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Homogenization with strong contrasting diffusivity in a circular oscillating domain with L^1 source term

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Abstract
In this article, we study the homogenization of an elliptic variational form with oscillating coefficients in a circular, highly oscillating domain, where the oscillatory part is made of two materials with high contrasting conductivity (or diffusivity) with the source term in L^1. We incorporate this phenomenon, namely, highly oscillating boundary, rapid oscillating coefficient, and the oscillating part made of high contrasting materials, which leads to non-uniform ellipticity as the oscillating parameter goes to 0. Further, due to the L^1 source term, the solutions are interpreted as renormalized solutions. To achieve our primary goal, we have proved the strong convergence results in the context of the L^2 source term in the first part (corrector results). In the second part, we have homogenized the renormalized variational form and established the relation between the ε-stage renormalized solution and the limit renormalized solution via convergence results. The unfolding operator for the polar coordinates is a central tool for the analysis.

Keywords Homogenization · Periodic unfolding · Oscillating boundary domain · Circular oscillating domain · Renormalized solution

Mathematics Subject Classification 80M40 · 80M35 · 35B27

1 Introduction

The homogenization of partial differential equations (PDEs) with strong contrasting diffusivity is important because it appears in the modeling of several multi-scale physical problems, such as the double porosity model, effective properties of composite materials

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with soft and hard cores, effective conductivity of composites made of high and low conductivity materials, effective behavior of composite materials bearing a high modulus of reinforcement and so on.

Due to the wide range of applicability, several mathematicians have worked in this direction. Panasenko [26] has one of the earliest works on the multi-scale analysis of high contrast composites. Several research works have been carried out in this direction in the last two decades. The homogenization of a double porosity model in a single-phase flow was studied in [4], which is an interesting piece of work. In [11], authors have considered the homogenization of a conductivity equation for a medium made up of highly conductive vertical fiber surrounded by another material assumed to be a poor conductor. In [29], the author investigates the homogenization of a stationary diffusion equation in a periodic composite medium made of two components with the ratio between the diffusion coefficients $O(\frac{1}{\alpha_\varepsilon^2})$ where ε is the size of the period and $\alpha_\varepsilon \to 0$. The homogenization of a hyperbolic PDE with strongly contrasting diffusive coefficients is performed in [24]. As a general reference for the homogenization procedure in composites made of materials with high contrasting behavior, see Ch. 7 of [5]. For further reading in this direction, we refer to the articles [6, 10, 20, 27, 28] and references therein.

All the articles mentioned above are on fixed domains. Not much work has been carried out on homogenization in oscillating domains with high contrasting diffusivity coefficients. In this context, one of the earliest works on an oscillating domain is given in [18], in which the authors consider an elliptic variational form on a pillar-type oscillating domain where an insulator-type material covers the high conductive core part of the reference pillar. In [18], the authors have used the pillar type shape very crucially. In [25], using the method of unfolding operator, the work in [18] is generalized from pillar type oscillating domain to a very general oscillating domain. Also, an interior optimal control problem is considered subject to a variational form having high contrasting diffusive coefficients. In [16], homogenization of an elliptic variational form was studied in a thin domain with a pillar-type reference cell made from materials with high contrasting conductivities. Depending on the ratio between the width of conductive parts and insulating parts, different limit problems are obtained in [16]. To see more about the homogenization of oscillating domains, we refer to the articles [2, 3, 13, 23] and references therein.

All the references mentioned above had their source terms in L^2, which means the homogenization method took place in a correct Hilbert space setup. As in this paper, the source term is in L^1 Banach space; one cannot expect the solution to be uniformly bounded in H^1. To resolve this issue, we will utilize R. J. DiPerna and P. L. Lions' description of the renormalized solution for the Boltzmann equation, which they have introduced in [14]. In addition, in [7, 12], the notion of the renormalized solution has been adopted for the elliptic equation. We refer to the articles [8, 9, 15, 19, 21] and references therein for further information on the use of renormalized solutions. In the context of homogenization with L^1 source term, in [17], the authors first time have studied the homogenization of a variational form in non-periodic pillar type oscillating domain with source term f in L^1. In comparison with [17], the domain under consideration in the current work exhibits periodic circular type oscillations with high contrast diffusivity coefficients.

In this paper, we consider a general second-order elliptic variational form with high contrasting diffusivity coefficients in a circular oscillatory domain $\mathcal{O}_\varepsilon$ (see Sect. 2 regarding the description of $\mathcal{O}_\varepsilon$). Here, we have analyzed the asymptotic behavior of the following model problem

$$\begin{cases} -\text{div}\big((\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-})A^\varepsilon \nabla u_\varepsilon\big) + u_\varepsilon = f \text{ in } \mathcal{O}_\varepsilon, \\ \qquad\qquad A^\varepsilon \nabla u_\varepsilon \cdot \nu^\varepsilon = 0 \text{ on } \partial\mathcal{O}_\varepsilon. \end{cases} \tag{1}$$

The novelty of our work is in three folds,

(i) The domain under consideration is a circular type oscillating domain having the reference cell made of materials with high contrasting behavior.
(ii) The source term f in the equation is in L^1.
(iii) Here, we are allowing an oscillating coefficient matrix with $O(\varepsilon)$ ellipticity constant in the insulating region $\mathcal{I}_\varepsilon$.

For the analysis, we require strong convergence with source term f in L^2, a significant result not available in the circular oscillating domain. Also, our approach to circumvent the difficulty due to the reference configuration by using the periodic unfolding method for the circular oscillating domain is a novelty, in the context of homogenization, compared to the existing literature. In the second part, we have homogenized the above problem (1) with source term f in L^1.

The main contributions of this article are summarized in Theorem 1, Theorem 2 and Theorem 4. Theorem 1 is about the weak convergence of the solutions and their derivatives when the source term f is in L^2. Using the weak convergence results, we have proved corrector results or strong convergence results which are available in Theorem 2 in the context of L^2 source term. In Theorem 4, the main homogenization results and convergence results for the renormalized variational form are described.

The rest of the article is organized as follows. In Sect. 2, we have discussed the configuration of the considered domain, the primary tool for the analysis that is polar unfolding operator, and the definition of some auxiliary functions needed for the analysis. In Sect. 3, we have homogenized the considered variational form and also shown the corrector result when the source term is in L^2. The central result that is the homogenization of the considered variational form with source term, f in L^1 is presented in Sect. 4.

2 Domain description and unfolding operator

2.1 Domain description

Let $0 < r_0 < r_1 < r_2$ be real numbers and for the simplicity of presentation, we take $\varepsilon = \frac{1}{n}, n \in \mathbb{N}$. Let Λ be a connected open subset of $\mathbb{R}^2$ which is contained in the annulus $\mathcal{O}^+ = \{(r,\theta) : r_0 < r < r_1\}$ with Lipschitz boundary which is the reference cell (See Fig. 1). It consists of two parts: namely insulating part $\mathcal{I}$ and high conductive region $\mathcal{C}$. Now define

$$\mathcal{I}_\varepsilon = \Big\{(r,\theta) \in \mathcal{O}^+ : \Big(r, \Big\{\frac{\theta}{\varepsilon}\Big\}_{2\pi}\Big) \in \mathcal{I}\Big\}, \quad \mathcal{C}_\varepsilon = \Big\{(r,\theta) \in \mathcal{O}^+ : \Big(r, \Big\{\frac{\theta}{\varepsilon}\Big\}_{2\pi}\Big) \in \mathcal{C}\Big\},$$
$$\mathcal{O}^+_\varepsilon = \text{int}\big(\overline{\mathcal{I}_\varepsilon \cup \mathcal{C}_\varepsilon}\big) = \Big\{(r,\theta) \in \mathcal{O} : \Big(r, \Big\{\frac{\theta}{\varepsilon}\Big\}_{2\pi}\Big) \in \Lambda\Big\}, \quad \mathcal{O}^- = \big\{(r,\theta) : r_1 < r < r_2\big\},$$
$$\mathcal{O}_\varepsilon = \text{int}\big(\overline{\mathcal{O}^+_\varepsilon \cup \mathcal{O}^-}\big) \quad \text{and} \quad \mathcal{O} = \text{int}\big(\overline{\mathcal{O}^+ \cup \mathcal{O}^-}\big),$$

where $\mathcal{O}^+_\varepsilon$ is the inner oscillating part with $\mathcal{I}_\varepsilon$ and $\mathcal{C}_\varepsilon$ as its insulating and conducting parts respectively. The domain $\mathcal{O}^-$ is the outer fixed part, $\mathcal{O}_\varepsilon$ is the oscillating domain and $\mathcal{O}$ is

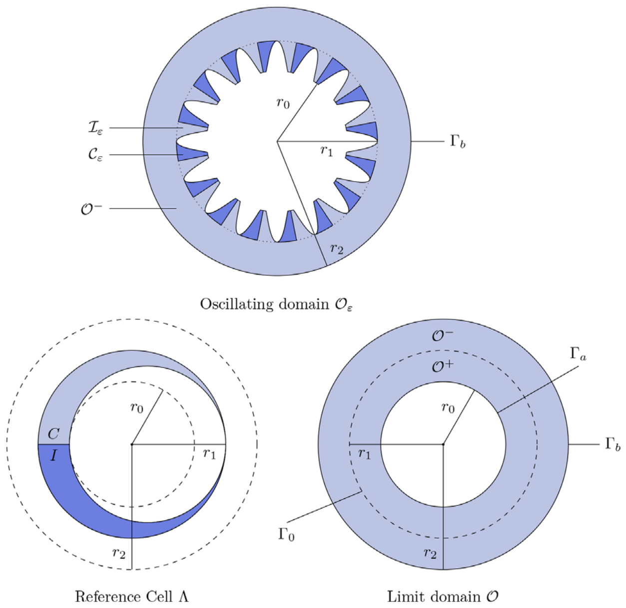


Fig. 1 Domain description

the limit domain. Also Γ_a, Γ_b are inner and outer boundaries of $\mathcal{O}$ and Γ_0 is the interface. Here $\left\{\frac{\theta}{\varepsilon}\right\}_{2\pi} = \frac{\theta}{\varepsilon} - \left[\frac{\theta}{2\pi\varepsilon}\right]2\pi$, where $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional parts. For $r \in (r_0, r_1)$, define

$$Y_{\mathcal{I}}(r) = \{\theta \in [0, 2\pi] : (r, \theta) \in \mathcal{I}\}, \quad Y_{\mathcal{C}}(r) = \{\theta \in [0, 2\pi] : (r, \theta) \in \mathcal{C}\},$$
$$Y(r) = Y_{\mathcal{I}}(r) \cup Y_{\mathcal{C}}(r) = \{\theta \in [0, 2\pi] : (r, \theta) \in \Lambda\}.$$

The domains mentioned above are required to define the unfolded domain and unfolding operator. We assume the following properties:

(1) The set $Y_{\mathcal{C}}(r)$ is connected for all $r \in (r_0, r_1)$.
(2) There exists $\rho > 0$ such that $0 < \rho \leq meas(Y_{\mathcal{C}}(r)) < 2\pi$ for all $r \in (r_0, r_1)$.

We now introduce the definition of unfolding operator on $\mathcal{O}_\varepsilon$ and its properties.

