Nonlinear Differ. Equ. Appl. (2025) 32:44

1021-9722/25/030001-30
published online March 28, 2025
https://doi.org/10.1007/s00030-025-01045-7

Nonlinear Differential Equations and Applications NoDEA

Optimal control and homogenization of semi-linear parabolic problem with highly oscillatory coefficients in an oscillating domain

A. K. Nandakumaran, Ritu Raj and Bidhan Chandra Sardar

Abstract. In this article, we explore the homogenization of an optimal control problem driven by a semi-linear parabolic equation within a two-dimensional oscillating domain, denoted as Ω_ϵ. The state equation and cost function in this scenario involve periodic coefficients, A^ϵ and B^ϵ, which exhibit significant oscillations. The objective of this study is to analyze the limiting behavior of both the optimal control and the corresponding state as the oscillations become increasingly fine. Furthermore, we aim to identify the optimal control problem that encapsulates the effects of these oscillating coefficients and to establish a corrector result for the state variable.

Mathematics Subject Classification. 35B27, 35B40, 49J20.

Keywords. Homogenization, Optimal control, Semi-linear parabolic equation, Unfolding operator.

1. Introduction

In this article, we consider the following optimal control problem (OCP):

$$\inf_{\theta_\epsilon \in L^2((0,T)\times\omega)} \left\{ J_\epsilon(\theta_\epsilon) = \frac{1}{2}\int_{\mathcal{Q}_\epsilon} B^\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon + \frac{\alpha}{2}\int_{\mathcal{Q}_\epsilon} \chi_{(0,T)\times\omega} |\theta_\epsilon|^2 \right\}, \quad (1.1)$$

subject to the constraint given by a semi-linear parabolic equation

$$\begin{cases} u_\epsilon' - div(A^\epsilon \nabla u_\epsilon) + F(u_\epsilon) + u_\epsilon = g + \chi_{(0,T)\times\omega}\theta_\epsilon & \text{in } \mathcal{Q}_\epsilon, \\ A^\epsilon \nabla u_\epsilon \cdot \eta_\epsilon = 0 & \text{on } (0,T)\times\partial\Omega_\epsilon, \\ u_\epsilon(0,x) = 0 & \text{in } \Omega_\epsilon, \end{cases} \quad (1.2)$$

where $\mathcal{Q}_\epsilon = (0,T) \times \Omega_\epsilon$ represents the oscillating domain over time and space, where Ω_ϵ is a bounded domain consisting of an oscillating upper section, Ω_ϵ^+, and a fixed lower section, Ω^-, as illustrated in Fig. 1. The function θ_ϵ represents the control function acting within an open subset $(0,T) \times \omega$, where $\omega \subset\subset \Omega^-$. The parameter $\alpha > 0$ is a regularization factor, and η_ϵ is the outward unit normal vector to the boundary $\partial\Omega_\epsilon$. The function $g \in L^2(\mathcal{Q})$ is a given data term. The function $F : \mathbb{R} \to \mathbb{R}$ is a smooth real-valued function satisfying $0 < c_1 \leq F'(s) \leq c_2, F(0) = 0$, and F'' is bounded.

The coefficient martix $A^\epsilon(t,x) = (a_{ij}^\epsilon(t,x))_{1 \leq i,j \leq 2}$ is given by $a_{ij}^\epsilon(t,x) = a_{ij}(t,x,\frac{x_1}{\epsilon})$. The coefficients satisfy the following conditions:

(A1) Each a_{ij} is in $L^\infty((0,T) \times \Omega \times \mathbb{R})$ and is 1-periodic in the last variable τ.
(A2) There exist constants $0 < M_1 < M_2$ such that:

$$M_1|\xi|^2 \leq a_{ij}(t,x,\tau)\xi_i\xi_j \leq M_2|\xi|^2, \text{ a.e. } (t,x,\tau) \in (0,T) \times \Omega \times (0,1), \ \forall\, \xi \in \mathbb{R}^2.$$

Similarly, the coefficient matrix B^ϵ in the cost functional is also structured similarly. The coefficients of B^ϵ also satisfy the conditions (A1) and (A2), and the matrix is symmetric.

Our objective is to analyze the limiting behavior of optimal solutions and to identify the homogenized equation, exploring the influence of A^ϵ and B^ϵ within an oscillating domain. Problems involving oscillating boundaries are common in various industrial applications. These include fluid flows in channels with rough walls, electromagnetic wave propagation in regions with uneven interfaces, heat transfer in winglets, and elasticity in materials with rough surfaces. Relevant studies and applications can be found in references such as [7,8,12,17,29,31].

In the context of the literature on OCP's in domains with oscillating boundaries, the work presented in [19] explores the homogenization of an OCP associated with a parabolic equation in a domain characterized by a set of cylinders placed ϵ-periodically over a fixed base. The authors examine an interior optimal control problem with a classical cost functional and derive a limiting optimal control problem featuring a different cost functional. In [18], the authors investigate a parabolic optimal control problem with Dirichlet boundary control, defined on a highly oscillating cylindrical domain. They utilize a classical L^2 cost functional and perform asymptotic analysis aided by an extension operator.

In [23], the authors analyze the homogenization problem related to a quasilinear optimal control problem defined on a thick junction domain of type 3 : 2 : 1. They consider a quasilinear boundary condition that depends on parameters ϵ, α, and δ. Using the abstract scheme developed by Buttazzo and Dal Maso, the authors obtain homogenization results for various values of α and δ. In [24], the authors conduct homogenization of an optimal control problem governed by the wave equation with oscillating coefficients in a pillar-like two-dimensional oscillating domain. They consider a Dirichlet cost functional involving oscillating coefficients and perform convergence analysis using the unfolding operator method. The study of oscillating boundaries has been extensively explored in the literature, including works such as [2,3,5,6,11,20,26,27,35,37,38], and [40].

The literature on nonlinear optimal control problems in oscillating domains is rather sparse. In this context, the work by the authors in [4] explores an optimal control problem governed by a semi-linear elliptic equation in a highly oscillating domain, with convergence analysis conducted using the unfolding operator method. They prove the optimality condition for the homogenized equation via a linear adjoint equation and establish corrector results. In [21], the authors investigate the homogenization of an optimal control problem associated with a nonlinear elliptic equation for a mixed Dirichlet-Neumann boundary value problem in an oscillating domain. They consider two distinct controls: one applied to the interior of the domain and the other to the boundary. The authors of [39] study the homogenization of a semi-linear elliptic equation with a matrix coefficient in a circular oscillating domain using the unfolding operator. They also examine an optimal control problem involving a Dirichlet cost functional that includes oscillating matrix coefficients. The Browder-Minty method is utilized to identify the limit for the nonlinear term. For further reading on homogenization problems related to nonlinear equations in highly oscillating domains, [15] discusses the homogenization of the p-Laplacian using the Gamma-convergence method, while [36] examines the homogenization problem associated with a semi-linear heat equation using the method of asymptotic expansion. For more literature on asymptotic analysis of non-linear problems, one can also look into [1,9,28] and [33].

This paper introduces a novel examination of a semi-linear parabolic equation with oscillating coefficients and a more complex Dirichlet-type cost functional, also characterized by oscillating coefficients that may differ from those in the heat equation's coefficient matrix. The complexity is heightened by the presence of two distinct oscillation matrices for the state equation and the cost functional within the oscillating domain. The primary objectives of this article are to define the control using an adjoint equation and to investigate the limiting behavior of the corresponding adjoint state and the optimal solution, which includes both the state and the optimal control. The coefficients in the limiting cost functional and adjoint state naturally incorporate contributions from both oscillation matrices, A^ϵ and B^ϵ, because the adjoint equation integrates both. To achieve homogenization results, we employ the unfolding operator method. This method was first introduced in [13] and later refined in [14]. Although several approaches, such as two-scale convergence, are available in the literature, the additional complexity introduced by the oscillating coefficients makes other methods less suitable for this particular problem. Therefore, we believe that the unfolding approach is well-suited to address the challenges presented in this paper. Although the current study concentrates on a 2D domain, the methods and conclusions given can be easily extended to higher-dimensional spaces, $\mathbb{R}^n$, for any $n > 2$. The main principles and procedures covered here can be used in higher-dimensional contexts with appropriate modification.

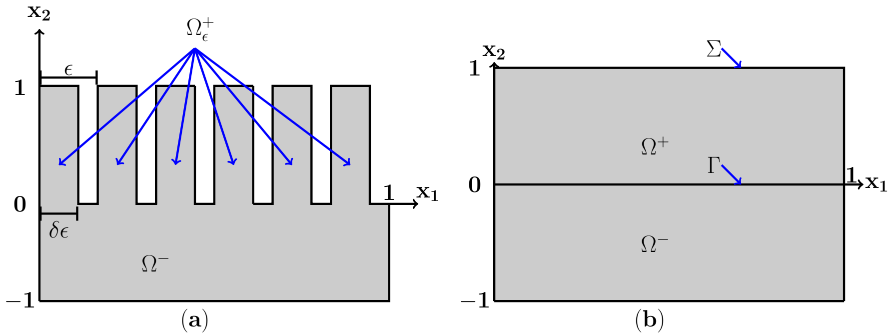

Figure 1. **a** Domain Ω_ϵ for $N = 6$, **b** Domain Ω

This article is organized as follows: Sect. 2 introduces the domain and provides a detailed characterization of the optimal control problem, including uniform estimates. Section 3 offers a concise description of the unfolding operator with some properties. Section 4 introduces the homogenized optimal control problem and provides a characterization of this homogenized problem. In Sect. 5, we present proofs for the convergence results. Finally, Sect. 6 is dedicated to proving a corrector result.

2. Domain illustrations and preliminary results

In this article, we consider the oscillating domain Ω_ϵ, which is similar to the one discussed in [16]. Oscillating domains of this type, often referred to as pillar-type domains, come in various forms (see [2,6,37] for examples). To provide a comprehensive understanding, we will now give a brief overview of the domain.

Let $0 < \delta < 1$ be a real number which is fixed. For $\epsilon = \frac{1}{N}$, where $N \in \mathbb{N}$, we define the oscillating part of the domain as Ω_ϵ^+, given by

$$\Omega_\epsilon^+ = \bigcup_{k=0}^{N-1} (k\epsilon, k\epsilon + \delta\epsilon) \times (0, 1).$$

The lower fixed part of the domain is denoted as Ω^- and is defined as $\Omega^- = (0,1)\times(-1,0)$. The complete domain Ω_ϵ is defined as $\Omega_\epsilon = \mathrm{int}(\overline{\Omega_\epsilon^+ \cup \Omega^-})$. An example of the domain Ω_ϵ for $N = 6$ is shown in Fig. 1a.

The limiting domain Ω is defined as $\Omega = (0,1) \times (-1,1)$. The top part of the domain Ω, represented by Ω^+, is defined as $\Omega^+ = (0,1) \times (0,1)$. The upper boundary of Ω, represented by $\sum$, is defined as $\sum = (0,1) \times \{1\}$. The common boundary separating Ω^+ and Ω^-, represented by Γ, is defined as $\Gamma = (0,1) \times \{0\}$. Figure 1b illustrates the domain Ω.
in the domain Ω, a point x is denoted as $x = (x_1, x_2)$. Additionally, we denote every point in the interval $(0,\delta)$ as z. In this study, we use the notation C to represent any constant that is independent of ϵ. We denote $\mathcal{Q}_\epsilon^+ = (0,T) \times \Omega_\epsilon^+$, $\mathcal{Q} = (0,T) \times \Omega$, $\mathcal{Q}^+ = (0,T) \times \Omega^+$ and $\mathcal{Q}^- = (0,T) \times \Omega^-$.

Our objective is to investigate the asymptotic behavior, as $\epsilon \to 0$, of the optimal control and the corresponding state for the problem defined by (1.1). We now introduce the variational formulation of the state equation (1.2).

Definition 2.1. A function $u_\epsilon \in W_\epsilon := \{u \in L^2(0,T;H^1(\Omega_\epsilon)) : u' \in L^2(0,T;(H^1(\Omega_\epsilon))^*)\}$ is a weak solution to (1.2) if

$$\int_{\mathcal{Q}_\epsilon} u_\epsilon'\psi + \int_{\mathcal{Q}_\epsilon} A^\epsilon \nabla u_\epsilon \cdot \nabla\psi + \int_{\mathcal{Q}_\epsilon} F(u_\epsilon)\psi + \int_{\mathcal{Q}_\epsilon} u_\epsilon\psi = \int_{\mathcal{Q}_\epsilon} (g + \chi_{(0,T)\times\omega}\theta_\epsilon)\psi, \tag{2.1}$$

for all $\psi \in L^2(0,T;H^1(\Omega_\epsilon))$, with $u_\epsilon(0,x) = 0$ in Ω_ϵ.

Remark 2.2. It's important to note that the solution u_ϵ of the problem (1.2) exhibits a higher regularity, specifically $u_\epsilon \in C([0,T];L^2(\Omega_\epsilon))$(see [25], Chapter IV). Therefore, the initial condition $u_\epsilon(0,x) = 0$ is well-defined and meaningful.

We recall that for any fixed ϵ, there exists a well-established result regarding the existence and uniqueness of the solution to the problem (1.2) (see [41] or [34]).

Theorem 2.3. *For every fixed ϵ, the problem (1.2) admits a unique weak solution $u_\epsilon \in W_\epsilon$. Moreover, there exists a constant C such that*

$$\|u_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))} + \|u_\epsilon'\|_{L^2(0,T;(H^1(\Omega_\epsilon))^*)} \le C\left(\|g\|_{L^2(\mathcal{Q})} + \|\theta_\epsilon\|_{L^2((0,T)\times\omega)}\right). \tag{2.2}$$

We will now provide a characterization of the optimal control through an adjoint system. Additionally, we will establish uniform estimates for both the state and control variables, which are essential for the convergence analysis to be discussed in the subsequent sections.

Let's recall the following well-known result regarding semi-linear optimal control problems (refer to [10,42] for more details).

