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Strong contrasting diffusivity in general oscillating domains: Homogenization of optimal control problems

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Abstract

The composites of materials with high contrasting properties is an interesting topic to study as it has applications. In this article, we wish to study problems in high oscillating domains, where the oscillatory part is made of two materials with high contrasting conductivities (or diffusivity). Thus the low contrast material acts as near insulation in-between the conducting materials. In the first part, we study the homogenization problem of an elliptic equation. The main discussion in this article is the study of optimal control problems based on the unfolding method. The interesting result is the difference in the limit behavior of the optimal control problem, which crucially depends on the action of the control, whether it is on the conductivity part or insulating part. In both cases, we derive the two-scale limit controls problems which are quite similar as far as analysis is concerned. But, if the controls are acting on the conductive region, a complete scale separation is available, whereas a complete separation is not visible in the insulating case due to the intrinsic nature of the problem. In this case, to obtain the limit optimal control problem in the macro scale, two cross-sectional cell problems are introduced. We do obtain the homogenized equation for the state, but the two-scale cost functional remains as it is.

MSC: 35B27; 35B40; 49J20; 80M40

Keywords: Contrasting diffusivity; Rough boundary; Optimal control; Unfolding operator; Homogenization

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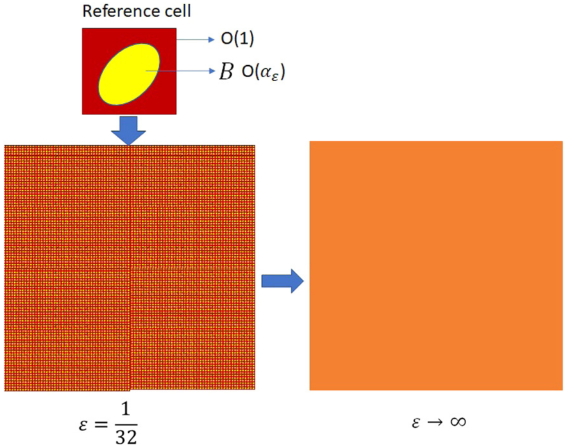

Fig. 1. Composite material.

1. Introduction

The study of partial differential equations (PDE) with strong contrasting diffusivity is important as it appears in modeling of several multi-scale physical problems such as the double porosity model, effective properties of composite material with soft and hard core, effective conductivity of composites made of materials having high and low conductivities, etc. Typically, for example, in a usual composite material, say consisting of two materials with conductivities α and β which oscillates in a small scale ε which tends to zero. In this situation, the uniform ellipticity of the elliptic system in the micro-scale is retained. In a strongly varying diffusivity problem, the domain (or composite) consisting of two materials which can be thought of as soft inclusions (say B_ε) distributed periodically in the small scale ($\varepsilon \to 0$) whose conductivity is of order $\alpha_\varepsilon \to 0$. The other material has conductivity $O(1)$ (see Fig. 1). This is the highly contrasting nature of the composite. Here note that the ellipticity coefficient is bounded below by α_ε and hence it is not uniformly elliptic like normal micro composites. Hence, we need to give special attention to study such problems.

The homogenization of PDEs with strong contrasting diffusivity terms appears in applications like the mathematical model of composites with high-modulus reinforcement. One of the earliest works on the multi-scale analysis of high contrast composites is available in G.P. Panasenko [1] and also see Ch. 7, [2] for the homogenization process. See also ([3,4]). In [5] authors have studied the homogenization of a double porosity model in a single-phase flow which is an interesting piece of work. In another work, Charef and Sili [6] have considered the homogenization of a conductivity equation for a medium made up of highly conductive vertical fiber surrounded by another material assumed to be a poor conductor whereas in [7], the author investigates homogenization of a stationary diffusion equation in a periodic composite medium made of two components with the ratio between the diffusion coefficients $O(\frac{1}{\alpha_\varepsilon^2})$ where ε is the size of the period and $\alpha_\varepsilon \to 0$. The homogenization of a hyperbolic PDE with strongly contrasting diffusive coefficients is performed in [8] and obtained the homogenized limit as a coupled two-scale system of macro and micro scales. We could also see the difficulty in scale separation in the case of optimal control problem in the present work as in [8], but eventually, we could achieve the separation by introducing a second cell problem. We also see the presence of the source term in the cost functional. Thus, strongly contrasting elliptic coefficients exhibit interesting behavior in certain cases. For further reading in this direction, we refer to the articles [9–11] and references therein.

Our aim is to consider domains with very general periodically oscillatory boundaries, where the oscillations with conducting materials are separated by nearly insulating materials together with an associated optimal control problem. The study of homogenization problems in oscillating boundary domains is a very active area for the last two decades and some relevant articles in this direction are ([12–19]) and references therein which definitely not exhaustive. Regarding optimal control problems in the oscillating domain, we refer to ([20–26]) and references therein and for general reading on homogenization, see ([27–32]).

Our domain under study consists of a fixed domain Ω^- together with a highly oscillating component Ω_ε^+. We intend to consider very general periodic oscillatory part Ω_ε^+ consisting of highly contrasting material (see Fig. 2). As a particular case, one can consider rapidly oscillating pillars (see Fig. 3) of base $O(\varepsilon)$ and height $O(1)$ consisting of a core material of conductivity $O(1)$ surrounded by another material with conductivity $O(\alpha_\varepsilon^2)$. In this direction, we cite the work of A. Gaudiello, and A. Sili [33]. They have considered homogenization of an elliptic PDE with strongly contrasting diffusivity term in a forest type or pillar type oscillating domain where a highly conductive pillar is covered by an insulator type material and the periodic distribution of pillars depends on the vanishing sequence ε.

The present article is devoted to the study of the homogenization of an elliptic PDE with strong contrasting diffusivity term in a general periodic oscillating domain and an associated interior optimal control problem. We consider an interior optimal control problem with a state or constraint equation where we apply control on the high conductive part and insulating (low conducting) part separately. Though the convergence analysis to obtain the two-scale limit system, is similar in both cases, the scale separation process is quite different. The first part is a non-trivial generalization of the work in [33], where the problem is in a pillar type oscillating domain. Here we are allowing the reference cell to be very general as long as the reference cells cross-section in x_1 direction is connected and having certain properties, see for typical example in Fig. 2. To get the limit problem, we have to define a family of cross-sectional cell problems (3). The unfolding operator developed in ([34,35]) is used for the analysis. To write the limit optimal control problem in scale separated form, another cross-sectional cell problem (28) is introduced. Using these cell problems, the separation of macro-scale and micro-scale is done in the limit optimal control problem when controls are acting on the high conducting part. It is surprising to see the appearance of the source term f in the limit cost functional, which is seen for the first time in our experience. On the other hand, if the controls act on the low conducting region, a complete scale separation is not possible as far as these authors are concerned. However, we are able to obtain the scale separation for the state, whereas the cost functional is not scale separated, which is a new feature of the problem.

The organization of the paper is as follows. In Section 2, we introduce the configuration of the domain and recall the unfolding operator and its properties. The problem statement is given in Section 3. In Section 3.1, limit function spaces are defined. The limiting analysis of the variational form and the well posedness of the limit system is discussed in Section 4. Homogenization of the interior optimal control problem with controls on the high conductive region is considered in Section 5. In Section 6, separation of macro and micro scale is performed on the limit optimal control problem. Finally, in Section 7, we study the analysis with controls acting on the low conductive (insulating) region.

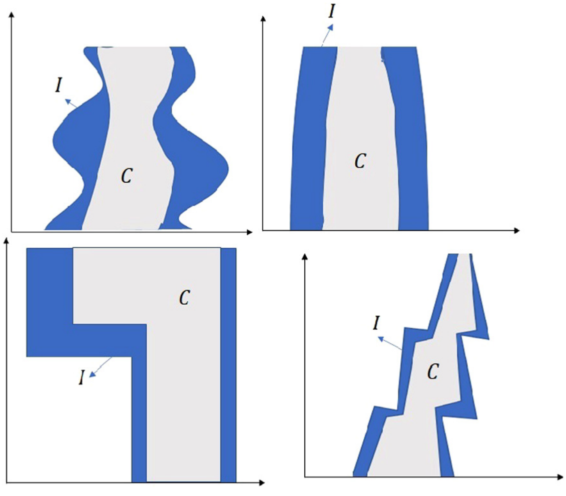

Fig. 2. Typical example of reference cells.

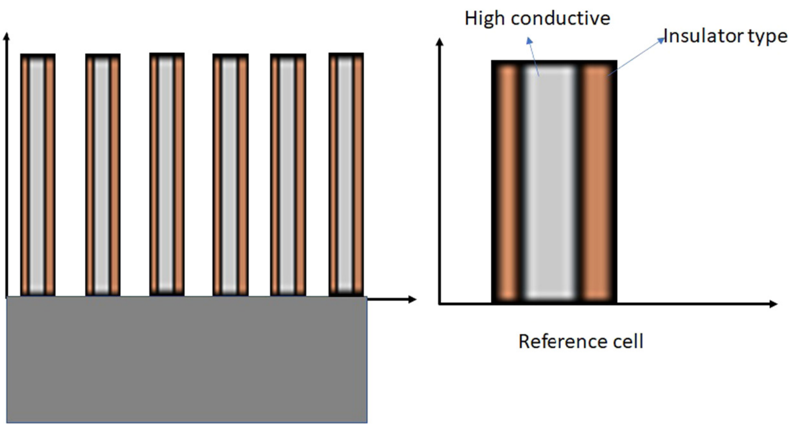

Fig. 3. Pillar type oscillating domain.

2. Domain description and unfolding operator

This work can be carried out in any finite dimension but for simpler presentation we will consider in 2-dimensional domain. For the same reason we are considering the reference cell as in Fig. 4, instead of figure as in Fig. 2. Now we will give description and geometrical assumptions on the reference cell mathematically. Let $\Lambda \subset (0, 1) \times (0, 1)$ (it is just for simplicity, one can consider $(0, L) \times (0, L)$ for any $L > 0$) and $C, I \subset \Lambda$. We divide Λ into two components C and I, that is $\overline{\Lambda} = \overline{C} \bigcup \overline{I}$, $C \bigcap I = \phi$ (empty set)and satisfies the following properties:

(i) Λ, C, I are Lipschitz domains.
(ii) The one-dimensional Lebesgue measure of $\overline{C} \cap \{(0, 1) \times 0\}$ and $\overline{C} \cap \{(0, 1) \times 1\}$ are strictly positive that is $|\overline{C} \cap \{(0, 1) \times 0\}| > 0$ and $|\overline{C} \cap \{(0, 1) \times 1\}| > 0$.

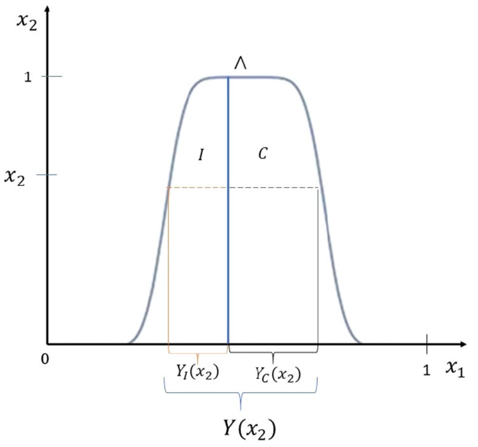

Fig. 4. Reference cell.

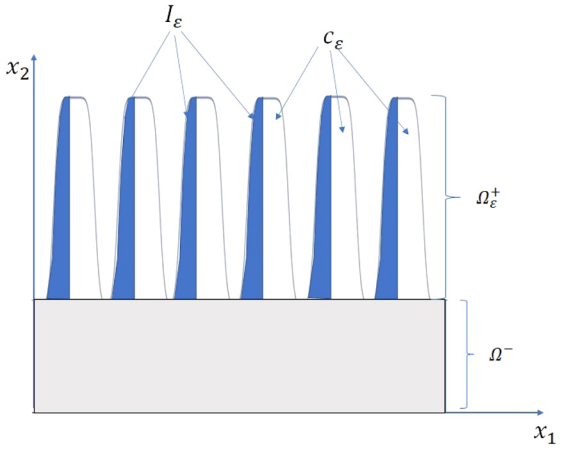

Fig. 5. Oscillating domains.

(iii) For $x_2 \in (0,1)$, let us define $Y(x_2) = \{y_1 \in (0,1) : (y_1, x_2) \in \Lambda\}$, $Y_{\mathrm{C}}(x_2) = \{y_1 \in (0,1) : (y_1, x_2) \in C\}$ and $Y_{\mathrm{I}}(x_2) = \{y_1 \in (0,1) : (y_1, x_2) \in I\}$. We assume that there exists $\delta > 0$, such that the Lebesgue measure of $Y(x_2)$, $Y_{\mathrm{C}}(x_2)$ and $Y_{\mathrm{I}}(x_2)$ are greater than δ that is $|Y(x_2)|, |Y_{\mathrm{C}}(x_2)|, |Y_{\mathrm{I}}(x_2)| > \delta$ for all $x_2 \in (0,1)$.