2.2 Periodic unfolding operator

Since the oscillations in $\mathcal{O}_\varepsilon$ are in a circular direction, we need unfolding operators in polar coordinates to do the analysis. Here we will recall the definition of a polar unfolding operator for $\mathcal{O}_\varepsilon^+$ and its properties without proof. For proof, one can see [1]. First, we will define the unfolded domains in which the unfolded functions are defined.

$$\mathcal{O}_U = \{(r,\theta,\tau) \mid \theta \in (0,2\pi), r \in (r_0,r_1), \tau \in Y(r)\},$$
$$\mathcal{O}_\mathcal{C} = \{(r,\theta,\tau) \mid \theta \in (0,2\pi), r \in (r_0,r_1), \tau \in Y_\mathcal{C}(r)\},$$
$$\mathcal{O}_\mathcal{I} = \{(r,\theta,\tau) \mid \theta \in (0,2\pi), r \in (r_0,r_1), \tau \in Y_\mathcal{I}(r)\}.$$

The subscript U in $\mathcal{O}_U$ refers to unfolding. Here $\mathcal{O}_U, \mathcal{O}_\mathcal{C}$ and $\mathcal{O}_\mathcal{I}$ are the unfolded domains corresponding to $\mathcal{O}_\varepsilon^+, \mathcal{C}_\varepsilon$ and $\mathcal{I}_\varepsilon$ respectively.

Let $\phi^\varepsilon : \mathcal{O}_U \to \mathcal{O}_\varepsilon^+$ be defined as $\phi^\varepsilon(\theta, r, \tau) = \left(r, \varepsilon\left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right)$. The ε - unfolding of a function $u : \mathcal{O}_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \mathcal{O}_U \to \mathbb{R}$. The operator which maps every function $u : \mathcal{O}_\varepsilon^+ \to \mathbb{R}$ to its ε - unfolding is called the unfolding operator. Let the unfolding operator be denoted by T^ε, that is,

$$T^\varepsilon : \{u : \mathcal{O}_\varepsilon^+ \to \mathbb{R}\} \to \{T^\varepsilon(u) : \mathcal{O}_U \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon(u)(r,\theta,\tau) = u\left(r, \varepsilon\left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right),$$

where $\left[\frac{\theta}{\varepsilon}\right]_{2\pi} = \left[\frac{\theta}{2\pi\varepsilon}\right]2\pi$. Analogously, we can define the unfolding operators

$$T_\mathcal{C}^\varepsilon : \{u : \mathcal{C}_\varepsilon \to \mathbb{R}\} \to \{T^\varepsilon(u) : \mathcal{O}_\mathcal{C} \to \mathbb{R}\}\text{and}$$
$$T_\mathcal{I}^\varepsilon : \{u : \mathcal{I}_\varepsilon \to \mathbb{R}\} \to \{T^\varepsilon(u) : \mathcal{O}_\mathcal{I} \to \mathbb{R}\}$$

Then, we have

$$T^\varepsilon(u) = \chi_{\mathcal{O}_\mathcal{C}} T_\mathcal{C}^\varepsilon + \chi_{\mathcal{O}_\mathcal{I}} T_\mathcal{I}^\varepsilon.$$

If $U \subset \mathbb{R}^2$ containing $\mathcal{O}_\varepsilon^+$ and u is a real-valued function on U, $T^\varepsilon(u)$ will mean, T^ε acting on the restriction of u to $\mathcal{O}_\varepsilon^+$. Some important properties of the circular unfolding operator are stated below. For each $\varepsilon > 0$,

(1) T^ε is linear. Further, if $u,\ v : \mathcal{O}_\varepsilon^+ \to \mathbb{R}$, then, $T^\varepsilon(uv) = T^\varepsilon(u)T^\varepsilon(v)$.
(2) Let $u \in L^1(\mathcal{O}_\varepsilon^+)$. Then,

$$\int_{\mathcal{O}_U} T^\varepsilon(u) = 2\pi \int_{\mathcal{O}_\varepsilon^+} u.$$

(3) Let $u \in L^2(\mathcal{O}_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2(\mathcal{O}_U)$ and $\|T^\varepsilon u\|_{L^2(\mathcal{O}_U)} = \sqrt{2\pi}\|u\|_{L^2(\mathcal{O}_\varepsilon^+)}$.
(4) Let $u, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta} \in L^2(\mathcal{O})$. Then, $T^\varepsilon u, \frac{\partial}{\partial r}T^\varepsilon u, \frac{\partial}{\partial \tau}T^\varepsilon u \in L^2(\mathcal{O}_U)$. Moreover,

$$\frac{\partial}{\partial r}T^\varepsilon u = T^\varepsilon \frac{\partial u}{\partial r} \quad \text{and} \quad \frac{\partial}{\partial \tau}T^\varepsilon u = \varepsilon T^\varepsilon \frac{\partial u}{\partial \theta}.$$

(5) Let $u \in L^2(\mathcal{O}^+)$. Then, $T^\varepsilon u \to u$ strongly in $L^2(\mathcal{O}_U)$. More generally, if $u_\varepsilon \to u$ strongly in $L^2(\mathcal{O}^+)$, then $T^\varepsilon u_\varepsilon \to u$ strongly in $L^2(\mathcal{O}_U)$.
(6) Let, for every ε, $u_\varepsilon \in L^2(\mathcal{O}_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\mathcal{O}_U)$. Then,

$$\widetilde{u_\varepsilon} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} u(r,\theta,\tau)d\tau \;\text{ weakly in } L^2(\mathcal{O}^+).$$

Here $\widetilde{u_\varepsilon}$ denotes the zero extension of u_ε to $\mathcal{O}$.

(7) Let, for every $\varepsilon > 0$, $u_\varepsilon \in H^1(\mathcal{O}_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ and $\frac{\partial}{\partial r}T^\varepsilon u_\varepsilon \rightharpoonup \frac{\partial u}{\partial r}$ weakly in $L^2(\mathcal{O}_U)$. Then,

$$\widetilde{u_\varepsilon} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} u d\tau \quad\text{and}\quad \widetilde{\frac{\partial u_\varepsilon}{\partial r}} \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} \frac{\partial u}{\partial r}\, d\tau \quad \text{weakly in } L^2(\mathcal{O}^+).$$

All the above properties are valid for $T^\varepsilon_{\mathcal{I}}$ and $T^\varepsilon_{\mathcal{C}}$.

2.3 Boundary unfolding operator

In order to get the interface conditions, we now introduce the following boundary unfolding operator T_0^ε on $\Gamma^\varepsilon_{\mathcal{C}}$. For every $\varepsilon > 0$, let us denote the unfolded boundary of $\Gamma^\varepsilon_{\mathcal{C}}$ by $\Gamma_{\mathcal{C}}$, defined by

$$\Gamma_{\mathcal{C}} = \{(r_1,\theta,\tau) : \theta \in (0,2\pi) \text{ and } \tau \in Y_{\mathcal{C}}(r_1)\}$$

Define boundary unfolding operator $T_0^\varepsilon : \{u : \Gamma^\varepsilon_{\mathcal{C}} \to \mathbb{R}\} \to \{T_0^\varepsilon(u) : \Gamma_{\mathcal{C}} \to \mathbb{R}\}$ as

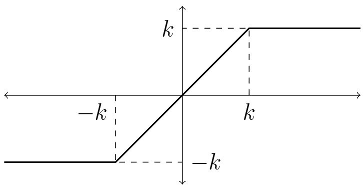
$$T_0^\varepsilon(u)(r_1,\theta,\tau) = u_\varepsilon\left(r_1, \varepsilon\left[\frac{\theta}{\varepsilon}\right]_{2\pi} + \varepsilon\tau\right).$$

Note that $T_0^\varepsilon(u) = T^\varepsilon(u)|_{r=r_1}$. Boundary unfolding operator also has similar properties as those of unfolding operator.

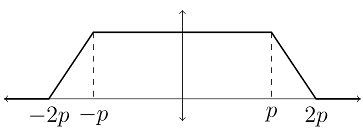
2.4 Auxiliary functions

Here we recall some auxiliary functions that are important in studying renormalized solutions and homogenization with L^1 data. The functions defined are standard and available in the literature. For details, we refer to [7, 12, 17, 22]. All the functions are defined from $\mathbb{R} \to \mathbb{R}$.

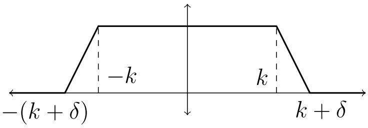
$$T_k(t) = \begin{cases} t & \text{if } |t| \le k, \\ k & \text{if } t > k, \\ -k & \text{if } t < -k. \end{cases}$$



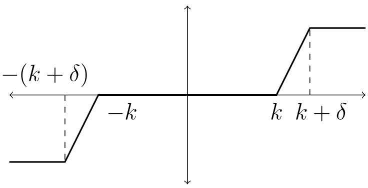
$$g_p(t) = \begin{cases} 1 & \text{if } |t| \le p, \\ 1 - \frac{|t|}{p} & \text{if } p < |t| < 2p, \\ 0 & \text{if } |t| \ge 2p. \end{cases}$$



$$g_\delta^k(t) = \begin{cases} 1 & \text{if } |t| \le k, \\ 1 - \frac{|t|-k}{\delta} & \text{if } k < |t| < k+\delta, \\ 0 & \text{if } |t| \ge k+\delta. \end{cases}$$



$$\hat{g}_\delta^k(t) = \begin{cases} 0 & \text{if } |t| \le k, \\ \frac{|t|-k}{\delta} & \text{if } k < t < k+\delta, \\ \frac{k-|t|}{\delta} & \text{if } -k-\delta < t < -k, \\ 1 & \text{if } t \ge k+\delta, \\ -1 & \text{if } t \le -k-\delta. \end{cases}$$



3 Homogenization with L^2 data

Let $A = [a_{i,j}]_{2\times 2}$, be a 2×2 matrix, where the entries $a_{ij} : \mathcal{O} \to \mathbb{R}$ are Caratheodory type functions. Also, A is uniformly elliptic and bounded in $\mathcal{O}$, that is, there exist $\alpha, \beta > 0$ such that

$$\langle A(x)\lambda, \lambda\rangle \ge \alpha|\lambda|^2 \quad \text{and} \quad |A(x)\lambda| \le \beta|\lambda|$$

for all $\lambda \in \mathbb{R}^2$ and $a.e$ in $\mathcal{O}$. Define

$$A^\varepsilon(r,\theta) = [a_{ij}^\varepsilon(r,\theta)]_{2\times 2} = \begin{cases} A\left(r, \frac{\theta}{\varepsilon}\right) & \text{if } (r,\theta) \in \mathcal{O}^+, \\ A(r,\theta) & \text{if } (r,\theta) \in \mathcal{O}^-. \end{cases}$$

Consider the following problem in the domain $\mathcal{O}_\varepsilon$:

$$\begin{cases} -\text{div}\big(\big(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\big)A^\varepsilon \nabla u_\varepsilon\big) + u_\varepsilon = f \text{ in } \mathcal{O}_\varepsilon, \\ \qquad\qquad A^\varepsilon \nabla u_\varepsilon \cdot \nu^\varepsilon = 0 \text{ on } \partial\mathcal{O}_\varepsilon. \end{cases} \tag{2}$$

Here $f \in L^2(\mathcal{O})$ is a given function, ν^ε is the outward normal vector on $\partial\mathcal{O}_\varepsilon$. The variational form corresponding to (2) is given as: Find $u_\varepsilon \in H^1(\mathcal{O}_\varepsilon)$ such that

$$\int_{\mathcal{O}_\varepsilon} \big(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\big)A^\varepsilon \nabla u_\varepsilon \nabla v + u_\varepsilon v = \int_{\mathcal{O}_\varepsilon} fv \quad \text{for all } v \in H^1(\mathcal{O}_\varepsilon). \tag{3}$$