Theorem 2.4. *For each fixed $\epsilon > 0$, the OCP (1.1) admits at least one solution. If $(\bar{u}_\epsilon, \bar{\theta}_\epsilon)$ is the optimal solution of (1.1), then the associated optimality system is given by*

$$\begin{cases} \bar{u}_\epsilon' - div(A^\epsilon\nabla\bar{u}_\epsilon) + F(\bar{u}_\epsilon) + \bar{u}_\epsilon = g + \chi_{(0,T)\times\omega}\bar{\theta}_\epsilon & in\ \mathcal{Q}_\epsilon, \\ \bar{v}_\epsilon' - div({}^tA^\epsilon\nabla\bar{v}_\epsilon) + F'(\bar{u}_\epsilon)\bar{v}_\epsilon + \bar{v}_\epsilon = -div(B^\epsilon\nabla\bar{u}_\epsilon) & in\ \mathcal{Q}_\epsilon, \\ A^\epsilon\nabla\bar{u}_\epsilon \cdot \eta_\epsilon = 0 & on\ (0,T)\times\partial\Omega_\epsilon, \\ ({}^tA^\epsilon\nabla\bar{v}_\epsilon - B^\epsilon\nabla\bar{u}_\epsilon)\cdot\eta_\epsilon = 0 & on\ (0,T)\times\partial\Omega_\epsilon, \\ \bar{u}_\epsilon(0,x) = 0,\ \bar{v}_\epsilon(T,x) = 0 & in\ \Omega_\epsilon, \\ \bar{\theta}_\epsilon = -\alpha^{-1}\chi_{(0,T)\times\omega}\bar{v}_\epsilon & in\ \mathcal{Q}^-. \end{cases} \tag{2.3}$$

The function $\bar{v}_\epsilon$ is referred to as the adjoint state corresponding to the state variable $\bar{u}_\epsilon$. It is important to note that $\bar{v}_\epsilon$ satisfies a linear equation. The following theorem provides a uniform estimate for the optimal control and state, which is essential for the convergence analysis.

Theorem 2.5. *Suppose* $(\bar{u}_\epsilon, \bar{\theta}_\epsilon)$ *is the optimal solution of* (1.1), *and let* $\bar{v}_\epsilon$ *be the adjoint state associated with* $\bar{u}_\epsilon$. *Then, there exists a constant* C *(independent of* ϵ*) such that:*

$$\|\bar{\theta}_\epsilon\|_{L^2((0,T)\times\omega)} \leq C,$$
$$\|\bar{u}_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))} \leq C,$$
$$\|\bar{v}_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))} \leq C.$$

Proof. Let $u_\epsilon(0)$ be the weak solution of (1.2) with $\theta_\epsilon = 0$. Applying the estimate (2.2), we get

$$\|u_\epsilon(0)\|_{L^2(0,T;H^1(\Omega_\epsilon))} + \|u'_\epsilon(0)\|_{L^2(0,T;(H^1(\Omega_\epsilon))^*)} \leq C\|g\|_{L^2(\mathcal{Q})}.$$

Thus, we conclude that the sequence $\|u_\epsilon(0)\|_{L^2(0,T;H^1(\Omega_\epsilon))}$ is uniformly bounded with respect to ϵ. Since $\bar{\theta}_\epsilon$ is the optimal solution of (1.1), we have, $J_\epsilon(\bar{\theta}_\epsilon) \leq J_\epsilon(0)$, i.e.

$$\frac{1}{2}\int_{\mathcal{Q}_\epsilon} B^\epsilon \nabla \bar{u}_\epsilon \cdot \nabla \bar{u}_\epsilon + \frac{\alpha}{2}\int_{\mathcal{Q}_\epsilon} \chi_{(0,T)\times\omega}|\bar{\theta}_\epsilon|^2 \leq \frac{1}{2}\int_{\mathcal{Q}_\epsilon} B^\epsilon \nabla u_\epsilon(0) \cdot \nabla u_\epsilon(0). \quad (2.4)$$

Utilizing the uniform bound derived for the sequence $u_\epsilon(0)$ and the assumptions on the matrix B^ϵ, inequality (2.4) establishes a uniform bound for the sequence $\|\bar{\theta}_\epsilon\|_{L^2((0,T)\times\omega)}$. Again, using the estimate (2.2), we conclude that $\|\bar{u}_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))}$ is uniformly bounded with respect to ϵ. The uniform bound for $\|\bar{v}_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon))}$ follows from the adjoint equation (2.3). □

This theorem ensures that the norms of the optimal control $\bar{\theta}_\epsilon$, the state $\bar{u}_\epsilon$, and the adjoint state $\bar{v}_\epsilon$ remain bounded uniformly with respect to ϵ. This uniform boundedness is crucial for analyzing the behavior of the solutions as $\epsilon \to 0$.

3. Unfolding operator and its properties

In homogenization theory, the periodic unfolding method is an important tool developed two decades ago for adressing problems from homogenization and we have used the tool extremely in the study of problems in oscillating domains. This section provides a brief overview of the definition and properties of unfolding operators in both the oscillating domain $\mathcal{Q}^+_\epsilon$ (denoted by T^+_ϵ) and the fixed domain $\mathcal{Q}^-$ (denoted by T^-_ϵ). For more comprehensive details on the unfolding operator, refer to sources such as [13, 14, 16]. For any point $x = (x_1, x_2)$, $[x] = ([x_1], [x_2])$, where $[x_i]$ represents the integer part of x_i.

The definition and properties of the unfolding operator T^+_ϵ on the oscillating domain $\mathcal{Q}^+_\epsilon$ are reviewed below.

Definition 3.1. [16] The ϵ- unfolding of a function $v : \mathcal{Q}^+_\epsilon \to \mathbb{R}$ is the function

$$T^+_\epsilon : \{v : \mathcal{Q}^+_\epsilon \to \mathbb{R}\} \to \{v : \mathcal{Q}^+ \times (0,\delta) \to \mathbb{R}\}$$

defined by

$$T^+_\epsilon(v)(t, x_1, x_2, z) = v\left(t, \epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon z, x_2\right).$$

The following proposition recalls the properties of the unfolding operator T_ϵ^+.

Proposition 3.2. *The unfolding operator T_ϵ^+ has the following properties:*

P1. T_ϵ^+ *is linear and let $u, v \in L^2(\mathcal{Q}_\epsilon^+)$. Then*

$$T_\epsilon^+(uv) = T_\epsilon^+(u) T_\epsilon^+(v).$$

P2. Let $v \in L^1(\mathcal{Q}_\epsilon^+)$, then

$$\int_{\mathcal{Q}^+ \times (0,\delta)} T_\epsilon^+(v) = \int_{\mathcal{Q}_\epsilon^+} v.$$

P3. Let us denote $\mathcal{D} = L^2\big((0,1) \times (0,T); H^1((0,1) \times (0,\delta))\big)$. Let $v \in L^2(0,T; H^1(\Omega_\epsilon^+))$ then $T_\epsilon^+(v) \in \mathcal{D}$. Moreover

$$\frac{\partial}{\partial x_2} T_\epsilon^+(v) = T_\epsilon^+\left(\frac{\partial v}{\partial x_2}\right),$$
$$\frac{\partial}{\partial z} T_\epsilon^+(v) = \epsilon T_\epsilon^+\left(\frac{\partial v}{\partial x_1}\right).$$

P4. Let $v \in L^2(\mathcal{Q}_\epsilon^+)$, then $T_\epsilon^+(v) \in L^2(\mathcal{Q}^+ \times (0,\delta))$ and

$$\int_{\mathcal{Q}^+ \times (0,\delta)} |T_\epsilon^+(v)|^2 dt = \int_{\mathcal{Q}_\epsilon^+} |v|^2.$$

P5. Let $v \in L^2(\mathcal{Q}^+)$, then

$$T_\epsilon^+(v) \to v \text{ in } L^2(\mathcal{Q}^+ \times (0,\delta)).$$

P6. Let $u_\epsilon \to u$ in $L^2(\mathcal{Q}^+)$. Then

$$T_\epsilon^+(u_\epsilon) \to u \text{ in } L^2(\mathcal{Q}^+ \times (0,\delta)).$$

P7. Let $u_\epsilon \in L^2(0,T; H^1(\Omega_\epsilon^+))$ be such that $T_\epsilon^+(u_\epsilon) \rightharpoonup u$ weakly in $\mathcal{D}$. Then

$$\widetilde{u_\epsilon} \rightharpoonup \int_0^\delta u dz \text{ in } L^2(\mathcal{Q}^+).$$

By employing thes above properties, we can effectively analyze the behavior of oscillating domains and establish uniform estimates crucial for convergence analysis in homogenization problems. To define the unfolding operator on the fixed part of the domain $\mathcal{Q}^-$, we introduce the following notations:

$$E_\epsilon = \{x \in \Omega^- : \epsilon(x + (0,1)^2) \subset \Omega^-\},$$
$$\hat{\Omega}_\epsilon^- = int\{\cup_{x \in E_\epsilon} \epsilon(x + (0,1)^2)\} \text{ and } \Lambda_\epsilon = \Omega^- \backslash \hat{\Omega}_\epsilon^-.$$

Definition 3.3. The ϵ- unfolding of a function $v : \mathcal{Q}^- \to \mathbb{R}$ is defined as a function

$$T_\epsilon^- : \{v : \mathcal{Q}^- \to \mathbb{R}\} \to \{v : (0,T) \times \hat{\Omega}_\epsilon^- \cup \Lambda_\epsilon \times (0,1)^2 \to \mathbb{R}\}$$

given by

$$T_\epsilon^-(v)(t,x,y) = \begin{cases} v\left(t, \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right) & (t,x,y) \in (0,T) \times \hat{\Omega}_\epsilon^- \times (0,1)^2, \\ 0 & (t,x,y) \in (0,T) \times \Lambda_\epsilon \times (0,1)^2. \end{cases}$$

The following theorem outlines an important result related to the unfolding operator T_ϵ^-:

Theorem 3.4. [14] *Let $\{v_\epsilon\}$ be a sequence in $L^2(0,T;H^1(\Omega^-))$ such that v_ϵ converges to v weakly in $L^2(0,T;H^1(\Omega^-))$. Then, there exist a subsequence of v_ϵ (still denoted by ϵ) and a $v_1 \in L^2((0,T)\times\Omega;H^1_{per}((0,1)^2))$, such that*

$$T_\epsilon^-(\nabla v_\epsilon) \rightharpoonup \nabla v + \nabla_y v_1 \text{ weakly in } L^2(\mathcal{Q}^- \times (0,1)^2).$$

This theorem establishes that the weak convergence of a sequence in the fixed domain can be expressed in terms of the unfolding operator T_ϵ^-, which decomposes the gradient into components corresponding to the macroscopic and microscopic scales.

4. Homogenized optimal control problem

In this section, we introduce the limit OCP characterized by a Dirichlet type cost functional in the domain Ω as specified in Eq. (4.1). Additionally, we use an adjoint equation to characterize the optimal control. The subsequent section will address the convergence analysis. For any function u defined on $\mathcal{Q}$, we denote

$$u^+(t,x) := u(t,x) \text{ if } (t,x) \in \mathcal{Q}^+,$$
$$u^-(t,x) := u(t,x) \text{ if } (t,x) \in \mathcal{Q}^-.$$

We define the limit OCP as:

$$\inf_{\theta\in L^2((0,T)\times\omega)} \left\{ J(\theta) = \frac{1}{2}\int_{\mathcal{Q}^+} b^* \left(\frac{\partial u^+}{\partial x_2}\right)^2 + \frac{1}{2}\int_{\mathcal{Q}^-} B^*\nabla u^- \cdot \nabla u^- + \frac{\alpha}{2}\int_{\mathcal{Q}} \chi_{(0,T)\times\omega}|\theta|^2 \right\}, \tag{4.1}$$

subject to the following constraint

$$\begin{cases} \delta(u^+)' - \dfrac{\partial}{\partial x_2}\left(a^*\dfrac{\partial u^+}{\partial x_2}\right) + \delta F(u^+) + \delta u^+ = \delta g \text{ in } \mathcal{Q}^+, \\ (u^-)' - div(A^*\nabla u^-) + F(u^-) + u^- = g + \chi_{(0,T)\times\omega}\theta \text{ in } \mathcal{Q}^-, \\ a^*\dfrac{\partial u^+}{\partial x_2} = 0 \text{ on } (0,T)\times\Sigma, \\ u^+ = u^-,\ a^*\dfrac{\partial u^+}{\partial x_2} = a^*_{21}\dfrac{\partial u^-}{\partial x_1} + a^*_{22}\dfrac{\partial u^-}{\partial x_2} \text{ on } (0,T)\times\Gamma, \\ A^*\nabla u^- \cdot \eta = 0 \text{ on } (0,T)\times\left(\partial\Omega^-\backslash\Gamma\right), \\ u(0,x) = 0 \text{ in } \Omega. \end{cases} \tag{4.2}$$

The coefficients b^* and a^* on $\mathcal{Q}^+$ are defined as follows

$$\begin{aligned} b^*(t,x) &= \int_0^\delta \left(b_{22} + b_{11}\left(\frac{a_{12}}{a_{11}}\right)^2 - 2\frac{a_{12}b_{12}}{a_{11}}\right)(t,x,z)dz, \\ a^*(t,x) &= \int_0^\delta \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)(t,x,z)dz. \end{aligned} \tag{4.3}$$

In the domain $\mathcal{Q}^-$, the coefficient matrices $B^* = (b^*_{ij})_{1\le i,j\le N}$ and $A^* = (a^*_{ij})_{1\le i,j\le N}$ are defined by

$$\begin{aligned} B^*(t,x) &= \int_{(0,1)^2} \big(B(t,x,y_1)\,(I+\nabla_y Z) + {}^{\mathsf{t}}A(t,x,y_1)\nabla_y X\big)\,dy, \\ A^*(t,x) &= \int_{(0,1)^2} A(t,x,y_1)\,(I+\nabla_y Z)\,dy, \end{aligned} \tag{4.4}$$

where, $Z=[Z_1,Z_2]$ and $X=[X_1,X_2]$ are such that each Z_i and X_i $(i=1,2)$ are defined on $(0,1)^2$ and satisfy the following cell problems:

$$\begin{cases} -\,div_y(A(t,x,y_1)\nabla Z_i) = div_y(A(t,x,y_1,)e_i) \text{ in } (0,1),^2 \\ -\,div_y({}^tA(t,x,y_1)\nabla X_i) = div_y(B(t,x,y_1,)(I+\nabla_y Z)e_i) \text{ in } (0,1)^2, \\ \displaystyle\int_{(0,1)^2} Z_i(y)dy = 0, \int_{(0,1)^2} X_i(y)dy = 0, \\ Z_i, X_i \text{ are 1 -periodic in y.} \end{cases} \tag{4.5}$$

Here, e_i denote canonical basis of $\mathbb{R}^2$. Let us define the function space $V(\Omega)$ as follows

$$V(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_2} \in L^2(\Omega^+), u \in H^1(\Omega^-), u^+ = u^- \text{ on } \Gamma \right\}. \tag{4.6}$$

The space $V(\Omega)$ is a Hilbert space with the norm given by

$$\|u\|^2_{V(\Omega)} = \|u\|^2_{L^2(\Omega^+)} + \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L^2(\Omega^+)} + \|u\|^2_{H^1(\Omega^-)}.$$

We consider the space

$$W = \left\{ u \in L^2(0,T;V(\Omega)) : u' \in L^2(0,T;(V(\Omega))^*) \right\}. \tag{4.7}$$

Remark 4.1. The triple $\big(V(\Omega), L^2(\Omega), (V(\Omega))^*\big)$ is an evolution triple as established in [22]; Lemma 3.7.