A 2-dimensional presentation of $\Lambda, I, C, Y_{\mathrm{C}}$, and Y_{I} are given in Fig. 4. For $\varepsilon = \frac{1}{m}$ where $m \in \mathbb{Z}^+$, (in fact, one can take any $\varepsilon \to 0$) define

$$C_\varepsilon = \bigcup_{k=0}^{m-1} \{(x_1, x_2) : \; x_1 \in (k\varepsilon + \varepsilon Y_{\mathrm{C}}(x_2)), \; x_2 \in (0,1)\},$$

$$I_\varepsilon = \bigcup_{k=0}^{m-1} \{(x_1, x_2) : \; x_1 \in (k\varepsilon + \varepsilon Y_{\mathrm{I}}(x_2)), \; x_2 \in (0,1)\}.$$

The upper oscillating part Ω_ε^+ and the lower fixed part Ω^- are given by (cf. Fig. 5)

$$\Omega_\varepsilon^+ = \left(\overline{I_\varepsilon \cup C_\varepsilon}\right)^o = \bigcup_{k=0}^{m-1}\{(x_1,x_2):\ x_1 \in (k\varepsilon + \varepsilon Y(x_2)), x_2 \in (0,1)\},\ \ \Omega^- = (0,1)\times(0,-1).$$

The oscillating domain under consideration is $\Omega_\varepsilon = \left(\overline{\Omega_\varepsilon^+ \cup \Omega^-}\right)^o$. The limit domain $\Omega = \left(\overline{\Omega^+ \cup \Omega^-}\right)^o$, where $\Omega^+ = (0,1)^2$. We denote the common boundary of C_ε and Ω^- by γ_C^ε:

$$\gamma_C^\varepsilon = \{(x_1,0) : (x_1,0) \in \overline{C}_\varepsilon\}.$$

The interface between Ω^+ and Ω^- is demoted by γ, which is given by

$$\gamma = \{(x_1,0):\ x_1 \in (0,1)\}.$$

For our analysis, unfolding operator will be used as a main tool which is going to be recalled now.

2.1. Interior unfolding operator

For our analysis we will make use of unfolding operator for general oscillating domain defined in [34,35]. For the sake of completeness we recall the definition and some properties of unfolding operator without proof.

The unfolded domain corresponding to the upper part Ω_ε^+ is given by

$$\Omega^u = \{(x_1,x_2,y_1) : (x_1,x_2) \in \Omega^+, y_1 \in Y(x_2)\}.$$

Definition 2.1. (The unfolding operator) Let $\phi^\varepsilon : \Omega^u \to \Omega_\varepsilon^+$ be defined as $\phi^\varepsilon(x_1,x_2,y_1) = \left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right)$. The ε- unfolding of a function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \Omega^u \to \mathbb{R}$. The operator which maps every function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. Let the unfolding operator is denoted by T^ε, that is,

$$T^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{T^\varepsilon u : \Omega^u \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon u(x_1,x_2,y_1) = u\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right) \text{ for all } (x_1,x_2,y_1) \in \Omega^u.$$

If $U \subset \mathbb{R}^2$ containing Ω_ε^+ and u is a real valued function on U, $T^\varepsilon u$ will mean, T^ε acting on the restriction of u to Ω_ε^+.

We denote T_C^ε as $T^\varepsilon|_{\Omega_C^u}$. Now we recall some crucial properties of unfolding operator without proof. For proof we refer to the reader to see [34].

Proposition 2.2. *For each $\varepsilon > 0$,*

(i) T^ε *is linear. Further, if* $u,\ v : \Omega_\varepsilon^+ \to \mathbb{R}$, *then,* $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.
(ii) Let $u \in L^1(\Omega_\varepsilon^+)$. *Then,*

$$\int_{\Omega^u} T^\varepsilon(u)dxdy_1 = \int_{\Omega_\varepsilon^+} udx.$$

(iii) Let $u \in L^2(\Omega_\varepsilon^+)$. *Then,* $T^\varepsilon u \in L^2(\Omega^u)$ *and* $\|T^\varepsilon u\|_{L^2(\Omega^u)} = \|u\|_{L^2(\Omega_\varepsilon^+)}$.
(iv) For $u \in H^1(C_\varepsilon)$, *we have* $T_c^\varepsilon u,\ \frac{\partial}{\partial x_2}(T_c^\varepsilon u) \in L^2(\Omega_c^u)$. *Moreover,*

$$\frac{\partial}{\partial x_2} T_c^\varepsilon u = T_c^\varepsilon \frac{\partial u}{\partial x_2} \text{ and } \frac{\partial}{\partial y_1} T_c^\varepsilon u = \varepsilon T_c^\varepsilon \frac{\partial u}{\partial x_1}.$$

(v) For any $u \in L^2(\Omega^+)$, $T_c^\varepsilon u \to u$ *strongly in* $L^2(\Omega_c^u)$. *More generally, if* $u_\varepsilon \to u$ *strongly in* $L^2(\Omega^+)$, *then,* $T_c^\varepsilon u_\varepsilon \to u$ *strongly in* $L^2(\Omega_c^u)$.

(vi) For any ϕ *defined on* Ω_ε^+ *or any subset of* Ω_ε^+, *we denote* $\tilde{\phi}$, *the extension of* ϕ *by* 0 *to the domain* Ω^+. *Let, for every* ε, $u_\varepsilon \in L^2(\Omega_\varepsilon^+)$ *be such that* $T^\varepsilon u_\varepsilon \rightharpoonup u$ *weakly in* $L^2(\Omega^u)$. *Then,*

$$\tilde{u}_\varepsilon \rightharpoonup \int_{Y(x_2)} u(x_1, x_2, y_1)dy_1 \ \text{ weakly in } \ L^2(\Omega^+).$$

2.2. Boundary unfolding operator:

In order to get the interface conditions, we now introduce the following boundary unfolding operator T_b^ε on γ_c^ε. For every $\varepsilon > 0$, let us denote the unfolded boundary of γ_c^ε by γ_c^u, defined by

$$\gamma_c^u = \{(x_1, 0, y_1) :\ x_1 \in (0,1), y_1 \in \bar{C} \cap \{(0,1) \times 0\}\}.$$

Define $\phi_{\gamma_c}^\varepsilon : \gamma_c^u \to \gamma_c^\varepsilon$ as

$$\phi_{\gamma_c}^\varepsilon(x_1, 0, y_1) = \left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, 0\right)$$

Define boundary unfolding operator $T_b^\varepsilon : \{u : \gamma_c^\varepsilon \to \mathbb{R}\} \to \{T_b^\varepsilon : \gamma_c^u \to \mathbb{R}\}$ as $T_b^\varepsilon u = u \circ \phi_{\gamma_c}^\varepsilon$.

Proposition 2.3. *The boundary unfolding operator* T_b^ε *enjoys the following properties:*

(i) T_b^ε *is linear. Further, if* $u,\ v : \gamma_C^\varepsilon \to \mathbb{R}$, *then,* $T_b^\varepsilon(uv) = T_b^\varepsilon(u)T_b^\varepsilon(v)$,
(ii) for $u \in L^2(\gamma_C^\varepsilon)$, *then* $\|T_b^\varepsilon u\|_{L^2(\gamma_C^u)} = \|u\|_{L^2(\gamma_C^\varepsilon)}$.
(iii) For any $\phi \in L^2(\gamma_C)$, $\|T_b^\varepsilon \phi - \phi\|_{L^2(\gamma_C^u)} \to 0$ *as* $\varepsilon \to 0$.
(iv) Let $\{\phi_\varepsilon\}$ *be a sequence in* $L^2(\gamma_C)$ *such that* $\|\phi_\varepsilon - \phi\|_{L^2(\gamma_C)} \to 0$, *then* $\|T_b^\varepsilon \phi_\varepsilon - \phi\|_{L^2(\gamma_C^u)} \to 0$,

Note: We do not require unfolding operators T_I^ε corresponding to the insulating part I_ε.

3. Problem description

In the above prescribed domain, firstly we want to consider the following ε dependent variational problem,

$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla u_\varepsilon \nabla \phi + \int_{\Omega_\varepsilon} u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi, \quad \text{for all } \phi \in H^1(\Omega_\varepsilon), \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$. The Lax-Milgram theorem ensures the existence and the uniqueness of the solution u_ε of the problem (1). Our aim is to analyze the asymptotic behavior of the above variational form as the oscillating parameter $\varepsilon \to 0$. Later in Section 5 an interior optimal control problem has been considered. Note that in C_ε, the normalized diffusivity is 1, whereas in I_ε, it is ε^2 which acts like an insulator. Of course, we have considered insulation only on one side of the main conducting material, but this is not an issue, we can apply insulation on both sides of C_ε. Thus, it is a standard Laplacian problem with coefficients 1 and ε^2 and hence it is not uniformly elliptic.

3.1. Limit function space and limit problem

In order to define the solution of the homogenized variational form, limit optimal control problem, and cell problems, we need appropriate function spaces which we will define now. For any function ϕ defined on Ω, we may write $\phi = \phi^+ \chi_{\Omega^+} + \phi^- \chi_{\Omega^-} = (\phi^+, \phi^-)$ throughout the article.

1. Define $H(\Omega) = \{\phi : \phi^+ \in L^2((0,1); H^1(0,1)), \phi^- \in H^1(\Omega^-), \phi^+ = \phi^- \, on \ \gamma\}$ with the following norm

$$\|\phi\|_{H(\Omega)} = \|\phi^-\|_{H^1(\Omega^-)} + \|\phi^+\|_{L^2(\Omega^+)} + \left\| \frac{\partial \phi^+}{\partial x_2} \right\|_{L^2(\Omega^+)}.$$

2. For any $x_2 \in (0,1)$, define $V^{x_2} = \{w = w(y_1) \in H^1(Y(x_2)) : w = 0 \text{ a.e. in } Y_c(x_2)\}$ with the following norm

$$\|w\|_{Y(x_2)} = \|w\|_{L^2(Y(x_2))} + \left\| \frac{\partial w}{\partial y_1} \right\|_{L^2(Y(x_2))}.$$

3. Finally, $V(\Omega) = \left\{ \psi = \psi(x, y_1) \in L^2(\Omega^u) : \psi = 0 \ in \ \Omega^u_c, \dfrac{\partial \psi}{\partial y_1} \in L^2(\Omega^u) \right\}$ with the following norm

$$\|\psi\|_{V(\Omega)} = \|\psi\|_{L^2(\Omega^u)} + \left\| \frac{\partial \psi}{\partial y_1} \right\|_{L^2(\Omega^u)}.$$

Then, the limit problem in variational form is: $u = (u^+, u^-) \in H(\Omega)$ solves uniquely

$$\begin{cases} \int\limits_{\Omega^+} |Y_c(x_2)| \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int\limits_{\Omega^+}\left(|Y(x_2)| - \int\limits_{Y_I(x_2)} \xi\, dy_1\right) u^+\phi + \int\limits_{\Omega^-}(\nabla u^- \nabla\phi + u\phi) \\ \qquad = \int\limits_{\Omega^+}\left(|Y(x_2)| - \int\limits_{Y_I(x_2)} \xi\, dy_1\right) f\phi + \int\limits_{\Omega^-} f\phi, \quad \text{for all } \phi \in H(\Omega). \end{cases} \tag{2}$$

We define the family of cell problems as follows: for $x_2 \in (0,1)$, define $\xi = \xi(x_2, \cdot)$ as the unique solution defined on the cell $Y(x_2)$ by

$$\begin{cases} \xi(x_2,\cdot) \in V^{x_2} \\ \int\limits_{Y(x_2)} \frac{\partial \xi(x_2,y_1)}{\partial y_1}\frac{\partial w(y_1)}{\partial y_1} dy_1 + \int\limits_{Y(x_2)} \xi(x_2,y_1) w(y_1)\, dy_1 = \int\limits_{Y(x_2)} w(y_1)\, dy_1 \quad \text{for all } w \in V^{x_2}. \end{cases} \tag{3}$$

4. Convergence and main theorem

This section is devoted to prove one of the main theorems of this article.

Theorem 4.1. *For every ε let u_ε be the unique solution to the variational problem (1). Let $H(\Omega)$ and V^{x_2} be defined as in Section 3.1 and $u = (u^+, u^-) \in H(\Omega)$ be the unique solution of the variational form (2). Then*

$$\begin{cases} u_\varepsilon^- \rightharpoonup u^- \text{ weakly in } H^1(\Omega^-), \\ \widetilde{u_\varepsilon^+} \rightharpoonup |Y(x_2)|u^+ + \int\limits_{Y_I(x_2)} (f - u^+)\xi(x_2,y_1)dy_1 \text{ weakly in } L^2(\Omega^+) \\ \chi_{C_\varepsilon^+}\widetilde{\frac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup 0, \quad \chi_{C_\varepsilon^+}\widetilde{\frac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup |Y_c(x_2)|\frac{\partial u^+}{\partial x_2} \text{ weakly in } L^2(\Omega^+) \\ \varepsilon\chi_{I_\varepsilon^+}\widetilde{\frac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup \int\limits_{Y_I(x_2)} \frac{\partial u_1}{\partial y_1} dy_1, \quad \varepsilon\chi_{I_\varepsilon^+}\widetilde{\frac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup 0 \text{ weakly in } L^2(\Omega^+) \end{cases} \tag{4}$$

as $\varepsilon \to 0$.

Proof. The proof will be accomplished in several steps.