The Lax-Milgram lemma guarantees the existence and uniqueness of u_ε for all $\varepsilon > 0$. We want to study the asymptotic behavior of u_ε as $\varepsilon \to 0$. Since the oscillations are in a circular fashion, to study the asymptotic behavior, we need to write the equation in polar form. Using polar identities, we get

$$
\begin{aligned}
A^\varepsilon \nabla u_\varepsilon \nabla v =& A^\varepsilon \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r}\cos(\theta) - \frac{1}{r}\frac{\partial u_\varepsilon}{\partial \theta}\sin(\theta) \\ \frac{\partial u_\varepsilon}{\partial r}\sin(\theta) + \frac{1}{r}\frac{\partial u_\varepsilon}{\partial \theta}\cos(\theta) \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial r}\cos(\theta) - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin(\theta) \\ \frac{\partial v}{\partial r}\sin(\theta) + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos(\theta) \end{bmatrix} \\
=& A^\varepsilon \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \frac{\partial u_\varepsilon}{\partial r}\frac{\partial v}{\partial r} + \frac{1}{r} A^\varepsilon \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \frac{\partial u_\varepsilon}{\partial r}\frac{\partial v}{\partial \theta} \\
&+ \frac{1}{r} A^\varepsilon \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \frac{\partial u_\varepsilon}{\partial \theta}\frac{\partial v}{\partial r} + \frac{1}{r^2} A^\varepsilon \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \frac{\partial u_\varepsilon}{\partial \theta}\frac{\partial v}{\partial \theta}.
\end{aligned}
$$

Now we can rewrite (3) as

$$
\int_{\mathcal{O}_\varepsilon^+} \left(\left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon}\right) \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial \theta} \end{bmatrix} + u_\varepsilon v \right) + \int_{\mathcal{O}^-} A \nabla u_\varepsilon \nabla v + u_\varepsilon v = \int_{\mathcal{O}_\varepsilon} fv, \quad (4)
$$

for all $v \in H^1(\mathcal{O}_\varepsilon)$, where

$$
\begin{aligned}
\alpha^\varepsilon &= A^\varepsilon \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \beta^\varepsilon = \frac{1}{r} A^\varepsilon \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}, \\
\gamma^\varepsilon &= \frac{1}{r} A^\varepsilon \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad \text{and} \quad \eta^\varepsilon = \frac{1}{r^2} A^\varepsilon \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}.
\end{aligned}
$$

On computation, we get $\det \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} = \frac{1}{r^2} \det A^\varepsilon$. Since A^ε is coercive, the matrix $\begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix}$ is also coercive uniformly in $\mathcal{O}^+$.

By the definition of the unfolding operator (see Sect. 2.2), we have $T^\varepsilon(A^\varepsilon)(r, \theta, \tau) = A(r, \tau)$. Since it is independent of ε, for simplicity, we denote $T^\varepsilon(A^\varepsilon)$ as A^0. Then, from the properties of the unfolding operator, we see that $T^\varepsilon(\alpha^\varepsilon), T^\varepsilon(\beta^\varepsilon), T^\varepsilon(\gamma^\varepsilon)$ and $T^\varepsilon(\eta^\varepsilon)$ converges to α, β, γ and η strongly in $L^2(\mathcal{O}_U)$, respectively, as $\varepsilon \to 0$, where

$$
\begin{aligned}
\alpha &= A^0 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \beta = \frac{1}{r} A^0 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}, \\
\gamma &= \frac{1}{r} A^0 \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad \text{and} \quad \eta = \frac{1}{r^2} A^0 \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}.
\end{aligned}
$$

Limit Problem: In order to define the solution of the homogenized variational form and cell problems, we need appropriate function spaces which we will define now. For any function ϕ defined on $\mathcal{O}$, we may write $\phi = \phi^+ \chi_{\mathcal{O}^+} + \phi^- \chi_{\mathcal{O}^-} = (\phi^+, \phi^-)$ throughout this article.

(1) Define $V(\mathcal{O}) = \left\{ \psi \in L^2(\mathcal{O}) : (x \cdot \nabla \psi) \in L^2(\mathcal{O}) \text{ and } \psi \in H^1(\mathcal{O}^-) \right\}$, with the inner product

$$
\langle \phi, \psi \rangle_{V(\mathcal{O})} = \langle \phi, \psi \rangle_{L^2(\mathcal{O}^+)} + \langle (x \cdot \nabla \phi), (x \cdot \nabla \psi) \rangle_{L^2(\mathcal{O}^+)} + \langle \phi, \psi \rangle_{H^1(\mathcal{O}^-)}.
$$

Note that since x is strictly away from origin, $V(\mathcal{O})$ is a Hilbert space. Also, since $x \cdot \nabla \psi \in L^2(\mathcal{O})$, we have $\psi^+ = \psi^-$ on Γ_0. Hence $V(\mathcal{O})$ can also be written as

$$
V(\mathcal{O}) = \left\{ (\psi^+, \psi^-) : (\psi^+, x \cdot \nabla \psi^+) \in L^2(\mathcal{O}^+) \times L^2(\mathcal{O}), \psi^- \in H^1(\mathcal{O}^-), \psi^+ = \psi^- \text{ on } \Gamma_0 \right\}
$$

(2) For any $r \in (r_0, r_1)$, define $V_r = \{ w \in H^1(Y(r)) : w = 0 \ a.e. \text{ in} Y_{\mathcal{C}}(r) \}$ with the norm

$$\|w\|_{Y(r)} = \|w\|_{L^2(Y(r))} + \left\| \frac{\partial w}{\partial \tau} \right\|_{L^2(Y(r))}.$$

(3) Finally $V_U(\mathcal{O}) = \left\{ \phi \in L^2(\mathcal{O}_U) \,:\, \phi = 0 \text{ in} \mathcal{O}_C, \frac{\partial \phi}{\partial \tau} \in L^2(\mathcal{O}_U) \right\}$ with the inner product

$$\langle u, v \rangle_{V_U(\mathcal{O})} = \langle u, v \rangle_{L^2(\mathcal{O}_U)} + \left\langle \frac{\partial u}{\partial \tau}, \frac{\partial v}{\partial \tau} \right\rangle_{L^2(\mathcal{O}_U)}.$$

Here also we use the subscript U because, $V_U(\mathcal{O})$ contains functions defined on the unfolded domain $\mathcal{O}_U$.

Now we are in a position to define the limit problem: Given $f \in L^2(\mathcal{O})$, consider the PDE

$$\begin{cases} -\text{div}\left(\frac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \right) + b_0 u^+ = b_0 f & \text{in} \quad \mathcal{O}^+, \\ -\text{div}(A(\nabla u^-)) + u^- = f & \text{in} \quad \mathcal{O}^-, \\ \frac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \cdot \nu = 0 & \text{on} \quad \Gamma_a, \\ A \nabla u^- \cdot \nu = 0 & \text{on} \quad \Gamma_b, \\ \frac{a_0(x)}{|x|^2} \left(x \cdot \nabla u^+ \right) x \cdot \nu = A \nabla u^- \cdot \nu & \text{on} \quad \Gamma_0, \\ u^+ = u^- & \text{on} \quad \Gamma_0, \end{cases} \tag{5}$$

where

$$a_0(r,\theta) = \int_{Y_C(r)} \left(\frac{\det(A(r,\tau))}{A(r,\tau) \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}} \right) d\tau \quad \text{and} \quad b_0(r) = \int_{Y(r)} (1-\xi) d\tau.$$

Here ξ is uniquely defined by the family of cell problems: For $r \in (r_0, r_1)$,

$$\begin{cases} \text{find } \xi(r,\cdot) \in V_r \text{ such that,} \\ \int_{Y(r)} \eta \frac{\partial \xi(r,\tau)}{\partial \tau} \frac{\partial w(\tau)}{\partial \tau}\, d\tau + \int_{Y(r)} \xi(r,\tau) w(\tau)\, d\tau = \int_{Y(r)} w(\tau)\, d\tau \quad \text{for all } \ w \in V_r. \end{cases} \tag{6}$$

The weak form of the limit problem (5) is given by: Find $u = u^+ \chi_{\mathcal{O}^+} + u^- \chi_{\mathcal{O}^-} \in V(\mathcal{O})$ such that

$$\int_{\mathcal{O}^+} \frac{a_0(x)}{|x|^2} (x \cdot \nabla u)(x \cdot \nabla \phi) + b_0 u \phi + \int_{\mathcal{O}^-} A \nabla u \nabla \phi + u \phi = \int_{\mathcal{O}^+} b_0 f \phi + \int_{\mathcal{O}^-} f \phi, \tag{7}$$

for all $\phi \in V(\mathcal{O})$. Since A is coercive, a_0 is strictly positive, that is $a_0 > k$ for some $k > 0$. Hence using Lax-Milgram lemma the PDE (7) has a unique solution in $V(\mathcal{O})$.

Using the polar transformation $r \frac{\partial}{\partial r} u = (x \cdot \nabla u)$, we can write the polar form of (7) as: Find $u \in V(\mathcal{O})$ such that

$$\int_{\mathcal{O}^+} a_0 \frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r} + b_0 u\phi + \int_{\mathcal{O}^-} A\nabla u \nabla \phi + u\phi = \int_{\mathcal{O}^+} b_0 f\phi + \int_{\mathcal{O}^-} f\phi, \quad \text{for all } \phi \in V(\mathcal{O}). \tag{8}$$

The convergence theorem in the classical case where $f \in L^2$ is given below.

Theorem 1 *Let u_ε, u and ξ be the solutions of* (3), (8) *and* (6) *respectively. Then as $\varepsilon \to 0$, we have the following convergences* (*for the whole sequence ε*):

$$\begin{aligned}
&u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(\mathcal{O}^-),\\
&\widetilde{u_\varepsilon} \rightharpoonup |Y(r)|u + (f-u)\int_{Y(r)} \xi d\tau \quad \text{weakly in } L^2(\mathcal{O}^+),\\
&\chi_{\mathcal{C}_\varepsilon}\widetilde{\frac{\partial u_\varepsilon}{\partial r}} \rightharpoonup |Y_{\mathcal{C}}(r)|\frac{\partial u}{\partial r}, \qquad \chi_{\mathcal{C}_\varepsilon}\widetilde{\frac{\partial u_\varepsilon}{\partial \theta}} \rightharpoonup -\frac{1}{2\pi}\frac{\partial u}{\partial r}\int_{Y_{\mathcal{C}}(r)} \frac{\beta}{\eta} d\tau \quad \text{weakly in } L^2(\mathcal{O}^+),\\
&\varepsilon\chi_{\mathcal{I}_\varepsilon}\widetilde{\frac{\partial u_\varepsilon}{\partial r}} \rightharpoonup 0, \qquad \varepsilon\chi_{\mathcal{I}_\varepsilon}\widetilde{\frac{\partial u_\varepsilon}{\partial \theta}} \rightharpoonup (f-u)\int_{Y(r)} \frac{\partial \xi}{\partial \tau} d\tau \quad \text{weakly in } L^2(\mathcal{O}^+).
\end{aligned}$$

Proof The proof will be accomplished in several steps.

Step 1: (Weak convergences of unfolded sequences) In (4) take $\phi = u_\varepsilon$ as a test function to get

$$\left\| \chi_{\mathcal{C}_\varepsilon} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \right\|_{L^2(\mathcal{O}_\varepsilon)} + \varepsilon \left\| \chi_{\mathcal{I}_\varepsilon} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \right\|_{L^2(\mathcal{O}_\varepsilon)} + \|\nabla u_\varepsilon\|_{L^2(\mathcal{O}^-)} + \|u_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon)} \leqslant \|f\|_{L^2(\mathcal{O}_\varepsilon)}. \tag{9}$$

Note that we do not have uniform bound of u_ε in $H^1(\mathcal{O}_\varepsilon)$ due to high contrasting diffusivity. More precisely, the bound of u_ε in $H^1(\mathcal{I}_\varepsilon)$ is of order ε^{-1}. Hence, we need to analyze the convergence of u_ε in $\mathcal{C}_\varepsilon$ and $\mathcal{I}_\varepsilon$ separately. This is done via the unfolding operator.