The coefficients a^* and b^* are strictly positive, which can be established using the fact that A and B belong to the set $\mathcal{M}$. The ellipticity of the matrix A^* is a well-known result, and Theorem 4.3 in [30] provides a proof for the ellipticity of the matrix B^*. Consequently, the Eq. (4.2) and and the cell problem (4.5) are both well-posed in the solution spaces $H^1((0,1)^2)$ and W, respectively.

By applying semilinear optimal control theory (see [4,10,42]), we state the following theorem regarding the optimality system for the problem (4.1).

Theorem 4.2. *For each fixed* $\epsilon > 0$, *the optimal control problem* (4.1) *admits at least one solution. Let* $(\bar{u}, \bar{\theta})$ *be the optimal solution of* (4.1). *Then the* *optimality system is given by*

$$\begin{cases} \delta(\bar{u}^+)' - \dfrac{\partial}{\partial x_2}\left(a^* \dfrac{\partial \bar{u}^+}{\partial x_2}\right) + \delta F(\bar{u}^+) + \delta \bar{u}^+ = \delta g & in\ \mathcal{Q}^+, \\ \delta(\bar{v}^+)' - \dfrac{\partial}{\partial x_2}\left(a^* \dfrac{\partial \bar{v}^+}{\partial x_2}\right) + \delta F'(\bar{u}^+)\bar{v}^+ + \delta \bar{v}^+ = \dfrac{\partial}{\partial x_2}\left(b^* \dfrac{\partial \bar{u}^+}{\partial x_2}\right) & in\ \mathcal{Q}^+, \\ (\bar{u}^-)' - div(A^* \nabla \bar{u}^-) + F(\bar{u}^-) + \bar{u}^- = g + \chi_{(0,T)\times\omega}\bar{\theta} & in\ \mathcal{Q}^-, \\ (\bar{v}^-)' - div({}^t A^* \nabla \bar{v}^-) + F'(\bar{u}^-)\bar{v}^- + \bar{v}^- = -div(B^* \nabla \bar{u}^-) & in\ \mathcal{Q}^-, \\ \bar{u}(0,x) = 0, \bar{v}(T,x) = 0 & in\ \Omega, \\ \bar{\theta} = -\alpha^{-1}\chi_{(0,T)\times\omega}\bar{v} & in\ \mathcal{Q}^-. \end{cases}$$

Additionally, the interface conditions are given by

$$\begin{cases} \bar{u}^+ = \bar{u}^-, \bar{v}^+ = \bar{v}^- & on\ (0,T)\times\Gamma, \\ a^* \dfrac{\partial \bar{u}^+}{\partial x_2} = a^*_{21}\dfrac{\partial \bar{u}^-}{\partial x_1} + a^*_{22}\dfrac{\partial \bar{u}^-}{\partial x_2} & on\ (0,T)\times\Gamma, \\ a^* \dfrac{\partial \bar{v}^+}{\partial x_2} + b^* \dfrac{\partial \bar{u}^+}{\partial x_2} = a^*_{12}\dfrac{\partial \bar{v}^-}{\partial x_1} + a^*_{22}\dfrac{\partial \bar{v}^-}{\partial x_2} - b^*_{21}\dfrac{\partial \bar{u}^-}{\partial x_1} - b^*_{22}\dfrac{\partial \bar{u}^-}{\partial x_2} & on\ (0,T)\times\Gamma. \end{cases}$$

The boundary conditions are specified as follows

$$\begin{cases} a^* \dfrac{\partial \bar{u}^+}{\partial x_2} = 0 & on\ (0,T)\times\Sigma, \\ a^* \dfrac{\partial \bar{v}^+}{\partial x_2} + b^* \dfrac{\partial \bar{u}^+}{\partial x_2} = 0 & on\ (0,T)\times\Sigma, \\ A^* \nabla \bar{u}^- \cdot \eta = 0, & on\ (0,T)\times\left(\partial\Omega^- \backslash \Gamma\right), \\ \left({}^t A^* \nabla \bar{v}^- - B^* \nabla \bar{u}^-\right)\cdot \eta = 0 & on\ (0,T)\times\left(\partial\Omega^- \backslash \Gamma\right). \end{cases}$$

The coefficients a^* and b^* are defined in equation (4.3), while A^* and B^* are defined in Eq. (4.4).

In the context of our optimal control problem, we denote the adjoint state associated with the state variable $\bar{u}$ as $\bar{v}$.

5. Convergence results

Here, we present the convergence results for the optimal control, the optimal state, and the corresponding adjoint state for problem (1.1).

Theorem 5.1. *Suppose $(\bar{u}_\epsilon, \bar{\theta}_\epsilon) \in W_\epsilon \times L^2((0,T)\times\omega)$ is the optimal solution of (1.1), and let $\bar{v}_\epsilon \in W_\epsilon$ be the adjoint state corresponding to this problem. Then, we have the following convergence results:*

$$\widetilde{\bar{u}^+_\epsilon} \rightharpoonup \delta u^+, \quad \widetilde{\frac{\partial \bar{u}^+_\epsilon}{\partial x_2}} \rightharpoonup \delta \frac{\partial u^+}{\partial x_2} \ \text{weakly in } L^2(\mathcal{Q}^+), \tag{5.1a}$$

$$\widetilde{\frac{\partial \bar{u}^+_\epsilon}{\partial x_1}} \rightharpoonup \left(-\int_0^\delta \frac{a_{12}}{a_{11}} dz\right) \frac{\partial u^+}{\partial x_2} \ \text{weakly in } L^2(\mathcal{Q}^+), \tag{5.1b}$$

$$\widetilde{\bar{v}_\epsilon^+} \rightharpoonup \delta v^+, \widetilde{\frac{\partial \bar{v}_\epsilon^+}{\partial x_2}} \rightharpoonup \delta \frac{\partial v^+}{\partial x_2} \text{ weakly in } L^2(\mathcal{Q}^+), \tag{5.1c}$$

$$\widetilde{\frac{\partial \bar{v}_\epsilon^+}{\partial x_1}} \rightharpoonup \left(\int_0^\delta \left[\frac{b_{12}}{a_{11}} - \frac{b_{11}a_{12}}{a_{11}^2}\right] dz\right) \frac{\partial u^+}{\partial x_2} - \left(\int_0^\delta \frac{a_{21}}{a_{11}} dz\right) \frac{\partial v^+}{\partial x_2} \text{ weakly in } L^2(\mathcal{Q}^+), \tag{5.1d}$$

$$\bar{u}_\epsilon^- \rightharpoonup u^-,\ \bar{v}_\epsilon^- \rightharpoonup v^- \text{ weakly in } L^2(0,T;H^1(\Omega^-)), \tag{5.1e}$$

$$\bar{\theta}_\epsilon \to \theta_0 = \alpha^{-1}\chi_{(0,T)\times\omega} v^- \text{ strongly } L^2(\mathcal{Q}^-). \tag{5.1f}$$

Here, $(u, \theta_0) \in W \times L^2(\mathcal{Q}^-)$ is the optimal solution of (4.1), and $v \in W$ is the adjoint state associated with the OCP (4.1).

Proof. The proof involves eight steps to demonstrate the convergence of the optimal control, the optimal state, and the corresponding adjoint state for the given problem. In the first step, we prove the convergence results for the optimal state $\bar{u}_\epsilon^+$ defined on Ω_ϵ^+. In the second step, we establish the convergence results for the optimal state $\bar{u}_\epsilon^-$ defined on Ω^-. The third critical step involves proving the convergence of the non-linear term $F(\bar{u}_\epsilon^+)$ under the unfolding operator. This is achieved using the Browder-Minty method by constructing a suitable inequality. In step four, we demonstrate strong convergence results for the optimal state $\bar{u}_\epsilon^+$, which is essential for identifying the limit problem for the adjoint state $\bar{v}_\epsilon^+$ and for proving the corrector results in the next section. Step five focuses on obtaining convergence results for the adjoint state $\bar{v}_\epsilon^+$. In step six, we derive the limit equation satisfied by the limits of $\bar{u}_\epsilon^+$ and $\bar{v}_\epsilon^+$ in the domain Ω^+. Step seven deals with obtaining convergence results for the adjoint state $\bar{v}_\epsilon^-$ in the lower domain Ω^-. Finally, in step eight, we prove that the limits obtained for $\bar{u}_\epsilon^+$ and $\bar{v}_\epsilon^+$ solve the optimality system as stated in Theorem 4.2.

Step 1: Convergence of $\bar{u}_\epsilon^+$

In this step, we will prove the convergence of $\bar{u}_\epsilon^+$ as outlined in the theorem. Using Proposition 3.2 (P1 and P2), we have the following estimate

$$\begin{aligned}
\|T_\epsilon^+(\bar{u}_\epsilon^+)\|_{\mathcal{D}}^2 &= \int_0^1 \int_0^T \|T_\epsilon^+(\bar{u}_\epsilon^+)\|_{H^1((0,1)\times(0,\delta))}^2 dx_1 dt \\
&= \int_{\mathcal{Q}^+\times(0,\delta)} \left[\epsilon^2 T_\epsilon^+\left(\left|\frac{\partial \bar{u}_\epsilon^+}{\partial x_1}\right|^2\right) + T_\epsilon^+\left(\left|\frac{\partial \bar{u}_\epsilon^+}{\partial x_2}\right|^2\right) + T_\epsilon^+(|\bar{u}_\epsilon^+|^2)\right] \\
&= \int_{\mathcal{Q}_\epsilon^+} \left[\epsilon^2\left|\frac{\partial \bar{u}_\epsilon^+}{\partial x_1}\right|^2 + \left|\frac{\partial \bar{u}_\epsilon^+}{\partial x_2}\right|^2 + |\bar{u}_\epsilon^+|^2\right] \le \|\bar{u}_\epsilon\|_{L^2(0,T;H^1(\Omega_\epsilon^+))}^2.
\end{aligned} \tag{5.2}$$

From Theorem 2.5 and inequality (5.2), we infer that $T_\epsilon^+(\bar{u}_\epsilon^+)$ is uniformly bounded in $\mathcal{D}$ independent of ϵ. Therefore, by the weak compactness argument, there exists a subsequence (still denoted by ϵ) and a function $u^+ \in \mathcal{D}$ such that

$$T_\epsilon^+(\bar{u}_\epsilon^+) \rightharpoonup u^+ \text{ weakly in } \mathcal{D}. \tag{5.3}$$

From (5.3), we obtain the following convergences:

$$T_\epsilon^+\left(\frac{\partial \bar{u}_\epsilon^+}{\partial x_2}\right) \rightharpoonup \frac{\partial u^+}{\partial x_2} \text{ weakly in } L^2(\mathcal{Q}^+ \times (0,\delta)), \tag{5.4a}$$

$$\frac{\partial}{\partial z} T_\epsilon^+(\bar{u}_\epsilon^+) \rightharpoonup \frac{\partial u^+}{\partial z} \text{ weakly in } L^2(\mathcal{Q}^+ \times (0,\delta)). \tag{5.4b}$$

Using Proposition 3.2 (P4), we have

$$\left\| T_\epsilon^+\left(\frac{\partial \bar{u}_\epsilon^+}{\partial x_1}\right) \right\|^2_{L^2(\mathcal{Q}^+\times(0,\delta))} = \left\| \frac{\partial \bar{u}_\epsilon^+}{\partial x_1} \right\|^2_{L^2(\mathcal{Q}_\epsilon^+)} \leq \|\bar{u}_\epsilon\|^2_{L^2(0,T;H^1(\Omega_\epsilon^+))}. \tag{5.5}$$

Using the inequality (5.5) and Theorem 2.5, we conclude that the sequence $T_\epsilon^+\left(\frac{\partial \bar{u}_\epsilon^+}{\partial x_1}\right)$ is uniformly bounded in $L^2\left(\mathcal{Q}^+ \times (0,\delta)\right)$. Hence, by the weak compactness argument, there exist a subsequence (still denoted by ϵ) and a fucntion $\xi_1 \in L^2(\mathcal{Q}^+ \times (0,\delta))$ such that

$$T_\epsilon^+\left(\frac{\partial \bar{u}_\epsilon^+}{\partial x_1}\right) \rightharpoonup \xi_1 \text{ weakly in } L^2\left(\mathcal{Q}^+ \times (0,\delta)\right). \tag{5.6}$$

Further, from (5.4b) and Proposition 3.2 (P3), we obtain

$$\epsilon T_\epsilon^+\left(\frac{\partial \bar{u}_\epsilon^+}{\partial x_1}\right) = \frac{\partial}{\partial z} T_\epsilon^+\left(\bar{u}_\epsilon^+\right) \rightharpoonup \frac{\partial u^+}{\partial z}. \tag{5.7}$$

From this convergence, we deduce that $\frac{\partial u^+}{\partial z} = 0$, implying that u^+ is independent of the z-variable. Consequently, u^+ is a function of (t, x_1, x_2) only. Using Eqs. (5.3), (5.4a), along with Proposition 3.2 (P7), we derive the following weak convergences

$$\widetilde{\bar{u}_\epsilon^+} \rightharpoonup \delta u^+ \text{ weakly in } L^2(\mathcal{Q}^+), \tag{5.8}$$

$$\widetilde{\frac{\partial \bar{u}_\epsilon^+}{\partial x_2}} \rightharpoonup \delta \frac{\partial u^+}{\partial x_2} \text{ weakly in } L^2(\mathcal{Q}^+). \tag{5.9}$$

Additionally, based on the assumptions on the function F and Theorem 2.5, the sequence $F(\bar{u}_\epsilon^+)$ is bounded in $L^2(\mathcal{Q}_\epsilon^+)$. Therefore, there exists a function $\xi_2 \in L^2(\mathcal{Q}^+ \times (0,\delta))$ such that

$$T_\epsilon^+(F(\bar{u}_\epsilon^+)) \rightharpoonup \xi_2 \text{ weakly in } L^2(\mathcal{Q}^+ \times (0,\delta)). \tag{5.10}$$

In subsequent steps, we will identify the limit ξ_2.