Step 1: (Uniform boundedness) In (1) take $\phi = u_\varepsilon$ as a test function to get

$$\|\chi_{C_\varepsilon^+}\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \varepsilon\|\chi_{I_\varepsilon^+}\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \|\nabla u_\varepsilon\|_{L^2(\Omega^-)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant \|f\|_{L^2(\Omega_\varepsilon)} \tag{5}$$

Thus observe a change in the order of the bound in the gradient in conducting and insulating materials, that is $\|\nabla u_\varepsilon\|_{L^2(C_\varepsilon^+)} \leq l$, $\|\nabla u_\varepsilon\|_{L^2(I_\varepsilon^+)} \leq l\varepsilon^{-1}$, where l is a generic constant. In essence, we do not have the uniform bound on the gradient, which is not surprising as the bound inversely depends on the ellipticity constant. Hence the idea is to effectively use the uniform bound wherever it is available.

Step 2: (Convergence of subsequences) From (5) and by the properties of unfolding operator and weak compactness of $H^1(\Omega^-)$, there exist $u_0(x, y_1) \in L^2(\Omega^u)$, $\eta(x, y_1) = (\eta_1, \eta_2)$, $z(x, y_1) = (z_1, z_2) \in (L^2(\Omega^u))^2$ and $u^- \in H^1(\Omega^-)$ such that, weakly

$$\begin{aligned}
&u_\varepsilon \rightharpoonup u^- \text{ in } H^1(\Omega^-)\\
&T^\varepsilon(u_\varepsilon^+) \rightharpoonup u_0(x, y_1) \text{ in } L^2(\Omega^u)\\
&T^\varepsilon(\chi_{C_\varepsilon^+}\widetilde{\nabla u_\varepsilon}) \rightharpoonup \chi_{\mathrm{C}}(y_1, x_2)\eta(x, y_1) = \chi_{\mathrm{C}}(y_1, x_2)(\eta_1, \eta_2) \text{ in } (L^2(\Omega^u_{\mathrm{C}}))^2\\
&T^\varepsilon(\varepsilon\chi_{I_\varepsilon^+}\widetilde{\nabla u_\varepsilon}) \rightharpoonup \chi_{\mathrm{I}}(y_1, x_2)z(x, y_1) = \chi_{\mathrm{I}}(y_1, x_2)(z_1, z_2) \text{ in } (L^2(\Omega^u))^2.
\end{aligned}$$

In the remaining steps, we identify $u_0, \eta_1, \eta_2, z_1, z_2$ and get properties enjoyed by these functions.

Step 3: In this step will show that u_0 is independent of y_1 in Ω^u_{C} and the existence of $u^+ \in L^2(\Omega^+)$, $u_1 \in L^2(\Omega^u)$ with $u_1 = 0$ a.e. in Ω^u_{C} such that

$$u_0(x, y) = u^+(x) + u_1(x, y_1). \tag{6}$$

First will show the existence of u^+. From (5) we have $\|u_\varepsilon^+\|_{H^1(C_\varepsilon)} \leq l$ where l is a generic constant independent of ε. Now using the definition of restricted unfolding operator $T_{\mathrm{C}}^\varepsilon$ we get $T_{\mathrm{C}}^\varepsilon u_\varepsilon = T^\varepsilon(u_\varepsilon|_{C_\varepsilon})$. Hence we have

$$T_{\mathrm{C}}^\varepsilon u_\varepsilon \rightharpoonup u_0(x, y_1)|_{\Omega^u_{\mathrm{C}}} \text{ weakly in } L^2(\Omega^u_{\mathrm{C}}).$$

Now from the properties of derivative of unfolding operator we get $\frac{\partial}{\partial y_1} T_{\mathrm{C}}^\varepsilon u_\varepsilon^+ = \varepsilon T_{\mathrm{C}}^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)$. Since we have uniform bound on $\left\|\frac{\partial u_\varepsilon^+}{\partial x_1}\right\|_{L^2(C_\varepsilon)}$, we deduce that $\frac{\partial}{\partial y_1}(u_0(x, y)) = 0$ in Ω^u_{C}. Hence $u_0(x, y) = u^+(x)$ a.e. in Ω^u_{C} for some $u^+ \in L^2(\Omega^+)$. Let $u_1 = u_0 - u^+$. Hence step 3 is completed.

Step 4: Here, we will show that

(i) $\frac{\partial u^+}{\partial x_2} \in L^2(\Omega^+)$.

(ii) $\eta_2(x, y_1) = \frac{\partial u^+}{\partial x_2}$ a.e. Ω^u_{C}.

(iii) $z_2(x, y_1) = 0$ a.e. in Ω^u_{I}.

Proof. (i) and (ii) follows from the properties of $T_{\mathrm{C}}^{\varepsilon}$.

Proof of (iii): The estimate (5), gives us $\varepsilon T^{\varepsilon} u_{\varepsilon}^{+} \to 0$ in $L^2(\Omega^u)$. Now, for $\psi \in C_c^{\infty}(\Omega^u)$, we have

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega^u} \varepsilon T^{\varepsilon}\left(\frac{\partial u_{\varepsilon}^{+}}{\partial x_2}\right)\psi(x,y_1) T^{\varepsilon}\chi_{\mathrm{I}_{\varepsilon}} = \int\limits_{\Omega^u} z_2(x,y_1)\psi(x,y_1)\chi_{\mathrm{I}} \tag{7}$$

But, using properties of derivative of unfolding operator and applying integration by parts in the left hand side of (7), we get

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega^u} \varepsilon T^{\varepsilon}\left(\frac{\partial u_{\varepsilon}^{+}}{\partial x_2}\right)\psi(x,y_1) T^{\varepsilon}\chi_{\mathrm{I}_{\varepsilon}} = -\lim_{\varepsilon\to 0} \int\limits_{\Omega^u} \varepsilon T^{\varepsilon} u_{\varepsilon}^{+}\left(\frac{\partial \psi}{\partial x_2}\right) T^{\varepsilon}\chi_{\mathrm{I}_{\varepsilon}} = 0.$$

This implies that $\int\limits_{\Omega^u} z_2(x,y_1)\psi(x,y_1)\chi_{\mathrm{I}} = \int\limits_{\Omega^u_{\mathrm{I}}} z_2(x,y_1)\psi(x,y_1) = 0$. Since $\psi \in C_c^{\infty}(\Omega^u)$ is arbitrary, we get the required the result. □

Step 5: This step is devoted to prove that $\eta_1 = 0$ a.e. in Ω^u_{C}. Taking $\psi_{\varepsilon}(x) = \varepsilon\psi\left(x, \frac{x_1}{\varepsilon}\right)$ as a test function in (1) where $\psi \in C_c^{\infty}(\Omega^{+}; C_{per}^{\infty}(0,1))$ and $\psi = 0$ on Ω^u_{I}, applying unfolding operator and letting $\varepsilon \to 0$ to get

$$\int\limits_{\Omega^u_{\mathrm{C}}} \eta_1 \frac{\partial \psi}{\partial y_1} = 0. \tag{8}$$

Since ψ is arbitrary implies that $\eta_1 = 0$ a.e. in Ω^u_{C}.

Step 6: Now, we will show $u^{+} = u^{-}$ on γ. Let $\phi \in C^{\infty}(\overline{\Omega^u_{\mathrm{C}}})$ such that $\phi = 0$ on $\partial\Omega^u_{\mathrm{C}} \backslash \gamma^u_{\mathrm{C}}$. A simple integration by parts gives the following

$$\int\limits_{\Omega^u_{\mathrm{C}}} T^{\varepsilon}\left(\frac{\partial u_{\varepsilon}^{+}}{\partial x_2}\right)\phi\, dx dy_1 = -\int\limits_{\Omega^u_{\mathrm{C}}} T^{\varepsilon} u_{\varepsilon}^{+}\frac{\partial \phi}{\partial x_2} dx dy_1 + \int\limits_{\gamma^u_{\mathrm{C}}} T_b^{\varepsilon}(u_{\varepsilon}^{+})\phi.$$

Since $u_{\varepsilon}^{+} = u_{\varepsilon}^{-}$ on $\gamma_{\mathrm{C}}^{\varepsilon}$ and with a simple computation, we get

$$\int\limits_{\gamma^u_{\mathrm{C}}} u^{+}\phi = \int\limits_{\gamma^u_{\mathrm{C}}} u^{-}\phi, \quad \text{for all } \phi \in C^{\infty}(\overline{\Omega^u_{\mathrm{C}}}) \text{ with } \phi = 0 \; on \; \partial\overline{\Omega^u_{\mathrm{C}}} \backslash \gamma^u_{\mathrm{C}}.$$

Hence we have $u^{+} = u^{-}$ on γ^u_{C}. Since u^{+} and u^{-} are independent of y_1, we have $u^{+} = u^{-}$ on γ.

Step 7: In this step, we will show that $z_1 = \dfrac{\partial u_1}{\partial y_1}$. Let $\psi_{\varepsilon}(x) = \psi\left(x, \frac{x_1}{\varepsilon}\right)$ where $\psi \in C_c^{\infty}(\Omega^u)$ with 1-periodic in y_1 and $\psi = 0$ on Ω^u_{C}. We also easily see that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+} \varepsilon\chi_{I_\varepsilon}\frac{\partial u_\varepsilon^+}{\partial x_1}\psi_\varepsilon = -\int_{\Omega^u}(u^+ + u_1(x,y_1))\chi_I(y_1,x_2)\frac{\partial \psi}{\partial y_1}.$$

On the other hand, we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+} \varepsilon\chi_{I_\varepsilon}\frac{\partial u_\varepsilon^+}{\partial x_1}\psi_\varepsilon = \lim_{\varepsilon\to 0}\int_{\Omega^u} T^\varepsilon\left(\varepsilon\chi_{I_\varepsilon}\frac{\partial u_\varepsilon^+}{\partial x_1}\right)T^\varepsilon\psi = \int_{\Omega^u}\chi_I(y_1,x_2)z_1(x,y_1)\psi(x,y_1)$$

Hence we have,

$$\int_{\Omega^u}\chi_I(y_1,x_2)z_1(x,y_1)\psi(x,y_1) = -\int_{\Omega^u}\chi_I(u^+ + u_1(x,y_1))\frac{\partial\psi}{\partial y_1}$$

Since ψ is arbitrary we get $z_1(x,y_1) = \dfrac{\partial u_1}{\partial y_1}$ a.e. in Ω_I^u.

Step 8: In this final step, we derive the limit equations using the results obtained in the previous steps. Let $\phi_\varepsilon(x) = \phi(x) + \phi_1\left(x, \frac{x_1}{\varepsilon}\right)$ where $\phi \in C^1(\bar{\Omega})$ and $\phi_1 \in C^\infty(\overline{\Omega^u})$ with 1 periodic in y_1 variable and $\phi_1 = 0$ on Ω_C^u. Now using ϕ_ε as a test function in (1), applying unfolding operator both side and letting $\varepsilon \to 0$ to get

$$\begin{aligned}\int_{\Omega_C^u}\frac{\partial u^+}{\partial x_2}\frac{\partial\phi}{\partial x_2} + \int_{\Omega_I^u}\frac{\partial u_1}{\partial y_1}\frac{\partial\phi_1}{\partial y_1} + \int_{\Omega^u}(u^+ + u_1)(\phi+\phi_1) + \int_{\Omega^-}(\nabla u^-\nabla\phi + u\phi)& \\ = \int_{\Omega^u} f(\phi+\phi_1) + \int_{\Omega^-} f\phi&\end{aligned} \tag{9}$$

Put $\phi = 0$ in the above equality to get,

$$\int_{\Omega_I^u}\frac{\partial u_1}{\partial y_1}\frac{\partial\phi_1}{\partial y_1} + \int_{\Omega^u}(u^+ + u_1)\phi_1 = \int_{\Omega^u} f\phi_1.$$

Now using the cell problem as in (3) and by uniqueness, we may write

$$u_1(x,y_1) = (f(x) - u^+(x))\xi(x_2,y_1).$$

Now if we put $\phi_1 = 0$ in (9), we get

$$\int_{\Omega_C^u}\frac{\partial u^+}{\partial x_2}\frac{\partial\phi}{\partial x_2} + \int_{\Omega^u}(u^+ + u_1)\phi + \int_{\Omega^-}(\nabla u^-\nabla\phi + u\phi) = \int_{\Omega^u} f\phi + \int_{\Omega^-} f\phi \tag{10}$$

Hence using the cell problem (3), the equation (9) reduces to

$$\int_{\Omega^u_{\rm C}} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u} (u^+ + (f-u^+)\xi)\phi + \int_{\Omega^-} (\nabla u^- \nabla \phi + u\phi) = \int_{\Omega^u} f\phi + \int_{\Omega^-} f\phi$$

which is same as

$$\int_{\Omega^+} |Y_{\rm c}(x_2)| \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \left(|Y(x_2)| - \int_{Y_{\rm I}(x_2)} \xi \, dy_1 \right) u^+\phi + \int_{\Omega^-} (\nabla u^- \nabla \phi + u\phi)$$

$$= \int_{\Omega^+} \left(|Y(x_2)| - \int_{Y_{\rm I}(x_2)} \xi \, dy_1 \right) f\phi + \int_{\Omega^-} f\phi.$$

This equation has a unique solution as $|Y_{\rm c}(x_2)| > \delta$ and $\left(|Y(x_2)| - \int_{Y_{\rm I}(x_2)} \xi\right) > |Y_{\rm c}(x_2)|$. To see the positivity of $\left(|Y(x_2)| - \int_{Y_{\rm I}(x_2)} \xi \right)$, take $w = \xi$ in the cell problem (3) to see that $\|\xi\|_{L^2(Y_{\rm I}(x_2))} < |Y_{\rm I}(x_2)|^{1/2}$. Now,

$$|Y(x_2)| - \int_{Y_{\rm I}(x_2)} \xi > (|Y(x_2)| - |Y_{\rm I}(x_2)|^{1/2} \|\xi\|_{L^2(Y_{\rm I}(x_2))}) > |Y(x_2)| - |Y_{\rm I}(x_2)|.$$

From the geometrical assumption on Λ, we have, $|Y(x_2)| - |Y_{\rm I}(x_2)| = Y_{\rm c}(x_2) > \delta$ for some $\delta > 0$. Hence our limit problem has unique solution. Thus, the proof of the theorem is completed. □

Remark 4.2.

