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Semi-linear optimal control problems with strong contrasting diffusivity in rough domains

A. K. Nandakumaran[1] · Abu Sufian[2] · Renjith Thazhathethil[3]

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Abstract

This article explores the homogenization of semi-linear elliptic partial differential equations (PDEs) with strong contrasting diffusivity in domains with highly oscillating boundary. The oscillations considered are general and can occur in multiple directions, and the oscillating part is made up of two distinct materials with strong contrasting diffusivity. Also, corrector results are established within this framework. The tool of analysis is periodic unfolding method and to deal with the non-linear term, we use the Browder-Mindy method. Furthermore, the research extends to the analysis of optimal control problems governed by semi-linear elliptic PDEs, where the cost functional and PDEs exhibit strong contrasting diffusivity. Later in this article we also make the similar studies in a circular oscillating boundary domain, where the oscillations are in circular fashion.

Keywords Homogenization · Optimal control · Periodic unfolding · Oscillating boundary · Circular oscillating domain

Mathematics Subject Classification 49J20 · 80M35 · 35B27

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✉ Abu Sufian
asufian@udec.cl

A. K. Nandakumaran
nands@iisc.ac.in

Renjith Thazhathethil
renjitht_pd@isibang.ac.in

1 Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

2 Universidad de Concepción, 4070409 Concepción, Chile

3 Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore Centre, Bangalore 560059, India

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1 Introduction

The study of the effective behavior of composite materials, composed of two or more pure materials with very high contrasts in their properties, necessitates the asymptotic analysis of partial differential equations (PDEs) with high-contrast diffusion coefficients. In this article, we consider the homogenization of two different kinds of domains, namely the pillar-type oscillating domain Ω_ε (see Figure 1(left)) and the circular-type oscillating domain $\mathcal{O}_\varepsilon$ (see Figure 1(right)). The domains Ω_ε and $\mathcal{O}_\varepsilon$ consist of two parts: fixed parts Ω^- and $\mathcal{O}^-$, and ε-dependent oscillating parts, denoted as Ω_ε^+ and $\mathcal{O}_\varepsilon^+$ respectively. The oscillating parts are composed of two materials, denoted as $\mathcal{C}_\varepsilon$ and $\mathcal{I}_\varepsilon$, which exhibit contrasting properties, as reflected in their diffusion coefficients. In these domains, we have considered following semi-linear PDE,

$$\begin{cases} -\operatorname{div}\left(\left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla u_\varepsilon\right) + k(u_\varepsilon) + u_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ \hfill A\nabla u \cdot \nu_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon. \end{cases}$$

It is evident that there is a contrasting nature in the diffusion coefficients of $\mathcal{C}\varepsilon$ and $\mathcal{I}\varepsilon$. Here A is a uniform elliptic matrix and k is a monotone type function. The same PDE is also considered in $\mathcal{O}_\varepsilon$. A detailed description is given in Subsection 2.4.

Our objective is to homogenize the aforementioned partial differential equation (PDE) along with its associated interior optimal control problem in both type of rough domains as the oscillating parameter approaches zero. We focus on general energy-type cost functionals. Further details regarding the problem formulations and descriptions is given in Subsection 2.5. and 3.3.

We shall briefly highlight some significant works in the literature that are relevant to our study. To the best of the authors' knowledge, the homogenization of PDEs with strongly contrasting diffusivity began in [39, 40], where the authors investigated the effective properties of composite materials containing high modulus reinforcements. Since then, numerous research works have been conducted in this direction. In [15], the authors explored the homogenization of a conductivity equation for a medium consisting of highly conductive vertical fibers embedded in another material, which is presumed to be a poor conductor.. For further reading in this direction we refer to [7, 8, 12, 42] and references therein.

The investigation of homogenization problems in oscillating boundary domains has been a highly active research area over the past two decades. Several significant contributions in this field can be found in [4, 6, 13, 14, 18, 20, 23, 24, 26], though this list is by no means exhaustive. For studies specifically addressing optimal control problems in oscillating domains, see [3, 19, 21, 22, 33, 35, 36] and the references therein.

The articles referenced above in rough domains primarily focus on problems having the ellipticity coefficients strictly away from zero and bounded. There has been limited research on homogenization in oscillating domains with high contrasting diffusivity coefficients. One of the pioneering studies in this direction is presented in [28], where the authors examine an elliptic variational problem on a pillar-type oscillating domain, with an insulator covering the highly conductive core of the pillar. The homogenization of a hyperbolic partial differential equation (PDE) with contrasting diffusive coefficients is discussed in [34], resulting in a homogenized limit described as a coupled two-scale system of macro and micro scales. In [36], the method of the unfolding operator is employed to extend the findings of [28] from the pillar-type oscillating domain to more general oscillating domains. This study also considers an interior optimal control problem involving a variational form with high contrasting diffusive coefficients. The present authors studied a homogenization problem with L^1-data with high-contrast coefficients in [37].

In the context of homogenizing non-linear problems within oscillating domains, significant insights can be found in [27], which analyses the asymptotic behavior of a monotone operator subject to nonlinear Signorini boundary conditions. The work in [5] addresses the homogenization of a nonlinear monotone problem in a locally periodic domain using the unfolding method. Furthermore, [31] employs the asymptotic expansion method to investigate a non-linear parabolic problem. For additional literature on this topic, refer to [9, 10, 25, 30] and the references therein.

The novelties of our work are as follows. The problems addressed in all the aforementioned references within high oscillating domains involve either linear problems with high contrast coefficients or semi-linear problems without contrast coefficients. To the best of the authors' knowledge, this work is the first to consider semi-linear problems with high contrast coefficients in high oscillating domains. One of the main difficulties is passing to the limit and identifying the limit of the semi-linear term. To overcome this difficulty, we must prove corrector-type results and employ the Browder-Minty method with suitable test functions. Furthermore, we have considered an interior optimal control problem with a very general type of energy cost functional. The corrector result is also crucially utilized in homogenization of the considered optimal control problems.

The primary methods utilized in this analysis are the unfolding operator and the monotone operator technique. The periodic unfolding method, initially introduced in [16], serves as a potent tool in homogenization theory. A modified version of this method was applied to homogenize problems in pillar-type oscillating domains, as detailed in [18]. The unfolding operator was subsequently generalized for general periodic oscillating domains in [1]. For further details on unfolding operators, refer to [17] and the references therein. The use of the monotone operator technique in homogenization is discussed in [2, 29, 32] and their references.

The article is structured as follows. As outlined in the introduction, we examine two distinct domain types: general oscillating and circular oscillating. These two types are explored in separate sections.

Section 2 focuses on the general oscillating domain, providing a detailed description and introducing the periodic unfolding operator and boundary unfolding operator in Subsections 2.1, 2.2, and 2.3.

Our primary goal is to homogenize an optimal control problem governed by a semi-linear elliptic PDE with strong contrasting diffusivity. To achieve this, we first address the homogenization aspect in Subsection 2.4. We prove the homogenization results in Theorem 1 and subsequently prove the corrector results in Theorem 2, which are essential for demonstrating the homogenization of the optimal control problem, our main objective. In Subsection 2.5, we delve into the homogenization of the aforementioned optimal control problem. Theorem 3 presents the homogenization results (corrector) for the state equation, while Theorem 4 provides the homogenization results for the adjoint equation.

Section 3 focuses on the circular oscillating domain, providing a detailed domain description in Subsection 3.1. As done in Section 2, we conduct the homogenization analysis in Subsection 3.2. We present homogenization and corrector results in Theorems 5 and 6, respectively. Our primary goal is to investigate the homogenization of the optimal control problem in Subsection 3.3. Theorems 7 and 8 provide the homogenization results for the corresponding state equation and adjoint equation, respectively.

2 General Oscillating Domain

2.1 Domain Description

For each $\varepsilon > 0$, we consider an ε-depended bounded domain $\Omega_\varepsilon \subset \mathbb{R}^n$ with oscillating boundary (see Figure 1]). Below, we describe the domain under consideration. Let $x = (x', x'') \in \mathbb{R}^n$ where $x' = (x_1, x_2, \cdots, x_m)$ and $x'' = (x_{m+1}, x_{m+2}, \cdots, x_n)$ with $1 \leqslant m \leqslant n-1$. Define

$$\Omega^+ = (0,1)^n, \quad \text{and} \quad Y^* = \prod_{i=1}^{m}(a_i, b_i) \times (0,1)^{n-m}$$

with $0 < a_i < b_i < 1$ for all $i = 1, 2, 3, \cdots, m$. Let Λ be a connected open subset of Y^* with Lipschitz boundary as our reference cell. It consists of two parts namely (nearly) insulating part $\mathcal{I}$ and conducting part $\mathcal{C}$. Now define

$$\mathcal{C}_\varepsilon = \left\{x \in \Omega^+ : \left(\left\{\frac{x'}{\varepsilon}\right\}, x''\right) \in \mathcal{C}\right\}, \quad \mathcal{I}_\varepsilon = \left\{x \in \Omega^+ : \left(\left\{\frac{x'}{\varepsilon}\right\}, x''\right) \in \mathcal{I}\right\},$$
$$\Omega_\varepsilon^+ = \left\{x \in \Omega^+ : \left(\left\{\frac{x'}{\varepsilon}\right\}, x''\right) \in \Lambda\right\} = \mathrm{int}\overline{(\mathcal{C}_\varepsilon \cup \mathcal{I}_\varepsilon)}$$

where $\left\{\frac{x'}{\varepsilon}\right\}$ denotes the fractional part of $\frac{x'}{\varepsilon}$. The lower fixed part is given by

$$\Omega^- = (0,1)^{n-1} \times (-1, 0).$$

The oscillating domain Ω_ε and limit domain Ω are defined as

$$\Omega_\varepsilon = \mathrm{int}\left(\overline{\Omega_\varepsilon^+ \cup \Omega^-}\right) \text{ and } \Omega = \mathrm{int}\left(\overline{\Omega^+ \cup \Omega^-}\right).$$

Here Ω_ε^+ is the upper oscillating part, Ω^- is the lower fixed part, Ω_ε is the oscillating domain and Ω is the limit domain. It is important to note that as per the definition of Ω_ε, the upper part Ω_ε^+ exhibits periodic oscillations. These oscillations involve a periodic arrangement of the reference cell Λ, which is scaled by ε in the x' variable and arranged in the x' direction with a period of ε. Also we denote Γ_a, Γ_b are upper and lower boundaries of Ω and Γ_0 is the

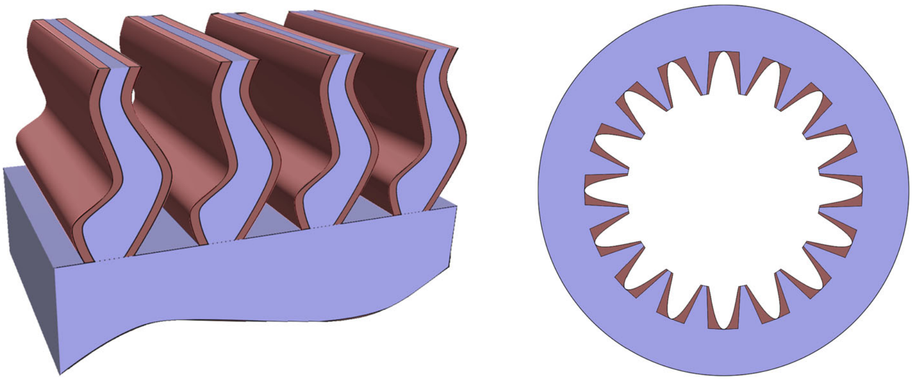

Fig. 1 Examples of Ω_ε (left) and $\mathcal{O}_\varepsilon$ (right)

interface defined as

$$\Gamma_a := (0,1)^{n-1} \times \{1\}, \quad \Gamma_b := (0,1)^{n-1} \times \{-1\}, \quad \Gamma_0 := (0,1)^{n-1} \times \{0\}.$$

For $x'' \in (0,1)^{n-m}$, define

$$Y_{\mathcal{C}}(x'') = \left\{ y \in (0,1)^m : (y, x'') \in \mathcal{C} \right\}, \quad Y_{\mathcal{I}}(x'') = \left\{ y \in (0,1)^m : (y, x'') \in \mathcal{I} \right\},$$
$$Y(x'') = \left\{ y \in (0,1)^m : (y, x'') \in \Lambda \right\} = Y_{\mathcal{C}}(x'') \cup Y_{\mathcal{I}}(x'').$$

Let $\left|Y(x'')\right|$ denote the m dimensional Lebesgue measure of $Y(x'')$. We assume the following properties on Λ:

(1) The set $Y_{\mathcal{C}}(x'')$ is connected for all $x'' \in (0,1)^{n-m}$.
(2) There exists $\rho > 0$ such that $0 < \rho \leq |Y_{\mathcal{C}}(x'')| < 1$ for all $x'' \in (0,1)^{n-m}$.
(3) The boundary part $\partial\Lambda \cap \Gamma_0$ is connected and have positive $n-1$ dimensional Lebesgue measure.

2.2 Periodic Unfolding Operator

Here we will define the periodic unfolding operator which is the main tool of our analysis. First, we will define the unfolded domains in which the unfolded functions are defined.

$$\Omega_{\mathcal{C}} = \left\{ (x, y) \mid x \in \Omega^+, y \in Y_{\mathcal{C}}(x'') \right\}, \quad \Omega_{\mathcal{I}} = \left\{ (x, y) \mid x \in \Omega^+, y \in Y_{\mathcal{I}}(x'') \right\},$$
$$\Omega_U = \left\{ (x, y) \mid x \in \Omega^+, y \in Y(x'') \right\} = \Omega_{\mathcal{C}} \cup \Omega_{\mathcal{I}}.$$

Let $\mathcal{G} = \{(x'', y) \mid x'' \in (0,1)^{n-m}, y \in Y(x'')\}$, then, one can write, $\Omega_U = (0,1)^m \times \mathcal{G}$. Let $\phi^\varepsilon : \Omega_U \to \Omega_\varepsilon^+$ be defined as $\phi^\varepsilon(x, y) = \left(\varepsilon \left[\frac{x'}{\varepsilon} \right] + \varepsilon y, x'' \right)$. The $\varepsilon-$ unfolding of a function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ is the function $u \circ \phi^\varepsilon : \Omega_U \to \mathbb{R}$. The operator which maps every function $u : \Omega_\varepsilon^+ \to \mathbb{R}$ to its ε-unfolding is called the unfolding operator. We denote the unfolding operator by T^ε, that is,

$$T^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{T^\varepsilon(u) : \Omega_U \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon(u)(x, y) = u\left(\varepsilon \left[\frac{x'}{\varepsilon} \right] + \varepsilon y, x'' \right).$$

Analogously we can define

$$T_{\mathcal{C}}^\varepsilon : \{u : \mathcal{C}_\varepsilon \to \mathbb{R}\} \to \{T_{\mathcal{C}}^\varepsilon(u) : \Omega_{\mathcal{C}} \to \mathbb{R}\} \quad \text{and}$$
$$T_{\mathcal{I}}^\varepsilon : \{u : \mathcal{I}_\varepsilon \to \mathbb{R}\} \to \{T_{\mathcal{I}}^\varepsilon(u) : \Omega_{\mathcal{I}} \to \mathbb{R}\}.$$

Then we have

$$T^\varepsilon = \chi_{\mathcal{C}_\varepsilon} T_{\mathcal{C}}^\varepsilon + \chi_{\mathcal{I}_\varepsilon} T_{\mathcal{I}}^\varepsilon$$

If $V \subset \mathbb{R}^N$ containing Ω_ε^+ and u is a real-valued function on V, $T^\varepsilon(u)$ means, that is T^ε acting on the restriction of u to Ω_ε^+. Some important properties of the unfolding operator are stated below. For each $\varepsilon > 0$,

(1) T^ε is linear. Further, if $u, \ v : \Omega_\varepsilon^+ \to \mathbb{R}$, then, $T^\varepsilon(uv) = T^\varepsilon(u) T^\varepsilon(v)$.
(2) Let $u \in L^1(\Omega_\varepsilon^+)$. then,

$$\int_{\Omega_U} T^\varepsilon(u) = \int_{\Omega_\varepsilon^+} u.$$

(3) Let $u \in L^2(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2(\Omega_U)$ and $\|T^\varepsilon u\|_{L^2(\Omega_U)} = \|u\|_{L^2(\Omega_\varepsilon^+)}$.