From (9) and by the properties of unfolding operator and weak compactness of Hilbert spaces, there exist $u_0, w_\theta, w_r, z_\theta, z_r \in L^2(\mathcal{O}_U)$ and $u^- \in H^1(\mathcal{O}^-)$ such that

$$\begin{aligned}
&T^\varepsilon(u_\varepsilon) \rightharpoonup u_0 \quad \text{weakly in } L^2(\mathcal{O}_U),\\
&T^\varepsilon_{\mathcal{C}}\left(\begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix}\right) \rightharpoonup \begin{bmatrix} w_r \\ w_\theta \end{bmatrix} \quad \text{weakly in } (L^2(\mathcal{O}_{\mathcal{C}}))^2,\\
&\varepsilon T^\varepsilon_{\mathcal{I}}\left(\begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix}\right) \rightharpoonup \begin{bmatrix} z_r \\ z_\theta \end{bmatrix} \quad \text{weakly in } (L^2(\mathcal{O}_{\mathcal{I}}))^2,\\
&u_\varepsilon \rightharpoonup u^- \quad \text{weakly in } H^1(\mathcal{O}^-).
\end{aligned} \tag{10}$$

To identify w_r, z_r and z_θ take $\phi \in C_c^\infty(\mathcal{O}_{\mathcal{C}})$ and $\psi \in C_c^\infty(\mathcal{O}_{\mathcal{I}})$. Then

$$\int_{\mathcal{O}_{\mathcal{C}}} w_r\phi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{C}}} T^{\varepsilon}_{\mathcal{C}}\left(\frac{\partial u_\varepsilon}{\partial r}\right)\phi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{C}}} \frac{\partial}{\partial r}T^{\varepsilon}_{\mathcal{C}}\left(u_\varepsilon\right)\phi$$
$$= -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{C}}} T^{\varepsilon}_{\mathcal{C}}(u_\varepsilon)\frac{\partial\phi}{\partial r} = -\int_{\mathcal{O}_{\mathcal{C}}} u_0\frac{\partial\phi}{\partial r} = \int_{\mathcal{O}_{\mathcal{C}}}\frac{\partial u_0}{\partial r}\phi,$$
$$\int_{\mathcal{O}_{\mathcal{I}}} z_r\psi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{I}}} \varepsilon T^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial r}\right)\psi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{I}}} \varepsilon\left(\frac{\partial}{\partial r}T^{\varepsilon}\left(u_\varepsilon\right)\right)\psi$$
$$= -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{I}}} \varepsilon T^{\varepsilon}\left(u_\varepsilon\right)\left(\frac{\partial\psi}{\partial r}\right) = 0,$$
$$\int_{\mathcal{O}_{\mathcal{I}}} z_\theta\psi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{I}}} \varepsilon T^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial \theta}\right)\psi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{I}}} \frac{\partial}{\partial \tau}T^{\varepsilon}\left(u_\varepsilon\right)\psi$$
$$= -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_{\mathcal{I}}} T^{\varepsilon}u_\varepsilon\left(\frac{\partial\psi}{\partial \tau}\right) = -\int_{\mathcal{O}_{\mathcal{I}}} u_0\frac{\partial\psi}{\partial\tau} = \int_{\mathcal{O}_{\mathcal{I}}}\frac{\partial u_0}{\partial\tau}\psi.$$

Since ϕ and ψ are arbitrary, we have

$$w_r = \frac{\partial u_0}{\partial r}, \quad z_r = 0 \quad \text{and} \quad z_\theta = \frac{\partial u_0}{\partial \tau}.$$

To identify w_θ, consider

$$\phi^\varepsilon(r,\theta) = \varepsilon\phi\left(r,\theta,\tfrac{\theta}{\varepsilon}\right), \tag{11}$$

where $\phi = \phi(r,\theta,\tau) \in C^\infty_c(\mathcal{O}^+;C^\infty_{per}(0,2\pi))$ and $\phi = 0$ on $\mathcal{O}_{\mathcal{I}}$. That is we have chosen ϕ^ε in such a way that it will vanish outside $\mathcal{C}_\varepsilon$. Then

$$\begin{aligned}
T^{\varepsilon}_{\mathcal{C}}\phi^\varepsilon(r,\theta,\tau) &= \varepsilon\phi\left(r,\varepsilon\left[\tfrac{\theta}{\varepsilon}\right]+\varepsilon\tau,\tau\right),\\
T^{\varepsilon}_{\mathcal{C}}\left(\frac{\partial\phi^\varepsilon}{\partial r}\right)(r,\theta,\tau) &= \varepsilon\frac{\partial\phi}{\partial r}\left(r,\varepsilon\left[\tfrac{\theta}{\varepsilon}\right]+\varepsilon\tau,\tau\right),\\
T^{\varepsilon}_{\mathcal{C}}\left(\frac{\partial\phi^\varepsilon}{\partial \theta}\right)(r,\theta,\tau) &= \varepsilon\frac{\partial\phi}{\partial \theta}\left(r,\varepsilon\left[\tfrac{\theta}{\varepsilon}\right]+\varepsilon\tau,\tau\right) + \frac{\partial\phi}{\partial \tau}\left(r,\varepsilon\left[\tfrac{\theta}{\varepsilon}\right]+\varepsilon\tau,\tau\right).
\end{aligned} \tag{12}$$

Now use ϕ^ε as a test function in (4). Since ϕ^ε vanishes outside $\mathcal{C}_\varepsilon$, we get

$$\int_{\mathcal{C}_\varepsilon}\begin{bmatrix}\alpha^\varepsilon & \gamma^\varepsilon\\ \beta^\varepsilon & \eta^\varepsilon\end{bmatrix}\begin{bmatrix}\frac{\partial u_\varepsilon}{\partial r}\\ \frac{\partial u_\varepsilon}{\partial\theta}\end{bmatrix}\begin{bmatrix}\frac{\partial \phi^\varepsilon}{\partial r}\\ \frac{\partial \phi\varepsilon}{\partial\theta}\end{bmatrix} + u_\varepsilon\phi = \int_{\mathcal{C}_\varepsilon} f\phi^\varepsilon,$$

Apply unfolding and pass to the limit as $\varepsilon\to 0$ to get

$$\int_{\mathcal{O}_{\mathcal{C}}}\left(\begin{bmatrix}\alpha & \gamma\\ \beta & \eta\end{bmatrix}\begin{bmatrix}w_r\\ w_\theta\end{bmatrix}\begin{bmatrix}0\\ \frac{\partial\phi}{\partial\tau}\end{bmatrix}\right) = \int_{\mathcal{O}_{\mathcal{C}}}\left(\beta w_r + \eta w_\theta\right)\frac{\partial\phi}{\partial\tau} = 0.$$

Since ϕ is arbitrary, we have

$$w_\theta = -\frac{\beta}{\eta}w_r = -\frac{\beta}{\eta}\frac{\partial u_0}{\partial r}.$$

Now for any $\phi \in C^\infty_c(\mathcal{O}_{\mathcal{C}})$, we have

$$\int_{\mathcal{O}_C} \frac{\partial u_0}{\partial \tau}\phi = -\int_{\mathcal{O}_C} u_0 \frac{\partial \phi}{\partial \tau} = -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_C} T^\varepsilon_C(u_\varepsilon)\frac{\partial \phi}{\partial \tau} = -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_C} \frac{\partial}{\partial \tau}T^\varepsilon_C(u_\varepsilon)\phi$$
$$= -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_C} \varepsilon T^\varepsilon_C\left(\frac{\partial u_\varepsilon}{\partial \theta}\right)\phi = 0,$$

which shows that u_0 is independent of τ in $\mathcal{O}_C$. That is, there exist $u^+ \in L^2(\mathcal{O}^+)$ such that $u_0 = u^+$ in $\mathcal{O}_C$. Define $u_1 = u_0 - u^+$. Then we can write u_0 as $u_0 = u^+ + u_1$ with $u^+ \in L^2(\mathcal{O}^+)$ and u_1 vanishes on $\mathcal{O}_C$. Also, we can rewrite (10) as

$$\begin{aligned}
&T^\varepsilon(u_\varepsilon) \rightharpoonup u^+ + u_1 \quad \text{weakly in } L^2(\mathcal{O}_U),\\
&T^\varepsilon_C\left(\begin{bmatrix}\frac{\partial u_\varepsilon}{\partial r}\\ \frac{\partial u_\varepsilon}{\partial \theta}\end{bmatrix}\right) \rightharpoonup \begin{bmatrix}1\\ -\frac{\beta}{\eta}\end{bmatrix}\frac{\partial u^+}{\partial r} \quad \text{weakly in } (L^2(\mathcal{O}_C))^2,\\
&\varepsilon T^\varepsilon_{\mathcal{I}}\left(\begin{bmatrix}\frac{\partial u_\varepsilon}{\partial r}\\ \frac{\partial u_\varepsilon}{\partial \theta}\end{bmatrix}\right) \rightharpoonup \begin{bmatrix}0\\ 1\end{bmatrix}\frac{\partial u_1}{\partial \tau} \quad \text{weakly in} (L^2(\mathcal{O}_{\mathcal{I}}))^2,\\
&u_\varepsilon \rightharpoonup u^- \quad \text{weakly in } H^1(\mathcal{O}^-).
\end{aligned}$$

Step 2 (Interface Condition): In this step, we are going to prove that $u^+ = u^-$ on Γ_0. By the continuity of trace operator and using properties of unfolding operator, we get

$$\int_{\Gamma_C} u^+\phi = \lim \int_{\Gamma_C} \left(T^\varepsilon_C(u_\varepsilon)\right)\Big|_{r=r_1} T^\varepsilon_0(\phi) = \lim \int_{\Gamma_C} \left(T^\varepsilon_0(u_\varepsilon|_{\mathcal{O}^+})\right)\Big|_{r=r_1} T^\varepsilon_0(\phi)$$
$$= \lim \int_{\Gamma_C} \left(T^\varepsilon_0(u_\varepsilon|_{\mathcal{O}}^-)\right)\Big|_{r=r_1} T^\varepsilon_0(\phi) = \int_{\Gamma_C} u^-\phi,$$

for any $\phi \in C^\infty_c(\Gamma_0)$. Hence, we have $u^+ = u^-$ on Γ_0. Define

$$u = \chi_{\mathcal{O}^+}u^+ + \chi_{\mathcal{O}}^- u^-.$$

Since $x \cdot \nabla u^+ = r\frac{\partial u^+}{\partial r} \in L^2(\mathcal{O}^+)$ and $u^- \in H^1(\mathcal{O}^-)$, the interface condition gives $u \in V(\mathcal{O})$.

Now we are going to characterize u_1 by cell problem (6) and then prove that u satisfies the limit problem.