1. In the above variational problem we have considered the contrasting diffusive coefficients as 1 and ε^2. In fact, we can consider the coefficient of the form $O(1)$ and α_ε^2, where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. According to the limit $k = \lim\limits_{\varepsilon\to 0} \frac{\alpha_\varepsilon}{\varepsilon}$, we will get three different limit problems for, $k = 0, k = \infty$ and $k \in (0, \infty)$. What we have studied is essentially the case, where $k \in (0, \infty)$, that is with $\alpha_\varepsilon = \varepsilon$ and hence the exact proof can be reproduced with minor changes. The coefficient of the second order term in the cell problem (3) will be k^2 instead of 1. The other two cases can also be handled with minor modifications of the proof of Theorem 4.1 which we do not present it here. Here we are presenting the case when $k = 1$, that is $\alpha_\varepsilon = \varepsilon$.
2. Here we have considered the PDE with principal part as a Laplace operator in 2-dimension. This is only to make the presentation simpler. We can carry out all the results in any finite dimension with more general linear elliptic PDE with principal part as div$(A(x) \cdot \nabla)$ where $A(x)$ are uniformly bounded and elliptic $n \times n$ matrices with high diffusivity coefficients. For this we have to make use of the Lemma 7.5 and 7.6, proven in one of our recent article [36].
3. ***Strong convergence:*** *If we assume M and C are regular enough to get the regularity of ξ, then we can actually prove a corrector result as follows. Let u_1 as defined in (6). Then, we have the following convergence as*

$$\lim_{\varepsilon\to 0}\left(\left\|\frac{\partial u_\varepsilon^+}{\partial x_2}-\frac{\partial u^+}{\partial x_2}\right\|_{L^2(C_\varepsilon)}+\left\|\frac{\partial u_\varepsilon^+}{\partial x_1}-\frac{\partial u^+}{\partial x_1}\right\|_{L^2(C_\varepsilon)}\right)=0,$$

$$\lim_{\varepsilon\to 0}\left\|\varepsilon\nabla u_\varepsilon-\begin{bmatrix}\frac{\partial u_1}{\partial y_1}\left(x,\left\{\frac{x_1}{\varepsilon}\right\}\right)\\ 0\end{bmatrix}\right\|_{L^2(I_\varepsilon)}=0,$$

$$\lim_{\varepsilon\to 0}\|u_\varepsilon^+-u^+\|_{L^2(C_\varepsilon)}=0,$$

$$\lim_{\varepsilon\to 0}\left\|u_\varepsilon^+-u^+-u_1\left(x,\left\{\frac{x_1}{\varepsilon}\right\}\right)\right\|_{L^2(I_\varepsilon)}=0.$$

The above convergence follows due to the energy convergence, but we do not have such strong convergence for the interior optimal control problem.

5. Optimal control problem

We now consider the optimal control problem associated with strong diffusivity terms in the coefficient. This is not done so far in the literature. We apply control in the strong diffusivity region C_ε. Let $u_d\in L^2(\Omega)$ be the desired state. For $\theta_\varepsilon\in L^2(C_\varepsilon)$ consider the cost functional

$$J_\varepsilon(u_\varepsilon,\theta_\varepsilon)=\frac{1}{2}\int_{\Omega_\varepsilon}|u_\varepsilon-u_d|^2+\frac{\beta}{2}\int_{C_\varepsilon}|\theta_\varepsilon|^2$$

where u_ε is the unique solution of the following variational problem: for $f\in L^2(\Omega)$

$$\begin{cases}\text{find } u_\varepsilon\in H^1(\Omega_\varepsilon) \text{ such that}\\ \int_{\Omega_\varepsilon}\left(\chi_{\Omega^-}+\chi_{C_\varepsilon}+\varepsilon^2\chi_{I_\varepsilon}\right)\nabla u_\varepsilon\nabla\phi+u_\varepsilon\phi=\int_{\Omega_\varepsilon}f\phi+\int_{\Omega_\varepsilon}\chi_{C_\varepsilon}\theta_\varepsilon\phi,\quad\text{for all }\phi\in H^1(\Omega_\varepsilon)\end{cases}\tag{11}$$

The optimal control problem is to find $(\bar u_\varepsilon,\bar\theta_\varepsilon)\in H^1(\Omega_\varepsilon)\times L^2(C_\varepsilon)$ such that

$$J_\varepsilon(\bar u_\varepsilon,\bar\theta_\varepsilon)=\inf\{J_\varepsilon(u_\varepsilon,\theta_\varepsilon)\}.\tag{12}$$

Here u_ε varies as in (11) as θ_ε varies in $L^2(C_\varepsilon)$.

Theorem 5.1. *For every $\varepsilon>0$ the optimal control problem* (12) *admits a unique solution* $(\bar u_\varepsilon,\ \bar\theta_\varepsilon)\in H^1(\Omega_\varepsilon)\times L^2(C_\varepsilon)$.

The above theorem is quite standard as we are considering quadratic cost functional together with the elliptic system. In order to analyze the asymptotic behavior of $(\bar u_\varepsilon,\bar\theta_\varepsilon)$ we will use the characterization of optimal control $\bar\theta_\varepsilon$ by introducing the adjoin state $\bar v_\varepsilon$ which is the solution of the following variational form

$$\begin{cases}\text{find } \bar v_\varepsilon\in H^1(\Omega_\varepsilon) \text{ such that}\\ \int_{\Omega_\varepsilon}\left(\chi_{\Omega^-}+\chi_{C_\varepsilon}+\varepsilon^2\chi_{I_\varepsilon}\right)\nabla\bar v_\varepsilon\nabla\phi+\bar v_\varepsilon\phi=\int_{\Omega_\varepsilon}(\bar u_\varepsilon-u_d)\phi,\quad\text{for all }\phi\in H^1(\Omega_\varepsilon).\end{cases}\tag{13}$$

Then, we have the following theorem which is also quite standard for each $\varepsilon > 0$.

Theorem 5.2. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (12) and $\bar{v}_\varepsilon$ be the unique solution of (13). Then $\bar{\theta}_\varepsilon$ is characterized by*

$$\bar{\theta}_\varepsilon = -\chi_{C_\varepsilon} \frac{1}{\beta} \bar{v}_\varepsilon. \tag{14}$$

Conversely, let $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ satisfy the following coupled system

$$\begin{cases} \int\limits_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla \hat{u}_\varepsilon \nabla \phi + \hat{u}_\varepsilon \phi = \int\limits_{\Omega_\varepsilon} f\phi + \int\limits_{\Omega_\varepsilon} \chi_{C_\varepsilon} \hat{\theta}_\varepsilon \phi, \\ \int\limits_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla \hat{v}_\varepsilon \nabla \phi + \hat{v}_\varepsilon \phi = \int\limits_{\Omega_\varepsilon} (\hat{u}_\varepsilon - u_d)\phi, \\ \hat{\theta}_\varepsilon = -\chi_{C_\varepsilon} \frac{1}{\beta} \hat{v}_\varepsilon, \end{cases} \tag{15}$$

for all $\phi \in H^1(\Omega_\varepsilon)$. Then $(\hat{u}_\varepsilon, \hat{\theta}_\varepsilon)$ is the optimal solution to the optimal control problem (12).

Hence, in order to study the asymptotic behavior of the optimal control problem (12), it is sufficient to analyze asymptotic behavior of the optimality system (15).

5.1. Limit control problem

In this subsection, we will state the limit control problem in the extended domain. For controls $\theta \in L^2(\Omega^+)$, consider the following L^2 cost functional

$$J(u, \theta) = \frac{1}{2} \int\limits_{\Omega^u} |u^+ + u_1 - u_d|^2 + \frac{1}{2} \int\limits_{\Omega^-} |u^- - u_d|^2 + \frac{\beta}{2} \int\limits_{\Omega^u_{\mathrm{C}}} |\theta|^2$$

where $(u, u_1) \in H(\Omega) \times V(\Omega)$ satisfies the following micro-macro state system

$$\begin{cases} \text{find } (u, u_1) \in H(\Omega) \times V(\Omega) \text{ such that} \\ \int\limits_{\Omega^u_{\mathrm{C}}} \frac{\partial u^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int\limits_{\Omega^u_{\mathrm{I}}} \frac{\partial u_1}{\partial y_1} \frac{\partial \phi_1}{\partial y_1} + \int\limits_{\Omega^u} (u^+ + u_1)(\phi + \phi_1) + \int\limits_{\Omega^-} (\nabla u^- \nabla \phi + u\phi) \\ \quad = \int\limits_{\Omega^u} (f + \chi_{\mathrm{C}}(y_1, x_2)\theta)(\phi + \phi_1) + \int\limits_{\Omega^-} f\phi, \quad \text{for all } (\phi, \phi_1) \in H(\Omega) \times V(\Omega). \end{cases} \tag{16}$$

Thus, the limit control system is defined as a two-scale system. We will separate the scales in the last section of the article to get the optimal control system in the macro variable. Lax-Milgram lemma guarantees the existence and uniqueness. To be more precise, here the continuous elliptic bi-linear form say $\mathscr{F} : (H(\Omega) \times V(\Omega)) \times (H(\Omega) \times V(\Omega)) \to \mathbb{R}$ is given by

$$\mathscr{F}((u,u_1),(\phi,\phi_1)) = \int_{\Omega^u_{\mathrm{C}}} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u_{\mathrm{I}}} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (u^+ + u_1)(\phi+\phi_1) + \int_{\Omega^-} (\nabla u^- \nabla \phi + u\phi),$$

and the continuous linear functional $F : H(\Omega) \times V(\Omega) \to \mathbb{R}$, given by

$$F(\phi,\phi_1) = \int_{\Omega^u} (f + \chi_{\mathrm{C}}(y_1,x_2)\theta)(\phi+\phi_1) + \int_{\Omega^-} f\phi.$$

The norm for any $(\phi,\phi_1) \in H(\Omega) \times V(\Omega)$ is given by $\|(\phi,\phi_1)\|_{H(\Omega)\times V(\Omega)} = \|\phi\|_{H(\Omega)} + \|\phi_1\|_{V(\Omega)}$.

Now the optimal control problem is to find $(\bar{u},\bar{u}_1,\bar{\theta}) \in H(\Omega) \times V(\Omega) \times L^2(\Omega^+)$ such that

$$J(\bar{u},\bar{u}_1,\bar{\theta}) = \inf\{J(u,u_1,\theta) : (u,u_1,\theta) \text{ satisfies (16)}\}. \tag{17}$$

A similar type of characterization theorem holds like ε level problem. The corresponding two-scale adjoint state equation for the above limit state equation is

$$\begin{cases} \text{find } (v,v_1) \in H(\Omega)\times V(\Omega) \text{ such that,} \\ \displaystyle\int_{\Omega^u_{\mathrm{C}}} \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u_{\mathrm{I}}} \frac{\partial v_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (v^+ + v_1)(\phi+\phi_1) + \int_{\Omega^-} (\nabla v^- \nabla\phi + v^-\phi) \\ \displaystyle\qquad = \int_{\Omega^u} (u^+ + u_1 - u_d)(\phi+\phi_1) + \int_{\Omega^-} (u^- - u_d)\phi. \end{cases} \tag{18}$$

Now we will state the characterization theorem and it can be proved in a very standard fashion and hence we omit it here.

Theorem 5.3. *Let $(\bar{u},\bar{u}_1,\bar{\theta})$ and $(\bar{v},\bar{v}_1)$ be the solutions of (17) and (18), then optimal control $\bar{\theta}$ is given by*

$$\chi_c(y_1,x_2)\bar{\theta}(x) = -\chi_c(y_1,x_2)\frac{1}{\beta}\bar{v}(x). \tag{19}$$

Conversely, suppose (u,u_1,θ), (v,v_1) satisfies the following system,

$$
\begin{cases}
\text{for all } (\phi,\phi_1)\in H(\Omega)\times V(\Omega),\\
\int\limits_{\Omega_C^u} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int\limits_{\Omega_I^u} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int\limits_{\Omega^u} (u^+ + u_1)(\phi+\phi_1) + \int\limits_{\Omega^-} (\nabla u^- \nabla \phi + u\phi)\\
\qquad\qquad = \int\limits_{\Omega^u} (f + \chi_c(y_1,x_2)\theta)(\phi+\phi_1) + \int\limits_{\Omega^-} f\phi,\\
\int\limits_{\Omega_C^u} \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int\limits_{\Omega_I^u} \frac{\partial v_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int\limits_{\Omega^u} (v^+ + v_1)(\phi+\phi_1) + \int\limits_{\Omega^-} (\nabla v^- \nabla \phi + v^-\phi)\\
\qquad\qquad = \int\limits_{\Omega^u} (u^+ + u_1 - u_d)(\phi+\phi_1) + \int\limits_{\Omega^-} (u^- - u_d)\phi,\\
\text{and } \theta = -\frac{1}{\beta} v^+ \text{ in } \Omega_c^u.
\end{cases}
\tag{20}
$$

Then, (u, u_1, θ) *is the optimal solution to the optimal control problem* (17).