(4) Let $u \in H^1(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \in L^2((0,1)^m; H^1(\mathcal{G}))$. Moreover,

$$\nabla_{x''} T^\varepsilon u = T^\varepsilon \nabla_{x''} u \quad \text{and} \quad \nabla_y T^\varepsilon u = \varepsilon T^\varepsilon \nabla_{x'} u.$$

(5) Let $u \in L^2(\Omega_\varepsilon^+)$. Then, $T^\varepsilon u \to u$ strongly in $L^2(\Omega_U)$. More generally, let $u_\varepsilon \to u$ strongly in $L^2(\Omega^+)$. Then, $T^\varepsilon u_\varepsilon \to u$ strongly in $L^2(\Omega_U)$.

(6) Let, for every ε, $u_\varepsilon \in L^2(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega_U)$. then,

$$\tilde{u}_\varepsilon \rightharpoonup \int_{Y(x'')} u(x,y)dy \quad \text{weakly in } L^2(\Omega^+).$$

(7) Let, for every $\varepsilon > 0$, $u_\varepsilon \in H^1(\Omega_\varepsilon^+)$ be such that $T^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $L^2((0,1)^m; H^1(\mathcal{G}))$. Then,

$$\widetilde{u}_\varepsilon \rightharpoonup \int_{Y(x'')} u(x,y)dy \text{ weakly in } L^2(\Omega^+) \text{ and}$$
$$\widetilde{\nabla_{x''} u_\varepsilon} \rightharpoonup \int_{Y(x'')} \nabla_{x''} u dy \quad \text{weakly in } L^2(\Omega^+)^{n-m}.$$

where $\widetilde{u}_\varepsilon$ denotes the extension by 0 of u_ε to Ω^+. This notation is used throughout the article.

All the above properties are valid for $T_{\mathcal{C}}^\varepsilon$ and $T_{\mathcal{I}}^\varepsilon$ also.

2.3 Boundary unfolding operator

In order to obtain the interface conditions, it is necessary to employ the boundary unfolding operator. For every $\varepsilon > 0$, let us denote the unfolded boundary of Γ^ε by $\Gamma_{\mathcal{C}}$, defined by

$$\Gamma_{\mathcal{C}} = \{(x,y) : x \in (0,1)^n, x_n = 0 \text{ and } y \in Y_{\mathcal{C}}(x'')\}.$$

Define the boundary unfolding operator $T_0^\varepsilon : \{u : \Gamma_{\mathcal{C}}^\varepsilon \to \mathbb{R}\} \to \{T_0^\varepsilon(u) : \Gamma_{\mathcal{C}} \to \mathbb{R}\}$ as

$$T_0^\varepsilon(u)(x,y) = u\left(\varepsilon\left[\frac{x'}{\varepsilon}\right] + \varepsilon y, x''\right).$$

Note that boundary unfolding operator also has similar properties as those of unfolding operator with obvious modification in the functional setup.

2.4 Homogenization

Here we are going to study the homogenization of a semi-linear elliptic PDE with strong contrasting diffusivity in the domain Ω_ε. Let $A(x) = [a_{ij}(x)]_{n\times n}$ be an $n \times n$ symmetric matrix, where the entries $a_{ij} \in L^\infty(\Omega)$. Also A is uniformly elliptic and bounded in Ω, that is, there exists $\alpha, \beta > 0$ such that

$$\langle A(x)\lambda, \lambda\rangle \geq \alpha|\lambda|^2 \quad \text{and} \quad |A(x)\lambda| \leq \beta|\lambda|$$

for all $\lambda \in \mathbb{R}^n$ and $a.e.$ in Ω. Let A_1, A_2, A_3, A_4 be sub-matrices of A such that

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where the orders of the sub-matrices are as follows:

$$A_1 : m \times m, \quad A_2 : m \times (n-m), \quad A_3 : (n-m) \times m, \quad A_4 : (n-m) \times (n-m).$$

Let $k : \mathbb{R} \to \mathbb{R}$ be a C^2 real-valued function such that

$$0 < C_1 \leqslant k'(t) \leqslant C_2, k(0) = 0 \text{ and } k'' \text{ is bounded.}$$

Consider the following problem in Ω_ε:

$$\begin{cases} -\text{div}\left(\left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A \nabla u_\varepsilon\right) + k(u_\varepsilon) + u_\varepsilon = f \text{ in } \Omega_\varepsilon, \\ \hfill A\nabla u \cdot \nu_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon. \end{cases}$$

Here $f \in L^2(\Omega)$ is a given function and ν^ε is the outward unit normal vector. The corresponding variational formulation is

$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that,} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A \nabla u_\varepsilon \nabla \phi + k(u_\varepsilon)\phi + u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi, \\ \text{for all } \phi \in H^1(\Omega_\varepsilon). \end{cases} \tag{1}$$

Observe that the conductivity within the nearly insulating region $\mathcal{I}_\varepsilon$ is on the order of ε^2, indicating a much lower electrical conductance. In contrast, the conductivity in the conducting region $\mathcal{C}_\varepsilon$ is approximately 1, justifying its name contrasting diffusivity. The Browder-Minty theorem ensures the existence and uniqueness of u_ε. Our aim is to examine the asymptotic behavior of u_ε as ε approaches 0.

Next, we will describe the resulting homogenized problem. Subsequently, we will validate that this indeed represents the homogenized limit problem by presenting convergence results.

To define the limit problem, we define the following Hilbert spaces:

(1) $W(\Omega) = \left\{\psi \in L^2(\Omega) : \nabla_{x''}\psi \in L^2(\Omega)^{n-m}, \psi|_{\Omega^-} \in H^1(\Omega^-)\right\}$ with inner product

$$\langle \phi, \psi \rangle_{W(\Omega)} = \langle \phi, \psi \rangle_{L^2(\Omega^+)} + \langle \nabla_{x''}\phi, \nabla_{x''}\psi \rangle_{L^2(\Omega^+)} + \langle \phi, \psi \rangle_{H^1(\Omega^-)}.$$

(2) $W(\Omega_U) = \left\{\phi \in L^2(\Omega_U) : \phi = 0 \text{ in } \Omega_{\mathcal{C}} \text{ and } \nabla_y \phi \in L^2(\Omega_U)\right\}$ with inner product

$$\langle \phi, \psi \rangle_{W(\Omega_U)} = \langle \phi, \psi \rangle_{L^2(\Omega_U)} + \left\langle \nabla_y \phi, \nabla_y \psi \right\rangle_{L^2(\Omega_U)}.$$

The homogenized limit problem, which is a two-scale problem is described as follows:

Limit Problem: Given $f \in L^2(\Omega)$, find $(u, u_1) \in W(\Omega) \times W(\Omega_U)$ such that

$$\begin{cases} \displaystyle\int_{\Omega^+} A_0 \nabla_{x''} u \nabla_{x''} \phi + |Y_{\mathcal{C}}(x'')|(k(u) + u)\phi + \int_{\Omega^-} A\nabla u \nabla \phi + (k(u) + u)\phi \\ \displaystyle\quad + \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y u_1 \nabla_y \phi_1 + (k(u + u_1) + u + u_1)(\phi + \phi_1) \\ \displaystyle\quad = \int_{\Omega^+} |Y_{\mathcal{C}}(x'')| f\phi + \int_{\Omega_{\mathcal{I}}} f(\phi + \phi_1) + \int_{\Omega^-} f\phi, \\ \text{for all } (\phi, \phi_1) \in W(\Omega) \times W(\Omega_U), \end{cases} \tag{2}$$

where

$$A_0(x) = A_0(x'') = |Y_{\mathcal{C}}(x'')| \left(\left[-A_3 A_1^{-1} \; I\right] A \left[-A_3 A_1^{-1} \; I\right]^t\right).$$

Now, we will present the convergence result that demonstrate that the limit problem we described is indeed the actual homogenized problem. The following two theorems establish the relationship between the solutions of (1) and (2).

Theorem 1 *Let u_ε and u be solutions of (1) and (2) respectively. Then as $\varepsilon \to 0$, we have the following convergences:*

$$\begin{aligned}
\widetilde{u_\varepsilon} &\rightharpoonup |Y(x'')|u + \int_{Y_{\mathcal{I}}(x'')} u_1 && \text{weakly in } L^2(\Omega^+),\\
\chi_{\mathcal{C}_\varepsilon}\widetilde{\nabla_{x'} u_\varepsilon} &\rightharpoonup |Y_{\mathcal{C}}(x'')|(-A_1^{-1}A_2)\nabla_{x''}u && \text{weakly in } L^2(\Omega^+)^m,\\
\chi_{\mathcal{C}_\varepsilon}\widetilde{\nabla_{x''} u_\varepsilon} &\rightharpoonup |Y_{\mathcal{C}}(x'')|\nabla_{x''}u && \text{weakly in } L^2(\Omega^+)^{n-m},\\
\varepsilon\chi_{\mathcal{I}_\varepsilon}\widetilde{\nabla_{x'} u_\varepsilon} &\rightharpoonup \int_{Y_{\mathcal{I}}(x'')} \nabla_y u_1 && \text{weakly in } L^2(\Omega^+)^m,\\
\varepsilon\chi_{\mathcal{I}_\varepsilon}\widetilde{\nabla_{x''} u_\varepsilon} &\rightharpoonup \mathbf{0} && \text{weakly in } L^2(\Omega^+)^{n-m},\\
\widetilde{k(u_\varepsilon)} &\rightharpoonup |Y_{\mathcal{C}}(x'')|k(u) + \int_{Y_{\mathcal{I}}(x'')} k(u+u_1) && \text{weakly in } L^2(\Omega^+),\\
u_\varepsilon &\rightharpoonup u && \text{weakly in } H^1(\Omega^-).
\end{aligned}$$

where $\widetilde{\cdot}$ *represents the extension by* 0.

Proof The proof will be accomplished in several steps.
Step 1: (Weak convergences of unfolded sequences) In (1) considering $\phi = u_\varepsilon$ as a test function and using the monotonicity property of k, we get

$$\left\|\chi_{\mathcal{C}_\varepsilon}\nabla u_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \varepsilon\left\|\chi_{\mathcal{I}_\varepsilon}\nabla u_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} + \|\nabla u_\varepsilon\|_{L^2(\Omega^-)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant \|f\|_{L^2(\Omega_\varepsilon)}. \quad (3)$$

Note that we do not have uniform bound of u_ε in $H^1(\Omega_\varepsilon)$ due to high contrasting diffusivity. More precisely, the bound of u_ε in $H^1(\mathcal{I}_\varepsilon)$ is of order ε^{-1}. Hence, we need to analyze the convergence of u_ε in $\mathcal{C}_\varepsilon$ and $\mathcal{I}_\varepsilon$ separately. This is done via the unfolding operator.

From (3) using the properties of unfolding operator and weak compactness of Hilbert spaces, there exist $u_0 \in L^2(\Omega_U)$, $\mathbf{w} \in L^2(\Omega_{\mathcal{C}})^n$, $\mathbf{z} \in L^2(\Omega_{\mathcal{I}})^n$ and $u^- \in H^1(\Omega^-)$ such that

$$\begin{aligned}
T^\varepsilon(u_\varepsilon) &\rightharpoonup u_0 && \text{weakly in } L^2(\Omega_U),\\
T^\varepsilon(k(u_\varepsilon)) &\rightharpoonup \xi && \text{weakly in } L^2(\Omega_U),\\
T^\varepsilon_{\mathcal{C}}(\nabla u_\varepsilon) &\rightharpoonup \mathbf{w} && \text{weakly in } (L^2(\Omega_{\mathcal{C}}))^n,\\
\varepsilon T^\varepsilon_{\mathcal{I}}(\nabla u_\varepsilon) &\rightharpoonup \mathbf{z} && \text{weakly in } (L^2(\Omega_{\mathcal{I}}))^n,\\
u_\varepsilon &\rightharpoonup u^- && \text{weakly in } H^1(\Omega^-).
\end{aligned} \quad (4)$$

To identify $\mathbf{w}$ and $\mathbf{z}$, take $\phi \in C_c^\infty(\Omega_{\mathcal{C}})$ and $\psi \in C_c^\infty(\Omega_{\mathcal{I}})$. For convenience, we denote $\mathbf{w} = (\mathbf{w}_{x'}, \mathbf{w}_{x''})$ and $\mathbf{z} = (\mathbf{z}_{x'}, \mathbf{z}_{x''})$. Then

$$\begin{aligned}
\int_{\Omega_{\mathcal{C}}} \mathbf{w}_{x_i''}\phi &= \lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} T^\varepsilon_{\mathcal{C}}\left(\frac{\partial u_\varepsilon}{\partial x_i''}\right)\phi = \lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} \frac{\partial}{\partial x_i''}T^\varepsilon_{\mathcal{C}}(u_\varepsilon)\,\phi\\
&= -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} T^\varepsilon_{\mathcal{C}}(u_\varepsilon)\frac{\partial\phi}{\partial x_i''} = -\int_{\Omega_{\mathcal{C}}} u_0\frac{\partial\phi}{\partial x_i''} = \int_{\Omega_{\mathcal{C}}}\frac{\partial u_0}{\partial x_i''}\phi,\\
\int_{\Omega_{\mathcal{I}}} \mathbf{z}_{x_i''}\psi &= \lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{I}}} \varepsilon T^\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_i''}\right)\psi = \lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{I}}} \varepsilon\left(\frac{\partial}{\partial x_i''}T^\varepsilon(u_\varepsilon)\right)\psi
\end{aligned}$$

$$= -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{I}}} \varepsilon T^{\varepsilon}(u_\varepsilon)\left(\frac{\partial \psi}{\partial x_i''}\right) = 0,$$

$$\int_{\Omega_{\mathcal{I}}} \mathbf{z}_{x_i'}\psi = \lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{I}}} \varepsilon T^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i'}\right)\psi = \lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{I}}} \frac{\partial}{\partial y_i} T^{\varepsilon}(u_\varepsilon)\psi$$

$$= -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{I}}} T^{\varepsilon} u_\varepsilon\left(\frac{\partial \psi}{\partial y_i}\right) = -\int_{\Omega_{\mathcal{I}}} u_0 \frac{\partial \psi}{\partial y_i} = \int_{\Omega_{\mathcal{I}}} \frac{\partial u_0}{\partial y_i}\psi.$$