Identification of ξ_1: To identify the limit ξ_1 obtained in (5.6), let us define the test function ψ_ϵ as follows

$$\psi_\epsilon = \epsilon \psi_1(t, x_1, x_2) \psi_2\left(\left\{\frac{x_1}{\epsilon}\right\}\right), \text{ where } \psi_1 \in C_c^\infty(\mathcal{Q}^+), \psi_2 \in C_{per}^\infty(0,1). \tag{5.11}$$

Note the following transformations for ψ_ϵ

$$T_\epsilon^+(\psi_\epsilon) = \epsilon \psi_1\left(t, \epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon z, x_2\right)\psi_2(z),$$

$$T_\epsilon^+\left(\frac{\partial \psi_\epsilon}{\partial x_1}\right) = \epsilon \frac{\partial \psi_1}{\partial x_1}\left(t, \epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon z, x_2\right)\psi_2(z) + \psi_1\left(t, \epsilon\left[\frac{x_1}{\epsilon}\right] + \epsilon z, x_2\right)\frac{\partial \psi_2}{\partial z},$$

$$T_\epsilon^+\left(\frac{\partial\psi_\epsilon}{\partial x_2}\right)=\epsilon\frac{\partial\psi_1}{\partial x_2}\left(t,\epsilon\left[\frac{x_1}{\epsilon}\right]+\epsilon z,x_2\right)\psi_2(z),$$
$$T_\epsilon^+\left(\psi_\epsilon'\right)=\epsilon\psi_1'\left(t,\epsilon\left[\frac{x_1}{\epsilon}\right]+\epsilon z,x_2\right)\psi_2(z).$$

As $\epsilon\to 0$, we observe the following convergences

$$\begin{aligned}
T_\epsilon^+(\psi_\epsilon)&\to 0 \text{ strongly in } L^2\left(\mathcal{Q}^+\times(0,\delta)\right),\\
T_\epsilon^+\left(\frac{\partial\psi_\epsilon}{\partial x_1}\right)&\to \psi_1(t,x_1,x_2)\frac{\partial\psi_2}{\partial z} \text{ strongly in } L^2\left(\mathcal{Q}^+\times(0,\delta)\right),\\
T_\epsilon^+\left(\frac{\partial\psi_\epsilon}{\partial x_2}\right)&\to 0 \text{ strongly in } L^2\left(\mathcal{Q}^+\times(0,\delta)\right),\\
T_\epsilon^+(\psi_\epsilon')&\to 0 \text{ strongly in } L^2\left(\mathcal{Q}^+\times(0,\delta)\right).
\end{aligned}\tag{5.12}$$

On taking ψ_ϵ as a test function in the weak formulation (2.1) corresponding to the weak solution $\bar{u}_\epsilon$ for $\bar{\theta}_\epsilon$ and by using the properties of the unfolding operator, we obtain the following equation

$$\begin{aligned}
&-\int_{\mathcal{Q}^+\times(0,\delta)}T_\epsilon^+(\bar{u}_\epsilon^+)T_\epsilon^+(\psi_\epsilon')+\int_{\mathcal{Q}^+\times(0,\delta)}T_\epsilon^+(A^\epsilon)T_\epsilon^+(\nabla\bar{u}_\epsilon^+)\cdot T_\epsilon^+(\nabla\psi_\epsilon)dt\\
&\quad+\int_{\mathcal{Q}^+\times(0,\delta)}T_\epsilon^+(F(\bar{u}_\epsilon^+))T_\epsilon^+(\psi_\epsilon)+\int_{\mathcal{Q}^+\times(0,\delta)}T_\epsilon^+(\bar{u}_\epsilon^+)T_\epsilon^+(\psi_\epsilon)\,dt\\
&=\int_{\mathcal{Q}^+\times(0,\delta)}T_\epsilon^+(g)T_\epsilon^+(\psi_\epsilon).
\end{aligned}\tag{5.13}$$

By employing the convergences (5.3), (5.4), (5.6) and (5.12), and taking the limit as $\epsilon\to 0$ in Eq. (5.13), we obtain

$$\int_{\mathcal{Q}^+\times(0,\delta)}\left(a_{11}\xi_1+a_{12}\frac{\partial u^+}{\partial x_2}\right)\psi_1\frac{\partial\psi_2}{\partial z}=0,$$

for all $\psi_1\in C_c^\infty(\mathcal{Q}^+)$ and $\psi_2\in C_{per}^\infty(0,1)$. From this, we conclude that

$$\xi_1=-\frac{a_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2}\text{ a.e. }(t,x,z)\in\mathcal{Q}^+\times(0,\delta).\tag{5.14}$$

Further, using Proposition 3.2, we obtain

$$\widetilde{\frac{\partial\bar{u}_\epsilon^+}{\partial x_1}}\rightharpoonup\left(-\int_0^\delta\frac{a_{12}}{a_{11}}dz\right)\frac{\partial u^+}{\partial x_2}\text{ weakly in } L^2(\mathcal{Q}^+).$$

Step 2: Convergences of $\bar{u}_\epsilon^-$

From Theorem 2.5, we know that $\|\bar{u}_\epsilon^-\|_{L^2(0,T;H^1(\Omega^-))}$ is uniformly bounded. Thus, there exists a subsequence (still denoted by ϵ) and $u^-\in L^2(0,T;H^1(\Omega^-))$ such that

$$\bar{u}_\epsilon^-\rightharpoonup u^-\text{ weakly in } L^2(0,T;H^1(\Omega^-)).\tag{5.15}$$

We also achieve strong convergence by the Aubin-Lions lemma:

$$\bar{u}_\epsilon^-\to u^-\text{ strongly in } L^2(\mathcal{Q}^-).\tag{5.16}$$

From Eq. (5.16), it follows that

$$T_\epsilon^-(\bar{u}_\epsilon^-) \to u^- \text{ in } L^2(\mathcal{Q}^- \times (0,1)^2). \tag{5.17}$$

With the strong convergence in (5.16), we obtain

$$F(\bar{u}_\epsilon^-) \to F(u^-) \text{ in } L^2(\mathcal{Q}^-). \tag{5.18}$$

Moreover, according to Theorem 3.4, there exist $u_1, \in L^2\left(\mathcal{Q}; H^1_{per}(0,1)^2\right)$ such that

$$T_\epsilon^-(\nabla \bar{u}_\epsilon^-) \rightharpoonup \nabla u^- + \nabla_y u_1 \text{ in } L^2(\mathcal{Q}^- \times (0,1)^2). \tag{5.19}$$

Additionally, from the definition of the unfolding operator and the assumption on the matrix A^ϵ, we have

$$T_\epsilon^-(A^\epsilon) \to A \text{ in } L^2(\mathcal{Q}^- \times (0,1)^2)^{2\times 2}. \tag{5.20}$$

Let us consider the test function

$$\psi^\epsilon = \psi_0(t,x) + \epsilon\psi_1\left(t, x, \frac{x}{\epsilon}\right),$$

where $\psi_0 \in C_c^\infty(\mathcal{Q}^-), \psi_1 \in C_c^\infty\left(\mathcal{Q}^-; C^\infty_{per}(0,1)^2\right)$. By employing ψ^ϵ as a test function in the weak formulation (2.1), and using the properties of the unfolding operator T_ϵ^-, we derive

$$\begin{aligned} -\int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(\bar{u}_\epsilon^-) T_\epsilon^-((\psi^\epsilon)') + \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(A^\epsilon) T_\epsilon^-(\nabla \bar{u}_\epsilon^-)\cdot T_\epsilon^-(\nabla\psi^\epsilon) \\ + \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(F(\bar{u}_\epsilon^-)) T_\epsilon^-(\psi^\epsilon) + \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(\bar{u}_\epsilon^-) T_\epsilon^-(\psi^\epsilon) \\ = \int_{\mathcal{Q}^-} \left(g + \chi_{(0,T)\times\omega}\bar{\theta}_\epsilon\right)\psi^\epsilon. \end{aligned} \tag{5.21}$$

Utilizing the limits obtained in (5.17)–(5.20) within Eq. (5.21), we obtain

$$\begin{aligned} \int_{\mathcal{Q}^-\times(0,1)^2} (u^-)'\psi_0 + \int_{\mathcal{Q}^-\times(0,1)^2} A(\nabla u^- + \nabla_y u_1)\cdot(\nabla\psi_0 + \nabla_y\psi_1) \\ + \int_{\mathcal{Q}^-\times(0,1)^2} F(u^-)\psi_0 + \int_{\mathcal{Q}^-\times(0,1)^2} u^-\psi_0 = \int_{\mathcal{Q}^-} \left(g + \theta_0\chi_{(0,T)\times\omega}\right)\psi_0. \end{aligned} \tag{5.22}$$

where θ_0 is the weak L^2- limit of $\bar{\theta}_\epsilon$. Setting $\psi_0 = 0$ in the equation (5.22), we obtain

$$\int_{\mathcal{Q}^-\times(0,1)^2} A\nabla_y u_1 \cdot \nabla_y \psi_1 = -\int_{\mathcal{Q}^-\times(0,1)^2} A\nabla u^- \cdot \nabla_y\psi_1. \tag{5.23}$$

Utilizing the solution of the cell problem (4.5) and Eq. (5.23), we can express u_1 as

$$u_1(t,x,y) = \sum_{i=1}^{2} \frac{\partial u^-}{\partial x_i} Z_i. \tag{5.24}$$

Setting $\psi_1 = 0$ and substituting u_1 from Eq. (5.24) into Eq. (5.22), we get

$$\int_{\mathcal{Q}^-\times(0,1)^2} (u^-)'\psi_0 + \int_{\mathcal{Q}^-\times(0,1)^2} A\left(\nabla u^- + \sum_{i=1}^{2}\frac{\partial u^-}{\partial x_i}\nabla_y Z_i\right)\cdot\nabla\psi_0 + \int_{\mathcal{Q}^-\times(0,1)^2} F(u^-)\psi_0 + \int_{\mathcal{Q}^-\times(0,1)^2} u^-\psi_0 = \int_{\mathcal{Q}^-}\left(g + \theta_0\chi_{(0,T)\times\omega}\right)\psi_0. \tag{5.25}$$

By leveraging the cell problem (4.5) and noting that u^- is independent of y, Eq. (5.25) can be rewritten as

$$\int_{\mathcal{Q}^-}(u^-)'\psi_0 + \int_{\mathcal{Q}^-}\left[\int_{(0,1)^2} A(x,y_1,t)\,(I+\nabla_y Z)\,dy\right]\nabla u^-\cdot\nabla\psi_0 + \int_{\mathcal{Q}^-}F(u^-)\psi_0 + \int_{\mathcal{Q}^-}u^-\psi_0 = \int_{\mathcal{Q}^-}\left(g+\theta_0\chi_{(0,T)\times\omega}\right)\psi_0. \tag{5.26}$$

Utilizing the definition of matrix A^* given in (4.4), Eq. (5.26) can be simplified to

$$\int_{\mathcal{Q}^-}(u^-)'\psi_0 + \int_{\mathcal{Q}^-}A^*\nabla u^-\cdot\nabla\psi_0 + \int_{\mathcal{Q}^-}F(u^-)\psi_0 + \int_{\mathcal{Q}^-}u^-\psi_0 = \int_{\mathcal{Q}^-}\left(g+\theta_0\chi_{(0,T)\times\omega}\right)\psi_0. \tag{5.27}$$

Define u as $u = \chi_{\Omega^+}u^+ + \chi_{\Omega^-}u^-$. By analogy with, for instance, [16], it can be shown that the traces of u^+ and u^- are identical on the interface Γ and hence u in $V(\Omega)$ and $u' \in L^2(0,T,(V(\Omega))^*)$ (see [32], page 107). It is straightforward to verify that u also satisfies the initial condition $u(0,x) = 0$.

Step 3: Identification of ξ_2 from (5.10) Given the weak convergence of $\bar{u}_\epsilon^+$ in the upper oscillating domain, determining the limit of $T_\epsilon^+(F(\bar{u}_\epsilon^+))$ presents a significant challenge. To address this, we employ the Browder-Minty method to identify ξ_2.

Due to the monotonicity of F, consider $\psi_0 \in L^2(0,T;C^1(\bar{\Omega}))$. We obtain the following inequality,

$$\frac{1}{2}\int_{\Omega_\epsilon}(\bar{u}_\epsilon(T,x)-\psi_0(T,x))^2\,dx + \int_{\mathcal{Q}_\epsilon^+}A^\epsilon\begin{pmatrix}\frac{\partial\bar{u}_\epsilon}{\partial x_1}+\frac{a_{12}}{a_{11}}\frac{\partial u}{\partial x_2}\\ \frac{\partial\bar{u}_\epsilon}{\partial x_2}-\frac{\partial\psi_0}{\partial x_2}\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial\bar{u}_\epsilon}{\partial x_1}+\frac{a_{12}}{a_{11}}\frac{\partial u}{\partial x_2}\\ \frac{\partial\bar{u}_\epsilon}{\partial x_2}-\frac{\partial\psi_0}{\partial x_2}\end{pmatrix} + \int_{\mathcal{Q}^-}A^\epsilon(\nabla\bar{u}_\epsilon-\nabla\psi_0)\cdot(\nabla\bar{u}_\epsilon-\nabla\psi_0) + \int_{\mathcal{Q}_\epsilon}(\bar{u}_\epsilon-\psi_0)^2 + \int_{\mathcal{Q}_\epsilon}(F(\bar{u}_\epsilon)-F(\psi_0))(\bar{u}_\epsilon-\psi_0) \geq 0. \tag{5.28}$$

Expanding the inequality above, we get

$$\frac{1}{2}\int_{\Omega_\epsilon}(\bar{u}_\epsilon(T,x))^2\,dx + \int_{\mathcal{Q}_\epsilon}A^\epsilon\nabla\bar{u}_\epsilon\cdot\nabla\bar{u}_\epsilon + \int_{\mathcal{Q}_\epsilon}(\bar{u}_\epsilon)^2 + \int_{\mathcal{Q}_\epsilon}F(\bar{u}_\epsilon)\bar{u}_\epsilon - \int_{\Omega_\epsilon}\bar{u}_\epsilon(T,x)\psi_0(x,T)dx - 2\int_{\mathcal{Q}_\epsilon}\bar{u}_\epsilon\psi_0 - \int_{\mathcal{Q}_\epsilon}F(\bar{u}_\epsilon)\psi_0 - \int_{\mathcal{Q}_\epsilon}F(\psi_0)\bar{u}_\epsilon$$

$$
\begin{aligned}
&-\int_{\mathcal{Q}^-} A^\epsilon \nabla \bar{u}_\epsilon \cdot \nabla \psi_0 - \int_{\mathcal{Q}^-} A^\epsilon \nabla \psi_0 \cdot \nabla \bar{u}_\epsilon \\
&+\int_{\mathcal{Q}_\epsilon^+} \left(a_{12} \frac{\partial \bar{u}_\epsilon}{\partial x_1} \frac{\partial u}{\partial x_2} + \frac{a_{12} a_{12}}{a_{11}} \frac{\partial \bar{u}_\epsilon}{\partial x_2} \frac{\partial u}{\partial x_2} - a_{21} \frac{\partial \bar{u}_\epsilon}{\partial x_1} \frac{\partial \psi_0}{\partial x_2} - a_{22} \frac{\partial \bar{u}_\epsilon}{\partial x_2} \frac{\partial \psi_0}{\partial x_2} \right) \\
&+\int_{\mathcal{Q}_\epsilon^+} \left(a_{12} \frac{\partial u}{\partial x_2} \frac{\partial \bar{u}_\epsilon}{\partial x_1} - a_{12} \frac{\partial \psi_0}{\partial x_2} \frac{\partial \bar{u}_\epsilon}{\partial x_1} - a_{22} \frac{\partial \psi_0}{\partial x_2} \frac{\partial \bar{u}_\epsilon}{\partial x_2} + \frac{a_{21} a_{12}}{a_{11}} \frac{\partial u}{\partial x_2} \frac{\partial \bar{u}_\epsilon}{\partial x_2} \right) \\
&+\frac{1}{2} \int_{\Omega_\epsilon} (\psi_0(T,x))^2 \, dx + \int_{\mathcal{Q}^-} A^\epsilon \nabla \psi_0 \cdot \nabla \psi_0 + \int_{\mathcal{Q}_\epsilon} (\psi_0)^2 + \int_{\mathcal{Q}_\epsilon} F(\psi_0) \psi_0 \\
&+\int_{\mathcal{Q}_\epsilon^+} \left[\frac{a_{12} a_{12}}{a_{11}} \left(\frac{\partial u^+}{\partial x_2} \right)^2 - \frac{a_{12} a_{12}}{a_{11}} \frac{\partial \psi_0}{\partial x_2} \frac{\partial u^+}{\partial x_2} - \frac{a_{21} a_{12}}{a_{11}} \frac{\partial \psi_0}{\partial x_2} \frac{\partial u^+}{\partial x_2} + a_{22} \left(\frac{\partial \psi_0}{\partial x_2} \right)^2 \right] \geq 0.
\end{aligned} \tag{5.29}
$$