Our aim is to pass to the limit in the limit in the optimality system (15) which will turn out to be the limit optimality system (20), which we state and prove below.

Theorem 5.4. *For every* $\varepsilon > 0$*, let* $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ *be the optimal solution to the optimal control problem* (12) *and* $\bar{v}_\varepsilon$ *be the adjoint state. Let* $H(\Omega)$ *and* $V(\Omega)$ *be defined as in Section 3. Then, we have the following convergences as* $\varepsilon \to 0$.

$$
\begin{cases}
\bar{u}_\varepsilon^- \rightharpoonup u^-,\ \bar{v}_\varepsilon^- \rightharpoonup v^- \text{ weakly in } H^1(\Omega^-),\\
T^\varepsilon \bar{u}_\varepsilon^+ \rightharpoonup u^+(x) + u_1(x,y_1),\ T^\varepsilon \bar{v}_\varepsilon^+ \rightharpoonup v^+(x) + v_1(x,y_1), \text{ weakly in } L^2(\Omega^u)\\
T^\varepsilon \chi_{C_\varepsilon} T^\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_2} \rightharpoonup \chi_c(y_1,x_2)\frac{\partial u^+}{\partial x_2},\ T^\varepsilon \chi_{C_\varepsilon} T^\varepsilon \frac{\partial v_\varepsilon^+}{\partial x_2} \rightharpoonup \chi_c(y_1,x_2)\frac{\partial u^+}{\partial x_2}, \text{ weakly in } L^2(\Omega^u)\\
\varepsilon T^\varepsilon \chi_{I_\varepsilon} T^\varepsilon \frac{\partial \bar{u}_\varepsilon^+}{\partial x_2} \rightharpoonup 0,\ \varepsilon T^\varepsilon \chi_{I_\varepsilon} T^\varepsilon \frac{\partial \bar{v}_\varepsilon^+}{\partial x_2} \rightharpoonup 0, \text{ weakly in } L^2(\Omega^u)\\
T^\varepsilon \chi_{C_\varepsilon} T^\varepsilon \frac{\partial \bar{u}_\varepsilon^+}{\partial x_1} \rightharpoonup 0,\ T^\varepsilon \chi_{C_\varepsilon} T^\varepsilon \frac{\partial \bar{v}_\varepsilon^+}{\partial x_1} \rightharpoonup 0, \text{ weakly in } L^2(\Omega^u)\\
\varepsilon T^\varepsilon \chi_{I_\varepsilon} T^\varepsilon \frac{\partial \bar{u}_\varepsilon^+}{\partial x_1} \rightharpoonup \frac{\partial u_1}{\partial y_1},\ \varepsilon T^\varepsilon \chi_{I_\varepsilon} T^\varepsilon \frac{\partial \bar{v}_\varepsilon^+}{\partial x_1} \rightharpoonup \frac{\partial v_1}{\partial y_1} \text{ weakly in } L^2(\Omega^u)\\
\chi_c(y_1,x_2) T^\varepsilon \bar{\theta}_\varepsilon \rightharpoonup -\chi_c(y_1,x_2)\frac{1}{\beta} v^+ \text{ weakly in } L^2(\Omega^u)
\end{cases}
\tag{21}
$$

Here $\left(\left(u, u_1, -\frac{1}{\beta}v^+\right), (v, v_1)\right)$ *is the unique solution to the optimality system* (20).

Proof. As usual, we present the proof in several steps for easy reading.

Step 1: (Uniform boundedness) As $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ is the optimal solution to the optimal control problem, we have $J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \leqslant J_\varepsilon(u_0, 0)$, where u_0 is the solution of (11) corresponding to $\theta_\varepsilon = 0$. This gives the uniform bound on $\|\bar{\theta}_\varepsilon\|_{L^2(C_\varepsilon)}$. In (11) choose $\phi = \bar{u}_\varepsilon$ to get

$$\begin{cases} \|\chi_{C_\varepsilon^+} \nabla \bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \varepsilon \|\chi_{I_\varepsilon^+} \nabla \bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \|\nabla \bar{u}_\varepsilon\|_{L^2(\Omega^-)} + \|\bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ \qquad \leqslant \|f\|_{L^2(\Omega_\varepsilon)} + \|\chi_{C_\varepsilon} \bar{\theta}_\varepsilon\|_{L^2(\Omega_\varepsilon^+)}. \end{cases} \tag{22}$$

Using the uniform bound on $\|\bar{\theta}_\varepsilon\|_{L^2(C_\varepsilon)}$, we obtain the uniform bounds on

$$\|\chi_{C_\varepsilon^+} \nabla \bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^+)}, \quad \varepsilon \|\chi_{I_\varepsilon^+} \nabla \bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^+)}, \quad \|\nabla \bar{u}_\varepsilon\|_{L^2(\Omega^-)}, \ \|\bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

Similarly, we have uniform bounds on

$$\|\chi_{C_\varepsilon^+} \nabla \bar{v}_\varepsilon\|_{L^2(\Omega_\varepsilon^+)}, \quad \varepsilon \|\chi_{I_\varepsilon^+} \nabla \bar{v}_\varepsilon\|_{L^2(\Omega_\varepsilon^+)}, \quad \|\nabla \bar{v}_\varepsilon\|_{L^2(\Omega^-)}, \quad \|\bar{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

Step 2: (Convergence of subsequences) Thus, by the uniform bounds and by the properties of unfolding operator there exist $u^-, v^- \in H^1(\Omega^-)$, $u_0(x, y_1), v_0(x, y_1) \in L^2(\Omega^u)$ and $\eta(x, y_1), \sigma(x, y_1)), z(x, y_1), \mu(x, y_1) \in (L^2(\Omega^u))^2$ such that

$$\begin{cases} \bar{u}_\varepsilon \rightharpoonup u^- \text{ in } H^1(\Omega^-), \\ T^\varepsilon(\bar{u}_\varepsilon^+) \rightharpoonup u_0(x, y_1) \text{ in } L^2(\Omega^u), \\ T^\varepsilon(\chi_{C_\varepsilon} \widetilde{\nabla \bar{u}_\varepsilon}) = T_{\mathrm{C}}^\varepsilon(\widetilde{\nabla \bar{u}_\varepsilon}) \rightharpoonup \eta(x, y_1) = (\eta_1, \eta_2) \text{ in } (L^2(\Omega_{\mathrm{C}}^u))^2, \\ T^\varepsilon(\varepsilon \chi_{I_\varepsilon} \widetilde{\nabla \bar{u}_\varepsilon}) \rightharpoonup \chi_{\mathrm{I}}(y_1, x_2) z(x, y_1) = \chi_{\mathrm{I}}(y_1, x_2)(z_1, z_2) \text{ in } (L^2(\Omega^u))^2, \end{cases}$$

$$\begin{cases} \bar{v}_\varepsilon \rightharpoonup v^- \text{ in } H^1(\Omega^-), \\ T^\varepsilon(\bar{v}_\varepsilon^+) \rightharpoonup v_0(x, y_1) \text{ in } L^2(\Omega^u), \\ T^\varepsilon(\chi_{C_\varepsilon} \widetilde{\nabla \bar{v}_\varepsilon}) = T_{\mathrm{C}}^\varepsilon(\widetilde{\nabla \bar{v}_\varepsilon}) \rightharpoonup \sigma(x, y_1) = (\sigma_1, \sigma_2) \text{ in } (L^2(\Omega_{\mathrm{C}}^u))^2, \\ T^\varepsilon(\varepsilon \chi_{I_\varepsilon} \widetilde{\nabla \bar{v}_\varepsilon}) \rightharpoonup \chi_{\mathrm{I}}(y_1, x_2) \mu(x, y_1) = \chi_{\mathrm{I}}(y_1, x_2)(\mu_1, \mu_2) \text{ in } (L^2(\Omega^u))^2. \end{cases}$$

Here $\sim$ denotes the extension by zero to all of Ω^+.

Step 3: As in the proof of Theorem 4.1, we may see that u_0 is independent of y_1 in Ω_{C}^u and there exist $u^+ \in L^2(\Omega^+)$, $u_1 \in L^2(\Omega^u)$ with $\dfrac{\partial u_1}{\partial y_1} \in L^2(\Omega^u)$ and $u_1 = 0$ a.e. in Ω_{C}^u such that

$$u_0(x, y) = u^+(x) + u_1(x, y_1).$$

Similarly v_0 is independent of y_1 in Ω_{C}^u and there exist $v^+ \in L^2(\Omega^+)$, $v_1 \in L^2(\Omega^u)$ with $\dfrac{\partial v_1}{\partial y_1} \in L^2(\Omega^u)$ and $v_1 = 0$ a.e. in Ω_{C}^u such that

$$v_0(x, y) = v^+(x) + v_1(x, y_1).$$

Step 4: In this step, we will show that

(i) $\frac{\partial u^+}{\partial x_2}, \frac{\partial v^+}{\partial x_2} \in L^2(\Omega^+)$.

(ii) $\eta_2(x, y_1) = \frac{\partial u^+}{\partial x_2}, \ \sigma_2(x, y_1) = \frac{\partial v^+}{\partial x_2}$ a.e. Ω^u_{C}.

(iii) $z_2(x, y_1) = 0, \ \mu_2(x, y_1) = 0$, a.e., in Ω^u_{I}.

Proof of this step is similar to the Step 4 in the proof of Theorem 4.1. For the sake of completeness, we will show it for the adjoint state. Proof for (i) and (ii) follows from the properties of $T^\varepsilon_{\mathrm{C}}$. For the proof of (iii), the estimate (22), gives $\varepsilon T^\varepsilon v^+_\varepsilon \to 0$ in $L^2(\Omega^u)$. Now, for $\psi \in C^\infty_c(\Omega^u)$, we have

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega^u} \varepsilon T^\varepsilon \left(\frac{\partial v^+_\varepsilon}{\partial x_2}\right) \psi(x, y_1) T^\varepsilon \chi_{\mathrm{I}_\varepsilon} = \int\limits_{\Omega^u} \mu_2(x, y_1)\psi(x, y_1)\chi_{\mathrm{I}} \tag{23}$$

But, using properties of derivative of unfolding operator and applying integration by parts on left hand side to get

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega^u} \varepsilon T^\varepsilon \left(\frac{\partial v^+_\varepsilon}{\partial x_2}\right) \psi(x, y_1) T^\varepsilon \chi_{\mathrm{I}_\varepsilon} = -\lim_{\varepsilon\to 0} \int\limits_{\Omega^u} \varepsilon T^\varepsilon v^+_\varepsilon \left(\frac{\partial \psi}{\partial x_2}\right) T^\varepsilon \chi_{\mathrm{I}_\varepsilon} = 0$$

This implies that $\int\limits_{\Omega^u} \mu_2(x, y_1)\psi(x, y_1)\chi_{\mathrm{I}} = \int\limits_{\Omega^u_{\mathrm{I}}} \mu_2(x, y_1)\psi(x, y_1) = 0$. The result (iii) follows as $\psi \in C^\infty_c(\Omega^u)$ is arbitrary.

Step 5: This step is devoted to prove that $\eta_1 = 0$, and $\sigma_1 = 0$ a.e. in Ω^u_{C}. Taking $\psi_\varepsilon(x) = \varepsilon\psi\left(x, \frac{x_1}{\varepsilon}\right)$ as a test function in (13), where $\psi \in C^\infty_c(\Omega^+; C^\infty_{per}(0, 1))$ and $\psi = 0$ on Ω^u_{I}, applying unfolding operator and letting $\varepsilon \to 0$ to get

$$\int\limits_{\Omega^u_{\mathrm{C}}} \sigma_1(x, y_1)\frac{\partial \psi}{\partial y_1} = 0. \tag{24}$$

The arbitrariness of ψ implies the claim. For η_1, the proof is same as in the proof of Theorem 4.1. Similarly as in Step 7 of Theorem 4.1, we have $z_1 = \frac{\partial u_1}{\partial y_1}, \ \mu_1 = \frac{\partial v_1}{\partial y_1}$.