Since ϕ and ψ are arbitrary, we have

$$\mathbf{w}_{x''} = \nabla_{x''} u_0, \quad \mathbf{z}_{x''} = 0 \quad \text{and} \quad \mathbf{z}_{x'} = \nabla_y u_0.$$

To identify $\mathbf{w}_{x'}$, following the idea from [28, 36], we consider

$$\phi^{\varepsilon}(x) = \varepsilon\phi(x)\psi\left(\tfrac{x'}{\varepsilon}\right), \tag{5}$$

where $\phi \in C_c^{\infty}(\Omega^+)$ and $\psi \in C_{per}^{\infty}((0,1)^m))$ with $\psi(y) = \lambda' \cdot y$ in $\mathcal{C}$, where $\lambda' \in \mathbb{R}^m$ is fixed but arbitrary. Then

$$T_{\mathcal{C}}^{\varepsilon}\phi^{\varepsilon}(x,y) = \varepsilon\phi\left(\varepsilon\left[\tfrac{x'}{\varepsilon}\right] + \varepsilon y, x''\right)\psi(y),$$

$$T_{\mathcal{C}}^{\varepsilon}\left(\nabla_{x''}\phi^{\varepsilon}\right)(x,y) = \varepsilon\nabla_{x''}\phi\left(\varepsilon\left[\tfrac{x'}{\varepsilon}\right] + \varepsilon y, x''\right)\psi(y),$$

$$T_{\mathcal{C}}^{\varepsilon}\left(\nabla_{x'}\phi^{\varepsilon}\right)(x,y) = \varepsilon\nabla_{x'}\phi\left(\varepsilon\left[\tfrac{x'}{\varepsilon}\right] + \varepsilon y, x''\right)\psi(y) + \phi\left(\varepsilon\left[\tfrac{x'}{\varepsilon}\right] + \varepsilon y, x''\right)\nabla_y\psi(y).$$

Now use ϕ^{ε} as a test function in (1) to get

$$\int_{\Omega_\varepsilon^+} A\nabla u_\varepsilon \nabla\phi^{\varepsilon} + (k(u_\varepsilon) + u_\varepsilon)\phi = \int_{\Omega_\varepsilon^+} f\phi^{\varepsilon}.$$

Apply unfolding operator and pass to the limit as $\varepsilon \to 0$ to get

$$\int_{\Omega_{\mathcal{C}}} \phi(x) A\mathbf{w}\begin{bmatrix}\lambda' \\ 0\end{bmatrix} = 0.$$

Since ϕ is arbitrary, we have $\int_{Y_{\mathcal{C}}(x'')} A_1\mathbf{w}_{x'} + A_2\mathbf{w}_{x''} = 0$ $a.e.$ in Ω^+, which implies

$$\int_{Y_{\mathcal{C}}(x'')} \mathbf{w}_{x'} = -\int_{Y_{\mathcal{C}}(x'')} A_1^{-1}A_2\mathbf{w}_{x''} = -\int_{Y_{\mathcal{C}}(x'')} A_1^{-1}A_2\nabla_{x''}u^{+}.$$

Now for any $\phi \in C_c^{\infty}(\Omega_{\mathcal{C}})$, we have

$$\int_{\Omega_{\mathcal{C}}} \frac{\partial u_0}{\partial y_i}\phi = -\int_{\Omega_{\mathcal{C}}} u_0\frac{\partial \phi}{\partial y_i} = -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} T_{\mathcal{C}}^{\varepsilon}(u_\varepsilon)\frac{\partial \phi}{\partial y_i} = -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} \frac{\partial}{\partial y_i} T_{\mathcal{C}}^{\varepsilon}(u_\varepsilon)\phi$$

$$= -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} \varepsilon T_{\mathcal{C}}^{\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_i'}\right)\phi = 0,$$

which shows that u_0 is independent of y in $\Omega_{\mathcal{C}}$. That is, there exist $u^+ \in L^2(\Omega^+)$ such that $u_0 = u^+$ in $\Omega_{\mathcal{C}}$. Define $u_1 = u_0 - u^+$. Then we can write u_0 as $u_0 = u^+ + u_1$ with

$u^+ \in L^2(\Omega^+)$ and u_1 vanishes on $\Omega_{\mathcal{C}}$. Also, we can rewrite (4) as

$$\begin{aligned} T^\varepsilon(u_\varepsilon) &\rightharpoonup u^+ + u_1 && \text{weakly in } L^2(\Omega_U), \\ \int_{Y_{\mathcal{C}}(x'')} T^\varepsilon_{\mathcal{C}}(\nabla u_\varepsilon) &\rightharpoonup |Y_{\mathcal{C}(x'')}| \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''} u^+ && \text{weakly in } (L^2(\Omega^+)^n, \\ \varepsilon T^\varepsilon_{\mathcal{I}}(\nabla u_\varepsilon) &\rightharpoonup \begin{bmatrix} \nabla_y u_1 \\ \mathbf{0} \end{bmatrix} && \text{weakly in } (L^2(\Omega_{\mathcal{I}}))^n, \\ u_\varepsilon &\rightharpoonup u^- && \text{weakly in } H^1(\Omega^-). \end{aligned} \tag{6}$$

Step 2 (Interface Condition): In this step, we are going to prove that $u^+ = u^-$ on Γ_0. By the continuity of trace operator and using properties of unfolding operator, we get

$$\begin{aligned} \int_{\Gamma_{\mathcal{C}}} u^+\phi &= \lim \int_{\Gamma_{\mathcal{C}}} \left(T^\varepsilon_{\mathcal{C}}(u_\varepsilon)\right)\big|_{x_n=0} T^\varepsilon_0(\phi) = \lim \int_{\Gamma_{\mathcal{C}}} \left(T^\varepsilon_0(u_\varepsilon|_{\Omega^+})\right)\big|_{x_n=0} T^\varepsilon_0(\phi) \\ &= \lim \int_{\Gamma_{\mathcal{C}}} \left(T^\varepsilon_0(u_\varepsilon|_{\Omega^-})\right)\big|_{x_n=0} T^\varepsilon_0(\phi) = \int_{\Gamma_{\mathcal{C}}} u^-\phi, \end{aligned}$$

for any $\phi \in C_c^\infty(\Gamma_0)$. Hence, we have $u^+ = u^-$ on Γ_0. Define $u = \chi_{\Omega^+}u^+ + \chi_{\Omega^-}u^-$. Since $\nabla_{x''}u^+ \in L^2(\Omega^+)^{n-m}$ and $u^- \in H^1(\Omega^-)$, the interface condition gives $u \in W(\Omega)$.

Step 3:(Identifying ξ) Now we are at the most crucial part of the proof of finding ξ. This calculation is intricate because it involves higher-order matrices, and we are using the Browder-Minty method. We will show that ξ is identified as:

$$\begin{aligned} \xi &= k(u + u_1) \text{ in } \Omega_I \\ \int_{Y_{\mathcal{C}}(x'')} \xi &= |Y_{\mathcal{C}}(x'')| k(u). \end{aligned} \tag{7}$$

Applying the Browder-Minty method involves selecting an appropriate integral J_ε, along with suitable test functions, which is the most challenging aspect. Let ϕ be in $C_c^1(\bar{\Omega})$ and ϕ_1 be in $C_c^\infty(\Omega_U)$, with 1-periodic in y and vanishing on $\Omega_{\mathcal{C}}$. Define

$$\phi_1^\varepsilon(x) = \phi_1^\varepsilon\left(x, \frac{x'}{\varepsilon}\right) \text{ and } \phi^\varepsilon = \phi + \phi_1^\varepsilon.$$

Consider the integral

$$\begin{aligned} J^\varepsilon = &\int_{\mathcal{C}_\varepsilon} A \begin{bmatrix} \nabla_{x'}u_\varepsilon - \left(-A_1^{-1}A_2\right)\nabla_{x''}u \\ \nabla_{x''}u_\varepsilon - \nabla_{x''}\phi \end{bmatrix} \begin{bmatrix} \nabla_{x'}u_\varepsilon - \left(-A_1^{-1}A_2\right)\nabla_{x''}u \\ \nabla_{x''}u_\varepsilon - \nabla_{x''}\phi \end{bmatrix} \\ &+ \int_{\mathcal{I}_\varepsilon} A \begin{bmatrix} \varepsilon\nabla_{x'}u_\varepsilon - \nabla_y\phi_1^\varepsilon \\ \varepsilon\nabla_{x''}u_\varepsilon \end{bmatrix} \begin{bmatrix} \varepsilon\nabla_{x'}u_\varepsilon - \nabla_y\phi_1^\varepsilon \\ \varepsilon\nabla_{x''}u_\varepsilon \end{bmatrix} \\ &+ \int_{\Omega_\varepsilon^+} \left(k(u_\varepsilon) - k(\phi^\varepsilon)\right)(u_\varepsilon - \phi^\varepsilon) + (u_\varepsilon - \phi^\varepsilon)^2 \\ &+ \int_{\Omega^-} A(\nabla u_\varepsilon - \nabla\phi)(\nabla u_\varepsilon - \nabla\phi) + (k(u_\varepsilon) - k(\phi)(u_\varepsilon - \phi)) + (u_\varepsilon - \phi)^2. \end{aligned}$$

Expand and rearrange to get

$$
\begin{aligned}
J^{\varepsilon} = & \int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + u_\varepsilon^2 \\
& + \int_{\mathcal{C}_\varepsilon} -A \begin{bmatrix} \nabla_{x'} u_\varepsilon \\ \nabla_{x''} u_\varepsilon \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} \phi \end{bmatrix} - A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} \phi \end{bmatrix} \begin{bmatrix} \nabla_{x'} u_\varepsilon \\ \nabla_{x''} u_\varepsilon \end{bmatrix} \\
& + \int_{\mathcal{C}_\varepsilon} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} \phi \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} \phi \end{bmatrix} \\
& + \int_{\mathcal{I}_\varepsilon} -A \begin{bmatrix} \varepsilon \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{bmatrix} \begin{bmatrix} \nabla_y \phi_1^\varepsilon \\ 0 \end{bmatrix} - A \begin{bmatrix} \nabla_y \phi_1^\varepsilon \\ 0 \end{bmatrix} \begin{bmatrix} \varepsilon \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{bmatrix} + A \begin{bmatrix} \nabla_y \phi_1^\varepsilon \\ 0 \end{bmatrix} \begin{bmatrix} \nabla_y \phi_1^\varepsilon \\ 0 \end{bmatrix} \\
& + \int_{\Omega_\varepsilon^+} -k(u_\varepsilon)\phi^\varepsilon - k(\phi^\varepsilon)u_\varepsilon + k(\phi^\varepsilon)\phi^\varepsilon - 2u_\varepsilon \phi^\varepsilon + (\phi^\varepsilon)^2 \\
& + \int_{\Omega^-} -A\nabla u_\varepsilon \nabla \phi - A\nabla \phi \nabla u_\varepsilon + A\nabla \phi \nabla \phi \\
& + \int_{\Omega^-} -k(u_\varepsilon)\phi - k(\phi)u_\varepsilon + k(\phi)\phi - 2u_\varepsilon \phi + \phi^2.
\end{aligned}
$$

Using (6), pass to the limit as $\varepsilon \to 0$ in the variational form (1) with test function ϕ^ε to get

$$
\begin{aligned}
\int_{\Omega_U} f(\phi + \phi_1) + \int_{\Omega^-} f\phi = & \int_{\Omega_{\mathcal{C}}} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \nabla \phi + \int_{\Omega_{\mathcal{I}}} A \begin{bmatrix} \nabla_y u_1 \\ 0 \end{bmatrix} \begin{bmatrix} \nabla_y \phi_1 \\ 0 \end{bmatrix} \\
& + \int_{\Omega_U} \xi(\phi + \phi_1) + (u + u_1)(\phi + \phi_1) \\
& + \int_{\Omega^-} A\nabla u \nabla \phi + k(u)\phi + u\phi \\
= & \int_{\Omega_U} \left(A_4 - A_3 A_1^{-1} A_2\right) \nabla_{x''} u \nabla_{x''} \phi + \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y u_1 \nabla_y \phi_1 \\
& + \int_{\Omega_U} \xi(\phi + \phi_1) + (u + u_1)(\phi + \phi_1) \\
& + \int_{\Omega^-} A\nabla u \nabla \phi + k(u)\phi + u\phi.
\end{aligned}
$$

By density arguments put $\phi = u$ and $\phi_1 = u_1$ to get

$$
\begin{aligned}
\int_{\Omega_{\mathcal{C}}} f(u + u_1) + \int_{\Omega^-} fu = & \int_{\Omega_{\mathcal{C}}} \left(A_4 - A_3 A_1^{-1} A_2\right) \nabla_{x''} u \nabla_{x''} u + \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y u_1 \nabla_y u_1 \\
& + \int_{\Omega_U} \xi(u + u_1) + (u + u_1)^2 + \int_{\Omega^-} A\nabla u \nabla u + k(u)u + u^2 \\
= & \int_{\Omega_{\mathcal{C}}} A \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \begin{bmatrix} -A_1^{-1} A_2 \nabla_{x''} u \\ \nabla_{x''} u \end{bmatrix} \\
& + \int_{\Omega_{\mathcal{I}}} A \begin{bmatrix} \nabla_y u_1 \\ 0 \end{bmatrix} \begin{bmatrix} \nabla_y u_1 \\ 0 \end{bmatrix} \\
& + \int_{\Omega_U} \xi(u + u_1) + (u + u_1)^2 + \int_{\Omega^-} A\nabla u \nabla u + k(u)u + u^2.
\end{aligned}
$$

Then from energy equality we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon}+\chi_{\mathcal{C}_\varepsilon}+\chi_{\Omega^-}\right)A\nabla u_\varepsilon\nabla u_\varepsilon+k(u_\varepsilon)u_\varepsilon+u_\varepsilon^2=\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f u_\varepsilon\\
&=\int_{\Omega_U} f(u+u_1)+\int_{\Omega^-} fu\\
&=\int_{\Omega_{\mathcal{C}}} A\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}u\end{bmatrix}\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}u\end{bmatrix}+\int_{\Omega_{\mathcal{I}}} A\begin{bmatrix}\nabla_y u_1\\ 0\end{bmatrix}\begin{bmatrix}\nabla_y u_1\\ 0\end{bmatrix}\\
&\quad+\int_{\Omega_U}\xi(u+u_1)+(u+u_1)^2+\int_{\Omega^-}A\nabla u\nabla u+k(u)u+u^2.
\end{aligned}
\tag{8}
$$

