^u} |\mathcal{T}^\varepsilon U_\varepsilon|^2 \leqslant C \int_{\Omega^u} |\mathcal{T}^\varepsilon \nabla_{x'} u_\varepsilon|^2 = C \int_{\Omega_\varepsilon^+} |\nabla_{x'} u_\varepsilon|^2 \leqslant C,$$

where C is a generic constant independent of ε. □

As $\|\mathcal{T}^\varepsilon U_\varepsilon\| \leqslant C$, by compactness there exist $U_1 \in L^2(\Omega^u)$ such that $\mathcal{T}^\varepsilon U_\varepsilon \rightharpoonup U_1$ in $L^2(\Omega^u)$. Then the following proposition holds.

Proposition 7.6. *Let for each* $\varepsilon > 0$, u_ε *be the unique solution of* (6.1). *Then*

$$\mathcal{T}^\varepsilon \nabla_{x'} u_\varepsilon \rightharpoonup \nabla_{y'} U_1 \ \textit{weakly in}\ (L^2(\Omega^u))^2.$$

Proof. Let $\psi(x,y') \in (C_c^\infty(\Omega^u))^2$. Let $T^\varepsilon(\nabla_{x'}u_\varepsilon) \rightharpoonup D'$ in $(L^2(\Omega^u))^2$. A simple integration by parts gives us the following,

$$\int_{\Omega^u} \mathcal{T}^\varepsilon(\nabla_{x'}u_\varepsilon)\psi(x,y') = \int_{\Omega^u} \frac{1}{\varepsilon}\nabla_{y'}(\mathcal{T}^\varepsilon u_\varepsilon)\psi(x,y') = -\int_{\Omega^u} \frac{1}{\varepsilon}\mathcal{T}^\varepsilon u_\varepsilon \mathrm{div}_{y'}\psi = -\int_{\Omega^u} \mathcal{T}^\varepsilon U_\varepsilon \mathrm{div}_{y'}\psi.$$

Now letting $\varepsilon \to 0$ in the above equation on both sides, we get

$$\int_{\Omega^u} D'\psi = -\int_{\Omega^u} U_1 \mathrm{div}_{y'}\psi.$$

Since ψ is arbitrary, implies $D' = \nabla_{y'}U_1$. $\square$

Further as in Theorem 4.4, we can write $\nabla_{y'}U_1$ in terms of $a_{i,j}$ and $\dfrac{\partial u^+}{\partial x_3}$ as

$$\nabla_{y'}U_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \frac{\partial u^+}{\partial x_3}.$$

Similar type of results hold for the adjoint state also.

Following the same path as in Sections 4 and 5, we get a similar type of theorem.

Theorem 7.7. *Let u_ε be the sequence of solution to (6.1). Then, following convergences hold,*

(i) $\widetilde{u}_\varepsilon^+ \rightharpoonup \omega(x)u^+$ *and* $\widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_3}} \rightharpoonup \omega(x)\dfrac{\partial u^+}{\partial x_3}$ *weakly in* $L^2(\Omega^+)$,
(ii) $u_\varepsilon^- \rightharpoonup u^-$ *weakly in* $H^1(\Omega^-)$.

Further, $u = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-}$ *belongs to* $H(\Omega,\omega)$ *and* u *is the unique solution of the following system*

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(A_0^+\dfrac{\partial u^+}{\partial x_3}\right) + \omega(x)u^+ = \omega(x)f \quad in \quad \Omega^+, \\ -\operatorname{div}(A_0^-\nabla u^-) + u^- = f \quad in \quad \Omega^-, \\ \quad \dfrac{\partial u^+}{\partial x_3}\cdot \nu_3 = 0 \quad on \quad \gamma_u, \\ \quad u^+ = u^-, \quad A_0^+\dfrac{\partial u^+}{\partial x_3}\sigma_3 - A_0^-\nabla u^-\cdot\sigma = 0 \quad on \quad \gamma_c, \\ A_0^-\nabla u\cdot\nu = 0 \quad on \quad \partial\Omega^-\backslash\gamma_c. \end{cases}$$

Here

$$A_0^+ = \int_{Y(x,x_3)} \frac{\det A}{\det A'}\mathrm{d}y', \quad and\ A_0^-(x) = \int_{(0,1)^3} A(x,y')\big(I + \nabla_y\chi\big)\mathrm{d}y,\ for\ x \in \Omega^-, \tag{7.5}$$

where χ is the solution of a similar type of cell problem defined in (4.3). Here A' is the 2×2 submatrix of A given by $A' = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$.