Step 6: (Convergence of optimal control) From the characterization of optimal control, we have $\bar{\theta}_\varepsilon(x) = -\frac{1}{\beta}\chi_{\mathrm{C}_\varepsilon}(y_1, x_2)\bar{v}_\varepsilon$. Hence by applying unfolding operator on both sides we have $\chi_{\mathrm{C}}(y_1, x_2)T^\varepsilon\bar{\theta} = -\frac{1}{\beta}\chi_{\mathrm{C}}(y_1, x_2)T^\varepsilon\bar{v}_\varepsilon$. Now from the definition of restricted unfolding operator we have $\chi_{\mathrm{C}}(y_1, x_2)T^\varepsilon\bar{\theta}_\varepsilon = -\frac{1}{\beta}\chi_{\mathrm{C}}(y_1, x_2)T^\varepsilon_{\mathrm{C}}\bar{v}_\varepsilon$. Hence we have

$$\chi_{\mathrm{C}}(y_1, x_2)T^\varepsilon\bar{\theta}_\varepsilon(x) \rightharpoonup -\frac{1}{\beta}\chi_{\mathrm{C}}(y_1, x_2)v^+(x) \ \text{ weakly in } L^2(\Omega^u).$$

Step 7: Let $\phi_\varepsilon(x) = \phi(x) + \phi_1\left(x, \frac{x_1}{\varepsilon}\right)$, where $\phi \in C^1(\bar{\Omega})$ and $\phi_1 \in C^\infty(\overline{\Omega^u})$ with 1 periodic in y_1 variable and $\phi_1 = 0$ on Ω^u_{C}. Now using ϕ_ε as a test function in (11) with $\theta_\varepsilon = \bar{\theta}_\varepsilon$ and letting $\varepsilon \to 0$ to get

$$\int_{\Omega^u_{\mathrm{C}}} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u_{\mathrm{I}}} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (u^+ + u_1)(\phi + \phi_1) + \int_{\Omega^-} (\nabla u^- \nabla \phi + u\phi)$$
$$= \int_{\Omega^u} \left(f - \chi_{\mathrm{C}}(y_1, x_2)\frac{1}{\beta}v^+\right)(\phi + \phi_1) + \int_{\Omega^-} f\phi. \tag{25}$$

Similarly for adjoint state we get,

$$\int_{\Omega^u_{\mathrm{C}}} \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u_{\mathrm{I}}} \frac{\partial v_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (v^+ + v_1)(\phi + \phi_1) + \int_{\Omega^-} (\nabla v^- \nabla \phi + v\phi)$$
$$= \int_{\Omega^u} (u^+ + u_1 - u_d)(\phi + \phi_1) + \int_{\Omega^-} (u^- - u_d)\phi. \tag{26}$$

By combining the equations (25) and (26) we obtain the optimality system (20). Since $(\phi, \phi_1) \in C^1(\bar{\Omega}) \times C^\infty(\overline{\Omega^u})$ is arbitrary, hence by density it is true for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$. □

6. Separation of scales

In Section 5, we have obtained the micro-macro two-scale system. We need to separate the scales x and y_1, and obtain the corresponding homogenized system and cell problem. That is, we have separate the cell problem in micro variable y_1 to obtain the homogenized system in macro variable. As we have mentioned in the introduction, such a separation may not be always possible as in a hyperbolic system (see [8]). In our case, we could separate the scales by introducing a second cell problem. First, recall the cell problem defined in Section 5 as:

$$\begin{cases} \text{find } \xi \in V^{x_2} \text{ such that} \\ \displaystyle\int_{Y(x_2)} \frac{\partial \xi}{\partial y_1}\frac{\partial w}{\partial y_1} dy_1 + \int_{Y(x_2)} \xi w dy_1 = \int_{Y(x_2)} w dy_1 \quad \text{for all } w \in V^{x_2}. \end{cases} \tag{27}$$

We now introduce another cell problem in the insulating region which gives contribution due to I. Let $w_1 = w_1(x_2, y_1)$ be the unique solution of the following cell problem

$$\begin{cases} w_1(x_2, y_1) \in V^{x_2} \text{ such that} \\ \displaystyle\int_I \frac{\partial w_1}{\partial y_1}\frac{\partial w}{\partial y_1} + \int_I w_1(x_2, y_1) w = \int_I \xi w \quad \text{for all } w \in V^{x_2}. \end{cases} \tag{28}$$

In the context of scale separation in the present article, the main difficulty arises due to the adjoint state. The limit state equation we can separate like Theorem 4.1. Let us first look into the limit state equation. Put $\phi = 0$ in (25) to get,

$$\int_{\Omega_{\mathrm{I}}^u} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (u^+ + u_1)\phi_1 = \int_{\Omega^u} f\phi_1 - \int_{\Omega^u} \frac{1}{\beta}\chi_C(y_1, x_2)v^+\phi_1.$$

Since $\phi_1 = 0$ a.e. in Ω_{C}^u, above equality can be written as

$$\int_{\Omega_{\mathrm{I}}^u} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega_{\mathrm{I}}^u} u_1\phi_1 = \int_{\Omega_{\mathrm{I}}^u} (f - u^+)\phi_1.$$

By uniqueness, we can write $u_1(x, y_1) = (f(x) - u^+(x))\xi(x_2, y_1)$. Now put $\phi_1 = 0$, in (25) to get

$$\begin{aligned}\int_{\Omega_{\mathrm{C}}^u} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u} (u^+ + u_1)\phi + \int_{\Omega^-} (\nabla u^- \nabla\phi + u\phi)\\ = \int_{\Omega^u} f\phi + \int_{\Omega^-} f\phi - \int_{\Omega^+} |Y_{\mathrm{c}}(x_2)|\frac{1}{\beta}v^+\phi.\end{aligned} \tag{29}$$

Hence using the cell problem (27), we can write (29) as

$$\begin{aligned}\int_{\Omega_{\mathrm{C}}^u} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u} (u^+ + (f - u^+)\xi)\phi + \int_{\Omega^-} (\nabla u^- \nabla\phi + u\phi)\\ = \int_{\Omega^u} f\phi + \int_{\Omega^-} f\phi - \int_{\Omega^+} |Y_{\mathrm{c}}(x_2)|\frac{1}{\beta}v^+\phi.\end{aligned} \tag{30}$$

Thus, we have

$$\begin{aligned}\int_{\Omega^+} |Y_{\mathrm{c}}(x_2)|\frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \left(|Y(x_2)| - \int_{Y_{\mathrm{I}}(x_2)} \xi\, dy_1\right) u^+\phi + \int_{\Omega^-} (\nabla u^- \nabla\phi + u\phi)\\ = \int_{\Omega^+} \left(|Y(x_2)| - \int_{Y_{\mathrm{I}}(x_2)} \xi\, dy_1\right) f\phi + \int_{\Omega^-} f\phi - \int_{\Omega^+} |Y_{\mathrm{c}}(x_2)|\frac{1}{\beta}v^+\phi\end{aligned} \tag{31}$$

Let us come to the limit adjoint state equation. By putting $\phi = 0$ in the adjoint state equation, we obtain

$$\int_{\Omega_{\mathrm{I}}^u} \frac{\partial v_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (v^+ + v_1)\phi_1 = \int_{\Omega^u} (u^+ + u_1 - u_d)\phi_1.$$

Above form can be written as

$$\int_{\Omega_I^u} \frac{\partial v_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (v^+ + v_1)\phi_1 = \int_{\Omega^u} (u^+ - u_d)\phi_1 + \int_{\Omega_I^u} (f - u^+)\xi\phi_1.$$

Hence we can write $v_1(x_2, y_1) = (u^+(x) - u_d(x) - v^+(x))\xi(x_2, y_1) + (f - u^+)w_1(x_2, y_1)$. Now put $\phi_1 = 0$ in (26) to get

$$\int_{\Omega_C^u} \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u} (v^+ + v_1)\phi + \int_{\Omega^-} (\nabla v^- \nabla\phi + v^-\phi)$$
$$= \int_{\Omega^u} (u^+ - u_d)\phi + \int_{\Omega^u} (f - u^+)\xi\phi + \int_{\Omega^-} (u^- - u_d)\phi.$$

Substituting the expression for v_1 in the above variational equality, we obtain

$$\int_{\Omega_C^u} \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u} [v^+ + (u^+ - u_d - v^+)\xi + (f - u^+)w_1]\phi + \int_{\Omega^-} (\nabla v^- \nabla\phi + v^-\phi)$$
$$= \int_{\Omega^u} (u^+ - u_d)\phi + \int_{\Omega^u} (f - u^+)\xi\phi + \int_{\Omega^-} (u^- - u_d)\phi.$$

By taking all the terms not involving v^+ to the right hand side, we get

$$\begin{aligned}\int_{\Omega_C^u} \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u} (1 - \xi)v^+\phi + \int_{\Omega^-} (\nabla v^- \nabla\phi + v^-\phi)\\ = \int_{\Omega^u} (u^+ - u_d)(1 - \xi)\phi + \int_{\Omega^u} (f - u^+)(\xi - w_1)\phi + \int_{\Omega^-} (u^- - u_d)\phi.\end{aligned} \tag{32}$$

Now, using w_1 as a test function in (27) and ξ as a test function in (28) we get

$$\int_{Y(x_2)} \frac{\partial \xi}{\partial y_1}\frac{\partial w_1}{\partial y_1} dy_1 + \int_{Y(x_2)} \xi w_1 dy_1 = \int_{Y(x_2)} w_1 dy_1,$$

and

$$\int_{Y(x_2)} \frac{\partial w_1}{\partial y_1}\frac{\partial \xi}{\partial y_1} dy_1 + \int_{Y(x_2)} w_1\xi dy_1 = \int_{Y(x_2)} \xi^2 dy_1.$$

Hence we get,

$$\int_{Y(x_2)} w_1 dy_1 = \int_{Y(x_2)} \xi^2 dy_1. \tag{33}$$

We use (33) in (32) and by integrating both sides with respect to y_1. The left hand side of (32) becomes

$$\int_{\Omega^+} |Y_c(x_2)| \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+}\left(|Y(x_2)| - \int_{Y_I(x_2)} \xi \, dy_1 \right) v^+\phi + \int_{\Omega^-} (\nabla v^- \nabla \phi + v^-\phi)$$

and the right hand side is

$$\int_{\Omega^+}\left[\left(\int_{Y(x_2)} (1-\xi)^2 dy_1 \right) u^+ - \left(\int_{Y(x_2)} (1-\xi) dy_1 \right) u_d + \left(\int_{Y_I(x_2)} (\xi - \xi^2) dy_1 \right) f \right]\phi$$
$$+ \int_{\Omega^-} (u^- - u_d)\phi.$$

Hence the optimality system (20) can be written in the scale separated form as

$$\text{Find } (u,v) \in H(\Omega)\times H(\Omega) \text{ such that}$$
$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_c(x_2)| \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+}\left(|Y(x_2)| - \int_{Y_I(x_2)} \xi \, dy_1 \right) u^+\phi + \int_{\Omega^-} (\nabla u^- \nabla \phi + u\phi) \\ \displaystyle\qquad = \int_{\Omega^+}\left(|Y(x_2)| - \int_{Y_I(x_2)} \xi \, dy_1 \right) f\phi - \int_{\Omega^+} |Y_c(x_2)| \frac{1}{\beta} v^+\phi + \int_{\Omega^-} f\phi, \end{cases} \tag{34}$$
$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_c(x_2)| \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+}\left(|Y(x_2)| - \int_{Y_I(x_2)} \xi \, dy_1 \right) v^+\phi + \int_{\Omega^-} (\nabla v^- \nabla \phi + v^-\phi) \\ \displaystyle = \int_{\Omega^+}\left[\left(\int_{Y(x_2)} (1-\xi)^2 dy_1 \right) u^+ - \left(\int_{Y(x_2)} (1-\xi) \right) u_d + \left(\int_{Y_I(x_2)} (\xi - \xi^2) dy_1 \right) f \right]\phi \\ \displaystyle\qquad + \int_{\Omega^-} (u^- - u_d)\phi. \end{cases}$$

The above optimality system corresponds to the following optimal control problem: For $f \in L^2(\Omega)$ and $\theta \in L^2(\Omega^+)$, consider the L^2 cost functional

$$\bar{J}(u,\theta) = \frac{1}{2}\int_{\Omega^+}\int_{Y(x_2)} \left|(1-\xi)u^+ + f\xi - u_d\right|^2 + \frac{1}{2}\int_{\Omega^-} |u^- - u_d|^2 + \frac{\beta}{2}\int_{\Omega^+} |Y_c(x_2)||\theta|^2$$

where $u = (u^+, u^-)$ satisfies the following state equation,

$$\int_{\Omega^+} |Y_c(x_2)| \frac{\partial u^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \left(|Y(x_2)| - \int_{Y_1(x_2)} \xi \, dy_1 \right) u^+ \phi + \int_{\Omega^-} (\nabla u^- \nabla \phi + u\phi)$$
$$= \int_{\Omega^+} \left(|Y(x_2)| - \int_{Y_1(x_2)} \xi \, dy_1 \right) f\phi + \int_{\Omega^-} f\phi + \int_{\Omega^+} |Y_c(x_2)| \theta \phi \quad \text{for all } \phi \in H(\Omega) \tag{35}$$

The optimal control problem is to find the $(\bar{u}, \bar{\theta}) \in H(\Omega) \times L^2(\Omega^+)$ such that

$$\bar{J}(\bar{u}, \bar{\theta}) = \inf\{J(u,\theta) : \ (u,\theta) \text{ obeys } (35)\}. \tag{36}$$

The strong form of the above optimality system can be written as follows: Let

$$\alpha(x) = |Y(x_2)| - \int_{Y_1(x_2)} \xi \, dy_1.$$

The state equations are

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(|Y_c(x_2)| \dfrac{\partial u^+}{\partial x_2} \right) + \alpha(x) u^+ = \alpha(x) f + |Y_c(x_2)| \theta \quad \text{in } \Omega^+ \\ -\Delta u^- + u^- = f \quad \text{in } \Omega^- \end{cases}$$

The adjoint system is

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(|Y_c(x_2)| \dfrac{\partial v^+}{\partial x_2} \right) + \alpha(x) v^+ = \left(\displaystyle\int_{Y(x_2)} (1-\xi)^2 dy_1 \right) u^+ - \left(\displaystyle\int_{Y(x_2)} (1-\xi) dy_1 \right) u_d \\ \qquad\qquad\qquad\qquad + \left(\displaystyle\int_{Y_1(x_2)} (\xi - \xi^2) dy_1 \right) f \quad \text{in } \Omega^+ \\ -\Delta v^- + v^- = (u^- - u_d) \quad \text{in } \Omega^- \end{cases}$$

The optimal control is

$$\theta = -\frac{1}{\beta} v^+ \quad \text{in } \Omega^+$$

The boundary and interface conditions are given by

$$\begin{cases} u^+ = u^-, v^+ = v^- \ \text{ on } \gamma_C \\ \nabla u^- \cdot \nu = \nabla v^- \cdot \nu = 0 \ \text{ on } \partial\Omega^- \backslash \gamma_C \\ |Y_C(0)| \dfrac{\partial u^+}{\partial x_2} = \dfrac{\partial u^-}{\partial x_2}, \ |Y_C(0)| \dfrac{\partial v^+}{\partial x_2} = \dfrac{\partial v^-}{\partial x_2} \ \text{ on } \gamma_C \\ \dfrac{\partial u^+}{\partial x_2} = 0, \ \dfrac{\partial v^+}{\partial x_2} = 0 \ \text{ on } \gamma_u \text{ (upper boundary).} \end{cases}$$

Remark 6.1. As in Remark 4.2, we can make the similar remark in the context of optimal control problem also.