Now pass to the limit as $\varepsilon\to 0$ in J_ε using (4) and (8) to get

$$
\begin{aligned}
\lim_{\varepsilon\to 0} J^\varepsilon &=\int_{\Omega_{\mathcal{C}}} A\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}u\end{bmatrix}\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}u\end{bmatrix}\\
&\quad+\int_{\Omega_{\mathcal{C}}} -A\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}u\end{bmatrix}\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}\phi\end{bmatrix}\\
&\quad+\int_{\Omega_{\mathcal{C}}} -A\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}\phi\end{bmatrix}\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}u\end{bmatrix}\\
&\quad+\int_{\Omega_{\mathcal{C}}} A\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}\phi\end{bmatrix}\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}\phi\end{bmatrix}\\
&\quad+\int_{\Omega_{\mathcal{I}}} A\begin{bmatrix}\nabla_y u_1\\ 0\end{bmatrix}\begin{bmatrix}\nabla_y u_1\\ 0\end{bmatrix}-A\begin{bmatrix}\nabla_y u_1\\ 0\end{bmatrix}\begin{bmatrix}\nabla_y \phi_1\\ 0\end{bmatrix}\\
&\quad+\int_{\Omega_{\mathcal{I}}} -A\begin{bmatrix}\nabla_y \phi_1\\ 0\end{bmatrix}\begin{bmatrix}\nabla_y u_1\\ 0\end{bmatrix}+A\begin{bmatrix}\nabla_y \phi_1\\ 0\end{bmatrix}\begin{bmatrix}\nabla_y \phi_1\\ 0\end{bmatrix}\\
&\quad+\int_{\Omega_U}\xi(u+u_1)-\xi(\phi+\phi_1)-k(\phi+\phi_1)(u+u_1)+k(\phi+\phi_1)(\phi+\phi_1)\\
&\quad+\int_{\Omega_U}(u+u_1)^2-2(u+u_1)(\phi+\phi_1)+(\phi+\phi_1)^2\\
&\quad+\int_{\Omega^-}A\nabla u\nabla u-A\nabla u\nabla\phi-A\nabla\phi\nabla u+A\nabla\phi\nabla\phi\\
&\quad+\int_{\Omega^-}k(u)u-k(u)\phi-k(\phi)u+k(\phi)\phi+u^2-2u\phi+\phi^2.
\end{aligned}
$$

By properly factorizing and simplifying we get,

$$
\begin{aligned}
\lim_{\varepsilon\to 0} J^\varepsilon &=\int_{\Omega_{\mathcal{C}}} A\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u+A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}u-\nabla_{x''}\phi\end{bmatrix}\begin{bmatrix}-A_1^{-1}A_2\nabla_{x''}u+A_1^{-1}A_2\nabla_{x''}u\\ \nabla_{x''}u-\nabla_{x''}\phi\end{bmatrix}\\
&\quad+\int_{\Omega_{\mathcal{I}}} A\begin{bmatrix}\nabla_y u_1-\nabla_y\phi_1\\ 0\end{bmatrix}\begin{bmatrix}\nabla_y u_1-\nabla_y\phi_1\\ 0\end{bmatrix}\\
&\quad+\int_{\Omega_U}(\xi-k(u+u_1))((u+u_1)-(\phi+\phi_1))+((u+u_1)-(\phi+\phi_1))^2\\
&\quad+\int_{\Omega^-}A(\nabla u-\nabla\phi)(\nabla u-\nabla\phi)+(k(u)-k(\phi))(u-\phi)+(u-\phi)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Omega_{\mathcal{C}}} A_4(\nabla_{x''} u - \nabla_{x''}\phi)(\nabla_{x''} u - \nabla_{x''}\phi) \\
&\quad + \int_{\Omega_{\mathcal{I}}} A_1(\nabla_y u_1 - \nabla_y \phi_1)(\nabla_y u_1 - \nabla_y \phi_1) \\
&\quad + \int_{\Omega_U} (\xi - k(\phi + \phi_1))((u + u_1) - (\phi - \phi_1)) + ((u + u_1) - (\phi - \phi_1))^2 \\
&\quad + \int_{\Omega^-} A(\nabla u - \nabla\phi)(\nabla u - \nabla\phi) + (k(u) - k(\phi))(u - \phi) + (u - \phi)^2.
\end{aligned}
$$

From the monotonicity of k, we have $J^\varepsilon \geq 0$ for all ε, which implies

$$
\begin{aligned}
\lim_{\varepsilon\to 0} J^\varepsilon &= \int_{\Omega_{\mathcal{C}}} A_4(\nabla_{x''} u - \nabla_{x''}\phi)(\nabla_{x''} u - \nabla_{x''}\phi) \\
&\quad + \int_{\Omega_{\mathcal{I}}} A_1(\nabla_y u_1 - \nabla_y \phi_1)(\nabla_y u_1 - \nabla_y \phi_1) \\
&\quad + \int_{\Omega_U} (\xi - k(\phi + \phi_1))((u + u_1) - (\phi + \phi_1)) + ((u + u_1) - (\phi + \phi_1))^2 \\
&\quad + \int_{\Omega^-} A(\nabla u - \nabla\phi)(\nabla u - \nabla\phi) + (k(u) - k(\phi))(u - \phi) + (u - \phi)^2. \geq 0.
\end{aligned}
$$

For $\lambda > 0$, choose $\phi = u + \lambda\psi$ and $\phi_1 = u_1 + \lambda\psi_1$, $\psi \in C^\infty\left(\overline{\Omega}\right)$, $\psi_1 \in C_c^\infty(\Omega_U)$ with 1-periodic in y and vanishes on $\Omega_{\mathcal{C}}$. Then, we get

$$
\begin{aligned}
&\int_{\Omega_{\mathcal{C}}} \lambda A_4 \nabla_{x''}\psi \nabla_{x''}\psi + \int_{\Omega_{\mathcal{I}}} \lambda A \nabla_y \psi_1 \nabla_y \psi_1 \\
&\quad + \int_{\Omega_U} (\xi - k(u + u_1 + \lambda(\psi + \psi_1)))(\psi + \psi_1) + \lambda(\psi + \psi_1)^2 \\
&\quad + \int_{\Omega^-} \lambda A \nabla\psi \nabla\psi + (k(u) - k(u - \lambda\psi))\psi + \lambda\psi^2 \geq 0.
\end{aligned}
$$

As $\lambda \to 0$,

$$
\int_{\Omega_U} (\xi - k(u + u_1))\,(\psi + \psi_1) \geq 0 \text{ for all } \psi, \psi_1.
$$

Hence,

$$
\int_{\Omega_U} (\xi - k(u + u_1))\,(\psi + \psi_1) = 0 \text{ for all } \psi, \psi_1.
$$

Now put $\psi = 0$ to get

$$
\xi = k(u + u_1) \text{ in } \Omega_I
$$

and put $\psi_1 = 0$, and since $u_1 = 0$ in $\Omega_{\mathcal{C}}$, we arrive at

$$
\int_{Y_{\mathcal{C}}(x'')} \xi = |Y_{\mathcal{C}}(x'')| k(u).
$$

Step 4 (Limit Problem): We now derive the limit equations using the results obtained in the previous steps. Let $\phi_\varepsilon(x) = \phi(x) + \phi_1\left(x, \frac{x'}{\varepsilon}\right)$, where $\phi \in C^1(\bar{\Omega})$ and $\phi_1 \in C_c^\infty(\Omega_U)$ with 1 periodic in y variable and $\phi_1 = 0$ on $\Omega_{\mathcal{C}}$. Now using ϕ_ε as a test function in (1), applying unfolding operator on both sides and letting $\varepsilon \to 0$ to get

$$\int_{\Omega_{\mathcal{C}}} \left(A_4 - A_3 A_1^{-1} A_2\right) \nabla_{x''} u \nabla_{x''} \phi + (\xi + u)\phi + \int_{\Omega^-} A \nabla u \nabla \phi + (k(u) + u)\phi$$
$$+ \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y u_1 \nabla_y \phi_1 + (k(u + u_1) + u + u_1)(\phi + \phi_1) \tag{9}$$
$$= \int_{\Omega_U} f(\phi + \phi_1) + \int_{\Omega^-} f\phi.$$

By density arguments we have (9) for all $(\phi, \phi_1) \in W(\Omega) \times W(\Omega_U)$.Average out and simplify to get the following variational form as the limit problem:

Find $(u, u_1) \in W(\Omega) \times W(\Omega_U)$ such that

$$\int_{\Omega^+} A_0 \nabla_{x''} u \nabla_{x''} \phi + |Y_{\mathcal{C}}(x'')|(k(u) + u)\phi + \int_{\Omega^-} A \nabla u \nabla \phi + (k(u) + u)\phi$$
$$+ \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y u_1 \nabla_y \phi_1 + (k(u + u_1) + u + u_1)(\phi + \phi_1) \tag{10}$$
$$= \int_{\Omega^+} |Y_{\mathcal{C}}(x'')| f\phi + \int_{\Omega^-} f\phi + \int_{\Omega_{\mathcal{I}}} f(\phi + \phi_1)$$

for all $(\phi, \phi_1) \in W(\Omega) \times W(\Omega_U)$, where $A_0(x) = |Y_{\mathcal{C}}(x'')| \left(A_4 - A_3 A_1^{-1} A_2\right)$.

Claim: A_0 is uniformly elliptic. To prove the claim, we can write A_0 in a more compact form as

$$A_0(x) = |Y_{\mathcal{C}}(x'')| \left(\left[-A_3 A_1^{-1} \ I\right] A \left[-A_3 A_1^{-1} \ I\right]^t,\right)$$

Let $\lambda \in \mathbb{R}^{n-m}$, then

$$\begin{aligned} \lambda^t A_0 \lambda &= |Y_{\mathcal{C}}(x'')| \lambda^t \left(\left[-A_3 A_1^{-1} \ I\right] A \left[-A_3 A_1^{-1} \ I\right]^t\right) \lambda \\ &= |Y_{\mathcal{C}}(x'')| \left(\begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix} \lambda\right)^t A \left(\begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix} \lambda\right) \\ &\geq \rho\alpha \left\| \begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix} \lambda \right\|^2 \geq \rho\alpha \|\lambda\|^2. \end{aligned}$$

Therefore A_0 is uniformly elliptic with ellipticity constant $\rho\alpha$.

From the convergence, we get the existence of the solution for the above problem. Now we have to prove the uniqueness. For that let $(u, u_1), (v, v_1)$ be 2 solutions of (10). On substituting each solution in (10) and subtracting we get

$$\int_{\Omega^+} A_0 \nabla_{x''}(u - v) \nabla_{x''} \phi + |Y_{\mathcal{C}}(x'')|(k(u) - k(v) + u - v)\phi$$
$$+ \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y (u_1 - v_1) \nabla_y \phi_1 + (k(u + u_1) - k(v + v_1) + u + u_1 - v - v_1)(\phi + \phi_1)$$
$$+ \int_{\Omega^-} A \nabla (u - v) \nabla \phi + (k(u) - k(v) + u - v)\phi = 0.$$

Since this holds true for all $(\phi, \phi_1) \in W(\Omega) \times W(\Omega_U)$, take $\phi = u - v$ and $\phi_1 = u_1 - v_1$ to get

$$\int_{\Omega^+} A_0 \nabla_{x''}(u-v)\nabla_{x''}(u-v) + |Y_{\mathcal{C}}(x'')|(k(u)-k(v))(u-v) + |Y_{\mathcal{C}}(x'')|(u-v)^2$$
$$+ \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y(u_1-v_1)\nabla_y(u_1-v_1) + (k(u+u_1)-k(v+v_1))(u+u_1-v-v_1)$$
$$+ (u+u_1-v-v_1)^2$$
$$+ \int_{\Omega^-} A\nabla(u-v)\nabla(u-v) + (k(u)-k(v))(u-v) + (u-v)^2 = 0.$$

Since k is monotone, we have $(k(u)-k(v))(u-v) \geq 0$ and $(k(u+u_1)-k(v+v_1))(u-v+u_1-v_1) \geq 0$. Then from ellipticity of the matrices A, A_1 and A_0, we have $u = v$ in $W(\Omega)$ and $u_1 = v_1$ in $W(\Omega_U)$. Hence (10) has a unique solution. □

Remark 1 Due to the semi-nonlinear nature of the system, achieving a complete separation of scales is not possible. The limiting problem leads to a coupled system involving both u and u_1, where u describes the macroscopic behavior, while u_1 captures the microscopic behavior. The primary difficulty can be seen when we set $\phi = 0$. The limiting variational form (2) becomes

$$\int_{\Omega_{\mathcal{I}}} A_1 \nabla u_1 \cdot \nabla \phi_1 + (k(u+u_1) + u + u_1)(\phi + \phi_1) = \int_{\Omega_{\mathcal{I}}} f\phi_1, \qquad \forall \phi_1 \in W(\Omega_U).$$

In this variational formulation, it is evident that u_1 cannot be separated from u due to the nonlinearity of k, resulting in the non-separability of the full macro-micro scale in the homogenized variational form. However, despite this non-separability, we achieve an almost complete separation, as can be seen from the structure of the variational formulation.