Step 3 (Limit Problem): We now derive the limit equations using the results obtained in the previous steps. Let $\phi_\varepsilon(r,\theta) = \phi(r,\theta) + \phi_1\left(r,\theta,\frac{\theta}{\varepsilon}\right)$, where $\phi \in C^1(\bar{\mathcal{O}})$ and $\phi_1 \in C^\infty(\overline{\mathcal{O}_U})$ with 1 periodic in τ variable and $\phi_1 = 0$ on $\mathcal{O}_C$. Now using ϕ_ε as a test function in (3), applying unfolding operator on both sides and letting $\varepsilon \to 0$ to get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathcal{O}_C}\left(\frac{1}{\eta}(\alpha\eta - \gamma\beta)\frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r} + u\phi\right) rdrd\theta d\tau\\
&\quad + \frac{1}{2\pi}\int_{\mathcal{O}_{\mathcal{I}}}\left(\eta\frac{\partial u_1}{\partial \tau}\frac{\partial \phi_1}{\partial \tau} + (u+u_1)(\phi+\phi_1)\right) rdrd\theta d\tau + \int_{\mathcal{O}^-} A\nabla u\nabla\phi + u\phi\, dx\\
&\quad = \frac{1}{2\pi}\int_{\mathcal{O}_U} f(\phi+\phi_1) rdrd\theta d\tau + \int_{\mathcal{O}^-} f\phi\, dx.
\end{aligned} \tag{13}$$

From density arguments, (13) holds true for all $\phi \in V(\mathcal{O})$ and $\phi_1 \in V_U(\mathcal{O})$. By averaging out, we see that $(u, u_1) \in V(\mathcal{O}) \times V_U(\mathcal{O})$ satisfies the variational form

$$\begin{aligned} &\int_{\mathcal{O}^+}\left(a_0\frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r}+\frac{|Y_C(r)|}{2\pi}u\phi\right)rdrd\theta+\int_{\mathcal{O}^-}(A\nabla u\nabla\phi+u\phi)dx\\ &\quad+\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{I}}\left(\eta\frac{\partial u_1}{\partial \tau}\frac{\partial \phi_1}{\partial \tau}+(u+u_1)(\phi+\phi_1)\right)rdrd\theta d\tau \\ &\quad=\int_{\mathcal{O}^+}\frac{|Y_C(r)|}{2\pi}f\phi dx+\int_{\mathcal{O}^-}f\phi\, rdrd\theta+\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{I}}f(\phi+\phi_1)rdrd\theta d\tau, \end{aligned} \tag{14}$$

for all $(\phi,\phi_1)\in V(\mathcal{O})\times V_U(\mathcal{O})$, where

$$a_0(r,\theta)=\int_{Y(r)}\frac{1}{\eta}(\alpha\eta-\gamma\beta)d\tau=\int_{Y(r)}\left(\frac{\det(A(r,\tau))}{A(r,\tau)\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}}\right)d\tau.$$

From the ellipticity properties of A, (14) has a unique solution. Now to evaluate u_1 in terms of u, put $\phi=0$ in (14) to get,

$$\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{I}}\eta\frac{\partial u_1}{\partial \tau}\frac{\partial \phi_1}{\partial \tau}+\int_{\mathcal{O}_\mathcal{I}}(u+u_1)\phi_1=\int_{\mathcal{O}_\mathcal{I}}f\phi_1.$$

Now using the cell problem as in (6) and by uniqueness, we may write

$$u_1(r,\theta,\tau)=(f(r,\theta)-u(r,\theta))\xi(r,\tau).$$

By replacing the expression for u_1 from above equality, we can simplify (14) by putting $\phi_1=0$ in (14), to get

$$\begin{aligned} &\int_{\mathcal{O}^+}a_0\frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r}+\frac{|Y_C(r)|}{2\pi}u\phi+\int_{\mathcal{O}^-}A\nabla u\nabla\phi+u\phi+\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{I}}(u+(f-u)\xi)\phi\\ &\qquad=\int_{\mathcal{O}^+}\frac{|Y_C(r)|}{2\pi}f\phi+\int_{\mathcal{O}^-}f\phi+\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{I}}f\phi. \end{aligned}$$

On simplifying, we can write the limit problem as

$$\int_{\mathcal{O}^+}a_0\frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r}+b_0u\phi+\int_{\mathcal{O}^-}A\nabla u\nabla\phi+u\phi=\int_{\mathcal{O}^+}b_0f\phi+\int_{\mathcal{O}^-}f\phi,\quad \text{for all }\ \phi\in V(\mathcal{O}).$$

with $b_0(r)=\int_{Y(r)}(1-\xi)d\tau$, where ξ is uniquely defined by the family of cell problems (6). Hence u is the solution of (8) and Theorem 1 is proved.

One of the central points of the current work is to homogenize (2) with L^1 data. Now we need to prove the following corrector results which are very crucial to establish the homogenization with L^1 data.

Theorem 2 (Corrector Results) *Let u_ε, u, and ξ be as in Theorem 1. Define*

$$\xi_\varepsilon : \mathcal{O}_\varepsilon^+ \to \mathbb{R} \quad \text{by} \quad \xi_\varepsilon(r,\theta) = \xi\Big(r, \Big\{\frac{\theta}{\varepsilon}\Big\}\Big) \quad \text{and}$$
$$\xi'_\varepsilon : \mathcal{O}_\varepsilon^+ \to \mathbb{R} \quad \text{by} \quad \xi'_\varepsilon(r,\theta) = \frac{\partial \xi}{\partial \tau}\Big(r, \Big\{\frac{\theta}{\varepsilon}\Big\}\Big).$$

Then, we have the following strong convergences.

$$\|u_\varepsilon - u\|_{H^1(\mathcal{O}^-)} \longrightarrow 0,$$
$$\Big\|u_\varepsilon - \big(u + \chi_{\mathcal{I}_\varepsilon}(f-u)\xi_\varepsilon\big)\Big\|_{L^2(\mathcal{O}_\varepsilon)} \longrightarrow 0,$$
$$\left\| \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} - \begin{bmatrix} 1 \\ -\frac{\beta^\varepsilon}{\eta^\varepsilon} \end{bmatrix} \frac{\partial u}{\partial r} \right\|_{L^2(\mathcal{C}_\varepsilon)^2} + \left\| \varepsilon \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} - \begin{bmatrix} 0 \\ (f-u)\xi'_\varepsilon \end{bmatrix} \right\|_{L^2(\mathcal{I}_\varepsilon)^2} \longrightarrow 0.$$

Proof Let us define,

$$\begin{aligned} J^\varepsilon = &\int_{\mathcal{C}_\varepsilon} \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} - \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} + \frac{\beta^\varepsilon}{\eta^\varepsilon}\frac{\partial u}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} - \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} + \frac{\beta^\varepsilon}{\eta^\varepsilon}\frac{\partial u}{\partial r} \end{bmatrix} + (u_\varepsilon - u)^2 \\ &+ \int_{\mathcal{I}_\varepsilon} \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} \varepsilon\frac{\partial u_\varepsilon}{\partial r} \\ \varepsilon\frac{\partial u}{\partial \theta} - (f-u)\xi'_\varepsilon \end{bmatrix} \begin{bmatrix} \varepsilon\frac{\partial u_\varepsilon}{\partial r} \\ \varepsilon\frac{\partial u}{\partial \theta} - (f-u)\xi'_\varepsilon \end{bmatrix} + \big(u_\varepsilon - u - (f-u)\xi_\varepsilon\big)^2 \\ &+ \int_{\mathcal{O}^-} A(\nabla u_\varepsilon - \nabla u)(\nabla u_\varepsilon - \nabla u) + (u_\varepsilon - u)^2. \end{aligned}$$

By expanding and rearranging, we get

$$J^\varepsilon = J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon,$$

where

$$\begin{aligned} J_1^\varepsilon =& \int_{\mathcal{O}_\varepsilon} \big(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\big) A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon + (u^\varepsilon)^2, \\ J_2^\varepsilon =& -\int_{\mathcal{C}_\varepsilon} \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{-\beta^\varepsilon}{\eta^\varepsilon} \end{bmatrix} \frac{\partial u}{\partial r} - \int_{\mathcal{C}_\varepsilon} \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} 1 \\ \frac{-\beta^\varepsilon}{\eta^\varepsilon} \end{bmatrix} \begin{bmatrix} \frac{\partial u_\varepsilon}{\partial r} \\ \frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \frac{\partial u}{\partial r} \\ &+ \int_{\mathcal{C}_\varepsilon} \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\beta^\varepsilon}{\eta^\varepsilon} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\beta^\varepsilon}{\eta^\varepsilon} \end{bmatrix} \Big(\frac{\partial u}{\partial r}\Big)^2 + \int_{\mathcal{C}_\varepsilon} (-2u_\varepsilon u + u^2), \\ J_3^\varepsilon =& -\int_{\mathcal{I}_\varepsilon} \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} \varepsilon\frac{\partial u_\varepsilon}{\partial r} \\ \varepsilon\frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \begin{bmatrix} 0 \\ (f-u)\xi'_\varepsilon \end{bmatrix} - \int_{\mathcal{I}_\varepsilon} \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} 0 \\ (f-u)\xi'_\varepsilon \end{bmatrix} \begin{bmatrix} \varepsilon\frac{\partial u_\varepsilon}{\partial r} \\ \varepsilon\frac{\partial u_\varepsilon}{\partial \theta} \end{bmatrix} \\ &+ \int_{\mathcal{I}_\varepsilon} \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} 0 \\ (f-u)\xi'_\varepsilon \end{bmatrix} \begin{bmatrix} 0 \\ (f-u)\xi'_\varepsilon \end{bmatrix} \\ &+ \int_{\mathcal{I}_\varepsilon} -2u_\varepsilon\big(u + (f-u)\xi_\varepsilon\big) + \big(u + (f-u)\xi_\varepsilon\big)^2, \\ J_4^\varepsilon =& \int_{\mathcal{O}^-} -A\nabla u_\varepsilon \nabla u - A\nabla u \nabla u_\varepsilon + A\nabla u \nabla u - 2u_\varepsilon u + u^2. \end{aligned}$$

On applying unfolding operator and passing to the limit as $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0} J_2^\varepsilon = -\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{C}}\left(\begin{bmatrix}\alpha & \gamma\\ \beta & \eta\end{bmatrix}\begin{bmatrix}1\\ \frac{-\beta}{\eta}\end{bmatrix}\begin{bmatrix}1\\ \frac{-\beta}{\eta}\end{bmatrix}\left(\frac{\partial u}{\partial r}\right)^2 + u^2\right).$$

Similarly since $T^\varepsilon_\mathcal{I}((f-u)\xi_\varepsilon) \to (f-u)\xi = u_1$ and $T^\varepsilon_\mathcal{I}((f-u)\xi'_\varepsilon) \to (f-u)\frac{\partial\xi}{\partial\tau} = \frac{\partial u_1}{\partial\tau}$ as $\varepsilon \to 0$, by using unfolding operator, we can arrive at

$$\lim_{\varepsilon\to 0} J_3^\varepsilon = -\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{I}}\left(\begin{bmatrix}\alpha & \gamma\\ \beta & \eta\end{bmatrix}\begin{bmatrix}0\\ \frac{\partial u_1}{\partial\tau}\end{bmatrix}\begin{bmatrix}0\\ \frac{\partial u_1}{\partial\tau}\end{bmatrix} + (u+u_1)^2\right).$$

Also, we have

$$\lim_{\varepsilon\to 0} J_4^\varepsilon = -\int_{\mathcal{O}^-}\left(A\nabla u\nabla u + u^2\right).$$

Using (3) and (14) we get

$$\begin{aligned}\lim_{\varepsilon\to 0} J_1^\varepsilon =& \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\varepsilon}\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right)A^\varepsilon\nabla u_\varepsilon\nabla u_\varepsilon + (u^\varepsilon)^2 = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\varepsilon} fu_\varepsilon\\ =&\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{C}} fu + \int_{\mathcal{O}_\mathcal{I}} f(u+u_1) + \int_{\mathcal{O}^-} fu\\ =&\frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{C}}\frac{1}{\eta}(\alpha\eta-\gamma\beta)\left(\frac{\partial u}{\partial r}\right)^2 + u^2 + \frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{I}}\eta\left(\frac{\partial u_1}{\partial\tau}\right)^2 + (u+u_1)^2\\ &+\int_{\mathcal{O}^-} A\nabla u\nabla u + u^2 \quad \left(\text{by taking } \phi = u \text{ and } \phi_1 = u_1 \text{ in } (14)\right)\\ =&-\left(\lim_{\varepsilon\to 0} J_2^\varepsilon + \lim_{\varepsilon\to 0} J_3^\varepsilon + \lim_{\varepsilon\to 0} J_4^\varepsilon\right).\end{aligned}$$

This implies that

$$\lim_{\varepsilon\to 0} J^\varepsilon = 0.$$

Now coercivity of the matrix A completes the proof of the Theorem 2.