7. Control on I_ε

In this section we will present the result on asymptotic behavior of optimal control and optimal state when the controls are applied on the insulating part I_ε. To prove these results, similar techniques will work as presented in Section 5. However, the major difference is the convergence of the control which is acting on the insulating part leading to difficulty in the separation of the scales. It seems to us that a complete separation is not possible. Nevertheless, we do obtain the state equation (homogenized equation) in macro-scale, but the cost functional is not able to separate in micro-scale. We will point out in the coming subsection the difficulty in separating the macro-micro scales. Let us prescribe the control problem. Let the desired state $u_d \in L^2(\Omega)$. For $\theta_\varepsilon \in L^2(I_\varepsilon)$, consider the following L^2-cost functional

$$J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2} \int_{I_\varepsilon} |\theta_\varepsilon|^2,$$

where u_ε is the unique solution of the following variational problem: for $f \in L^2(\Omega)$

$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla u_\varepsilon \nabla \phi + u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi + \int_{\Omega_\varepsilon} \chi_{I_\varepsilon} \theta_\varepsilon \phi, \end{cases} \tag{37}$$

for all $\phi \in H^1(\Omega_\varepsilon)$. The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(I_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon) : \ (u_\varepsilon, \theta_\varepsilon) \text{ satisfies } (37)\}. \tag{38}$$

The following results, for fixed $\varepsilon > 0$, are standard.

Theorem 7.1. *For every $\varepsilon > 0$ the optimal control problem (38) admits a unique solution $(\bar{u}_\varepsilon, \ \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(I_\varepsilon)$.*

Theorem 7.2. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (38) and $\bar{v}_\varepsilon$ be the unique solution of the adjoint state. Then $\bar{\theta}_\varepsilon$ can be written as $\bar{\theta}_\varepsilon = -\chi_{I_\varepsilon} \frac{1}{\beta} \bar{v}_\varepsilon$. Conversely, let $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ satisfy the following variational system,*

$$\begin{cases} \int\limits_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla \hat{u}_\varepsilon \nabla \phi + \hat{u}_\varepsilon \phi = \int\limits_{\Omega_\varepsilon} f\phi + \int\limits_{\Omega_\varepsilon} \chi_{I_\varepsilon} \hat{\theta}_\varepsilon \phi, \\ \int\limits_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla \hat{v}_\varepsilon \nabla \phi + \hat{v}_\varepsilon \phi = \int\limits_{\Omega_\varepsilon} (\hat{u}_\varepsilon - u_d)\phi, \\ \hat{\theta}_\varepsilon = -\chi_{I_\varepsilon} \frac{1}{\beta} \hat{v}_\varepsilon, \end{cases} \tag{39}$$

for all $\phi \in H^1(\Omega_\varepsilon)$. Then $(\hat{u}_\varepsilon, \hat{\theta}_\varepsilon)$ is the optimal solution to the optimal control problem (38).

Uniform boundedness: As $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ is the optimal solution to the optimal control problem, we have $J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \leqslant J_\varepsilon(u^0_\varepsilon, 0)$, where u^0_ε is the state corresponding to the control $\theta_\varepsilon = 0$. This gives the uniform bound on $\|\bar{\theta}_\varepsilon\|_{L^2(I_\varepsilon)}$. In (37) choose $\phi = \bar{u}_\varepsilon$ to get

$$\|\chi_{C^+_\varepsilon} \nabla \bar{u}_\varepsilon\|_{L^2(\Omega^+_\varepsilon)} + \varepsilon \|\chi_{I^+_\varepsilon} \nabla \bar{u}_\varepsilon\|_{L^2(\Omega^+_\varepsilon)} + \|\nabla \bar{u}_\varepsilon\|_{L^2(\Omega^-)} + \|\bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant \|f\|_{L^2(\Omega_\varepsilon)} + \|\chi_{I_\varepsilon} \bar{\theta}_\varepsilon\|. \tag{40}$$

Using the uniform bound on $\|\bar{\theta}_\varepsilon\|_{L^2(I_\varepsilon)}$ we deduce the uniform bounds on the states as $\|\chi_{C^+_\varepsilon} \nabla \bar{u}_\varepsilon\|_{L^2(\Omega^+_\varepsilon)}$, $\varepsilon \|\chi_{I^+_\varepsilon} \nabla \bar{u}_\varepsilon\|_{L^2(\Omega^+_\varepsilon)}$, $\|\nabla \bar{u}_\varepsilon\|_{L^2(\Omega^-)}$, $\|\bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}$. Similarly we have uniform bounds on $\|\chi_{C^+_\varepsilon} \nabla \bar{v}_\varepsilon\|_{L^2(\Omega^+_\varepsilon)}$, $\varepsilon \|\chi_{I^+_\varepsilon} \nabla \bar{v}_\varepsilon\|_{L^2(\Omega^+_\varepsilon)}$, $\|\nabla \bar{v}_\varepsilon\|_{L^2(\Omega^-)}$, and $\|\bar{v}_\varepsilon\|_{L^2(\Omega_\varepsilon)}$.

Theorem 7.3. *For every $\varepsilon > 0$, let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (38) and $\bar{v}_\varepsilon$ be the adjoint state. Let $H(\Omega)$ and $V(\Omega)$ be defined as in Section 3. Then, we have the following convergences as $\varepsilon \to 0$*

$$\begin{cases} \bar{u}^-_\varepsilon \rightharpoonup u^-, \ \bar{v}^-_\varepsilon \rightharpoonup v^- \text{ weakly in } H^1(\Omega^-), \\ T^\varepsilon \bar{u}^+_\varepsilon \rightharpoonup u^+(x) + u_1(x, y_1), \ T^\varepsilon \bar{v}^+_\varepsilon \rightharpoonup v^+(x) + v_1(x, y_1), \ \text{ weakly in } L^2(\Omega^u) \\ T^\varepsilon \chi_{C_\varepsilon} T^\varepsilon \dfrac{\partial u^+_\varepsilon}{\partial x_2} \rightharpoonup \chi_c(y_1, x_2) \dfrac{\partial u^+}{\partial x_2}, \ T^\varepsilon \chi^+_{C_\varepsilon} T^\varepsilon \dfrac{\partial v^+_\varepsilon}{\partial x_2} \rightharpoonup \chi_c(y_1, x_2) \dfrac{\partial u^+}{\partial x_2}, \text{ weakly in } L^2(\Omega^u) \\ \varepsilon T^\varepsilon \chi^+_{I_\varepsilon} T^\varepsilon \dfrac{\partial \bar{u}^+_\varepsilon}{\partial x_2} \rightharpoonup 0, \ \varepsilon T^\varepsilon \chi^+_{I_\varepsilon} T^\varepsilon \dfrac{\partial \bar{v}^+_\varepsilon}{\partial x_2} \rightharpoonup 0, \text{ weakly in } L^2(\Omega^u) \\ T^\varepsilon \chi_{C_\varepsilon} T^\varepsilon \dfrac{\partial \bar{u}^+_\varepsilon}{\partial x_1} \rightharpoonup 0, \ T^\varepsilon \chi^+_{C_\varepsilon} T^\varepsilon \dfrac{\partial \bar{v}^+_\varepsilon}{\partial x_1} \rightharpoonup 0, \ \text{ weakly in } L^2(\Omega^u) \\ \varepsilon T^\varepsilon \chi_{I_\varepsilon} T^\varepsilon \dfrac{\partial \bar{u}^+_\varepsilon}{\partial x_1} \rightharpoonup \dfrac{\partial u_1}{\partial y_1}, \ \varepsilon T^\varepsilon \chi^+_{I_\varepsilon} T^\varepsilon \dfrac{\partial \bar{v}^+_\varepsilon}{\partial x_1} \rightharpoonup \dfrac{\partial v_1}{\partial y_1} \text{ weakly in } L^2(\Omega^u) \\ \chi_I(y_1, x_2) T^\varepsilon \bar{\theta}_\varepsilon \rightharpoonup -\chi_I(y_1, x_2) \dfrac{1}{\beta}(v^+ + v_1) \text{ weakly in } L^2(\Omega^u) \end{cases} \tag{41}$$

where $(u, u_1, v, v_1) \in (H(\Omega) \times V(\Omega))^2$ satisfies the following 2-scale optimality system

$$
\begin{cases}
\int\limits_{\Omega_C^u} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int\limits_{\Omega_I^u} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int\limits_{\Omega^u} (u^+ + u_1)(\phi + \phi_1) + \int\limits_{\Omega^-} (\nabla u^- \nabla \phi + u\phi) \\
\qquad = \int\limits_{\Omega^u} (f + \chi_I(y_1, x_2)(\theta + \theta_1)(\phi + \phi_1) + \int\limits_{\Omega^-} f\phi, \\
\int\limits_{\Omega_C^u} \frac{\partial v^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int\limits_{\Omega_I^u} \frac{\partial v_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int\limits_{\Omega^u} (v^+ + v_1)(\phi + \phi_1) + \int\limits_{\Omega^-} (\nabla v^- \nabla \phi + v^-\phi) \\
\qquad = \int\limits_{\Omega^u} (u^+ + u_1 - u_d)(\phi + \phi_1) + \int\limits_{\Omega^-} (u^- - u_d)\phi, \\
\textit{for all } (\phi, \phi_1) \in H(\Omega) \times V(\Omega) \textit{ and} \\
\theta + \theta_1 = -\frac{1}{\beta}(v^+ + v_1) \textit{ in } \Omega_I^u.
\end{cases}
\tag{42}
$$

If we look at the convergence (21) in the case of controls acting on the non-insulating region, the limit control has only one component $\theta \in L^2(\Omega^+)$, but in the present case the limit control has the form $\theta + \theta_1$, where θ_1 also depends on the micro-variable. Interestingly, we do give a formula for θ_1 in terms of θ and two-scale part of the adjoint solution. This is the difficulty in getting a complete separation of the scales in the cost functional. The optimality system (42) corresponds to the following optimal control problem; For the source term $f \in L^2(\Omega)$ and control $(\theta, \theta_1) \in L^2(\Omega^+) \times L^2(\Omega_I^u)$ (or one can think $\theta_1 \in L^2(\Omega^u)$ with $\theta_1 = 0$ a.e. in Ω_C^u), consider the following L^2-cost functional

$$
J(u, u_1, \theta, \theta_1) = \frac{1}{2}\int\limits_{\Omega^u} (u^+ + u_1 - u_d)^2 + \frac{1}{2}\int\limits_{\Omega^-} (u^- - u_d)^2 + \frac{\beta}{2}\int\limits_{\Omega_I^u} (\theta + \theta_1)^2 \tag{43}
$$

where $(u, u_1) \in H(\Omega) \times V(\Omega)$ satisfies the following variational form

$$
\begin{aligned}
&\int\limits_{\Omega_C^u} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int\limits_{\Omega_I^u} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int\limits_{\Omega^u} (u^+ + u_1)(\phi + \phi_1) + \int\limits_{\Omega^-} (\nabla u^- \nabla \phi + u\phi) \\
&\qquad = \int\limits_{\Omega^u} (f + \chi_I(y_1, x_2)(\theta + \theta_1))(\phi + \phi_1) + \int\limits_{\Omega^-} f\phi,
\end{aligned}
\tag{44}
$$

for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$. Now the optimal control problem is to find $(\bar{u}, \bar{u}_1, \bar{\theta}, \bar{\theta}_1) \in H(\Omega) \times V(\Omega) \times L^2(\Omega^+) \times L^2(\Omega_I^u)$ such that

$$
J(\bar{u}, \bar{u}_1, \bar{\theta}, \bar{\theta}_1) = \inf\{J(u, u_1, \theta, \theta_1) : (u, u_1, \theta, \theta_1) \text{ satisfies } (44)\}. \tag{45}
$$

Using the standard method of calculus of variations, we can prove the unique existence of the two-scale system which is stated below.

Theorem 7.4. *The optimal control problem* (45) *has a unique solution* $(\bar{u}, \bar{u}_1, \bar{\theta}, \bar{\theta}_1) \in H(\Omega) \times V(\Omega) \times L^2(\Omega^+) \times L^2(\Omega_I^u)$.