Theorem 2 *(Corrector results) Let u_ε, (u, u_1), be solutions of (1),(2) respectively. Define*

$$p_\varepsilon : \Omega_\varepsilon^+ \to \mathbb{R} \quad by \quad p_\varepsilon(x) = u(x) + u_1\left(x, \left\{\tfrac{x'}{\varepsilon}\right\}\right) \quad and$$
$$\mathbf{p}_\varepsilon : \Omega_\varepsilon^+ \to \mathbb{R}^m \quad by \quad \mathbf{p}_\varepsilon(x) = \nabla_y u_1\left(x, \left\{\tfrac{x'}{\varepsilon}\right\}\right).$$

Then as $\varepsilon \to 0$, we have

$$\left\| \nabla u_\varepsilon - \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''} u \right\|_{L^2(\mathcal{C}_\varepsilon)^n} + \left\| \varepsilon \nabla u_\varepsilon - \begin{bmatrix} \mathbf{p}_\varepsilon \\ \mathbf{0} \end{bmatrix} \right\|_{L^2(\mathcal{I}_\varepsilon)^n} \longrightarrow 0$$
$$\|u_\varepsilon - p_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \|u_\varepsilon - u\|_{H^1(\Omega^-)} \longrightarrow 0.$$

Proof Let us define,

$$J^\varepsilon = \int_{\mathcal{C}_\varepsilon} A\left(\nabla u_\varepsilon - \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''} u\right)\left(\nabla u_\varepsilon - \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''} u\right)$$
$$+ \int_{\mathcal{C}_\varepsilon} (k(u_\varepsilon) - k(u))(u_\varepsilon - u) + (u_\varepsilon - u)^2$$
$$+ \int_{\mathcal{I}_\varepsilon} A\left(\varepsilon \nabla u_\varepsilon - \begin{bmatrix} \mathbf{p}_\varepsilon \\ \mathbf{0} \end{bmatrix}\right)\left(\varepsilon \nabla u_\varepsilon - \begin{bmatrix} \mathbf{p}_\varepsilon \\ \mathbf{0} \end{bmatrix}\right)$$
$$+ \int_{\mathcal{I}_\varepsilon} (k(u_\varepsilon) - k(p_\varepsilon))(u_\varepsilon - p_\varepsilon) + (u_\varepsilon - p_\varepsilon)^2$$
$$+ \int_{\Omega^-} A(\nabla u_\varepsilon - \nabla u)(\nabla u_\varepsilon - \nabla u) + (u_\varepsilon - u)^2.$$

By expanding and rearranging, we get

$$J^{\varepsilon} = J_1^{\varepsilon} + J_2^{\varepsilon} + J_3^{\varepsilon} + J_4^{\varepsilon},$$

where

$$\begin{aligned}
J_1^{\varepsilon} &= \int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + (u^\varepsilon)^2,\\
J_2^{\varepsilon} &= \int_{\mathcal{C}_\varepsilon} -A\nabla u_\varepsilon \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''}u - A\begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''}u\nabla u_\varepsilon\\
&\quad + \int_{\mathcal{C}_\varepsilon} A\begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix}\begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''}u\nabla_{x''}u\\
&\quad + \int_{\mathcal{C}_\varepsilon} -k(u_\varepsilon)u - k(u)u_\varepsilon + k(u)u - 2u_\varepsilon u + u^2,\\
J_3^{\varepsilon} &= \int_{\mathcal{I}_\varepsilon} -\varepsilon A\nabla u_\varepsilon \begin{bmatrix} \mathbf{p}_\varepsilon \\ \mathbf{0} \end{bmatrix} - \varepsilon A\begin{bmatrix} \mathbf{p}_\varepsilon \\ \mathbf{0} \end{bmatrix}\nabla u_\varepsilon + A\begin{bmatrix} \mathbf{p}_\varepsilon \\ \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{p}_\varepsilon \\ \mathbf{0} \end{bmatrix}\\
&\quad + \int_{\mathcal{I}_\varepsilon} -k(u_\varepsilon)p_\varepsilon - k(p_\varepsilon)u_\varepsilon + k(p_\varepsilon)p_\varepsilon - 2u_\varepsilon p_\varepsilon + (p_\varepsilon)^2, ,\\
J_4^{\varepsilon} &= \int_{\Omega^-} -A\nabla u_\varepsilon\nabla u - A\nabla u\nabla u_\varepsilon + A\nabla u\nabla u\\
&\quad + \int_{\Omega^-} -k(u_\varepsilon)u - k(u)u_\varepsilon + k(u)u - 2u_\varepsilon u + u^2.
\end{aligned}$$

On applying unfolding operator and passing to the limit as $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0} J_2^{\varepsilon} = \int_{\Omega_{\mathcal{C}}} -(A_4 - A_3A_1^{-1}A_2)\nabla_{x''}u\nabla_{x''}u - \xi u - u^2.$$

Similarly since $T_{\mathcal{I}}^{\varepsilon}(p_\varepsilon) \to p = u + u_1$ and $T_{\mathcal{I}}^{\varepsilon}(\mathbf{p}_\varepsilon) \to \nabla_y p = \nabla_y u_1$ as $\varepsilon \to 0$. By using unfolding operator, we can arrive at

$$\lim_{\varepsilon\to 0} J_3^{\varepsilon} = \int_{\Omega_{\mathcal{I}}} -A_1\nabla_y u_1\nabla_y u_1 - k(u+u_1)(u+u_1) - (u+u_1)^2.$$

Also, we have

$$\lim_{\varepsilon\to 0} J_4^{\varepsilon} = \int_{\Omega^-} -A\nabla u\nabla u - k(u)u - u^2.$$

Using (1) and (9) we get

$$\begin{aligned}
\lim_{\varepsilon\to 0} J_1^{\varepsilon} &= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla u_\varepsilon \nabla u_\varepsilon + k(u_\varepsilon)u_\varepsilon + u_\varepsilon^2\\
&= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f u_\varepsilon = \int_{\Omega_{\mathcal{C}}} fu + \int_{\Omega_{\mathcal{I}}} f(u+u_1) + \int_{\Omega^-} fu\\
&= \int_{\Omega_{\mathcal{C}}} (A_4 - A_3A_1^{-1}A_2)\nabla_{x''}u\nabla_{x''}u + \xi u + u^2\\
&\quad + \int_{\Omega_{\mathcal{I}}} A_1\nabla_y u_1\nabla_y u_1 + k(u+u_1)(u+u_1) + (u+u_1)^2\\
&\quad + \int_{\Omega^-} A\nabla u\nabla u + k(u)u + u^2 \quad \text{(by taking } \phi = u \text{ and } \phi_1 = u_1 \text{ in (9))}
\end{aligned}$$

$$= -\left(\lim_{\varepsilon\to 0} J_2^\varepsilon + \lim_{\varepsilon\to 0} J_3^\varepsilon + \lim_{\varepsilon\to 0} J_4^\varepsilon\right).$$

This implies that

$$\lim_{\varepsilon\to 0} J^\varepsilon = 0.$$

Now coercivity of the matrix "A" completes the proof of the Theorem 2. □

Given the significant role of corrector results in homogenization theory, their inclusion here is primarily driven by their necessity in proving the homogenization of optimal control problems, a topic addressed in the following section.

2.5 Optimal Control

Here we are going to study the homogenization of an optimal control problem in the general oscillating domain Ω_ε. Define A as in Section 2.4 and also define

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

in a way as A defined with similar properties. Let $\omega \subset\subset \Omega^-$ be an open set and admissible control set is $L^2(\omega)$. Consider the following minimization problem: Minimize

$$J_\varepsilon(u,\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) B\nabla u \nabla u + \frac{\beta}{2}\int_{\Omega_\varepsilon} \chi_\omega |\theta|^2, \tag{11}$$

where (u,θ) satisfies the following system

$$\begin{cases} -\mathrm{div}\left(\left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla u\right) + k(u) + u \;=\; f + \chi_\omega \theta & \text{in } \Omega_\varepsilon, \\ A\nabla u \cdot \nu_\varepsilon \;=\; 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $f \in L^2(\Omega)$. From the semi-linear optimal control theory, we have atleast existence of one optimal solution $(u_\varepsilon, \theta_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(\omega)$ (see [2, 11, 43]).

We intend to investigate the asymptotic behavior of $(u_\varepsilon, \theta_\varepsilon)$ as $\varepsilon \to 0$. Referring to established findings on semi-linear optimal control problems (see [11, 43]), we can transform a minimization problem into a system of PDEs known as the optimality system corresponding to said minimization problem. In our analysis we deal with the optimal solution that comes from the optimality system. Let $(u_\varepsilon, \theta_\varepsilon)$ denotes the solution (optimal solution) of (11) that is coming from solving the optimality system, that is $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$ will satisfy the following optimality system.

$$\begin{cases} -\,\mathrm{div}\left(\left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla u_\varepsilon\right) + k(u_\varepsilon) + u_\varepsilon \;=\; f + \chi_\omega \theta_\varepsilon & \text{in } \Omega_\varepsilon, \\ -\,\mathrm{div}\left(\left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla v_\varepsilon\right) + k'(u_\varepsilon) v_\varepsilon + v_\varepsilon & \\ \quad = -\mathrm{div}\left(\left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) B\nabla v_\varepsilon\right) & \text{in } \Omega_\varepsilon, \\ A\nabla u_\varepsilon \cdot \nu_\varepsilon \;=\; 0, \quad A\nabla v_\varepsilon \cdot \nu_\varepsilon \;=\; B\nabla u_\varepsilon & \text{on } \partial\Omega_\varepsilon, \\ \theta_\varepsilon \;=\; -\dfrac{1}{\beta} v_\varepsilon & \text{in } \omega. \end{cases}$$

Corresponding variational form is: Given $f \in L^2(\Omega)$,

$$\begin{cases} \text{find } (u_\varepsilon, v_\varepsilon) \in H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla u_\varepsilon \nabla\psi + (k(u_\varepsilon) + u_\varepsilon)\psi = \int_{\Omega_\varepsilon} (f + \chi_\omega \theta_\varepsilon)\psi, \\ \displaystyle\int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla v_\varepsilon \nabla\phi + (k'(u_\varepsilon)v_\varepsilon + v_\varepsilon)\phi \\ \displaystyle\quad = \int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) B\nabla u_\varepsilon \nabla\phi, \\ \text{for all } (\psi, \phi) \in H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \text{ with } \theta_\varepsilon = -\dfrac{1}{\beta}\chi_\omega v_\varepsilon. \end{cases} \tag{12}$$

Therefore, our goal, which initially involved homogenizing optimal control problems, has now shifted to homogenizing the system of PDEs. Specifically, we aim to investigate how $(u_\varepsilon, v_\varepsilon)$ behaves as ε approaches 0.

Next, we will explain the resulting homogenized problem and its corresponding minimization problem. Then, we will demonstrate that this indeed represents the homogenized limit problem by providing convergence results. The two scale homogenized limit problem is as follows:

Limit optimality system: Given $f \in L^2(\Omega)$, find $((u, u_1), (v, v_1)) \in (W(\Omega) \times W(\Omega_U))^2$ such that

$$\begin{cases} \displaystyle\int_{\Omega^+} A_0 \nabla_{x''} u \nabla_{x''}\phi + |Y_{\mathcal{C}}(x'')|(k(u) + u)\phi + \int_{\Omega^-} A\nabla u \nabla\phi + (k(u) + u)\phi \\ \displaystyle\qquad + \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y u_1 \nabla_y \phi_1 + (k(u + u_1) + u + u_1)(\phi + \phi_1) \\ \displaystyle\quad = \int_{\Omega^+} |Y_{\mathcal{C}}(x'')| f\phi + \int_{\Omega_{\mathcal{I}}} f(\phi + \phi_1) + \int_{\Omega^-} (f + \chi_\omega\theta)\phi, \\ \displaystyle\int_{\Omega^+} A_0 \nabla_{x''} v \nabla_{x''}\psi + |Y_{\mathcal{C}}(x'')|(k'(u)v + v)\psi + \int_{\Omega^-} A\nabla v \nabla\psi + (k'(u)v + v)\psi \\ \displaystyle\qquad + \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y v_1 \nabla_y \phi_1 + (k'(u + u_1)(v + v_1) + v + v_1)(\phi + \phi_1) \\ \displaystyle\quad = \int_{\Omega^+} B_0 \nabla_{x''} u \nabla_{x''}\psi + \int_{\Omega_{\mathcal{I}}} B_1 \nabla_y u_1 \nabla_y \psi_1 + \int_{\Omega^-} B\nabla u \nabla\psi \end{cases} \tag{13}$$

for all $(\phi, \psi) \in W(\Omega) \times W(\Omega)$ with $\theta = -\frac{1}{\beta}\chi_\omega v$. Here

$$A_0 = \left|Y_{\mathcal{C}}(x'')\right| \left(\left[-A_3 A_1^{-1} \; I\right] A \left[-A_3 A_1^{-1} \; I\right]^t\right) \quad \text{and}$$
$$B_0 = \left|Y_{\mathcal{C}}(x'')\right| \left(\left[-A_3 A_1^{-1} \; I\right] B \left[-A_3 A_1^{-1} \; I\right]^t\right).$$

The optimal control problem corresponding to (13) is given by:

Limit optimal control problem: The above optimality system corresponds to the following limit optimal control problem: Find $(\bar{u}, \bar{u}_1, \bar{\theta}) \in W(\Omega) \times W(\Omega_U) \times L^2(\omega)$ such that

$$J(\bar{u}, \bar{u}_1, \bar{\theta}) = \inf_{\theta \in L^2(\omega)} J(u, u_1, \theta),$$

where

$$J(u, u_1, \theta) = \int_{\Omega^+} B_0 \nabla u^+ \nabla u^+ \int_{\Omega_I} B_1 \nabla_y u_1 \nabla_y u_1 + \frac{\beta}{2}\int_\omega |\theta|^2$$

and (u, u_1, θ) satisfies the following variational form,

$$\int_{\Omega^+} A_0 \nabla_{x''} u \nabla_{x''} \phi + |Y_{\mathcal{C}}(x'')|(k(u)+u)\phi + \int_{\Omega^-} A\nabla u \nabla \phi + (k(u)+u)\phi$$
$$+ \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y u_1 \nabla_y \phi_1 + (k(u+u_1)+u+u_1)(\phi+\phi_1)$$
$$= \int_{\Omega^+} |Y_{\mathcal{C}}(x'')| f\phi + \int_{\Omega_{\mathcal{I}}} f(\phi+\phi_1) + \int_{\Omega^-} (f+\chi_\omega \theta)\phi$$

for all $(\phi, \phi_1) \in W(\Omega) \times W(\Omega_U)$.

Remark 2 The limit optimal control problem, while not immediately appearing standard, possesses elliptic and monotone properties within the solution space. Given that B, B_0, A, and A_0 are elliptic operators, the associated cost functional is coercive. Consequently, the existence theory can be established in a manner analogous to the ε-stage optimal control problem which is well studied optimal control problem.

Now, it is time to provide convergence analysis to demonstrate that the homogenized limit problem we described is indeed the actual limit problem. The following two theorems provide us with the convergence results of our ε-stage problem (12) to the limit problem (13).