Substituting $\bar{u}_\epsilon$ for ψ in the weak formulation(2.1) and integrating by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2} \int_{\Omega_\epsilon} (\bar{u}_\epsilon(T,x))^2 \, dx + \int_{\mathcal{Q}_\epsilon} A^\epsilon \nabla \bar{u}_\epsilon \cdot \nabla \bar{u}_\epsilon + \int_{\mathcal{Q}_\epsilon} (\bar{u}_\epsilon)^2 + \int_{\mathcal{Q}_\epsilon} F(\bar{u}_\epsilon) \bar{u}_\epsilon \\
&\quad = \int_{\mathcal{Q}_\epsilon} g \bar{u}_\epsilon + \int_{\mathcal{Q}^-} \chi_{(0,T)\times\omega} \bar{\theta}_\epsilon \bar{u}_\epsilon.
\end{aligned}
$$

Applying the unfolding operator, we can rewrite the above equation as

$$
\begin{aligned}
&\frac{1}{2} \int_{\mathcal{Q}^+ \times (0,\delta)} \left(T_\epsilon^+ (\bar{u}_\epsilon(T,x)) \right)^2 dt + \frac{1}{2} \int_{\Omega^-} \left[\bar{u}_\epsilon(T,x) \right]^2 dx \\
&\quad + \int_{\mathcal{Q}^+ \times (0,\delta)} T_\epsilon^+ (A^\epsilon) T_\epsilon^+ (\nabla \bar{u}_\epsilon) \cdot T_\epsilon^+ (\nabla \bar{u}_\epsilon) + \int_{\mathcal{Q}^-} A^\epsilon \nabla \bar{u}_\epsilon \cdot \nabla \bar{u}_\epsilon \\
&\quad + \int_{\mathcal{Q}^+ \times (0,\delta)} \left(T_\epsilon^+ (\bar{u}_\epsilon) \right)^2 + \int_{\mathcal{Q}^-} (\bar{u}_\epsilon)^2 + \int_{\mathcal{Q}^-} F(\bar{u}_\epsilon) \bar{u}_\epsilon \\
&\quad + \int_{\mathcal{Q}^+ \times (0,\delta)} T_\epsilon^+ (F(\bar{u}_\epsilon)) T_\epsilon^+ (\bar{u}_\epsilon) dt = \int_{\mathcal{Q}^+ \times (0,\delta)} T_\epsilon^+ (g) T_\epsilon^+ (\bar{u}_\epsilon) \\
&\quad + \int_{\mathcal{Q}^-} (g + \chi_{(0,T)\times\omega} \bar{\theta}_\epsilon) \bar{u}_\epsilon.
\end{aligned} \tag{5.30}
$$

Using the convergence obtained in (5.3) and fact that u^+ is independent of z, we pass the limit $\epsilon \to 0$ in Eq. (5.30) to obtain

$$
\begin{aligned}
&\lim_{\epsilon \to 0} \frac{1}{2} \int_{\mathcal{Q}^+ \times (0,\delta)} \left[T_\epsilon^+ (\bar{u}_\epsilon(T,x)) \right]^2 dt + \frac{1}{2} \int_{\Omega^-} \left[\bar{u}_\epsilon(T,x) \right]^2 dx \\
&\quad + \int_{\mathcal{Q}^+ \times (0,\delta)} T_\epsilon^+ (A^\epsilon) T_\epsilon^+ (\nabla \bar{u}_\epsilon) \cdot T_\epsilon^+ (\nabla \bar{u}_\epsilon) + \int_{\mathcal{Q}^-} A^\epsilon \nabla \bar{u}_\epsilon \cdot \nabla \bar{u}_\epsilon \\
&\quad + \int_{\mathcal{Q}^+ \times (0,\delta)} \left(T_\epsilon^+ (\bar{u}_\epsilon) \right)^2 + \int_{\mathcal{Q}^-} (\bar{u}_\epsilon)^2 + \int_{\mathcal{Q}^-} F(\bar{u}_\epsilon) \bar{u}_\epsilon \\
&\quad + \int_{\mathcal{Q}^+ \times (0,\delta)} T_\epsilon^+ (F(\bar{u}_\epsilon)) T_\epsilon^+ (\bar{u}_\epsilon) dt = \delta \int_{\mathcal{Q}^+} g u^+ + \int_{\mathcal{Q}^-} (g + \chi_{(0,T)\times\omega} \theta_0) u^-.
\end{aligned} \tag{5.31}
$$

Now, by applying the unfolding operator to the weak formulation (2.1), we obtain

$$
\begin{aligned}
&-\int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{u}_\epsilon^+)T_\epsilon^+(\psi') + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(A^\epsilon)T_\epsilon^+(\nabla\bar{u}_\epsilon^+)\cdot T_\epsilon^+(\nabla\psi)\\
&\quad+\int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(F(\bar{u}_\epsilon))T_\epsilon^+(\psi) + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{u}_\epsilon^+)T_\epsilon^+(\psi)\\
&\quad-\int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(\bar{u}_\epsilon^-)T_\epsilon^-((\psi)') + \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(A^\epsilon)T_\epsilon^-(\nabla\bar{u}_\epsilon^-)\cdot T_\epsilon^-(\nabla\psi)\\
&\quad+\int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(F(\bar{u}_\epsilon^-))T_\epsilon^-(\psi) + \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(\bar{u}_\epsilon^-)T_\epsilon^-(\psi)\\
&\quad=\int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(g)T_\epsilon^+(\psi) + \int_{\mathcal{Q}^-}\left(g+\chi_{(0,T)\times\omega}\bar{\theta}_\epsilon\right)\psi.
\end{aligned}
\tag{5.32}
$$

Utilizing the convergence results from the previous steps, we can take the limit as $\epsilon \to 0$ in the above equation to obtain

$$
\begin{aligned}
&\delta\int_{\mathcal{Q}^+}(u^+)'\psi + \int_{\mathcal{Q}^+}\left[\int_0^\delta\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)dz\right]\frac{\partial u^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + \int_{\mathcal{Q}^+\times(0,\delta)}\xi_2\psi\\
&\quad+\delta\int_{\mathcal{Q}^+}u^+\psi + \int_{\mathcal{Q}^-}(u^-)'\psi + \int_{\mathcal{Q}^-}A^*\nabla u^-\cdot\nabla\psi + \int_{\mathcal{Q}^-}F(u^-)\psi\\
&\quad+\int_{\mathcal{Q}^-}u^-\psi = \delta\int_{\mathcal{Q}^+}g\psi + \int_{\mathcal{Q}^-}\left(g+\theta_0\chi_{(0,T)\times\omega}\right)\psi.
\end{aligned}
\tag{5.33}
$$

By substituting ψ with u in (5.33) and integrating by parts the first term, we obtain

$$
\begin{aligned}
&\frac{\delta}{2}\int_{\Omega^+}\left(u^+(T,x)\right)^2dx + \int_{\mathcal{Q}^+}\left[\int_0^\delta\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)dz\right]\left(\frac{\partial u^+}{\partial x_2}\right)^2\\
&\quad+\int_{\mathcal{Q}^+\times(0,\delta)}\xi_2u^+ + \delta\int_{\mathcal{Q}^+}\left(u^+\right)^2 + \int_{\mathcal{Q}^-}\left(u^-(T,x)\right)^2 + \int_{\mathcal{Q}^-}A^*\nabla u^-\cdot\nabla u^-\\
&\quad+\int_{\mathcal{Q}^-}F(u^-)u^- + \int_{\mathcal{Q}^-}\left(u^-\right)^2 = \delta\int_{\mathcal{Q}^+}gu^+ + \int_{\mathcal{Q}^-}\left(g+\theta_0\chi_{(0,T)\times\omega}\right)u^-.
\end{aligned}
\tag{5.34}
$$

From (5.31) and (5.34), we get the following convergence

$$
\begin{aligned}
&\lim_{\epsilon\to0}\frac{1}{2}\int_{\mathcal{Q}^+\times(0,\delta)}\left(T_\epsilon^+(\bar{u}_\epsilon(T,x))\right)^2dt + \frac{1}{2}\int_{\Omega^-}(\bar{u}_\epsilon(T,x))^2\\
&\quad+\int_{\mathcal{Q}^+\times(0,\delta)}T_\epsilon^+(A^\epsilon)T_\epsilon^+(\nabla\bar{u}_\epsilon)\cdot T_\epsilon^+(\nabla\bar{u}_\epsilon) + \int_{\mathcal{Q}^-}A^\epsilon\nabla\bar{u}_\epsilon\cdot\nabla\bar{u}_\epsilon\\
&\quad+\int_{\mathcal{Q}^+\times(0,\delta)}\left(T_\epsilon^+(\bar{u}_\epsilon)\right)^2 + \int_{\mathcal{Q}^-}(\bar{u}_\epsilon)^2 + \int_{\mathcal{Q}^-}F(\bar{u}_\epsilon)\bar{u}_\epsilon + \int_{\mathcal{Q}^+\times(0,\delta)}T_\epsilon^+(F(\bar{u}_\epsilon))T_\epsilon^+(\bar{u}_\epsilon)\\
&=\frac{\delta}{2}\int_{\Omega^+}\left(u^+(T,x)\right)^2dx + \int_{\mathcal{Q}^+}\left[\int_0^\delta\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)dz\right]\left(\frac{\partial u^+}{\partial x_2}\right)^2 + \int_{\mathcal{Q}^+\times(0,\delta)}\xi_2u^+
\end{aligned}
$$

$$+\delta \int_{\mathcal{Q}^+} (u^+)^2 + \int_{\mathcal{Q}^-} (u^-(T,x))^2 + \int_{\mathcal{Q}^-} A^*\nabla u^- \cdot \nabla u^- + \int_{\mathcal{Q}^-} F(u^-)u^- + \int_{\mathcal{Q}^-} (u^-)^2 . \tag{5.35}$$

Next, we apply the convergence results from (5.35) to take the limit in the first four terms of inequality (5.29). For the remaining terms, we use the convergence results established in the previous two steps. After rearranging the terms, we obtain

$$\begin{aligned}&\frac{\delta}{2}\int_{\Omega^+} \left(u^+(T,x) - \psi_0(T,x)\right)^2 dx + \frac{1}{2}\int_{\Omega^-} \left(u^-(T,x) - \psi_0(T,x)\right)^2 dx \\ &+ \delta \int_{\mathcal{Q}^+} \left(u^+ - \psi_0\right)^2 + \int_{\mathcal{Q}^-} \left(u^- - \psi_0\right)^2 \\ &+ \int_0^\delta \int_{\mathcal{Q}^+} \left(\xi_2 - F(\psi_0)\right)\left(u^+ - \psi_0\right) + \int_{\mathcal{Q}^-} \left(F(u^-) - F(\psi_0)\right)\left(u^- - \psi_0\right) \\ &+ \int_0^\delta \int_{\mathcal{Q}^+} a_{22}\left(\frac{\partial u^+}{\partial x_2} - \frac{\partial \psi_0}{\partial x_2}\right)^2 + \int_{\mathcal{Q}^-} A^*\left(\nabla u^- - \nabla\psi_0\right)\cdot\left(\nabla u^- - \nabla\psi_0\right) \geq 0.\end{aligned} \tag{5.36}$$

Now, for $\phi \in L^2(0,T;C^1(\bar{\Omega}))$, take $\psi_0 = u - \lambda\phi, \lambda > 0$, in the inequality (5.36), to obtain

$$\begin{aligned}&\frac{\delta}{2}\lambda^2 \int_{\Omega^+} (\phi(T,x))^2 dx + \frac{1}{2}\lambda^2 \int_{\Omega^-} (\phi(T,x))^2 dx + \delta\lambda^2 \int_{\mathcal{Q}^+} \phi^2 + \lambda^2 \int_{\mathcal{Q}^-} \phi^2 \\ &+ \lambda \int_0^\delta \int_{\mathcal{Q}^+} \left(\xi_2 - F(u^+ - \lambda\phi)\right)\phi + \lambda \int_{\mathcal{Q}^-} \left(F(u^-) - F(u^- - \lambda\phi)\right)\phi \\ &+ \lambda^2 \int_0^\delta \int_{\mathcal{Q}^+} a_{22}\left(\frac{\partial\phi}{\partial x_2}\right)^2 + \lambda^2 \int_{\mathcal{Q}^-} A^*\nabla\phi\cdot\nabla\phi \geq 0,\ \forall\phi \in L^2(0,T;C^1(\bar{\Omega})).\end{aligned} \tag{5.37}$$

By taking limit $\lambda \to 0$ in (5.37), to obtain

$$\int_{\mathcal{Q}^+} \left(\xi_2 - F(u^+)\right)\phi \geq 0,\ \forall\phi \in L^2(0,T;C^1(\bar{\Omega})).$$

Thus, we conclude that $\xi_2 = F(u^+)$.