We are now in a position to study homogenization with L^1 data.

4 Homogenization with L^1 data

Now we will consider the variational form with L^1 data. Let A and A^ε be defined as in Sect. 3. Consider the following ε dependent problem:

$$\begin{cases} -\operatorname{div}\left(\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right)A^\varepsilon\nabla u_\varepsilon\right) + u_\varepsilon = f \text{ in } \mathcal{O}_\varepsilon,\\ A^\varepsilon\nabla u_\varepsilon\cdot\nu^\varepsilon = 0 \text{ on } \partial\mathcal{O}_\varepsilon.\end{cases} \tag{15}$$

Here, $f \in L^1(\mathcal{O})$ is a given function, ν^ε is the outward unit normal vector on $\partial\mathcal{O}_\varepsilon$. As it is well known, we remark that the solution is not defined in the usual weak formulation but using the concept of renormalized solution. Recall the auxiliary function T_k defined as in Sect. 2.4. A function u_ε is called a renormalized solution of (15) if

$$\begin{cases} u_\varepsilon \in L^1(\mathcal{O}_\varepsilon) \text{ such that } T_k(u_\varepsilon) \in H^1(\mathcal{O}_\varepsilon), \text{ for all } k > 0, \\ \frac{1}{k}\|T_k(u_\varepsilon)\|^2_{H^1(\mathcal{O}_\varepsilon)} \to 0 \text{ as } k \to \infty, \\ \int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_k(u_\varepsilon) \nabla(\psi g(u_\varepsilon)) + u_\varepsilon \psi g(u_\varepsilon) = \int_{\mathcal{O}_\varepsilon} f \psi g(u_\varepsilon), \\ \text{for all } k > 0, \psi \in H^1(\mathcal{O}_\varepsilon) \cap L^\infty(\mathcal{O}_\varepsilon), g \in PC^1_c(\mathbb{R}) \text{ with supp}\,(g) \subset [-k, k]. \end{cases} \tag{16}$$

Here $PC^1_c(\mathbb{R})$ denotes the set of all Lipschitz continuous functions which are piece-wise differentiable on $\mathbb{R}$ with compact support. In polar form, we can write (16) as

$$\begin{cases} \text{find } u_\varepsilon \in L^1(\mathcal{O}_\varepsilon) \text{ such that } T_k(u_\varepsilon) \in H^1(\mathcal{O}_\varepsilon), \text{ for all } k > 0, \\ \frac{1}{k}\|T_k(u)\|^2_{H^1(\mathcal{O}_\varepsilon)} \to 0 \text{ as } k \to \infty, \\ \int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon}) \begin{bmatrix} \alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} T_k(u_\varepsilon) \\ \frac{\partial}{\partial \theta} T_k(u_\varepsilon) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r}(\psi g(u_\varepsilon)) \\ \frac{\partial}{\partial \theta}(\psi g(u_\varepsilon)) \end{bmatrix} + T_k(u_\varepsilon)\psi g(u_\varepsilon) \\ \qquad + \int_{\mathcal{O}^-} A \nabla T_k(u_\varepsilon) \nabla(\psi g(u_\varepsilon)) + T_k(u_\varepsilon)\psi g(u_\varepsilon) = \int_{\mathcal{O}_\varepsilon} f \psi g(u_\varepsilon), \\ \text{for all } k > 0, \psi \in H^1(\mathcal{O}_\varepsilon) \cap L^\infty(\mathcal{O}_\varepsilon), g \in PC^1_c(\mathbb{R}) \text{ with supp}\,(g) \subset [-k, k]. \end{cases}$$

We want to study the asymptotic behavior of u_ε as $\varepsilon \to 0$. We prove that the limit problem nothing but (5) with $f \in L^1(\mathcal{O})$. The corresponding formulation in polar coordinates is given below:

Limit problem: Given $f \in L^1(\mathcal{O})$, consider the problem:

$$\begin{cases} \text{find } u \in L^1(\mathcal{O}) \text{ such that } T_k(u) \in V(\mathcal{O}) \text{ for all } k > 0, \\ \frac{1}{k}\|T_k(u)\|^2_{V(\mathcal{O})} \to 0 \; as \; k \to \infty, \\ \int_{\mathcal{O}^+} a_0 \frac{\partial}{\partial r} T_k(u) \frac{\partial(\psi g(u))}{\partial r} + b_0 T_k(u)\psi g(u) + \int_{\mathcal{O}^-} A(\nabla T_k(u) \nabla(\psi g(u)) + T_k(u)\psi g(u) \\ \qquad = \int_{\mathcal{O}^+} b_0 f \psi g(u) + \int_{\mathcal{O}^-} f \psi g(u), \\ \text{for all } k > 0, \psi \in V(\mathcal{O}) \cap L^\infty(\mathcal{O}), \; g \in PC^1_c(\mathbb{R}) \text{ with supp}\,(g) \subset [-k, k]. \end{cases} \tag{17}$$

where a_0 and b_0 as in (5).

We recall the properties of renormalized solutions without proof. The proofs are available in [17].

(I) **(Existence and uniqueness)** There exist unique renormalized solutions for (16) and (17).

(II) **(Energy equality)** Following energy equalities hold for (16) and (17) respectively:

$$\int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon}) A^\varepsilon \nabla T_k(u_\varepsilon) \nabla T_k(u_\varepsilon) + u_\varepsilon T_k(u_\varepsilon) = \int_{\mathcal{O}_\varepsilon} f T_k(u_\varepsilon),$$

$$\int_{\mathcal{O}^+} a_0 \Big(\frac{\partial}{\partial r} T_k(u)\Big)^2 + b_0 u T_k(u) + \int_{\mathcal{O}^-} A \nabla T_k(u) \nabla T_k(u) + u T_k(u) = \int_{\mathcal{O}^+} b_0 f T_k(u) + \int_{\mathcal{O}^-} f T_k(u).$$

(III) **(Lipschitz Property)** Let $f_1, f_2 \in L^1(\mathcal{O})$ and u_1, u_2 be renormalized solutions of (17) with $f = f_1$ and $f = f_2$ respectively. Then

$$\|u_1 - u_2\|_{L^1(\mathcal{O})} \le \|f_1 - f_2\|_{L^1(\mathcal{O})}.$$

Analogous result holds for (16) also.

Now we are going to prove an equivalent form of renormalized solution defined above.

Lemma 1 *The renormalized formulation* (16) *is equivalent to the following formulation*:

$$\begin{cases} \text{find } u_\varepsilon \in L^1(\mathcal{O}_\varepsilon) \text{ such that } T_k(u_\varepsilon) \in H^1(\mathcal{O}_\varepsilon), \text{ for all } k > 0, \\ \frac{1}{k} \|T_k(u)\|^2_{H^1(\mathcal{O}_\varepsilon)} \to 0 \textit{ as } k \to \infty, \\ \int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_k(u_\varepsilon) \nabla w + u_\varepsilon w = \int_{\mathcal{O}_\varepsilon} f w, \\ \text{for all } k > 0, w \in H^1(\mathcal{O}_\varepsilon) \cap L^\infty(\mathcal{O}_\varepsilon) \text{ such that } \nabla w = 0 \text{ when } |u_\varepsilon| > k. \end{cases} \tag{18}$$

That is, u_ε *is a solution of* (16) *if and only if it is a solution of* (18).

Proof Let u_ε be a solution of (18). Clearly for ψ and g as in (16), $vg(u_\varepsilon)$ will satisfy the conditions for w in (18). Hence u_ε is a solution of (16).

Conversely, suppose that u_ε is a solution of (16). Let w be a test function as in (18). Choose $\psi = w$ and $g = g_p$ for $2p > k$ (defined in Sect. 2.4) in (16). Then it follows that

$$\int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_{2p}(u_\varepsilon) \nabla w g_p(u_\varepsilon) + \int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_{2p}(u_\varepsilon) \nabla T_{2p}(u_\varepsilon) w g_p'(u_\varepsilon) + u_\varepsilon w g_p(u_\varepsilon) = \int_{\mathcal{O}_\varepsilon} f w g_p(u_\varepsilon).$$

Since $\nabla w = 0$ when $|u_\varepsilon| > k$, we have $\nabla T_{2p}(u_\varepsilon) = \nabla T_k(u_\varepsilon)$. Then using the Lebesgue dominated convergence theorem, we have

$$\lim_{p\to\infty} \int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_{2p}(u_\varepsilon) \nabla w g_p(u_\varepsilon) + u_\varepsilon w g_p(u_\varepsilon) = \int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_k(u_\varepsilon) \nabla w + u_\varepsilon w$$

and

$$\lim_{p\to\infty} \int_{\mathcal{O}_\varepsilon} f w g_p(u_\varepsilon) = \int_{\mathcal{O}_\varepsilon} f w.$$

Now from the third equality in (16), we have

$$\left|\int_{\mathcal{O}_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) A^\varepsilon \nabla T_{2p}(u_\varepsilon) \nabla T_{2p}(u_\varepsilon) w g_p'(u_\varepsilon)\right| \\ \leq \|w\|_{L^\infty(\mathcal{O}_\varepsilon)} \frac{1}{p} \int_{\mathcal{O}_\varepsilon} |\nabla T_{2p}(u_\varepsilon)|^2 \to 0 \text{ as } p \to \infty.$$

Since this is true for all $k > 0$, we see that u_ε satisfies (18).

Now, we require a lemma which gives the boundedness of ξ given by the family of cell problems (6).

Lemma 2 *Let the function $\xi = \xi(r, \tau)$ be given by the family of the cell problems* (6). *Then there exists a constant C independent r, τ such that,*

$$|\xi(r, \tau)| \leqslant C \quad \text{for all } \; r \in (r_1, r_2), \tau \in Y(r).$$

Proof By choosing $\xi(r, \cdot)$ as test function in (6), and using the Young's inequality in the right hand side, we deduce,

$$\int_{Y(r)} \eta \left(\frac{\partial \xi(r, \tau)}{\partial \tau}\right)^2 + (\xi(r, \tau))^2 \, d\tau = \int_{Y(r)} \xi(r, \tau) \, d\tau \leq \frac{1}{2} \int_{Y(r)} (\xi(r, \tau))^2 \, d\tau + \frac{1}{2} \int_{Y(r)} 1 \, d\tau.$$

Then the coercivity of A ensures that $\eta > \delta$ for some $\delta > 0$. Hence, there exists a constant C independent of r such that

$$\|\xi(r, \cdot)\|_{H^1(Y(r))} \leq C|Y(r)| \leq 2\pi C.$$

Now, for each $r \in (r_0, r_1)$, $\xi(r, \cdot) \in V_r \subset H^1(Y(r))$. Note that $Y(r)$ is an open bounded subset of $\mathbb{R}$. Hence $\xi(r, \cdot)$ is absolutely continuous and there exist a constant K independent of r such that

$$\|\xi(r, \cdot)\|_{L^\infty(Y(r))} \leq K\|\xi(r, \cdot)\|_{H^1(Y(r))} \leq 2\pi KC.$$

Since C and K are independent of r, lemma is proved.