Theorem 3 *Let $(u_\varepsilon, v_\varepsilon)$, $((u, u_1), (v, v_1))$ be solutions of (12), (13) respectively. Define*

$$p_\varepsilon : \Omega_\varepsilon^+ \to \mathbb{R} \quad by \quad p_\varepsilon(x) = u(x) + u_1\left(x, \left\{\frac{x'}{\varepsilon}\right\}\right) \quad and$$
$$\mathbf{p}_\varepsilon : \Omega_\varepsilon^+ \to \mathbb{R}^m \quad by \quad \mathbf{p}_\varepsilon(x) = \nabla_y u_1\left(x, \left\{\frac{x'}{\varepsilon}\right\}\right).$$

Then as $\varepsilon \to 0$, we have the following convergences.

$$\left\| \nabla u_\varepsilon - \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''} u \right\|_{L^2(\mathcal{C}_\varepsilon)^2} + \left\| \varepsilon \nabla u_\varepsilon - \begin{bmatrix} \mathbf{p}_\varepsilon \\ \mathbf{0} \end{bmatrix} \right\|_{L^2(\mathcal{I}_\varepsilon)^2} \longrightarrow 0,$$
$$\|u_\varepsilon - u\|_{H^1(\Omega^-)} + \|u_\varepsilon - p_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} \longrightarrow 0.$$

Proof The proof will be the same as Theorem 2. The only extra term is $\chi_\omega \theta_\varepsilon$. Since ω is compactly contained in Ω^-, and $\|\theta^\varepsilon\|_{H^1(\omega)} \leqslant C$, it won't create any difficulties in any steps of the proof we had given in the case of homogenization. □

Theorem 4 *Let $(u_\varepsilon, v_\varepsilon)$ and $((u, u_1), (v, v_1))$ be the solutions of (12) and (13) respectively. Then as $\varepsilon \to 0$, we have the following convergences weakly in respective spaces:*

$$\widetilde{v_\varepsilon} \rightharpoonup |Y(x'')|v + \int_{Y_{\mathcal{I}}(x'')} v_1 \qquad in\ L^2(\Omega^+),$$
$$\chi_{\mathcal{C}_\varepsilon} \widetilde{\nabla_{x'} v_\varepsilon} \rightharpoonup |Y_{\mathcal{C}}(x'')| A_1^{-1} \left(\left(-B_1 A_1^{-1} A_2 + B_2 \right) \nabla_{x''} u - A_2 \nabla_{x''} v \right) \qquad in\ L^2(\Omega^+)^m,$$
$$\chi_{\mathcal{C}_\varepsilon} \widetilde{\nabla_{x''} v_\varepsilon} \rightharpoonup |Y_{\mathcal{C}}(x'')| \nabla_{x''} v \qquad in\ L^2(\Omega^+)^{n-m},$$
$$\chi_{\mathcal{I}_\varepsilon} \widetilde{\nabla_{x'} v_\varepsilon} \rightharpoonup \int_{Y_{\mathcal{I}}(x'')} \nabla_y v_1 \qquad in\ L^2(\Omega^+)^m,$$
$$\chi_{\mathcal{I}_\varepsilon} \widetilde{\nabla_{x''} v_\varepsilon} \rightharpoonup \mathbf{0} \qquad in\ L^2(\Omega^+)^{n-m},$$
$$v_\varepsilon \rightharpoonup v \qquad in\ H^1(\Omega^-).$$

For u_ε the convergence results are exactly same as in Theorem 1.

Proof The proof will be accomplished in several steps.
Step 1: (Weak convergences of unfolded sequences) Since the convergence results for state are exactly similar to Theorem 1, we only focus on convergence results for the adjoint state v_ε.

To obtain required bound, in the adjoint variational form consider $\phi = v_\varepsilon$ as a test function to get

$$\begin{aligned}\int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) A\nabla v_\varepsilon \nabla v_\varepsilon + (k'(u_\varepsilon)v_\varepsilon + v_\varepsilon)v_\varepsilon \\ = \int_{\Omega_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\Omega^-}\right) B\nabla u_\varepsilon \nabla v_\epsilon\end{aligned}$$

Now using the uniform bound of B and Young's inequality in RHS, and the lower bound for k' in LHS, we get

$$\begin{aligned}&\left\| \chi_{\mathcal{C}_\varepsilon} \nabla v_\varepsilon \right\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^2 \left\| \chi_{\mathcal{I}_\varepsilon} \nabla v_\varepsilon \right\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla v_\varepsilon\|^2_{L^2(\Omega^-)} + \|v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \\ &\leqslant C(\varepsilon^2 \|\chi_{\mathcal{I}_\varepsilon} \nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\chi_{\mathcal{C}_\varepsilon} \nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}) + \frac{1}{2}(\left\| \chi_{\mathcal{C}_\varepsilon} \nabla v_\varepsilon \right\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^2 \left\| \chi_{\mathcal{I}_\varepsilon} \nabla v_\varepsilon \right\|^2_{L^2(\Omega_\varepsilon)}).\end{aligned}$$

Now using the bound for the state u_ε in the above inequality, we obtain

$$\left\| \chi_{\mathcal{C}_\varepsilon} \nabla v_\varepsilon \right\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^2 \left\| \chi_{\mathcal{I}_\varepsilon} \nabla v_\varepsilon \right\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla v_\varepsilon\|^2_{L^2(\Omega^-)} + \|v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leqslant C. \tag{14}$$

As we did earlier, we do not have uniform bound of v_ε in $H^1(\Omega_\varepsilon)$ due to high contrasting diffusivity. More precisely, the bound of v_ε in $H^1(\mathcal{I}_\varepsilon)$ is of order ε^{-1}. Hence, we need to analyze the convergence of v_ε in $\mathcal{C}_\varepsilon$ and $\mathcal{I}_\varepsilon$ separately. This is done via the unfolding operator.

From (14) and by the properties of unfolding operator and weak compactness of Hilbert spaces, there exist $v_0 \in L^2(\Omega_U)$, $\mathbf{w} \in L^2(\Omega_{\mathcal{C}})^n$, $\mathbf{z} \in L^2(\Omega_{\mathcal{I}})^n$ and $u^- \in H^1(\Omega^-)$ such that

$$\begin{aligned} T^\varepsilon(v_\varepsilon) &\rightharpoonup v_0 && \text{weakly in } L^2(\Omega_U),\\ T^\varepsilon_{\mathcal{C}}(\nabla v_\varepsilon) &\rightharpoonup \mathbf{w} && \text{weakly in } (L^2(\Omega_{\mathcal{C}}))^n,\\ \varepsilon T^\varepsilon_{\mathcal{I}}(\nabla v_\varepsilon) &\rightharpoonup \mathbf{z} && \text{weakly in } (L^2(\Omega_{\mathcal{I}}))^n,\\ v_\varepsilon &\rightharpoonup v^- && \text{weakly in } H^1(\Omega^-).\end{aligned} \tag{15}$$

For convenience, we denote $\mathbf{w} = (\mathbf{w}_{x'}, \mathbf{w}_{x''})$ and $\mathbf{z} = (\mathbf{z}_{x'}, \mathbf{z}_{x''})$. Following the same path as in the *Step 1* of the proof of Theorem 1, we have

$$\mathbf{w}_{x''} = \nabla_{x''} v_0, \quad \mathbf{z}_{x''} = 0 \quad \text{and} \quad \mathbf{z}_{x'} = \nabla_y v_0.$$

To identify $\mathbf{w}_{x'}$, following the same idea as in *Step 1* of the proof of Theorem 1, we use the following test function $\phi^\varepsilon(x) = \varepsilon\phi(x)\psi\left(\frac{x'}{\varepsilon}\right)$, where $\phi \in C_c^\infty(\Omega^+)$ and $\psi \in C^\infty_{per}((0,1)^m))$ with $\psi(y) = \lambda' \cdot y$ in $\mathcal{C}$ where $\lambda' \in \mathbb{R}^m$ is fixed but arbitrary, as a test function in the adjoint varitional form to obtain

$$\int_{\Omega_{\mathcal{C}}} \phi(x) A\mathbf{w} \begin{bmatrix}\lambda' \\ 0\end{bmatrix} = \int_{\Omega_{\mathcal{C}}} B \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix} \nabla_{x''} u \begin{bmatrix}\lambda' \\ 0\end{bmatrix}.$$

Since $\phi \in C_c^\infty(\Omega^+)$ and $\lambda \in \mathbb{R}^{m-1}$ are arbitrary, we have

$$\int_{Y_{\mathcal{C}}(x'')} \mathbf{w}_{x'} = \int_{Y_{\mathcal{C}}(x'')} A_1^{-1}\left((B_2 - B_1 A_1^{-1} A_2)\nabla_{x''} u - A_2 \nabla_{x''} v_0\right).$$

Now for any $\phi \in C_c^\infty(\Omega_{\mathcal{C}})$, we have

$$\int_{\Omega_{\mathcal{C}}} \frac{\partial v_0}{\partial y_i}\phi = -\int_{\Omega_{\mathcal{C}}} v_0 \frac{\partial \phi}{\partial y_i} = -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} T_{\mathcal{C}}^{\varepsilon}(v_\varepsilon)\frac{\partial \phi}{\partial y_i} = -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} \frac{\partial}{\partial y_i} T_{\mathcal{C}}^{\varepsilon}(v_\varepsilon)\phi$$
$$= -\lim_{\varepsilon\to 0}\int_{\Omega_{\mathcal{C}}} \varepsilon T_{\mathcal{C}}^{\varepsilon}\left(\frac{\partial v_\varepsilon}{\partial x_i'}\right)\phi = 0,$$

which shows that v_0 is independent of y in $\Omega_{\mathcal{C}}$. That is, there exists $v^+ \in L^2(\Omega^+)$ such that $v_0 = v^+$ in $\Omega_{\mathcal{C}}$. Define $v_1 = v_0 - v^+$. Then we can write v_0 as $v_0 = v^+ + v_1$ with $v^+ \in L^2(\Omega^+)$ and v_1 vanishes on $\Omega_{\mathcal{C}}$.

Step 2 (Interface Condition): Similar to the trace equality at the interface derived in the proof of Theorem 1, we also have $v^+ = v^-$ on Γ_0. Define

$$v = \chi_{\Omega^+}v^+ + \chi_{\Omega^-}v^-.$$

Since $\nabla_{x''}v^+ \in L^2(\Omega^+)^{n-m}$ and $v^- \in H^1(\Omega^-)$, the interface condition gives $v \in W(\Omega)$.

Hence we can rewrite (15) as

$$\begin{aligned} T^{\varepsilon}(v_\varepsilon) &\rightharpoonup v + v_1 && \text{weakly in } L^2(\Omega_U),\\ \varepsilon T_{\mathcal{I}}^{\varepsilon}(\nabla v_\varepsilon) &\rightharpoonup \begin{bmatrix}\nabla_y v_1\\ \mathbf{0}\end{bmatrix} && \text{weakly in } (L^2(\Omega_{\mathcal{I}}))^n,\\ v_\varepsilon &\rightharpoonup v && \text{weakly in } H^1(\Omega^-), \end{aligned}$$

$$\& \int_{Y_{\mathcal{C}}(x'')} T_{\mathcal{C}}^{\varepsilon}(\nabla v_\varepsilon) \rightharpoonup |Y_{\mathcal{C}}(x'')|\left(\begin{bmatrix}A_1^{-1}(B_2 - B_1A_1^{-1}A_2)\\ \mathbf{0}\end{bmatrix}\nabla_{x''}u + \begin{bmatrix}-A_1^{-1}A_2\\ I\end{bmatrix}\nabla_{x''}v\right)$$

weakly in $(L^2(\Omega^+))^n$.

Step 4 (Limit Problem): We now derive the limit equations using the results obtained in the previous steps. Let $\phi_\varepsilon(x) = \phi(x) + \phi_1\left(x, \frac{x'}{\varepsilon}\right)$, where $\phi \in C^1(\bar{\Omega})$ and $\phi_1 \in C_c^\infty(\Omega_U)$ with 1 periodic in y variable and $\phi_1 = 0$ on $\Omega_{\mathcal{C}}$. Now using ϕ_ε as a test function in (1), applying unfolding operator on both sides and letting $\varepsilon \to 0$ to get

$$\begin{aligned} &\int_{\Omega_{\mathcal{C}}} A\left(\begin{bmatrix}A_1^{-1}(B_2 - B_1A_1^{-1}A_2)\\ \mathbf{0}\end{bmatrix}\nabla_{x''}u + \begin{bmatrix}-A_1^{-1}A_2\\ I\end{bmatrix}\nabla_{x''}v\right)\nabla\phi + (k'(u)v + v)\phi\\ &\quad + \int_{\Omega_{\mathcal{I}}} A_1\nabla_y v_1\nabla_y\phi_1 + (k'(u+u_1)(v+v_1) + v + v_1)(\phi+\phi_1)\\ &\quad + \int_{\Omega^-} A\nabla v\nabla\phi + (k'(u)v + v)\phi\\ &= \int_{\Omega_{\mathcal{C}}} B\begin{bmatrix}-A_1^{-1}A_2\\ I\end{bmatrix}\nabla_{x''}u\nabla\phi + \int_{\Omega_{\mathcal{I}}} B_1\nabla_y u_1\nabla_y\phi_1 + \int_{\Omega^-} B\nabla u\nabla\phi. \end{aligned}$$

By simplifying, we get

$$\begin{aligned}
&\int_{\Omega_{\mathcal{C}}} (A_4 - A_3 A_1^{-1} A_2)\nabla_{x''} v \nabla_{x''}\phi + (k'(u)v + v)\phi \\
&\quad + \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y v_1 \nabla_y \phi_1 + (k'(u+u_1)(v+v_1) + v + v_1)(\phi + \phi_1) \\
&\quad + \int_{\Omega^-} A\nabla v \nabla \phi + (k'(u)v + v)\phi \qquad (16) \\
&= \int_{\Omega_{\mathcal{C}}} \left(-B_3 A_1^{-1} A_2 + A_3 A_1^{-1} B_1 A_1^{-1} A_2 + B_4 - A_3 A_1^{-1} B_2\right) \nabla_{x''} u \nabla \phi \\
&\quad + \int_{\Omega_{\mathcal{I}}} B_1 \nabla_y u_1 \nabla_y \phi_1 + \int_{\Omega^-} B\nabla u \nabla \phi.
\end{aligned}$$

By density arguments we have (16) for all $(\phi, \phi_1) \in W(\Omega) \times W(\Omega_U)$.

Average out and simplify to get the variational form: Find $(v, v_1) \in W(\Omega) \times W(\Omega_U)$ such that

$$\begin{aligned}
&\int_{\Omega^+} A_0 \nabla_{x''} v \nabla_{x''}\phi + |Y_{\mathcal{C}}(x'')|(k'(u)v + v)\phi + \int_{\Omega^-} A\nabla v \nabla \phi + (k'(u)v + v)\phi \\
&\quad + \int_{\Omega_{\mathcal{I}}} A_1 \nabla_y v_1 \nabla_y \phi_1 + (k'(u+u_1)(v+v_1) + v + v_1)(\phi + \phi_1) \\
&= \int_{\Omega^+} B_0 \nabla_{x''} u \nabla \phi + \int_{\Omega_{\mathcal{I}}} B_1 \nabla_y u_1 \nabla_y \phi_1 + \int_{\Omega^-} B\nabla u \nabla \phi
\end{aligned}$$

for all $(\phi, \phi_1) \in W(\Omega) \times W(\Omega_U)$, where the coefficients A_0 and B_0 are given by

$$\begin{aligned}
A_0 &= \left|Y_{\mathcal{C}}(x'')\right| \left(A_4 - A_3 A_1^{-1} A_2\right), \\
B_0 &= \left|Y_{\mathcal{C}}(x'')\right| \left(-B_3 A_1^{-1} A_2 + A_3 A_1^{-1} B_1 A_1^{-1} A_2 + B_4 - A_3 A_1^{-1} B_2\right).
\end{aligned}$$

We already proved in Theorem 1 that A_0 is elliptic by rewriting it in a different form. We can rewrite B_0 also in a compact form as

$$B_0 = \left|Y_{\mathcal{C}}(x'')\right| \left(\left[-A_3 A_1^{-1} \;\; I\right] B \left[-A_3 A_1^{-1} \;\; I\right]^t\right)$$

and continue the same steps as we do to get the ellipticity of A_0 and we can prove that B_0 is uniformly elliptic. □

3 Circular Oscillating Domain

In this section we consider the homogenization of optimal control problem on a circular oscillating domain. In this section we are dealing with only two dimensional domain. For $x = (x_1, x_2)$, we denote and define $\bar{x}$ as $\bar{x} = (-x_2, x_1)$.