Step 4: We will prove the following strong convergences in this step:

$$\begin{aligned}&||\bar{u}_\epsilon^+ - u^+||_{L^2(\mathcal{Q}_\epsilon^+)} + \left|\left|\frac{\partial \bar{u}_\epsilon^+}{\partial x_2} - \frac{\partial u^+}{\partial x_2}\right|\right|_{L^2(\mathcal{Q}_\epsilon^+)} \\ &+ \left|\left|\frac{\partial \bar{u}_\epsilon^+}{\partial x_1} + \left(\frac{a_{12}}{a_{11}}\right)\frac{\partial u^+}{\partial x_2}\right|\right|_{L^2(\mathcal{Q}_\epsilon^+)} + ||\bar{u}_\epsilon^- - u^-||_{L^2(0,T;H^1(\Omega^-))} \to 0.\end{aligned} \tag{5.38}$$

Above convergence will be used for the convergence analysis for adjoint states. Consider

$$
\begin{aligned}
L_\epsilon =& \frac{1}{2}\int_{\mathcal{Q}_\epsilon^+} \left[\bar{u}_\epsilon^+(T,x) - u^+(T,x)\right]^2 + \int_{\mathcal{Q}_\epsilon^+} (\bar{u}_\epsilon^+ - u^+)^2 \\
&+ \int_{\mathcal{Q}_\epsilon^+} A^\epsilon \begin{pmatrix} \frac{\partial \bar{u}_\epsilon^+}{\partial x_1} - \left(\frac{-a_{12}}{a_{11}}\right)\frac{\partial u^+}{\partial x_2} \\ \frac{\partial \bar{u}_\epsilon^+}{\partial x_2} - \frac{\partial u^+}{\partial x_2} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \bar{u}_\epsilon^+}{\partial x_1} - \left(\frac{-a_{12}}{a_{11}}\right)\frac{\partial u^+}{\partial x_2} \\ \frac{\partial \bar{u}_\epsilon^+}{\partial x_2} - \frac{\partial u^+}{\partial x_2} \end{pmatrix} \\
&+ \int_{\mathcal{Q}_\epsilon^+} \left[F(\bar{u}_\epsilon^+) - F(u^+)\right]\left[\bar{u}_\epsilon^+ - u^+\right] + \frac{1}{2}\int_{\mathcal{Q}^-} (\bar{u}_\epsilon^-(T,x) - u^-(T,x))^2 \\
&+ \int_{\mathcal{Q}^-} A^\epsilon \left[\nabla \bar{u}_\epsilon^- - \nabla u^-\right] \cdot \left[\nabla \bar{u}_\epsilon^- - \nabla u^-\right] \\
&+ \int_{\mathcal{Q}^-} (\bar{u}_\epsilon^- - u^-)^2 + \int_{\mathcal{Q}^-} \left[F(\bar{u}_\epsilon^-) - F(u^-)\right]\left[\bar{u}_\epsilon^- - u^-\right].
\end{aligned}
\tag{5.39}
$$

Upon expanding (5.39), we get

$$
L_\epsilon = L_\epsilon^1 + L_\epsilon^2 + L_\epsilon^3,
$$

where

$$
\begin{aligned}
L_\epsilon^1 =& \frac{1}{2}\int_{\mathcal{Q}_\epsilon^+} \left[(\bar{u}_\epsilon^+(T,x)\right]^2 + \frac{1}{2}\int_{\mathcal{Q}^-} \left[\bar{u}_\epsilon^-(T,x)\right]^2 + \int_{\mathcal{Q}_\epsilon^+} A^\epsilon \nabla \bar{u}_\epsilon^+ \cdot \nabla \bar{u}_\epsilon^+ \\
&+ \int_{\mathcal{Q}^-} A^\epsilon \nabla \bar{u}_\epsilon^- \cdot \nabla \bar{u}_\epsilon^- + \int_{\mathcal{Q}_\epsilon^+} (\bar{u}_\epsilon^+)^2 + \int_{\mathcal{Q}^-} (\bar{u}_\epsilon^-)^2 \\
&+ \int_{\mathcal{Q}^-} F(\bar{u}_\epsilon^-)\bar{u}_\epsilon^- + \int_{\mathcal{Q}_\epsilon^+} F(\bar{u}_\epsilon^+)\bar{u}_\epsilon^+,
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
L_\epsilon^2 =& -\int_{\mathcal{Q}_\epsilon^+} \bar{u}_\epsilon^+(T,x) u^+(T,x) - \int_{\mathcal{Q}^-} \bar{u}_\epsilon^-(T,x) u^-(T,x) - 2\int_{\mathcal{Q}_\epsilon^+} \bar{u}_\epsilon^+ u^+ \\
&- 2\int_{\mathcal{Q}^-} \bar{u}_\epsilon^- u^- - \int_{\mathcal{Q}_\epsilon^+} F(\bar{u}_\epsilon^+) u^+ - \int_{\mathcal{Q}^-} F(\bar{u}_\epsilon^-) u^- - \int_{\mathcal{Q}_\epsilon^+} F(u^+)\bar{u}_\epsilon^+ \\
&- \int_{\mathcal{Q}^-} F(u^-)\bar{u}_\epsilon^- - \int_{\mathcal{Q}^-} A^\epsilon \nabla \bar{u}_\epsilon^- \cdot \nabla u^- - \int_{\mathcal{Q}^-} A^\epsilon \nabla u^- \cdot \nabla \bar{u}_\epsilon^- \\
&- \int_{\mathcal{Q}_\epsilon^+} A^\epsilon \begin{pmatrix} \frac{\partial \bar{u}_\epsilon^+}{\partial x_1} \\ \frac{\partial \bar{u}_\epsilon^+}{\partial x_2} \end{pmatrix} \cdot \begin{pmatrix} \left(\frac{-a_{12}}{a_{11}}\right)\frac{\partial u^+}{\partial x_2} \\ \frac{\partial u^+}{\partial x_2} \end{pmatrix} - \int_{\mathcal{Q}_\epsilon^+} A^\epsilon \begin{pmatrix} \left(\frac{-a_{12}}{a_{11}}\right)\frac{\partial u^+}{\partial x_2} \\ \frac{\partial u^+}{\partial x_2} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \bar{u}_\epsilon^+}{\partial x_1} \\ \frac{\partial \bar{u}_\epsilon^+}{\partial x_2} \end{pmatrix},
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
L_\epsilon^3 =& \frac{1}{2}\int_{\mathcal{Q}_\epsilon^+} \left[u^+(T,x)\right]^2 + \frac{1}{2}\int_{\mathcal{Q}^-} \left[u^-(T,x)\right]^2 \\
&+ \int_{\mathcal{Q}_\epsilon^+} A^\epsilon \begin{pmatrix} \left(\frac{-a_{12}}{a_{11}}\right)\frac{\partial u^+}{\partial x_2} \\ \frac{\partial u^+}{\partial x_2} \end{pmatrix} \cdot \begin{pmatrix} \left(\frac{-a_{12}}{a_{11}}\right)\frac{\partial u^+}{\partial x_2} \\ \frac{\partial u^+}{\partial x_2} \end{pmatrix} + \int_{\mathcal{Q}^-} A^\epsilon \nabla u^- \cdot \nabla u^- \\
&+ \int_{\mathcal{Q}_\epsilon^+} (u^+)^2 + \int_{\mathcal{Q}^-} (u^-)^2 + \int_{\mathcal{Q}^-} F(u^-)u^- \int_{\mathcal{Q}_\epsilon^+} F(u^+)u^+.
\end{aligned}
\tag{5.42}
$$

Using the unfolding operator, the convergence obtained in (5.35) and $\xi_2 = F(v^+)$, we get

$$\lim_{\epsilon\to 0} L^1_\epsilon = \frac{\delta}{2}\int_{\Omega^+}\left[u^+(T,x)\right]^2 + \int_{\mathcal{Q}^+}\left[\int_0^\delta\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)dz\right]\left(\frac{\partial u^+}{\partial x_2}\right)^2$$
$$+\int_{\mathcal{Q}^+\times(0,\delta)} F(u^+)u^+ + \delta\int_{\mathcal{Q}^+}\left(u^+\right)^2 + \int_{\mathcal{Q}^-}\left[u^-(T,x)\right]^2$$
$$+\int_{\mathcal{Q}^-} A^*\nabla u^-\cdot\nabla u^- + \int_{\mathcal{Q}^-} F(u^-)u^- + \int_{\mathcal{Q}^-}\left(u^-\right)^2$$

Utilizing the convergence results obtained in step 1 and step 2, we can easily obtained the limits for L^2_ϵ and L^3_ϵ, and after rearrangements, we obtain

$$\lim_{\epsilon\to 0}\left(L^2_\epsilon + L^3_\epsilon\right) = -\frac{\delta}{2}\int_{\Omega^+}\left[u^+(T,x)\right]^2 - \int_{\mathcal{Q}^+}\left[\int_0^\delta\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)dz\right]\left(\frac{\partial u^+}{\partial x_2}\right)^2$$
$$-\int_{\mathcal{Q}^+\times(0,\delta)} F(u^+)u^+ - \delta\int_{\mathcal{Q}^+}\left(u^+\right)^2 - \int_{\mathcal{Q}^-}\left[u^-(T,x)\right]^2$$
$$-\int_{\mathcal{Q}^-} A^*\nabla u^-\cdot\nabla u^- - \int_{\mathcal{Q}^-} F(u^-)u^- - \int_{\mathcal{Q}^-}\left(u^-\right)^2 = -\lim_{\epsilon\to 0} L^1_\epsilon.$$

This implies $\lim_{\epsilon\to 0}\left(L^1_\epsilon + L^2_\epsilon + L^3_\epsilon\right) = 0$. Thus, we get $\lim_{\epsilon\to 0} L_\epsilon = 0$. Now, using the assumption that matrix A is coercive and F is monotonic, (5.38) is proved.

From the strong convergence proved in (5.38), using triangle inequality and Proposition 3.2, we have following convergence under unfolding

$$\begin{aligned}
&\|T^+_\epsilon(\bar{u}^+_\epsilon) - u^+\|_{L^2(\mathcal{Q}^+\times(0,\delta))} \to 0,\\
&\left\|T^+_\epsilon\left(\frac{\partial \bar{u}^+_\epsilon}{\partial x_2}\right) - \frac{\partial u^+}{\partial x_2}\right\|_{L^2(\mathcal{Q}^+\times(0,\delta))} \to 0,\\
&\left\|T^+_\epsilon\left(\frac{\partial \bar{u}^+_\epsilon}{\partial x_1}\right) + \left(\frac{a_{12}}{a_{11}}\right)\frac{\partial u^+}{\partial x_2}\right\|_{L^2(\mathcal{Q}^+\times(0,\delta))} \to 0.
\end{aligned} \tag{5.43}$$

Also, from the strong convergence in (5.43) and the assumptions on F, we get following convergence for non linear term under unfolding

$$\begin{aligned}
&T^+_\epsilon(F'(\bar{u}^+_\epsilon)) = F'(T^+_\epsilon(\bar{u}^+_\epsilon)) \to F'(u^+) \text{ strongly in } L^2\left(\mathcal{Q}^+\times(0,\delta)\right),\\
&T^-_\epsilon(F'(\bar{u}^-_\epsilon)) = F'(T^-_\epsilon(\bar{u}^-_\epsilon)) \to F'(u^-) \text{ strongly in } L^2(\mathcal{Q}^-\times(0,1)^2).
\end{aligned} \tag{5.44}$$

Step 5: In this step, we obtain the convergence results for the adjoint state $\bar{v}_\epsilon$ in the upper domain. Similar to Step 1, we derive the following convergences for the adjoint state $\bar{v}_\epsilon$, i.e., there exists $v^+ \in \mathcal{D}$ and $\xi_3 \in L^2\left(\mathcal{Q}^+\times(0,\delta)\right)$ such that, up to a subsequence,

$$T^+_\epsilon(\bar{v}^+_\epsilon) \rightharpoonup v^+ \text{ weakly in } \mathcal{D}, \tag{5.45a}$$

$$T^+_\epsilon\left(\frac{\partial \bar{v}^+_\epsilon}{\partial x_2}\right) \rightharpoonup \frac{\partial v^+}{\partial x_2} \text{ weakly in } L^2(\mathcal{Q}^+\times(0,\delta)), \tag{5.45b}$$

$$T_\epsilon^+\left(\frac{\partial \bar{v}_\epsilon^+}{\partial x_1}\right) \rightharpoonup \xi_3 \text{ weakly in } L^2(\mathcal{Q}^+ \times (0,\delta)), \tag{5.45c}$$

where v^+ is independent of the z variable. Furthermore, due to Proposition 3.2(P7), and convergence results obtained in (5.45), we conclude that

$$\widetilde{\bar{v}_\epsilon^+} \rightharpoonup \delta v^+ \text{ weakly in } L^2(\mathcal{Q}^+), \tag{5.46}$$

$$\widetilde{\frac{\partial \bar{v}_\epsilon^+}{\partial x_2}} \rightharpoonup \delta \frac{\partial v^+}{\partial x_2} \text{ weakly in } L^2(\mathcal{Q}^+). \tag{5.47}$$

Identification of ξ_3: Using the properties of the unfolding operator and ψ_ϵ defined in (5.11)as the test function in the weak formulation of the adjoint equation in (2.4), we obtain

$$\begin{aligned} &-\int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{v}_\epsilon^+)T_\epsilon^+(\psi_\epsilon') + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+({}^tA^\epsilon)T_\epsilon^+(\nabla \bar{v}_\epsilon^+)\cdot T_\epsilon^+(\nabla\psi_\epsilon) \\ &\quad + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(F'(\bar{u}_\epsilon))T_\epsilon^+(\bar{v}_\epsilon^+)T_\epsilon^+(\psi_\epsilon) + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{v}_\epsilon^+)T_\epsilon^+(\psi_\epsilon) \\ &= \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(B^\epsilon)T_\epsilon^+(\nabla \bar{u}_\epsilon^+)\cdot T_\epsilon^+(\nabla\psi_\epsilon). \end{aligned} \tag{5.48}$$

By utilizing the convergence obtained in (5.3), (5.4), (5.6), (5.12), (5.44) and (5.45), we pass the limit $\epsilon \to 0$ in Eq. (5.48) to obtain

$$\int_{\mathcal{Q}^+\times(0,\delta)} \left(a_{11}\xi_3 + a_{21}\frac{\partial v^+}{\partial x_2}\right)\psi_1\frac{\partial \psi_2}{\partial z} = \int_{\mathcal{Q}^+\times(0,\delta)} \left(-b_{11}\frac{a_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2} + b_{12}\frac{\partial v^+}{\partial x_2}\right)\psi_1\frac{\partial \psi_2}{\partial z},$$

which can be rewritten as

$$\int_{\mathcal{Q}^+\times(0,\delta)} \left(a_{11}\xi_3 + a_{21}\frac{\partial v^+}{\partial x_2} + b_{11}\frac{a_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2} - b_{12}\frac{\partial u^+}{\partial x_2}\right)\psi_1\frac{\partial \psi_2}{\partial z} = 0, \tag{5.49}$$

for all $\psi_1 \in C_c^\infty(\mathcal{Q}^+)$ and $\psi_2 \in C_{per}^\infty(0,1)$. Hence, from equation (5.49), we get

$$\xi_3 = \frac{b_{12}}{a_{11}}\frac{\partial u^+}{\partial x_2} - \frac{b_{11}a_{12}}{a_{11}^2}\frac{\partial u^+}{\partial x_2} - \frac{a_{21}}{a_{11}}\frac{\partial v^+}{\partial x_2} \text{ a.e. } (t,x,z) \in \mathcal{Q}^+ \times (0,\delta). \tag{5.50}$$