Theorem 3 (Corrector Results) *Let u_ε, u be the unique renormalized solutions of* (16) *and* (17), *and ξ_ε be as in Theorem* 2 $\left(\xi_\varepsilon(r, \theta) = \xi\left(r, \left\{\frac{\theta}{\varepsilon}\right\}\right)\right)$. *Then as $\varepsilon \to 0$, we have*

$$\left\|u_\varepsilon - \left(u + \chi_{\mathcal{I}_\varepsilon}(f - u)\xi_\varepsilon\right)\right\|_{L^1(\mathcal{O}_\varepsilon)} \longrightarrow 0.$$

Proof Let f^n be a sequence in $L^2(\mathcal{O})$ such that $f^n \to f$ in $L^1(\mathcal{O})$. Let u_ε^n, u^n be the renormalized solutions of (16) and (17) with source term f^n. Then

$$\begin{aligned} &\left\|u_\varepsilon - \left(u + \chi_{\mathcal{I}_\varepsilon}(f - u)\xi_\varepsilon\right)\right\|_{L^1(\mathcal{O}_\varepsilon)} \leq \\ &\quad \|u_\varepsilon - u_\varepsilon^n\|_{L^1(\mathcal{O}_\varepsilon)} + \left\|u_\varepsilon^n - \left(u^n + \chi_{\mathcal{I}_\varepsilon}(f^n - u^n)\xi_\varepsilon\right)\right\|_{L^1(\mathcal{O}_\varepsilon)} \\ &\qquad + \left\|\left(u^n + \chi_{\mathcal{I}_\varepsilon}(f^n - u^n)\xi_\varepsilon\right) - \left(u + \chi_{\mathcal{I}_\varepsilon}(f - u)\xi_\varepsilon\right)\right\|_{L^1(\mathcal{O}_\varepsilon)}. \end{aligned} \tag{19}$$

Now we can investigate each term on the right hand side of the above expression. From the Lipschitz Property of renormalized solutions (see [17]), we have

$$\left\|u_\varepsilon - u_\varepsilon^n\right\|_{L^1(\mathcal{O}_\varepsilon)} \le \|f^n - f\|_{L^1(\mathcal{O}_\varepsilon)} \longrightarrow 0 \text{ as } n \to \infty.$$

From Theorem 2 with $f = f^n$, we have

$$\left\|u_\varepsilon^n - \left(u^n + \chi_{\mathcal{I}_\varepsilon}(f^n - u^n)\xi_\varepsilon\right)\right\|_{L^2(\mathcal{O}_\varepsilon)} \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

Then again using Lipschitz property of renormalized solutions, we get

$$\begin{aligned}\left\|\left(u^n + \chi_{\mathcal{I}_\varepsilon}(f^n - u^n)\xi_\varepsilon\right) - \left(u + \chi_{\mathcal{I}_\varepsilon}(f - u)\xi_\varepsilon\right)\right\|_{L^1(\mathcal{O}_\varepsilon)} \\ \le \|u^n - u\|_{L^1(\mathcal{O}_\varepsilon)} + \left\|(f^n - u^n - f + u)\xi_\varepsilon\right\|_{L^1(\mathcal{I}_\varepsilon)} \\ \le \|f^n - f\|_{L^1(\mathcal{O}_\varepsilon)} + \left\|(f^n - u^n - f + u))\xi_\varepsilon\right\|_{L^1(\mathcal{I}_\varepsilon)}.\end{aligned}$$

By using Lemma 2, we have

$$\|f^n - f\|_{L^1(\mathcal{O}_\varepsilon)} + \left\|(f^n - u^n - f + u))\xi_\varepsilon\right\|_{L^1(\mathcal{I}_\varepsilon)} \longrightarrow 0 \text{ as } n \to \infty.$$

So all terms on the right hand side of (19 converge to 0 and hence we have Theorem 3.

Since $T_k(u_\varepsilon) \in H^1(\mathcal{O}_\varepsilon)$ and $T_k(u) \in V(\mathcal{O})$ for all $k > 0$, the study of truncation functions will be more interesting. The following theorem gives the convergence results of truncation function and their derivatives.

Theorem 4 *Let u_ε, u be the unique renormalized solutions of* (16) *and* (17), *and ξ be given by the family of cell problems* (6). *Then as $\varepsilon \to 0$, we have the following weak convergences in $L^2(\mathcal{O}^+)$:*

$$\left.\begin{aligned}&\chi_{\mathcal{O}_\varepsilon} T_k(u_\varepsilon) \rightharpoonup \frac{1}{2\pi}\int_{Y(r)} T_k(u + (f - u)\xi)d\tau, \qquad \chi_{\mathcal{C}_\varepsilon}\frac{\partial}{\partial r}T_k(u_\varepsilon) \rightharpoonup \frac{|Y_{\mathcal{C}}(r)|}{2\pi}\frac{\partial}{\partial r}T_k(u),\\ &\chi_{\mathcal{C}_\varepsilon}\frac{\partial}{\partial\theta}T_k(u_\varepsilon) \rightharpoonup -\frac{1}{2\pi}\frac{\partial}{\partial r}T_k(u)\int_{Y_{\mathcal{C}}(r)}\frac{\beta}{\eta}d\tau,\\ &\varepsilon\chi_{\mathcal{I}_\varepsilon}\frac{\partial}{\partial\theta}T_k(u_\varepsilon) \rightharpoonup \frac{1}{2\pi}\int_{Y_{\mathcal{I}}(r)}\frac{\partial}{\partial\tau}T_k(u + (f - u)\xi)d\tau \qquad \text{and} \qquad \varepsilon\chi_{\mathcal{I}_\varepsilon}\frac{\partial u_\varepsilon}{\partial r} \rightharpoonup 0.\end{aligned}\right\} \tag{20}$$

Also, we have the following weak convergence in $H^1(\mathcal{O}^-)$, namely

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } H^1(\mathcal{O}^-). \tag{21}$$

Proof From Theorem 3, directly follows that

$$\left\|T_k(u_\varepsilon) - T_k\left(u + \chi_{\mathcal{I}_\varepsilon}(f - u)\xi_\varepsilon\right)\right\|_{L^2(\mathcal{O}_\varepsilon)} \longrightarrow 0.$$

On applying unfolding, we get

$$T^\varepsilon u_\varepsilon - T^\varepsilon(u + \chi_{\mathcal{I}_\varepsilon}(f - u)\xi_\varepsilon) \longrightarrow 0 \text{ strongly in } L^1(\mathcal{O}_U) \text{ as } \varepsilon \to 0.$$

On the other hand

$$T^{\varepsilon}(u + \chi_{\mathcal{I}_\varepsilon}(f - u)\xi_\varepsilon) \longrightarrow u + \chi_{\mathcal{O}_{\mathcal{I}}}(f - u)\xi \;\text{ strongly in }\; L^1(\mathcal{O}_U) \;\text{ as }\; \varepsilon \to 0.$$

Define $u_1 = (f - u)\xi$, which is a measurable function vanish on $\mathcal{O}_{\mathcal{C}}$. Hence, we have

$$T^{\varepsilon} u_\varepsilon \longrightarrow u + \chi_{\mathcal{O}_{\mathcal{I}}} u_1 = u + u_1 \;\text{ strongly in }\; L^1(\mathcal{O}_U) \;\text{ as }\; \varepsilon \to 0. \tag{22}$$

Since $\left(T_k(u_\varepsilon)\right)^2 \le u_\varepsilon T_k(u_\varepsilon)$ and A is coercive, from energy equality of (16), we can deduce that

$$\left\|\chi_{\mathcal{C}_\varepsilon} \nabla T_k(u_\varepsilon)\right\|_{L^2(\mathcal{O}_\varepsilon)} + \varepsilon \left\|\chi_{\mathcal{I}_\varepsilon} \nabla T_k(u_\varepsilon)\right\|_{L^2(\mathcal{O}_\varepsilon)} + \|\nabla T_k(u_\varepsilon)\|_{L^2(\mathcal{O}^-)} + \|T_k(u_\varepsilon)\|_{L^2(\mathcal{O}_\varepsilon)} \leqslant \|f\|_{L^1(\mathcal{O})}.$$

Now from the properties of unfolding operators and weak compactness of Hilbert spaces, we have the following convergences:

$$T^{\varepsilon}_{\mathcal{C}}\left(\begin{bmatrix} T_k(u_\varepsilon) \\ \frac{\partial}{\partial r} T_k(u_\varepsilon) \\ \frac{\partial}{\partial \theta} T_k(u_\varepsilon) \end{bmatrix}\right) \rightharpoonup \begin{bmatrix} w \\ w_r \\ w_\theta \end{bmatrix} \text{ weakly in } (L^2(\mathcal{O}_{\mathcal{C}}))^3,$$

$$T^{\varepsilon}_{\mathcal{I}}\left(\begin{bmatrix} T_k(u_\varepsilon) \\ \varepsilon \frac{\partial}{\partial r} T_k(u_\varepsilon) \\ \varepsilon \frac{\partial}{\partial \theta} T_k(u_\varepsilon) \end{bmatrix}\right) \rightharpoonup \begin{bmatrix} z \\ z_r \\ z_\theta \end{bmatrix} \text{ weakly in } (L^2(\mathcal{O}_{\mathcal{I}}))^3,$$

for some w, w_r, w_θ, z, z_r and z_θ, which we need to identify. Since $T^{\varepsilon}(T_k(u_\varepsilon)) = T_k(T^{\varepsilon}(u_\varepsilon))$, using the Lipschitz property of truncation function T_k, we have

$$\begin{aligned} \int_{\mathcal{O}_U} |T^{\varepsilon}(T_k(u_\varepsilon)) - T_k(u + u_1)|^2 &\le \int_{\mathcal{O}_U} |T_k(T^{\varepsilon}(u_\varepsilon)) - T_k(u + u_1)|^2 \\ &\le \int_{\mathcal{O}_U} 2k|T^{\varepsilon}(u_\varepsilon) - (u + u_1)|. \end{aligned}$$

Then from (22), we have

$$T^{\varepsilon}(T_k(u_\varepsilon)) \longrightarrow T_k(u + u_1) \;\text{ strongly in }\; L^2(\mathcal{O}_U).$$

Hence we have $w = T_k(u)$ and $z = T_k(u + u_1)$. Now to identify w_r, z_r and z_θ, take $\phi \in C^\infty_c(\mathcal{O}_{\mathcal{C}})$ and $\psi \in C^\infty_c(\mathcal{O}_{\mathcal{I}})$. Then

$$\int_{\mathcal{O}_\mathcal{C}} w_r\phi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{C}} T_\mathcal{C}^\varepsilon\Big(\frac{\partial}{\partial r}T_k(u_\varepsilon)\Big)\phi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{C}} \frac{\partial}{\partial r}T_\mathcal{C}^\varepsilon\big(T_k(u_\varepsilon)\big)\phi$$
$$= -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{C}} T_\mathcal{C}^\varepsilon(T_k(u_\varepsilon))\frac{\partial\phi}{\partial r} = -\int_{\mathcal{O}_\mathcal{C}} T_k(u)\frac{\partial\phi}{\partial r} = \int_{\mathcal{O}_\mathcal{C}} \frac{\partial}{\partial r}T_k(u)\phi,$$
$$\int_{\mathcal{O}_\mathcal{I}} z_r\psi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{I}} \varepsilon T^\varepsilon\Big(\frac{\partial}{\partial r}T_k(u_\varepsilon)\Big)\psi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{I}} \varepsilon\frac{\partial}{\partial r}\big(T^\varepsilon T_k(u_\varepsilon)\big)\psi$$
$$= -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{I}} \varepsilon T^\varepsilon T_k(u_\varepsilon)\left(\frac{\partial\psi}{\partial r}\right) = 0,$$
$$\int_{\mathcal{O}_\mathcal{I}} z_\theta\psi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{I}} \varepsilon T^\varepsilon\Big(\frac{\partial}{\partial\theta}T_k(u_\varepsilon)\Big)\psi = \lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{I}} \frac{\partial}{\partial\tau}T^\varepsilon T_k\big(u_\varepsilon\big)\psi$$
$$= -\lim_{\varepsilon\to 0}\int_{\mathcal{O}_\mathcal{I}} T^\varepsilon T_k(u_\varepsilon)\left(\frac{\partial\psi}{\partial\tau}\right) = -\int_{\mathcal{O}_\mathcal{I}} T_k(u+u_1)\frac{\partial\psi}{\partial\tau} = \int_{\mathcal{O}_\mathcal{I}} \frac{\partial}{\partial\tau}T_k(u+u_1)\psi.$$

Since ϕ and ψ are arbitrary we have $w_r = \frac{\partial}{\partial r}T_k(u)$, $z_r = 0$ and $z_\theta = \frac{\partial}{\partial\tau}T_k(u+u_1)$.