3.1 Domain description

For each $\varepsilon > 0$, we consider an ε-depended bounded domain $\mathcal{O}_\varepsilon \subset \mathbb{R}^n$ with oscillating boundary (see Figure 1]). Below, we describe the domain under consideration.

Let $0 < r_0 < r_1 < r_2$ be real numbers and for the simplicity of presentation, we take $\varepsilon = \frac{1}{n}, n \in \mathbb{N}$. Let Λ be a connected open subset of $\mathbb{R}^2$ which is contained in the annulus $\mathcal{O}^+ = \{(r,\theta) : r_0 < r < r_1\}$ with Lipschitz boundary is the reference cell. It consists of two parts: namely insulating part $\mathcal{I}$ and high conductive region $\mathcal{C}$. Now define

$$\begin{aligned}
&\mathcal{I}_\varepsilon = \left\{(r,\theta) \in \mathcal{O}^+ : \left(r, \left\{\tfrac{\theta}{\varepsilon}\right\}_{2\pi}\right) \in \mathcal{I}\right\}, \quad \mathcal{C}_\varepsilon = \left\{(r,\theta) \in \mathcal{O}^+ : \left(r, \left\{\tfrac{\theta}{\varepsilon}\right\}_{2\pi}\right) \in \mathcal{C}\right\},\\
&\mathcal{O}_\varepsilon^+ = \mathrm{int}\overline{(\mathcal{I}_\varepsilon \cup \mathcal{C}_\varepsilon)} = \left\{(r,\theta) \in \mathcal{O} : \left(r, \left\{\tfrac{\theta}{\varepsilon}\right\}_{2\pi}\right) \in \Lambda\right\}, \quad \mathcal{O}^- = \{(r,\theta) : r_1 < r < r_2\},\\
&\mathcal{O}_\varepsilon = \mathrm{int}\overline{\left(\mathcal{O}_\varepsilon^+ \cup \mathcal{O}^-\right)} \quad \text{and} \quad \mathcal{O} = \mathrm{int}\overline{\left(\mathcal{O}^+ \cup \mathcal{O}^-\right)},
\end{aligned}$$

where $\mathcal{O}_\varepsilon^+$ is the inner oscillating part with $\mathcal{I}_\varepsilon$ and $\mathcal{C}_\varepsilon$ as its insulating and conducting parts respectively. The domain $\mathcal{O}^-$ is the outer fixed part, $\mathcal{O}_\varepsilon$ is the oscillating domain (see Figure 1) and $\mathcal{O}$ is the limit domain. Also Γ_a, Γ_b are inner and outer boundaries of $\mathcal{O}$ and Γ_0 is the interface. Here $\left\{\frac{\theta}{\varepsilon}\right\}_{2\pi} = \frac{\theta}{\varepsilon} - \left[\frac{\theta}{2\pi\varepsilon}\right]2\pi$, where $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional parts. For $r \in (r_0, r_1)$, define

$$\begin{aligned}
&Y_{\mathcal{I}}(r) = \{\theta \in [0,2\pi] : (r,\theta) \in \mathcal{I}\}, \quad Y_{\mathcal{C}}(r) = \{\theta \in [0,2\pi] : (r,\theta) \in \mathcal{C}\},\\
&Y(r) = Y_{\mathcal{I}}(r) \cup Y_{\mathcal{C}}(r) = \{\theta \in [0,2\pi] : (r,\theta) \in \Lambda\}.
\end{aligned}$$

The domains mentioned above are required to define the unfolded domain and unfolding operator. We assume the following properties:

(1) The set $Y_{\mathcal{C}}(r)$ is connected for all $r \in (r_0, r_1)$.
(2) There exists $\rho > 0$ such that $0 < \rho \le |(Y_{\mathcal{C}}(r))| < 2\pi$ for all $r \in (r_0, r_1)$.

We also define the following domains which is needed to define the homogenized limit problem:

$$\begin{aligned}
\mathcal{O}_U &= \{(r,\theta,\tau) \mid \theta \in (0,2\pi), r \in (r_0,r_1), \tau \in Y(r)\},\\
\mathcal{O}_{\mathcal{C}} &= \{(r,\theta,\tau) \mid \theta \in (0,2\pi), r \in (r_0,r_1), \tau \in Y_{\mathcal{C}}(r)\},\\
\mathcal{O}_{\mathcal{I}} &= \{(r,\theta,\tau) \mid \theta \in (0,2\pi), r \in (r_0,r_1), \tau \in Y_{\mathcal{I}}(r)\}.
\end{aligned}$$

3.2 Homogenization

Let $A = [a_{ij}]_{2\times 2}$
be a 2×2 matrix where the entries $a_{ij} \in L^\infty(\mathcal{O})$. Also A is uniformly elliptic and bounded in $\mathcal{O}$, that is, there exist constants $\alpha, \beta > 0$ such that

$$\langle A(x)\lambda, \lambda\rangle \ge \alpha|\lambda|^2 \quad \text{and} \quad |A(x)\lambda| \le \beta|\lambda|$$

for all $\lambda \in \mathbb{R}^2$ and $a.e$ in $\mathcal{O}$. Let $k : \mathbb{R} \to \mathbb{R}$ be a C^2 real-valued function such that

$$0 < C_1 \leqslant k'(t) \leqslant C_2, k(0) = 0 \text{ and } k'' \text{ is bounded.}$$

Consider the following problem in the domain $\mathcal{O}_\varepsilon$:

$$\begin{cases}
-\mathrm{div}\left(\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right)A\nabla u_\varepsilon\right) + k(u_\varepsilon) + u_\varepsilon = f \text{ in } \mathcal{O}_\varepsilon,\\
\qquad\qquad\qquad\qquad\qquad\qquad A\nabla u_\varepsilon \cdot \nu^\varepsilon = 0 \text{ on } \partial\mathcal{O}_\varepsilon.
\end{cases}$$

Here $f \in L^2(\mathcal{O})$ is a given function, ν^ε is the outward normal vector on $\partial\mathcal{O}_\varepsilon$. Corresponding variational form is given by

$$\begin{cases} \text{find } u_\varepsilon \in H^1(\mathcal{O}_\varepsilon), \text{ such that} \\ \displaystyle\int_{\mathcal{O}_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) A\nabla u_\varepsilon \nabla\phi + k(u_\varepsilon)\phi + u_\varepsilon\phi = \int_{\mathcal{O}_\varepsilon} f\phi \\ \text{for all } \phi \in H^1(\mathcal{O}_\varepsilon). \end{cases} \tag{17}$$

For each $\varepsilon > 0$, we have the existence of unique $u_\varepsilon \in H^1(\mathcal{O}_\varepsilon)$ by the Browder - Minty theorem (see [41]). Our aim is to study the asymptotic behavior of u_ε as $\varepsilon \to 0$. Let us describe the limit problem.

Limit problem: To define the solution of the homogenized variational form, we need appropriate function spaces, which we will define now. For any function ϕ defined on $\mathcal{O}$, we may write $\phi = \phi^+\chi_{\mathcal{O}^+} + \phi^-\chi_{\mathcal{O}^-} = (\phi^+, \phi^-)$ throughout this article. Define

(1) $V(\mathcal{O}) = \left\{\psi \in L^2(\mathcal{O}) : (x\cdot\nabla\psi) \in L^2(\mathcal{O}) \text{ and } \psi \in H^1(\mathcal{O}^-)\right\}$, with the inner product

$$\langle\phi,\psi\rangle_{V(\mathcal{O})} = \langle\phi,\psi\rangle_{L^2(\mathcal{O}^+)} + \langle(x\cdot\nabla\phi), (x\cdot\nabla\psi)\rangle_{L^2(\mathcal{O}^+)} + \langle\phi,\psi\rangle_{H^1(\mathcal{O}^-)}.$$

(2) $V(\mathcal{O}_U) = \left\{\phi \in L^2(\mathcal{O}_U) : \phi = 0 \text{ in } \mathcal{O}_\mathcal{C} \text{ and } \frac{\partial\phi}{\partial\tau} \in L^2(\mathcal{O}_U)\right\}$ with inner product

$$\langle\phi,\psi\rangle_{V(\mathcal{O}_U)} = \langle\phi,\psi\rangle_{L^2(\mathcal{O}_U)} + \left\langle\frac{\partial\phi}{\partial\tau}, \frac{\partial\psi}{\partial\tau}\right\rangle_{L^2(\mathcal{O}_U)}.$$

Note that since x is strictly away from the origin, $V(\mathcal{O})$ is a Hilbert space. Now we are in a position to define the two-scale limit problem: Given $f \in L^2(\mathcal{O})$, find $(u, u_1) \in V(\mathcal{O}) \times V(\mathcal{O}_U)$ such that

$$\begin{cases} \displaystyle\int_{\mathcal{O}^+} \frac{a_0(x)}{|x|^2}(x\cdot\nabla u)(x\cdot\nabla\phi) + |Y_\mathcal{C}(|x|)|(k(u)+u)\phi \\ \displaystyle\quad + \int_{\Omega_\mathcal{I}} \frac{\bar{x}^t A\bar{x}}{|x|^2}\frac{\partial u_1}{\partial\tau}\frac{\partial\phi}{\partial\tau} + (k(u+u_1)+u+u_1)(\phi+\phi_1) \\ \displaystyle\quad + \int_{\mathcal{O}^-} A\nabla u\nabla\phi + (k(u)+u)\phi \\ \displaystyle = \int_{\mathcal{O}^+} |Y_\mathcal{C}(|x|)| f\phi + \int_{\mathcal{O}_\mathcal{I}} f(\phi+\phi_1) + \int_{\mathcal{O}^-} f\phi, \end{cases} \tag{18}$$

for all $(\phi, \phi_1) \in V(\mathcal{O}) \times V(\mathcal{O}_U)$, where

$$a_0(x) = |Y_\mathcal{C}(|x|)|\left(\frac{\det(A)}{\bar{x}^t A\bar{x}}\right).$$

Theorem 5 *Let u_ε and (u, u_1) be the solutions of (17) and (18) respectively. Then as $\varepsilon \to 0$, we have the following convergences (for the whole sequence ε) weakly in $L^2(\mathcal{O}^+)$*

$$\begin{aligned}
\widetilde{u_\varepsilon} &\rightharpoonup |Y(|x|)|u + \int_{Y_{\mathcal{I}}(|x|)} u_1, \\
\chi_{\mathcal{C}_\varepsilon}\left(x \cdot \widetilde{\nabla u_\varepsilon}\right) &\rightharpoonup |Y_{\mathcal{C}}(|x|)|\,(x \cdot \nabla u)\,, \\
\chi_{\mathcal{C}_\varepsilon}\left(\bar{x} \cdot \widetilde{\nabla u_\varepsilon}\right) &\rightharpoonup - Y_{\mathcal{C}}(|x|)\frac{\bar{x}^t A x}{\bar{x}^t A \bar{x}}\,(x \cdot \nabla u)\,, \\
\varepsilon\chi_{\mathcal{I}_\varepsilon}\left(x \cdot \widetilde{\nabla u_\varepsilon}\right) &\rightharpoonup 0, \\
\varepsilon\chi_{\mathcal{I}_\varepsilon}\left(\bar{x} \cdot \widetilde{\nabla u_\varepsilon}\right) &\rightharpoonup \int_{Y_{\mathcal{I}}(|x|)} \frac{\partial u_1}{\partial \tau} d\tau.
\end{aligned}$$

Also we have

$$u_\varepsilon \rightharpoonup u \text{ weakly in } H^1(\mathcal{O}^-).$$

Proof The proof largely follows the structure of Theorem 1, but we must adapt our approach to account for the circular oscillations in the domain. To begin, we will express equation (17) in polar coordinates as follows: Find $u_\varepsilon \in H^1(\mathcal{O}_\varepsilon)$ such that

$$\begin{aligned}
&\int_{\mathcal{O}_\varepsilon^+}\left(\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right)\bar{A}\begin{bmatrix}\frac{\partial u_\varepsilon}{\partial r}\\ \frac{\partial u_\varepsilon}{\partial \theta}\end{bmatrix}\begin{bmatrix}\frac{\partial \phi}{\partial r}\\ \frac{\partial \phi}{\partial \theta}\end{bmatrix} + k(u_\varepsilon)\phi + u_\varepsilon\phi\right) r dr d\theta \\
&\quad + \int_{\mathcal{O}^-} A\nabla u_\varepsilon \nabla\phi + \left(k(u_\varepsilon) + u_\varepsilon\right)\phi dx = \int_{\mathcal{O}_\varepsilon} f\phi dx,
\end{aligned} \tag{19}$$

for all $\phi \in H^1(\mathcal{O}_\varepsilon)$, with $\bar{A} = [\bar{a}_{ij}]_{2\times 2} = X^t A X$, where

$$X = \begin{bmatrix}\cos\theta & -\frac{1}{r}\sin\theta\\ \sin\theta & \frac{1}{r}\cos\theta\end{bmatrix} = \begin{bmatrix}\frac{x_1}{|x|} & -\frac{x_2}{|x|^2}\\ \frac{x_2}{|x|} & \frac{x_1}{|x|^2}\end{bmatrix} = \begin{bmatrix}\frac{x}{|x|} & \frac{\bar{x}}{|x|^2}\end{bmatrix}.$$

Now the idea is to pass to the limit as $\varepsilon \to 0$ in (19) using polar unfolding operator(see [38]). The main difficulty is to pass to the limit in the non linear term. Ultimately, we will arrive at the following limit problem in polar form, along with the corresponding convergences in polar coordinates.