Moreover, using proposition (3.2), we get

$$\widetilde{\frac{\partial \bar{v}_\epsilon^+}{\partial x_1}} \rightharpoonup \left(\int_0^\delta \left[\frac{b_{12}}{a_{11}} - \frac{b_{11}a_{12}}{a_{11}^2}\right]dz\right)\frac{\partial u^+}{\partial x_2} - \left(\int_0^\delta \frac{a_{21}}{a_{11}}dz\right)\frac{\partial v^+}{\partial x_2} \text{ weakly in } L^2(\mathcal{Q}^+).$$

Step 6: In this step, we identify the limit equations satisfied by u^+ and v^+ in the upper domain Ω^+. In the weak formulation (2.1) for $\bar{u}_\epsilon$, by replacing ψ with $\phi \in C_c^\infty(\mathcal{Q}^+)$ and further utilizing the properties of the unfolding operator, we obtain

$$\begin{aligned}
&-\int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{u}_\epsilon^+)T_\epsilon^+(\phi') + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(A^\epsilon)T_\epsilon^+(\nabla\bar{u}_\epsilon^+)\cdot T_\epsilon^+(\nabla\phi) \\
&\quad + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(F(\bar{u}_\epsilon^+))T_\epsilon^+(\phi) + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{u}_\epsilon^+)T_\epsilon^+(\phi) \\
&= \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(g)T_\epsilon^+(\phi).
\end{aligned} \tag{5.51}$$

We pass the limit $\epsilon \to 0$ in Eq. (5.51) using the convergence results obtained in (5.3), (5.4), (5.6), and (5.10) and noting that u^+ is independent of z, we obtain

$$\begin{aligned}
&\delta\int_{\mathcal{Q}^+}(u^+)'\phi + \int_{\mathcal{Q}^+}\left[\int_0^\delta\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)dz\right]\frac{\partial u^+}{\partial x_2}\frac{\partial\phi}{\partial x_2} \\
&\quad + \delta\int_{\mathcal{Q}^+}F(u^+)\phi + \delta\int_{\mathcal{Q}^+}u^+\phi = \delta\int_{\mathcal{Q}^+}g\phi.
\end{aligned} \tag{5.52}$$

Using the value of a^* from (4.3), Eq. (5.52) can be rewritten as

$$\delta\int_{\mathcal{Q}^+}(u^+)'\phi + \int_{\mathcal{Q}^+}a^*\frac{\partial u^+}{\partial x_2}\frac{\partial\phi}{\partial x_2} + \delta\int_{\mathcal{Q}^+}F(u^+)\phi + \delta\int_{\mathcal{Q}^+}u^+\phi = \delta\int_{\mathcal{Q}^+}g\phi. \tag{5.53}$$

Next, by taking $\psi \in C_c^\infty(\mathcal{Q}^+)$ as a test function in the weak formulation of the adjoint equation satisfied by $\bar{v}_\epsilon^+$, and again utilizing the properties of the unfolding operator, we get

$$\begin{aligned}
&-\int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{v}_\epsilon^+)T_\epsilon^+(\psi') + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+({}^tA^\epsilon)T_\epsilon^+(\nabla\bar{v}_\epsilon^+)\cdot T_\epsilon^+(\nabla\psi) \\
&\quad + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(F'(\bar{u}_\epsilon^+))T_\epsilon^+(\bar{v}_\epsilon^+)T_\epsilon^+(\psi) + \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{v}_\epsilon^+)T_\epsilon^+(\psi) \\
&= \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(B^\epsilon)T_\epsilon^+(\nabla\bar{u}_\epsilon^+)\cdot T_\epsilon^+(\nabla\psi).
\end{aligned} \tag{5.54}$$

Passing to the limit $\epsilon \to 0$ in equation (5.54), using the convergence results from (5.3), (5.4), (5.6), (5.44), and (5.45). Thus, we obtain

$$\begin{aligned}
&\delta\int_{\mathcal{Q}^+}(v^+)'\psi + \int_{\mathcal{Q}^+}\left[\int_0^\delta\left(a_{22}-\frac{a_{12}a_{21}}{a_{11}}\right)dz\right]\frac{\partial v^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + \delta\int_{\mathcal{Q}^+}F'(u^+)v^+\psi \\
&\quad + \delta\int_{\mathcal{Q}^+}v^+\psi = \int_{\mathcal{Q}^+}\left[\int_0^\delta\left(b_{22}+b_{11}\left(\frac{a_{12}}{a_{11}}\right)^2 - 2\frac{a_{12}b_{12}}{a_{11}}\right)dz\right]\frac{\partial u^+}{\partial x_2}\frac{\partial\psi}{\partial x_2}.
\end{aligned} \tag{5.55}$$

Using the values of a^* and b^* from (4.3), equation (5.55) can be rewritten as

$$\delta\int_{\mathcal{Q}^+}(v^+)'\psi + \int_{\mathcal{Q}^+}a^*\frac{\partial v^+}{\partial x_2}\frac{\partial\psi}{\partial x_2} + \delta\int_{\mathcal{Q}^+}F'(u^+)v^+\psi + \delta\int_{\mathcal{Q}^+}v^+\psi = \int_{\mathcal{Q}^+}b^*\frac{\partial u^+}{\partial x_2}\frac{\partial\psi}{\partial x_2}. \tag{5.56}$$

Step 7: In this step, we will derive the convergence results for the adjoint state $\bar{v}_\epsilon$ in the lower domain. According to Theorem 2.5, we know that $\|\bar{v}_\epsilon^-\|_{L^2(0,T;H^1(\Omega^-))}$ is uniformly bounded. Consequently, there exists a subsequence (still denoted by ϵ) and $v^- \in L^2(0,T;H^1(\Omega^-))$ such that

$$\bar{v}_\epsilon^- \rightharpoonup v^- \text{ weakly in } L^2(0,T;H^1(\Omega^-)). \tag{5.57}$$

The strong convergence is established using the Aubin-Lions lemma as

$$\bar{v}_\epsilon^- \to v^- \text{ strongly in } L^2(\mathcal{Q}^-). \tag{5.58}$$

Furthermore, using the unfolding technique and the result from (5.58), we obtain

$$T_\epsilon^-(\bar{v}_\epsilon^-) \to v^- \text{ in } L^2(\mathcal{Q}^- \times (0,1)^2). \tag{5.59}$$

Moreover, by Theorem 3.4, there exists $u_1 \in L^2\left(\mathcal{Q}; H^1_{per}(0,1)^2\right)$ such that

$$T_\epsilon^-(\nabla\bar{v}_\epsilon^-) \rightharpoonup \nabla v^- + \nabla_y v_1 \text{ in } L^2(\mathcal{Q}^- \times (0,1)^2). \tag{5.60}$$

From the optimality condition in (2.4), we know that

$$\bar{\theta}_\epsilon = -\alpha^{-1}\chi_{(0,T)\times\omega}\bar{v}_\epsilon. \tag{5.61}$$

Using the convergence from (5.58), we can pass the limit in (5.61) to get

$$\bar{\theta}_\epsilon \to -\alpha^{-1}\chi_{(0,T)\times\omega}v^- \text{ in } L^2(\mathcal{Q}^-). \tag{5.62}$$

Thus, the weak limit θ_0 in Eq. (5.22) is identified as $\theta_0 = -\alpha^{-1}v^-$. Next, we apply the unfolding operator to the adjoint equation in the lower domain $\mathcal{Q}^-$, using the following test function

$$\psi^\epsilon = \psi_0(t,x) + \epsilon\psi_1\left(t,x,\frac{x}{\epsilon}\right),$$

where $\psi_0 \in C_c^\infty(\mathcal{Q}^-), \psi_1 \in C_c^\infty\left(\mathcal{Q}^-; C^\infty_{per}(0,1)^2\right)$. We obtain

$$\begin{aligned}
&-\int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(\bar{v}_\epsilon^-)T_\epsilon^-((\psi^\epsilon)') + \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-({}^tA^\epsilon)T_\epsilon^-(\nabla\bar{v}_\epsilon^-)\cdot T_\epsilon^-(\nabla\psi^\epsilon)\\
&\quad + \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(F'(\bar{u}_\epsilon^-))T_\epsilon^-(\bar{v}_\epsilon^-)T_\epsilon^-(\psi^\epsilon) + \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(\bar{u}_\epsilon^-)T_\epsilon^-(\psi^\epsilon)\\
&= \int_{\mathcal{Q}^-\times(0,1)^2} T_\epsilon^-(B^\epsilon)T_\epsilon^-(\nabla\bar{u}_\epsilon^-)\cdot T_\epsilon^-(\nabla\psi^\epsilon).
\end{aligned} \tag{5.63}$$

Using the limits obtained in (5.17), (5.19), (5.44),(5.59) and (5.60) in the equation (5.63), we get

$$\begin{aligned}
&\int_{\mathcal{Q}^-\times(0,1)^2} (v^-)'\psi_0 + \int_{\mathcal{Q}^-\times(0,1)^2} ({}^tA)(\nabla v^- + \nabla_y v_1)\cdot(\nabla\psi_0 + \nabla_y\psi_1)\\
&\quad + \int_{\mathcal{Q}^-\times(0,1)^2} F'(u^-)v^-\psi_0 + \int_{\mathcal{Q}^-\times(0,1)^2} v^-\psi_0\\
&= \int_{\mathcal{Q}^-\times(0,1)^2} B(\nabla u^- + \nabla_y u_1)\cdot(\nabla\psi_0 + \nabla_y\psi_1).
\end{aligned} \tag{5.64}$$

Putting $\psi_0 = 0$ and u_1 from (5.24) in the equation (5.64), we get

$$\begin{aligned}\int_{\mathcal{Q}^- \times (0,1)^2} ({}^t A)\nabla_y v_1 \cdot \nabla_y \psi_1 \\ = \int_{\mathcal{Q}^- \times (0,1)^2} \left[B\left(I + \nabla_y Z\right)\nabla u^- \cdot \nabla_y \psi_1 - ({}^t A)\nabla v^- \cdot \nabla_y \psi_1\right].\end{aligned} \tag{5.65}$$

With the help aid of solution of the cell problem (4.5) and Eq. (5.65), we can write v_1 as

$$v_1(t, x, y) = \sum_{i=1}^{2}\left(\frac{\partial v^-}{\partial x_i} Z_i - \frac{\partial u^-}{\partial x_i} X_i\right). \tag{5.66}$$

Now putting $\psi_1 = 0$, u_1 from (5.24) and v_1 from (5.66) in Eq. (5.64), we get

$$\begin{aligned}&\int_{\mathcal{Q}^- \times (0,1)^2} (v^-)'\psi_0 + \int_{\mathcal{Q}^- \times (0,1)^2} ({}^t A)\left[\nabla v^- + \sum_{i=1}^{2}\left(\frac{\partial v^-}{\partial x_i}\nabla_y Z_i - \frac{\partial u^-}{\partial x_i}\nabla_y X_i\right)\right] \cdot \nabla\psi_0 \\ &\quad + \int_{\mathcal{Q}^- \times (0,1)^2} F'(u^-)v^-\psi_0 + \int_{\mathcal{Q}^- \times (0,1)^2} v^-\psi_0 \\ &= \int_{\mathcal{Q}^- \times (0,1)^2} B\left(\nabla u^- + \sum_{i=1}^{2}\frac{\partial u^-}{\partial x_i} Z_i\right) \cdot \nabla\psi_0.\end{aligned} \tag{5.67}$$

Employing the cell problem (4.5), and the fact that v^- and u^- are independent of y, we can rewrite equation (5.67) as

$$\begin{aligned}&\int_{\mathcal{Q}^-} (v^-)'\psi_0 + \int_{\mathcal{Q}^-}\left[\int_{(0,1)^2} ({}^t A)(x, y_1, t)\left(I + \nabla_y Z\right) dy\right]\nabla v^- \cdot \nabla\psi_0 \\ &\quad + \int_{\mathcal{Q}^-} F'(u^-)v^-\psi_0 + \int_{\mathcal{Q}^-} v^-\psi_0 \\ &= \int_{\mathcal{Q}^-}\left[\int_{(0,1)^2}\left(B(x, y_1, t)\left(I + \nabla_y Z\right) + ({}^t A)(x, y_1, t)\nabla_y X\right) dy\right]\nabla u^- \cdot \nabla\psi_0.\end{aligned} \tag{5.68}$$

By employing the definitions of matrix A^* and B^* as provided in Eq. (4.4), we could rewrite Eq. (5.68) as follows:

$$\begin{aligned}\int_{\mathcal{Q}^-} (v^-)'\psi_0 + \int_{\mathcal{Q}^-} ({}^t A^*)\nabla v^- \cdot \nabla\psi_0 + \int_{\mathcal{Q}^-} F'(u^-)v^-\psi_0 \\ + \int_{\mathcal{Q}^-} v^-\psi_0 = \int_{\mathcal{Q}^-} B^*\nabla u^- \cdot \nabla\psi_0.\end{aligned} \tag{5.69}$$

Define v as $v = \chi_{\Omega^+}v^+ + \chi_{\Omega^-}v^-$. By analogy with, for instance, [16], it can be shown that the traces of v^+ and v^- are identical on the interface Γ and hence v in $V(\Omega)$ and $v' \in L^2(0, T, (V(\Omega))^*)$ (see [32], page 107). It is straightforward to verify that v also satisfies the initial condition $v(T, x) = 0$.

Step 8: In this step, we demonstrate that $u = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-}$ and $v := v^+\chi_{\Omega^+} + v^-\chi_{\Omega^-}$ satisfy the limit optimality system given by Theorem 4.2, in the weak sense.