7.1. Separation of scales

Here, as in the earlier optimal control problem on C_ε, we are unable to apply the procedure for the complete separation of scales in the limit state equation. This may be due to the intrinsic nature of the problem. However, as mentioned earlier, we do obtain the homogenized equation for the limit state in the macro variable. In this case, if we put $\phi = 0$ in the limit state equation (44), we get the following

$$\int_{\Omega_1^u} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (u^+ + u_1)\phi_1 = \int_{\Omega^u} (f + \chi_\text{I}(y_1, x_2)(\theta + \theta_1)\phi_1$$

Since $(\theta + \theta_1) = -\chi_\text{I}\frac{1}{\beta}(v^+ + v_1)$, hence we cannot separate like earlier as v_1 is a function of y_1 also. We separate as much as can by splitting u_1 into two parts u_{11} and u_{12}, where in some sense u_{12} does not involve micro scale, but u_{11} has dependency on both micro and macro scales. The idea behind the separation of scale here to look u and u_1 closely and try to see how they are coupled with each other.

In the limit optimality system (42), in the state equation part put $\phi_1 = 0$, and keep $\phi \in H(\Omega)$ arbitrary, and then put $\phi = 0$ and keep $\phi_1 \in V(\Omega)$ arbitrary, we will obtain the following variational system

$$\begin{cases} \displaystyle\int_{\Omega^+} \left(|Y_\text{c}(x_2)|\frac{\partial u^+}{\partial x_2}\right)\frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+} |Y(x_2)u^+\phi^+ = \int_{\Omega^+}\int_{Y(x_2)} (f + (\theta + \theta_1) - u_1)\phi^+ \\ \displaystyle\int_{\Omega^u} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} u_1\phi_1 = \int_{\Omega^u} [(\theta + \theta_1) + (f - u^+)]\phi_1 \\ \displaystyle\int_{\Omega^-} \nabla u^- \nabla \phi^- + \int_{\Omega^-} u^-\phi = \int_{\Omega^-} f\phi^- \end{cases}$$

for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$. In the above, we have assumed $\text{supp}(\phi^+) \subset \Omega^+$ and $\text{supp}(\phi^-) \subset \Omega^-$. This is really not necessary, but for the convenience of writing the equations in Ω^+ and Ω^- separately. Let us write $u_1 = u_{11} + u_{12}$, where u_{11} and u_{12} satisfy the following variational forms,

$$\int_{\Omega^u} \frac{\partial u_{11}}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} u_{11}\phi_1 = \int_{\Omega^u} (\theta + \theta_1)\phi_1 \tag{46}$$

$$\int_{\Omega^u} \frac{\partial u_{12}}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} u_{12}\phi_1 = \int_{\Omega^u} (f - u^+)\phi_1 \tag{47}$$

The second equation motivates us to introduce the cell problem as follows: Find $\xi \in V(\Omega)$ which satisfies the following variational form

$$\int_{\Omega^u} \frac{\partial \xi}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} \xi\phi_1 = \int_{\Omega^u} \phi_1. \tag{48}$$

Then, we can write the solution of (47) as $u_{12} = (f - u^+)\xi$. Note that for each $x_2 \in (0,1)$, $\xi(x_2,\cdot)$ is the solution of the cell problem (3). Hence the state equation in the upper part becomes

$$\int_{\Omega^+} |Y_c(x_2)|\frac{\partial u^+}{\partial x_2}\frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+} |Y(x_2)|u^+\phi^+ = \int_{\Omega^+}\int_{Y(x_2)} (f + (\theta+\theta_1) - u_{11} - (f-u^+)\xi)\phi^+.$$

This implies

$$\int_{\Omega^+} |Y_c(x_2)|\frac{\partial u^+}{\partial x_2}\frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+} \alpha(x)u^+\phi^+ = \int_{\Omega^+}\int_{Y(x_2)} ((1-\xi)f + (\theta+\theta_1))\phi^+ - \int_{\Omega^+}\int_{Y(x_2)} u_{11}\phi^+,$$

where $\alpha(x) = \left(|Y(x_2)| - \int_{Y_1(x_2)} \xi\, dy_1\right)$. Now using $\phi^+\xi$ as a test function in (46) and $u_{11}\phi^+$ in (48), we get the following

$$\int_{\Omega^u} \phi^+\frac{\partial u_{11}}{\partial y_1}\frac{\partial \xi}{\partial y_1} + \int_{\Omega^u} u_{11}\phi^+\xi = \int_{\Omega^u} (\theta+\theta_1)\phi^+\xi$$

$$\int_{\Omega^u} \phi^+\frac{\partial \xi}{\partial y_1}\frac{\partial u_{11}}{\partial y_1} + \int_{\Omega^u} \xi\phi^+u_{11} = \int_{\Omega^u} \phi^+u_{11}$$

It follows that $\int_{\Omega^+}\int_{Y(x_2)} \phi^+u_{11} = \int_{\Omega^+}\int_{Y(x_2)} \xi(\theta+\theta_1)\phi^+$ for all $\phi^+ \in C^\infty(\bar{\Omega}^+)$. Now using this equality and density, the two-scale state equation reduces to the following

$$\left\{\begin{aligned} &\int_{\Omega^+} |Y_c(x_2)|\frac{\partial u^+}{\partial x_2}\frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+} \alpha(x)u^+\phi^+ + \int_{\Omega^-} \nabla u^-\nabla\phi^- + \int_{\Omega^-} u^-\phi \\ &\quad = \int_{\Omega^+}\int_{Y(x_2)} ((1-\xi)f + (1-\xi)(\theta+\theta_1))\phi^+ + \int_{\Omega^-} f\phi^-, \\ &\int_{\Omega^u} \frac{\partial u_{11}}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} u_{11}\phi_1 = \int_{\Omega^u} (\theta+\theta_1)\phi_1, \end{aligned}\right. \tag{49}$$

for all $(\phi,\phi_1) \in H(\Omega)\times V(\Omega)$. The L^2-cost functional reduces to

$$J(u,u_{11},\theta,\theta_1) = \frac{1}{2}\int_{\Omega^-} (u^- - u_d)^2 + \frac{1}{2}\int_{\Omega^+}\int_{Y(x_2)} ((1-\xi)u^+ + \xi f + u_{11} - u_d)^2 + \frac{\beta}{2}\int_{\Omega^+}\int_{Y(x_2)} (\theta+\theta_1)^2$$

Hence the limit optimal control is to find $(\bar{u}, \bar{u}_{11}, \bar{\theta}, \bar{\theta}_1) \in H(\Omega) \times V(\Omega) \times L^2(\Omega^+) \times L^2(\Omega^u_{\mathrm{I}})$ such that

$$J(\bar{u}, \bar{u}_{11}, \bar{\theta}, \bar{\theta}_1) = \inf\{J(u, u_{11}, \theta, \theta_1) : (u, u_{11}, \theta, \theta_1) \text{ satisfies } (49)\}. \tag{50}$$

Let the adjoint state $(v, v_{11}) \in H(\Omega) \times V(\Omega)$ satisfies the following system,

$$\begin{cases} \int\limits_{\Omega^+} |Y_c(x_2)| \frac{\partial \bar{v}^+}{\partial x_2} \frac{\partial \phi^+}{\partial x_2} + \int\limits_{\Omega^+} \alpha(x) \bar{v}^+ \phi^+ + \int\limits_{\Omega^-} \nabla v^- \nabla \phi^- + \int\limits_{\Omega^-} u^- \phi^- = \int\limits_{\Omega^-} (\bar{u}^- - u_d)\phi^- + \\ \qquad \int\limits_{\Omega^+} \int\limits_{Y(x_2)} \Big[(1-\xi)^2 \bar{u}^+ + \xi(1-\xi) f\Big] \phi^+ + \int\limits_{\Omega^+} \int\limits_{Y(x_2)} [(1-\xi)\bar{u}_{11} - (1-\xi)u_d] \phi^+, \\ \int\limits_{\Omega^u} \frac{\partial \bar{v}_{11}}{\partial y_1} \frac{\partial \phi_1}{\partial y_1} + \int\limits_{\Omega^u} \bar{v}_{11} \phi_1 = \int\limits_{\Omega^u} [(1-\xi)\bar{u}^+ + \xi f + \bar{u}_{11} - u_d] \phi_1, \end{cases} \tag{51}$$

for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$.

Theorem 7.5. *Let $(\bar{v}, \bar{v}_{11}) \in H(\Omega) \times V(\Omega)$ adjoint state that is satisfies (51) and $(\bar{\theta}, \bar{\theta}_1) \in L^2(\Omega^+) \times L^2(\Omega^u_I)$ is the optimal control. Then the optimal control is given by*

$$(\bar{\theta} + \bar{\theta}_1) = -\frac{1}{\beta}((1-\xi)\bar{v}^+ + \bar{v}_{11}) \text{ a.e. in } \Omega^u_I$$

Proof. Let $(\theta, \theta_1) \in L^2(\Omega^+) \times L^2(\Omega^u_{\mathrm{I}})$ fixed but arbitrary. Define $(\theta^\lambda, \theta_1^\lambda) = (\bar{\theta}, \bar{\theta}_1) + \lambda(\theta, \theta_1)$, corresponding to $(\theta^\lambda, \theta_1^\lambda)$ the state denoted by $(u^\lambda, u_{11}^\lambda)$ and the cost by J_λ. Also denote the cost corresponding to $(\bar{\theta}, \bar{\theta}_1)$ by $\bar{J}$. Now

$$\lim_{\lambda \to 0} \frac{J_\lambda - \bar{J}}{\lambda} = \int\limits_{\Omega^u} [(1-\xi)\bar{u}^+ + \xi f + \bar{u}_{11} - u_d][(1-\xi)w^+ + w_{11}] + \int\limits_{\Omega^-} (\bar{u}^- - u_d) w^- + \beta \int\limits_{\Omega^u_I} (\theta + \theta_1)(\bar{\theta} + \bar{\theta}_1)$$

Hence we have

$$J'(\bar{\theta}, \bar{\theta}_1)(\theta, \theta_1) = \int\limits_{\Omega^u} [(1-\xi)\bar{u}^+ + \xi f + \bar{u}_{11} - u_d][(1-\xi)w^+ + w_{11}] + \int\limits_{\Omega^-} (\bar{u}^- - u_d) w^- + \beta \int\limits_{\Omega^u_I} (\theta + \theta_1)(\bar{\theta} + \bar{\theta}_1) \tag{52}$$

where $(w, w_{11}) \in H(\Omega) \times V(\Omega)$ satisfies the following variational system

$$\begin{cases} \int\limits_{\Omega^+} |Y_c(x_2)| \frac{\partial w^+}{\partial x_2}\frac{\partial \phi^+}{\partial x_2} + \int\limits_{\Omega^+} \alpha(x) w^+\phi^+ \int\limits_{\Omega^-} \nabla w^- \nabla \phi^- + \int\limits_{\Omega^-} w^-\phi^- = \int\limits_{\Omega_I^u} (1-\xi)(\theta+\theta_1)\phi^+ \\ \int\limits_{\Omega^u} \frac{\partial w_{11}}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int\limits_{\Omega^u} w_{11}\phi_1 = \int\limits_{\Omega_I^u} (\theta+\theta_1)\phi_1. \end{cases} \tag{53}$$

Hence using $(\bar{v}, \bar{v}_{11})$ as a test function in (53) and (w, w_{11}) in (51), we get

$$\int\limits_{\Omega^u} [(1-\xi)\bar{u}^+ + \xi f + \bar{u}_{11} - u_d][(1-\xi)w^+ + w_{11}] + \int\limits_{\Omega^-} (\bar{u}^- - u_d)w^- \\ = \int\limits_{\Omega_I^u} (1-\xi)(\theta+\theta_1)\bar{v}^+ + \int\limits_{\Omega^u} (\theta+\theta_1)\bar{v}_{11}$$

Using the above equality in (52), we get

$$J'(\bar{\theta},\bar{\theta}_1)(\theta,\theta_1) = \beta \int\limits_{\Omega_I^u} (\theta+\theta_1)(\bar{\theta}+\bar{\theta}_1) + \int\limits_{\Omega_I^u} (\theta+\theta_1)[(1-\xi)\bar{v}^+ + \bar{v}_{11}] \tag{54}$$

As $(\bar{\theta},\bar{\theta}_1)$ is the optimal control we have $J'(\bar{\theta},\bar{\theta}_1)(\theta,\theta_1) = 0$. Since (θ,θ_1) was fixed but arbitrary, gives us the following equality

$$\bar{\theta} + \bar{\theta}_1 = -\frac{1}{\beta}[(1-\xi)\bar{v}^+ + \bar{v}_{11}] \text{ in } \Omega_I^u. \quad \square$$

Remark 7.6. Though, we do not have further decomposition of the cost functional, the control has a partial separation as follows. From the above characterization we can have the explicit information about the optimal control. Recall that $\bar{v}_{11}$, ξ and θ_1 are supported in Ω_I^u. Hence we have

$$\bar{\theta}(x) = -\frac{1}{\beta}\bar{v}^+(x) \text{ for all } (x, y_1) \in \Omega_c^u,$$

and hence $\bar{\theta} = -\frac{1}{\beta}\bar{v}^+$. This implies that $\bar{\theta}_1 = -\frac{1}{\beta}(-\xi\bar{v}^+ + \bar{v}_{11})$ in Ω_I^u.

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