Optimal control problem: Let $B^\varepsilon = B\left(x, \frac{x'}{\varepsilon}\right) = (b_{i,j}(x))_{i,j=1}^3$ be a 3×3 matrix valued function having the same property as A defined in Section 6. We also assume that A and B are symmetric. Let us consider the following cost functional

$$J_\varepsilon = J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B^\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \mathrm{d}x + \frac{\beta}{2}\int_{\Omega_\varepsilon} |\theta_\varepsilon|^2 \mathrm{d}x. \tag{7.6}$$

$\beta > 0$ being a fixed constant, $(u_\varepsilon, \theta_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ satisfies the following PDE,

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_\varepsilon) + u_\varepsilon = f + \theta_\varepsilon \quad \text{in} \quad \Omega_\varepsilon, \\ A^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{7.7}$$

The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon) : \ (u_\varepsilon, \theta_\varepsilon) \text{ obeys } (7.7)\}. \tag{7.8}$$

7.2. Homogenized problem

Let

$$W(x, y') = \begin{bmatrix} \dfrac{a_{12}a_{23}}{-a_{12}^2 + a_{11}a_{22}} - \dfrac{a_{13}a_{22}}{-a_{12}^2 + a_{11}a_{22}} \\[2ex] \dfrac{a_{12}a_{13}}{-a_{12}^2 + a_{11}a_{22}} - \dfrac{a_{11}a_{23}}{-a_{12}^2 + a_{11}a_{22}} \\[2ex] 1 \end{bmatrix},$$

then define a 1×1 matrix $B_\#^+$ as

$$B_\#^+(x) = \int_{Y(x,x_3)} B(x, y') W(x, y') \cdot W(x, y') \mathrm{d}y'$$

and

$$B_\#^-(x) = \int_{(0,1)^3} (B(x, y')(I + \nabla_y \chi) + A(x, y') \nabla_y N) \mathrm{d}y, \qquad \text{for } x \in \Omega^-$$

where χ and N are the solutions of a similar type of cell problems defined in (5.10).

The limit state equation for the control $\theta \in L^2(\Omega)$ is given by

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(A_0^+ \dfrac{\partial u^+}{\partial x_3}\right) + \omega(x) u^+ = \omega(x) f + \omega(x)\theta \quad \text{in} \quad \Omega^+, \\ -\mathrm{div}(A_0^- \nabla u^-) + u^- = f + \theta \quad \text{in} \quad \Omega^-, \\ \quad \dfrac{\partial u^+}{\partial x_3} \cdot \eta_3 = 0 \quad \text{on} \quad \gamma_u, \\ \quad u^+ = u^-, \quad A_0^+ \dfrac{\partial u^+}{\partial x_3} \sigma_3 - A_0^- \nabla u^- \cdot \sigma = 0 \quad \text{on} \quad \gamma_c, \\ A_0^- \nabla u^- \cdot \nu = 0 \quad \text{on} \quad \partial\Omega^- \backslash \gamma_c. \end{cases} \tag{7.9}$$

Then, the limit optimal control problem is : find $(\bar{u},\bar{\theta}) \in H(\Omega,\omega)\times L^2(\Omega)$ such that

$$J(\bar{u},\bar{\theta}) = \inf\{J(u,\theta) :\ (u,\theta)\in H(\Omega,\omega)\times L^2(\Omega) \text{ satisfies } (7.9)\}, \tag{7.10}$$

where the cost functional J is defined as

$$J(u,\theta) = \frac{1}{2}\int_{\Omega^+} B^+_{\#}\left(\frac{\partial u^+}{\partial x_3}\right)^2 \mathrm{d}x + \frac{1}{2}\int_{\Omega^-} B^-_{\#}\nabla u^-\cdot\nabla u^- \mathrm{d}x + \frac{\beta}{2}\int_{\Omega}|\theta|^2\mathrm{d}x. \tag{7.11}$$

Theorem 7.8. *Let $(\bar{u},\bar{\theta})$ be the optimal solution to the optimal control problem (7.10), the optimal control is characterized by*