By employing the function $\psi \in C_c^\infty(0,T;H^1(\Omega))$ as a test function in the weak formulation (2.1) and taking the limit as $\epsilon \to 0$, decompose each integral in the weak formulation over $\mathcal{Q}_\epsilon^+$ and $\mathcal{Q}^-$. This decomposition is achieved using results from Step 2 for the lower domain and Step 6 for the upper domain. Consequently, we obtain the following result

$$\begin{aligned}\delta \int_{\mathcal{Q}^+}(u^+)'\psi &+ \int_{\mathcal{Q}^+} a^* \frac{\partial u^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + \delta \int_{\mathcal{Q}^+} F(u^+)\psi + \delta \int_{\mathcal{Q}^+} u^+\psi + \int_{\mathcal{Q}^-}(u^-)'\psi \\ &+ \int_{\mathcal{Q}^-} A^*\nabla u^- \cdot \nabla \psi + \int_{\mathcal{Q}^-} F(u^-)\psi + \int_{\mathcal{Q}^-} u^-\psi \\ &= \delta \int_{\mathcal{Q}^+} g\psi + \int_{\mathcal{Q}^-}\left(g - \alpha^{-1}\chi_{(0,T)\times\omega} v^-\right)\psi.\end{aligned} \tag{5.70}$$

Similarly, by using $\psi \in C_c^\infty(0,T;H^1(\Omega))$ as a test function in the weak formulation of the adjoint equation in (2.3), and then taking the limit as $\epsilon \to 0$ with the help of results from Step 6 for the upper domain and Step 7 for the lower domain, we obtain

$$\begin{aligned}\delta \int_{\mathcal{Q}^+}(v^+)'\psi &+ \int_{\mathcal{Q}^+} a^* \frac{\partial v^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + \delta \int_{\mathcal{Q}^+} F'(u^+)v^+\psi + \delta \int_{\mathcal{Q}^+} v^+\psi + \int_{\mathcal{Q}^-}(v^-)'\psi \\ &+ \int_{\mathcal{Q}^-} ({}^tA^*)\nabla v^- \cdot \nabla \psi + \int_{\mathcal{Q}^-} F'(u^-)v^-\psi + \int_{\mathcal{Q}^-} v^-\psi \\ &= \int_{\mathcal{Q}^+} b^* \frac{\partial u^+}{\partial x_2}\frac{\partial \psi}{\partial x_2} + \int_{\mathcal{Q}^-} B^*\nabla u^- \cdot \nabla \psi.\end{aligned} \tag{5.71}$$

from the Eqs. (5.70) and (5.71), it is clear that (u, v) solves the optimality system given by Theorem 4.2 in the weak sense. This concludes the proof of the theorem. □

6. Corrector results

In Eq. (5.38), we have shown that $\bar{u}_\epsilon^-$ convergences to u^- stongly in $L^2(0,T;$ $H^1(\Omega^-))$. Now, we improve the weak convergence reasults $\bar{u}_\epsilon^+$ obtained in Step 1 by proving the following corrector results.

Theorem 6.1. *Let* $\bar{u}_\epsilon$,

be as defined in Theorem 5.1. Then, we have the following strong convergences

$$\begin{aligned}&\widetilde{\bar{u}_\epsilon^+} - \chi_{\Omega_\epsilon^+} u^+ \to 0 \quad \textit{strongly in} \quad L^2(\mathcal{Q}^+),\\ &\widetilde{\frac{\partial \bar{u}_\epsilon^+}{\partial x_2}} - \chi_{\Omega_\epsilon^+}\frac{\partial u^+}{\partial x_2} \to 0 \quad \textit{strongly in} \quad L^2(\mathcal{Q}^+),\\ &\widetilde{\frac{\partial \bar{u}_\epsilon^+}{\partial x_1}} + \left(\frac{a_{12}}{a_{11}}\right)\chi_{\Omega_\epsilon^+}\frac{\partial u^+}{\partial x_2} \to 0 \quad \textit{strongly in} \quad L^2(\mathcal{Q}^+).\end{aligned} \tag{6.1}$$

Proof. Using the triangle inequality, we can write

$$\begin{aligned} &\|T_\epsilon^+(\bar{u}_\epsilon^+) - T_\epsilon^+(u^+)\|_{L^2(\mathcal{Q}^+\times(0,\delta))} \\ &\quad \le \|T_\epsilon^+(\bar{u}_\epsilon^+) - u^+\|_{L^2(\mathcal{Q}^+\times(0,\delta))} + \|u^+ - T_\epsilon^+(u^+)\|_{L^2(\mathcal{Q}^+\times(0,\delta))}. \end{aligned} \tag{6.2}$$

Using (5.43) and Proposition 3.2, both terms on the right-hand side of (6.2) onverge to zero. Thus, we obtain

$$\|T_\epsilon^+(\bar{u}_\epsilon^+) - T_\epsilon^+(u^+)\|_{L^2(\mathcal{Q}^+\times(0,\delta))} \to 0. \tag{6.3}$$

Note that

$$\begin{aligned} \|T_\epsilon^+(\bar{u}_\epsilon^+) - T_\epsilon^+(u^+)\|_{L^2(\mathcal{Q}^+\times(0,\delta))} &= \int_{\mathcal{Q}^+\times(0,\delta)} |T_\epsilon^+(\bar{u}_\epsilon^+) - T_\epsilon^+(u^+)|^2 \\ &= \int_{\mathcal{Q}^+\times(0,\delta)} T_\epsilon^+(\bar{u}_\epsilon^+ - u^+)^2 = \int_{\mathcal{Q}_\epsilon^+} (\bar{u}_\epsilon^+ - u^+)^2. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4), we obtain

$$\widetilde{\bar{u}_\epsilon^+} - \chi_{\Omega_\epsilon^+} u^+ \to 0 \ \text{ strongly in } \ L^2(\mathcal{Q}^+).$$

Similarly, we can prove the convergence results for $\widetilde{\frac{\partial \bar{v}_\epsilon^+}{\partial x_2}}$ and $\widetilde{\frac{\partial \bar{v}_\epsilon^+}{\partial x_1}}$. This completes the proof of the theorem. □

Acknowledgements

1 The first author would like to thank the Department of Science & Technology, Government of India for the support under FIST (No:SR/FST/MS-II/2021/97) to the Department of Mathematics, Indian Institute of Science (IISC), Bangalore, India and the project is partially supported by the the Department of Science & Technology, Government of India under project No. CRG/2021/000458. The second author expresses gratitude to the Council of Scientific & Industrial Research (CSIR) for providing the Research Fellowship under grant number 09/1005(0035)/2020-EMR-I. The third author acknowledges the support received from the Science & Engineering Research Board (SERB), Government of India, under project number EEQ/2023/000531.

Author contributions AKN, RR and BCS wrote the main article and prepared the figure, and all authors reviewed the manuscript.

Data Availability Statement No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] Aiyappan, S., Cardone, G., Perugia, C., Prakash, R.: Homogenization of a nonlinear monotone problem in a locally periodic domain via unfolding method. Nonlinear Anal. Real World Appl. **66**, 103537 (2022)

[2] Aiyappan, S., Nandakumaran, A.K.: Optimal control problem in a domain with branched structure and homogenization. Math. Methods Appl. Sci. **40**(8), 3173–3189 (2017)

[3] Aiyappan, S., Nandakumaran, A.K., Prakash, R.: Generalization of unfolding operator for highly oscillating smooth boundary domains and homogenization. Calc. Var. Part. Differ. Equ. **57**, 1–30 (2018)

[4] Aiyappan, S., Nandakumaran, A.K., Prakash, R.: Semi-linear optimal control problem on a smooth oscillating domain. Commun. Contemp. Math. **22**(4), 1950029, 26 (2020)

[5] Aiyappan, S., Nandakumaran, A.K., Sufian, A.: Asymptotic analysis of a boundary optimal control problem on a general branched structure. Math. Methods Appl. Sci. **42**, 6407–6434 (2019)

[6] Aiyappan, S., Sardar, B.C.: Biharmonic equation in a highly oscillating domain and homogenization of an associated control problem. Appl. Anal. **98**(16), 2783–2801 (2019)

[7] Amirat, Y., Simon, J.: Riblets and drag minimization. In: Optimization Methods in Partial Differential Equations (South Hadley, MA, 1996), Contemporary Mathematics, vol. 209, pp. 9–17. American Mathematical Society, Providence (1997)

[8] Birnir, B., Hou, S., Wellander, N.: Derivation of the viscous Moore-Greitzer equation for aeroengine flow. J. Math. Phys. **48**(6), 06520931 (2007)

[9] Blanchard, D., Gaudiello, A.: Homogenization of highly oscillating boundaries and reduction of dimension for a monotone problem. ESAIM Control Optim. Calc. Var. **9**, 449–460 (2003)

[10] Bonnans, J.F.: Second-order analysis for control constrained optimal control problems of semilinear elliptic systems. Appl. Math. Optim. **38**(3), 303–325 (1998)

[11] Brizzi, R., Chalot, J.: Boundary homogenization and Neumann boundary value problem. Ricerche mat. **46**, 341–388 (1997)

[12] Casado-Dıaz, J., Fernández-Cara, E., Simon, J.: Why viscous fluids adhere to rugose walls: a mathematical explanation. J. Differ. Equ. **189**(2), 526–537 (2003)

[13] Cioranescu, D., Damlamian, A., Griso, G.: Periodic unfolding and homogenization. C. R. Math. Acad. Sci. Paris **335**(1), 99–104 (2002)

[14] Cioranescu, D., Damlamian, A., Griso, G.: The periodic unfolding method in homogenization. SIAM J. Math. Anal. **40**(4), 1585–1620 (2008)

[15] Corbo Esposito, A., Donato, P., Gaudiello, A., Picard, C.: Homogenization of the p-Laplacian in a domain with oscillating boundary. Commun. Appl. Nonlinear Anal. **4**, 1–23 (1997)

[16] Damlamian, A., Pettersson, K.: Homogenization of oscillating boundaries. Discrete Contin. Dyn. Syst. **23**(1–2), 197–219 (2009)

[17] D'Angelo, C., Panasenko, G., Quarteroni, A.: Asymptotic-numerical derivation of the robin type coupling conditions for the macroscopic pressure at a reservoir-capillaries interface. Appl. Anal. **92**(1), 158–171 (2013)

[18] D'Apice, C., De Maio, U., Kogut, P.I.: Gap phenomenon in the homogenization of parabolic optimal control problems. IMA J. Math. Control Inf. **25**(4), 461–489 (2008)

[19] De Maio, U., Gaudiello, A., Lefter, C.: Optimal control for a parabolic problem in a domain with highly oscillating boundary. Appl. Anal. **83**(12), 1245–1264 (2004)

[20] De Maio, U., Gaudiello, A., Lefter, C.G.: Null internal controllability for a Kirchhoff-Love plate with a comb-like shaped structure. SIAM J. Control. Optim. **62**(5), 2456–2474 (2024)

[21] De Maio, U., Kogut, P.I., Manzo, R.: Asymptotic analysis of an optimal boundary control problem for ill-posed elliptic equation in domains with Rugous boundary. Asymptot. Anal. **118**(3), 209–234 (2020)

[22] Durante, T., Faella, L., Perugia, C.: Homogenization and behaviour of optimal controls for the wave equation in domains with oscillating boundary. NoDEA Nonlinear Differ. Equ. Appl. **14**(5–6), 455–489 (2007)

[23] Durante, T., Mel'nyk, T.A.: Homogenization of quasilinear optimal control problems involving a thick multilevel junction of type 3:2:1. ESAIM Control Optim. Calc. Var. **18**(2), 583–610 (2012)

[24] Faella, L., Raj, R., Sardar, B.C.: Optimal control problem governed by wave equation in an oscillating domain and homogenization. Z. Angew. Math. Phys. **75**(2), 1–22 (2024)

[25] Gajewski, H., Gröger, K., Zacharias, K.: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Mathematische Lehrbücher und Monographien, II. Abteilung, Mathematische Monographien, vol. Band 38, Akademie-Verlag, Berlin (1974)

[26] Gaudiello, A., Guibé, O., Cois Murat, F.: Homogenization of the brush problem with a source term in L^1. Arch. Ration. Mech. Anal. **225**, 1–64 (2017)

[27] Gaudiello, A., Lenczner, M.: A two-dimensional electrostatic model of interdigitated comb drive in longitudinal mode. SIAM J. Appl. Math. **80**, 792–813 (2020)

[28] Gaudiello, A., Mel'nyk, T.: Homogenization of a nonlinear monotone problem with nonlinear Signorini boundary conditions in a domain with highly rough boundary. J. Differ. Equ. **265**, 5419–5454 (2018)

[29] Jäger, W.I., Mikelić, A.: Couette flows over a rough boundary and drag reduction. Commun. Math. Phys. **232**(3), 429–455 (2003)

[30] Kesavan, S., Saint Jean Paulin, J.: Homogenization of an optimal control problem. SIAM J. Control. Optim. **35**(5), 1557–1573 (1997)

[31] Lenczner, M.: Multiscale model for atomic force microscope array mechanical behavior. Appl. Phys. Lett. **90**, 9 (2007)

[32] Lions, J.-L.: Optimal control of systems governed by partial differential equations. Die Grundlehren der mathematischen Wissenschaften, vol. Band 170, Springer, New York (1971) (Translated from the French by S. K. Mitter)

[33] Mahadevan, R., Nandakumaran, A.K., Prakash, R.: Homogenization of an elliptic equation in a domain with oscillating boundary with non-homogeneous non-linear boundary conditions. Appl. Math. Optim. **82**, 245–278 (2020)

[34] Mel'nik, T.A., Sivak, O.A.: Asymptotic Analysis of a Parabolic Semilinear Problem with Nonlinear Boundary Multiphase Interactions in a Perforated Domain, vol. 164. Problems in Mathematical Analysis, No. 43, pp. 427–453 (2010)

[35] Mel'nyk, T., Sadovyi, D.: Multiple-scale Analysis of Boundary-Value Problems in Thick Multi-level Junctions of Type 3:2:2. Springer Briefs in Mathematics, Springer, Cham (2019)

[36] Mel'nyk, T.A.: Asymptotic approximation for the solution to a semi-linear parabolic problem in a thick junction with the branched structure. J. Math. Anal. Appl. **424**(2), 1237–1260 (2015)

[37] Nandakumaran, A.K., Prakash, R., Sardar, B.C.: Periodic controls in an oscillating domain: controls via unfolding and homogenization. SIAM J. Control. Optim. **53**(5), 3245–3269 (2015)

[38] Nandakumaran, A.K., Sufian, A.: Oscillating PDE in a rough domain with a curved interface: homogenization of an optimal control problem. ESAIM Control Optim. Calc. Var. **27**, S4, 37 (2021)

[39] Nandakumaran, A.K., Sufian, A., Thazhathethil, R.: Homogenization of semilinear optimal control problems on oscillating domains with matrix coefficients. Appl. Math. Optim. **89**(2), 46, 42 (2024)

[40] Sardar, B.C., Sufian, A.: Homogenization of a boundary optimal control problem governed by Stokes equations. Complex Var. Elliptic Equ. **67**(12), 2944–2974 (2022)

[41] Showalter, R.E.: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations. Mathematical Surveys and Monographs, vol. 49. American Mathematical Society, Providence (1997)

[42] Tröltzsch, F.: Optimal control of partial differential equations. Grad. Stud. Math. **112**, 399 (2010)

A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science, Bangalore
Bengaluru 560012
India
e-mail: nands@iisc.ac.in

Ritu Raj
Department of Mathematics
Indian Institute of Technology Ropar
Bara Phool 140001
India
e-mail: ritu.20maz0003@iitrpr.ac.in

Bidhan Chandra Sardar
Department of Mathematics
Indian Institute of Technology Madras
Chennai 600036
India
e-mail: bidhan.math@iitm.ac.in

Received: 19 August 2024.
Revised: 6 January 2025.
Accepted: 20 February 2025.