Now the remaining one is w_θ. Consider ϕ^ε as in (11) and g_δ^k as in Sect. 2.4. Take $\psi = \phi^\varepsilon$ and $g = g_\delta^k$ in (16) to get

$$\int_{\mathcal{O}_\varepsilon} (\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-})A^\varepsilon\nabla T_{k+1}(u_\varepsilon)\nabla T_{k+1}(u_\varepsilon)\phi^\varepsilon(g_\delta^k)'(u_\varepsilon)$$
$$+\int_{\mathcal{O}_\varepsilon} (\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-})A^\varepsilon\nabla T_{k+1}(u_\varepsilon)\nabla\phi^\varepsilon g_\delta^k(u_\varepsilon) + \int_{\mathcal{O}_\varepsilon} u_\varepsilon\phi^\varepsilon g_\delta^k(u_\varepsilon) = \int_{\mathcal{O}_\varepsilon} f\phi^\varepsilon g_\delta^k(u_\varepsilon).$$

Now since $g_\delta^k(u_\varepsilon) \to \chi_{\{|u_\varepsilon|\le k\}}$ $a.e$ as $\delta\to 0$, by Lebesgue dominated convergence theorem, as $\delta\to 0$, we obtain:

$$\int_{\mathcal{O}_\varepsilon} (\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-})A^\varepsilon\nabla T_{k+1}(u_\varepsilon)\nabla\phi^\varepsilon g_\delta^k(u_\varepsilon)$$
$$\longrightarrow \int_{\mathcal{O}_\varepsilon} (\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-})A^\varepsilon\nabla T_{k+1}(u_\varepsilon)\nabla\phi^\varepsilon\chi_{\{|u_\varepsilon|\le k\}}$$
$$= \int_{\mathcal{O}_\varepsilon} (\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-})A^\varepsilon\nabla T_k(u_\varepsilon)\nabla\phi^\varepsilon,$$

$$\int_{\mathcal{O}_\varepsilon} u_\varepsilon\phi^\varepsilon g_\delta^k(u_\varepsilon) \longrightarrow \int_{\mathcal{O}_\varepsilon} u_\varepsilon\phi^\varepsilon\chi_{\{|u_\varepsilon|\le k\}} \quad \text{and} \quad \int_{\mathcal{O}_\varepsilon} f\phi^\varepsilon g_\delta^k(u_\varepsilon) \longrightarrow \int_{\mathcal{O}_\varepsilon} f\phi^\varepsilon\chi_{\{|u_\varepsilon|\le k\}}.$$

Therefore, we have

$$\limsup_{\delta\to 0}\int_{\mathcal{O}_\varepsilon} (\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-})A^\varepsilon\nabla T_{k+1}(u_\varepsilon)\nabla T_{k+1}(u_\varepsilon)\phi^\varepsilon(g_\delta^k)'(u_\varepsilon)$$
$$+\int_{\mathcal{O}_\varepsilon} (\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-})A^\varepsilon\nabla T_k(u_\varepsilon)\nabla\phi^\varepsilon + \int_{\mathcal{O}_\varepsilon} u_\varepsilon\phi^\varepsilon\chi_{\{|u_\varepsilon|\le k\}} = \int_{\mathcal{O}_\varepsilon} f\phi^\varepsilon\chi_{\{|u_\varepsilon|\le k\}}. \tag{23}$$

The last two terms terms in (23) will converge to 0 as $\varepsilon\to 0$ from the definition of ϕ^ε. Now, we look into the first two terms. To handle the first term in (23), let $\psi = 1$ and $g = \hat{g}_\delta^k$

(as defined in Sect. 2.4) in (16). Here g is not compactly supported, but still we can use it as a test function in (16) due to Lemma 1. Thus

$$\int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_{k+1}(u_\varepsilon) \nabla T_{k+1}(u_\varepsilon) (\hat{g}^k_\delta)'(u_\varepsilon) + \int_{\mathcal{O}_\varepsilon} u_\varepsilon \hat{g}^k_\delta(u_\varepsilon) = \int_{\mathcal{O}_\varepsilon} f \hat{g}^k_\delta(u_\varepsilon).$$

Since $(\hat{g}^k_\delta)' = \frac{1}{\delta}\chi_{\{k\le|u_\varepsilon|\le k+\delta\}}$ and $u_\varepsilon \hat{g}^k_\delta(u_\varepsilon) \ge 0$, we have

$$\frac{1}{\delta}\int_{\mathcal{O}_\varepsilon} A^\varepsilon \nabla T_{k+1}(u_\varepsilon) \nabla T_{k+1}(u_\varepsilon) \chi_{\{k\le|u_\varepsilon|\le k+\delta\}} \le \|f\|_{L^1(\mathcal{O})}.$$

Therefore, we have

$$\begin{aligned}
&\limsup_{\delta\to0}\left|\int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_{k+1}(u_\varepsilon) \nabla T_{k+1}(u_\varepsilon) \phi^\varepsilon (g^k_\delta)'(u_\varepsilon)\right| \\
&\le \varepsilon \limsup_{\delta\to0}\frac{1}{\delta}\int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_{k+1}(u_\varepsilon) \nabla T_{k+1}(u_\varepsilon) \chi_{\{k\le|u_\varepsilon|\le k+\delta\}} \\
&\le \ \varepsilon\|f\|_{L^1(\mathcal{O})},
\end{aligned}$$

which implies

$$\lim_{\varepsilon\to0}\left(\limsup_{\delta\to0}\int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_{k+1}(u_\varepsilon) \nabla T_{k+1}(u_\varepsilon) \phi^\varepsilon (g^k_\delta)'(u_\varepsilon)\right) = 0.$$

Hence from (23), we get

$$\lim_{\varepsilon\to0}\int_{\mathcal{O}_\varepsilon} (\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}) A^\varepsilon \nabla T_k(u_\varepsilon) \nabla \phi^\varepsilon = 0.$$

Since ϕ^ε vanishes outside $\mathcal{C}_\varepsilon$, we have

$$\lim_{\varepsilon\to0}\int_{\mathcal{C}_\varepsilon} A^\varepsilon \nabla T_k(u_\varepsilon)\nabla\phi^\varepsilon = \int_{\mathcal{C}_\varepsilon} \begin{bmatrix}\alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon\end{bmatrix} \begin{bmatrix}\frac{\partial}{\partial r}T_k(u_\varepsilon) \\ \frac{\partial}{\partial\theta}T_k(u_\varepsilon)\end{bmatrix} \begin{bmatrix}\frac{\partial}{\partial r}\phi^\varepsilon \\ \frac{\partial}{\partial\theta}\phi^\varepsilon\end{bmatrix} = 0.$$

Apply unfolding operator and pass to the limit as $\varepsilon \to 0$ using (12) to get

$$\begin{aligned}
0 = \int_{\mathcal{C}_\varepsilon} \begin{bmatrix}\alpha^\varepsilon & \gamma^\varepsilon \\ \beta^\varepsilon & \eta^\varepsilon\end{bmatrix} \begin{bmatrix}\frac{\partial}{\partial r}T_k(u_\varepsilon) \\ \frac{\partial}{\partial\theta}T_k(u_\varepsilon)\end{bmatrix} \begin{bmatrix}\frac{\partial}{\partial r}\phi^\varepsilon \\ \frac{\partial}{\partial\theta}\phi^\varepsilon\end{bmatrix} &= \frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{C}} \begin{bmatrix}\alpha & \gamma \\ \beta & \eta\end{bmatrix}\begin{bmatrix}w_r \\ w_\theta\end{bmatrix}\begin{bmatrix}0 \\ \frac{\partial\phi}{\partial\tau}\end{bmatrix} \\
&= \frac{1}{2\pi}\int_{\mathcal{O}_\mathcal{C}} \left(\beta w_r + \eta w_\theta\right)\frac{\partial\phi}{\partial\tau}.
\end{aligned}$$

Since ϕ is arbitrary,

$$w_\theta \ = \ -\frac{\beta}{\eta}w_r \ = \ -\frac{\beta}{\eta}\frac{\partial}{\partial r}T_k(u).$$

Therefore we have

$$T_{\mathcal{C}}^{\varepsilon}\left(\begin{bmatrix} T_k(u_\varepsilon) \\ \frac{\partial}{\partial r}T_k(u_\varepsilon) \\ \frac{\partial}{\partial \theta}T_k(u_\varepsilon) \end{bmatrix}\right) \rightharpoonup \begin{bmatrix} T_k(u) \\ \frac{\partial}{\partial r}T_k(u) \\ -\frac{\beta}{\eta}\frac{\partial}{\partial r}T_k(u) \end{bmatrix} \text{ weakly in } (L^2(\mathcal{O}_{\mathcal{C}}))^3,$$

$$T_{\mathcal{I}}^{\varepsilon}\left(\begin{bmatrix} T_k(u_\varepsilon) \\ \varepsilon\frac{\partial}{\partial r}T_k(u_\varepsilon) \\ \varepsilon\frac{\partial}{\partial \theta}T_k(u_\varepsilon) \end{bmatrix}\right) \rightharpoonup \begin{bmatrix} T_k(u+(f-u)\xi) \\ 0 \\ \frac{\partial}{\partial \tau}T_k(u+(f-u)\xi) \end{bmatrix} \text{ weakly in } (L^2(\mathcal{O}_{\mathcal{I}}))^3.$$

Then from property (6) of unfolding operator we have (20).

Since $\{T_k(u_\varepsilon)\}$ is bounded in $H^1(\mathcal{O}^-)$, by weak compactness there exist some $v \in H^1(\mathcal{O}^-)$ such that

$$T_k(u_\varepsilon) \to v \text{ weakly in } H^1(\mathcal{O}^-).$$

But from Theorem 3, using the Lipschitz property of truncation function T_k we get

$$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } L^2(\mathcal{O}^-),$$

which gives $v = T_k(u)$. Thus we have (21) and hence Theorem 4.

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