$$\bar{\theta} = -\frac{1}{\beta}\bar{v},$$

where the adjoint state $\bar{v}$ satisfies the following adjoint PDE

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(A^+_0\dfrac{\partial \bar{v}^+}{\partial x_3}\right)+\omega(x)\bar{v}^+ = -\dfrac{\partial}{\partial x_3}\left(B^+_{\#}\dfrac{\partial \bar{u}^+}{\partial x_3}\right) \quad in\ \ \Omega^+ \\ -div(A^-_0\nabla\bar{v}^-)+\bar{v}^- = -div(B^-_{\#}\nabla\bar{v}^-) \quad in\ \ \Omega^- \\ \quad \dfrac{\partial\bar{v}^+}{\partial x_3}\nu_3 = 0 \quad on\ \ \gamma_u \\ \quad \bar{v}^+ = \bar{v}^-, \quad \left(A^+_0\dfrac{\partial\bar{v}^+}{\partial x_3} - B^+_{\#}\dfrac{\partial u^+}{\partial x_3}\right)\sigma_3 - (A^-_0\nabla\bar{v}^- - B^-_{\#}\nabla\bar{u}^-)\cdot\sigma = 0 \quad on\ \ \gamma_c \\ A^-_0\nabla\bar{v}\cdot\nu = 0 \quad on\ \ \partial\Omega^-\backslash\gamma_c. \end{cases} \tag{7.12}$$

Conversely, assume that $(u,v)\in H(\Omega,\omega)\times H(\Omega,\omega)$ satisfies the following system,

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(A^+_0\dfrac{\partial u^+}{\partial x_3}\right)+\omega(x)u^+ = \omega(x)f - \omega(x)\dfrac{1}{\beta}v^+ \quad in\ \ \Omega^+, \\ -\dfrac{\partial}{\partial x_3}\left(A^+_0\dfrac{\partial v^+}{\partial x_3}\right)+\omega(x)v^+ = -\dfrac{\partial}{\partial x_3}\left(B^+_{\#}\dfrac{\partial u^+}{\partial x_3}\right) \quad in\ \ \Omega^+, \\ -div(A^-_0\nabla u^-)+u^- = f - \dfrac{1}{\beta}v^- \quad in\ \ \Omega^-, \\ -div(A^-_0\nabla v^-)+v^- = -div(B^-_{\#}\nabla u^-) \quad in\ \ \Omega^-, \end{cases} \tag{7.13}$$

together with the boundary conditions

$$\begin{cases} \dfrac{\partial u^+}{\partial x_3}\cdot\nu_3 = 0,\ \dfrac{\partial v^+}{\partial x_3}\cdot\nu_3 = 0 \quad on\ \ \gamma_u, \\ A^-_0\nabla v\cdot\nu = 0 \quad on\ \ \partial\Omega^-\backslash\gamma_c, \end{cases}$$

and the interface conditions

$$\begin{cases} u^+ = u^-,\ v^+ = v^- \quad on\ \ \gamma_c, \\ \dfrac{\partial u^+}{\partial x_3}\sigma_3 - A^-\nabla u^-\cdot\sigma = 0, \\ \left(A^+_0\dfrac{\partial\bar{v}^+}{\partial x_3} - B^+_{\#}\dfrac{\partial u^+}{\partial x_3}\right)\sigma_3 - (A^-_0\nabla v^- - B^-_{\#}\nabla u^-)\cdot\sigma = 0 \quad on\ \ \gamma_c. \end{cases}$$

Then the pair $(u, -\frac{1}{\beta}v)$ is the optimal solution to the optimal control problem (7.10).

Theorem 7.9. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ and $(\bar{u}, \bar{\theta})$ be the optimal solution to (7.8) and (7.10) respectively. Also let $\bar{v}_\varepsilon$ and $\bar{v}$ are the adjoint states corresponding to the optimal control problem (7.8) and (7.10) respectively. Then*

$$\begin{aligned}
&\widetilde{\bar{u}_\varepsilon^+} \rightharpoonup \omega(x)\bar{u}^+, \ \widetilde{\frac{\partial u_\varepsilon^+}{\partial x_3}} \rightharpoonup \omega(x)\frac{\partial \bar{u}^+}{\partial x_3} \quad \textit{weakly in} \quad L^2(\Omega^+),\\
&\widetilde{\bar{v}_\varepsilon^+} \rightharpoonup \omega(x)\bar{v}^+, \ \widetilde{\frac{\partial v_\varepsilon^+}{\partial x_2}} \rightharpoonup \omega(x)\frac{\partial \bar{v}^+}{\partial x_3} \quad \textit{weakly in} \quad L^2(\Omega^+),\\
&\bar{u}_\varepsilon^- \rightharpoonup \bar{u}^-, \ \bar{v}_\varepsilon^- \rightharpoonup \bar{v}^- \textit{weakly in } H^1(\Omega^-)\\
&\widetilde{\bar{\theta}}_\varepsilon \rightharpoonup -\omega(x)\frac{1}{\beta}\bar{v}, \quad \textit{weakly in} \quad L^2(\Omega).
\end{aligned} \tag{7.14}$$

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