Limit problem polar form: Find $(u, u_1) \in V(\mathcal{O}) \times V(\mathcal{O}_U)$

$$\begin{aligned}
&\int_{\mathcal{O}^+} a_0\frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r} + |Y_{\mathcal{C}}(r)|(k(u) + u)\phi + \int_{\mathcal{O}^-} A\nabla u\nabla\phi + (k(u) + u)\phi \\
&\quad + \int_{\mathcal{O}_{\mathcal{I}}} \bar{a}_{22}\frac{\partial u_1}{\partial \tau}\frac{\partial \phi_1}{\partial \tau} + (k(u + u_1) + u + u_1)(\phi + \phi_1) \\
&= \int_{\mathcal{O}^+} |Y_{\mathcal{C}}(r)| f\phi + \int_{\mathcal{O}^-} f\phi + \int_{\mathcal{O}_{\mathcal{I}}} f(\phi + \phi_1),
\end{aligned} \tag{20}$$

for all $(\phi, \phi_1) \in V(\mathcal{O}) \times V(\mathcal{O}_U)$ with

$$a_0(x) = |Y_{\mathcal{C}}(r)|\left(\frac{\det(A)}{\bar{a}_{22}}\right).$$

Convergences in polar form: Let u_ε, (u, u_1) be solutions of (19),(20), then we can arrive the following convergences weakly in $L^2(\mathcal{O})^+$ as $\varepsilon \to 0$

$$\widetilde{u_\varepsilon} \rightharpoonup |Y(r)|u + \int_{Y_{\mathcal{I}}(r)} u_1,$$
$$\chi_{\mathcal{C}_\varepsilon} \frac{\widetilde{\partial u_\varepsilon}}{\partial r} \rightharpoonup |Y_{\mathcal{C}}(r)| \frac{\partial u}{\partial r}, \qquad \chi_{\mathcal{C}_\varepsilon} \frac{\widetilde{\partial u_\varepsilon}}{\partial \theta} \rightharpoonup -Y_{\mathcal{C}}(r) \frac{\bar{a}_{21}}{\bar{a_{22}}} \frac{\partial u}{\partial r},$$
$$\varepsilon \chi_{\mathcal{I}_\varepsilon} \frac{\widetilde{\partial u_\varepsilon}}{\partial r} \rightharpoonup 0, \qquad \varepsilon \chi_{\mathcal{I}_\varepsilon} \frac{\widetilde{\partial u_\varepsilon}}{\partial \theta} \rightharpoonup \int_{Y_{\mathcal{I}}(r)} \frac{\partial u_1}{\partial \tau} d\tau.$$

Also we have

$$u_\varepsilon \rightharpoonup u^- \text{ weakly in } H^1(\mathcal{O}^-).$$

Now use the polar transformations

$$r \frac{\partial u}{\partial r} = x \cdot \nabla u \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \bar{x} \cdot \nabla u$$

in the limit problem and convergences in polar form to limit problem (18) and Theorem 5. □

Theorem 6 ***(Corrector Results)*** *Let u_ε and (u, u_1) be the solutions of (17) and (18) respectively. Define*

$$p_\varepsilon : \mathcal{O}_\varepsilon^+ \to \mathbb{R} \;\; by \;\; p_\varepsilon(r, \theta) = u(r, \theta) + u_1\left(r, \theta, \left\{\tfrac{\theta}{\varepsilon}\right\}\right) \;\; and$$
$$\mathbf{p}_\varepsilon : \mathcal{O}_\varepsilon^+ \to \mathbb{R} \;\; by \;\; \mathbf{p}_\varepsilon(r, \theta) = \frac{\partial u_1}{\partial \tau}\left(r, \theta, \left\{\tfrac{\theta}{\varepsilon}\right\}\right).$$

Then, we have the following norm convergences as $\varepsilon \to 0$,

$$\left\| X^{-1} \nabla u_\varepsilon - \begin{bmatrix} 1 \\ \frac{\bar{x}^t A x}{\bar{x}^t A \bar{x}} \end{bmatrix} (x \cdot \nabla u) \right\|_{L^2(\mathcal{C}_\varepsilon)^2} + \left\| \varepsilon X^{-1} \nabla u_\varepsilon - \begin{bmatrix} 0 \\ \mathbf{p}_\varepsilon \end{bmatrix} \right\|_{L^2(\mathcal{I}_\varepsilon)^2} \longrightarrow 0,$$
$$\& \quad \|u_\varepsilon - p_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \|u_\varepsilon - u\|_{H^1(\Omega^-)} \longrightarrow 0.$$

3.3 Optimal Control

An optimal control problem in $\mathcal{O}_\varepsilon$ governed by a semi-linear elliptic PDE as described in the previous section can also be investigated. We briefly describe it without elaborating the details. Let $A = [a_{ij}]_{2\times 2}$ and $B = [b_{ij}]_{2\times 2}$ be 2×2 symmetric matrices that are uniformly elliptic and bounded. Let $\omega \subset\subset \mathcal{O}^-$ be an open set and admissible control set is $L^2(\omega)$. Consider the following minimization problem in $\mathcal{O}_\varepsilon$

$$\text{Minimize: } J_\varepsilon(u, \theta) = \frac{1}{2} \int_{\mathcal{O}_\varepsilon} \left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) B \nabla u \nabla u + \frac{\beta}{2} \int_{\mathcal{O}_\varepsilon} \chi_\omega |\theta|^2, \tag{21}$$

where (u, θ) satisfies the following system,

$$\begin{cases} -\text{div}\left(\left(\varepsilon^2 \chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) A \nabla u\right) + k(u) + u = f + \chi_\omega \theta \text{ in } \mathcal{O}_\varepsilon, \\ \qquad\qquad A \nabla u \cdot \nu_\varepsilon = 0 \text{ on } \partial \mathcal{O}_\varepsilon, \end{cases}$$

with $f \in L^2(\mathcal{O})$.

As we did in the previous section, our approach to addressing the problem involves transforming the minimization problem into an optimality system and then homogenize the system.

From findings on semi-linear optimal control problems (refer to [11, 43]), the solutions $(u_\varepsilon, \theta_\varepsilon)$ of (21) satisfy the optimality system:

$$\begin{cases} -\operatorname{div}\left(\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) A\nabla u_\varepsilon\right) + k(u_\varepsilon) + u_\varepsilon = f + \chi_\omega\theta_\varepsilon & \text{in } \mathcal{O}_\varepsilon, \\ -\operatorname{div}\left(\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) A\nabla v_\varepsilon\right) + k'(u_\varepsilon)v_\varepsilon + v_\varepsilon & \\ \quad = -\operatorname{div}\left(\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) B\nabla u_\varepsilon\right) & \text{in } \mathcal{O}_\varepsilon, \\ A\nabla u_\varepsilon \cdot \nu_\varepsilon = 0, \quad A\nabla v_\varepsilon \cdot \nu_\varepsilon = B\nabla u_\varepsilon & \text{on } \partial\mathcal{O}_\varepsilon, \\ \theta_\varepsilon = -\chi_\omega \frac{1}{\beta} v_\varepsilon. & \end{cases}$$

Corresponding variational form is as follows: Given $f \in L^2(\mathcal{O})$, find $(u_\varepsilon, v_\varepsilon) \in H^1(\mathcal{O}_\varepsilon) \times H^1(\mathcal{O}_\varepsilon)$ such that

$$\begin{cases} \int_{\mathcal{O}_\varepsilon} \left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) A\nabla u_\varepsilon \nabla\phi + (k(u_\varepsilon) + u_\varepsilon)\phi = \int_{\mathcal{O}_\varepsilon} (f + \chi_\omega\theta_\varepsilon)\phi, \\ \int_{\mathcal{O}_\varepsilon} \left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) A\nabla v_\varepsilon \nabla\psi + (k'(u_\varepsilon)v_\varepsilon + v_\varepsilon)\psi \\ \quad = \int_{\mathcal{O}_\varepsilon} \left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) B\nabla u_\varepsilon \nabla\psi, \\ \theta_\varepsilon = -\frac{1}{\beta}\chi_\omega v_\varepsilon. \end{cases} \tag{22}$$

for all $(\phi, \psi) \in H^1(\mathcal{O}_\varepsilon) \times H^1(\mathcal{O}_\varepsilon)$.

Our task now is to homogenize the optimality system (22). In other words, we aim to investigate the asymptotic behavior of $(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \to 0$. Similar to our previous approach, we will first establish the limit problem and subsequently present the homogenization results.

Limit system: Given $f \in L^2(\Omega)$, find $((u, u_1), (v, v_1)) \in (V(\mathcal{O}) \times V(\mathcal{O}_U))^2$ such that

$$\begin{cases} \int_{\mathcal{O}^+} \frac{a_0(x)}{|x|^2} (x \cdot \nabla u)(x \cdot \nabla\phi) + |Y_{\mathcal{C}}(|x|)|(k(u) + u)\phi \\ \quad + \int_{\Omega_{\mathcal{I}}} \frac{\bar{x}^t A\bar{x}}{|x|^2} \frac{\partial u_1}{\partial\tau}\frac{\partial\phi}{\partial\tau} + (k(u + u_1) + u + u_1)(\phi + \phi_1) \\ \quad + \int_{\mathcal{O}^-} A\nabla u\nabla\phi + (k(u) + u)\phi \\ \quad = \int_{\mathcal{O}^+} |Y_{\mathcal{C}}(|x|)| f\phi + \int_{\mathcal{O}_{\mathcal{I}}} f(\phi + \phi_1) \int_{\mathcal{O}^-} (f + \chi_\omega\theta)\phi, \\ \int_{\mathcal{O}^+} \frac{a_0\,(x)}{|x|^2} (x \cdot \nabla v)(x \cdot \nabla\phi) + |Y(|x|)| \left(k'(u)v + v\right)\phi \\ \quad + \int_{\Omega_{\mathcal{I}}} \frac{\bar{x}^t A\bar{x}}{|x|^2} \frac{\partial v_1}{\partial\tau}\frac{\partial\phi}{\partial\tau} + (k'(u + u_1)(v + v_1) + v + v_1)(\phi + \phi_1) \\ \quad + \int_{\mathcal{O}^-} A\nabla v\nabla\phi + (k'(u)v + v)\phi \\ \quad = \int_{\mathcal{O}^+} \frac{b_0\,(x)}{|x|^2} (x \cdot \nabla u)(x \cdot \nabla\phi) + \int_{\mathcal{O}_{\mathcal{I}}} \frac{\bar{x}^t B\bar{x}}{|x|^2} \frac{\partial u_1}{\partial\tau}\frac{\partial\phi}{\partial\tau} + \int_{\mathcal{O}^-} B\nabla u\nabla\phi, \end{cases} \tag{23}$$

for all $(\psi, \phi) \in V(\mathcal{O}) \times V(\mathcal{O})$ with $\theta = -\frac{1}{\beta}\chi_\omega v$. Here

$$\begin{aligned} a_0 =& |Y_{\mathcal{C}}(|x|)| \left(\frac{\det(A)}{\bar{x}^t A\bar{x}}\right), \\ b_0 =& |Y_{\mathcal{C}}(|x|)| \left(X\begin{bmatrix} 1 \\ \frac{\bar{x}^t Ax}{\bar{x}^t A\bar{x}}\end{bmatrix}\right)^t B \left(X\begin{bmatrix} 1 \\ \frac{\bar{x}^t Ax}{\bar{x}^t A\bar{x}}\end{bmatrix}\right). \end{aligned}$$

The limit cost functional is given by:

$$J(u, u_1, \theta) = \int_{\mathcal{O}^+} \frac{a_0(x)}{|x|^2} (x\cdot\nabla u)(x\cdot\nabla u) + \int_{\Omega_{\mathcal{I}}} \frac{\bar{x}^t A\bar{x}}{|x|^2} \frac{\partial u_1}{\partial\tau}\frac{\partial u_1}{\partial\tau} + \frac{\beta}{2}\int_\Omega \chi_\omega|\theta|^2.$$

Theorem 7 *Let $(u_\varepsilon, v_\varepsilon)$, and $((u, u_1), (v, v_1))$, be solutions of (22) and (23) respectively. Define*

$$\begin{aligned} p_\varepsilon &: \mathcal{O}_\varepsilon^+ \to \mathbb{R} \quad by \quad p_\varepsilon(r,\theta) = u(r,\theta) + u_1\left(r, \theta, \left\{\tfrac{\theta}{\varepsilon}\right\}\right) \quad and \\ \mathbf{p}_\varepsilon &: \mathcal{O}_\varepsilon^+ \to \mathbb{R} \quad by \quad \mathbf{p}_\varepsilon(r,\theta) = \frac{\partial u_1}{\partial\tau}\left(r, \theta, \left\{\tfrac{\theta}{\varepsilon}\right\}\right). \end{aligned}$$

Then as $\varepsilon \to 0$, we have the following convergences.

$$\begin{aligned} \left\| X^{-1}\nabla u_\varepsilon - \begin{bmatrix} 1 \\ \frac{\bar{x}^t Ax}{\bar{x}^t A\bar{x}}\end{bmatrix}(x\cdot\nabla u)\right\|_{L^2(\mathcal{C}_\varepsilon)^2} + \left\|\varepsilon X^{-1}\nabla u_\varepsilon - \begin{bmatrix} 0 \\ \mathbf{p}_\varepsilon\end{bmatrix}\right\|_{L^2(\mathcal{I}_\varepsilon)^2} &\longrightarrow 0, \\ \|u_\varepsilon - p_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \|u_\varepsilon - u\|_{H^1(\Omega^-)} &\longrightarrow 0. \end{aligned}$$

Theorem 8 *Let $(u_\varepsilon, v_\varepsilon)$ and $((u, u_1), (v, v_1))$, be solutions of (22) and (23) respectively. Then as $\varepsilon \to 0$, we have the following convergences weakly in $L^2(\mathcal{O}^+)$:*

$$\begin{aligned} \widetilde{v_\varepsilon} &\rightharpoonup |Y(|x|)|v + \int_{Y_{\mathcal{I}}(|x|)} v_1, \\ \chi_{\mathcal{C}_\varepsilon}\left(x\cdot\widetilde{\nabla v_\varepsilon}\right) &\rightharpoonup |Y_{\mathcal{C}}(|x|)|\,(x\cdot\nabla v), \\ \chi_{\mathcal{C}_\varepsilon}\left(\bar{x}\cdot\widetilde{\nabla v_\varepsilon}\right) &\rightharpoonup |Y_{\mathcal{C}}(|x|)|\,(\gamma\,(x\cdot\nabla u) + \eta\,(x\cdot\nabla v)), \\ \chi_{\mathcal{I}_\varepsilon}\left(x\cdot\widetilde{\nabla v_\varepsilon}\right) &\rightharpoonup 0, \\ \chi_{\mathcal{I}_\varepsilon}\left(\bar{x}\cdot\widetilde{\nabla v_\varepsilon}\right) &\rightharpoonup \int_{Y_{\mathcal{I}}(|x|)} \frac{\partial v_1}{\partial\tau}, \end{aligned}$$

where

$$\gamma = \frac{\bar{x}^t\left(Bx\bar{x}^t A - Ax\bar{x}^t B\right)\bar{x}}{(\bar{x}^t A\bar{x})^2} \quad and \quad \eta = -\frac{\bar{x}^t Ax}{\bar{x}^t A\bar{x}}.$$

Moreover we have $v_\varepsilon \rightharpoonup v$ in $H^1(\mathcal{O}^-)$. The convergence results for u_ε are exactly same as in Theorem 5.

Proof Here also the proof follows similar analysis as we have done in theorem 5, we convert the adjoint equation in (22) to polar form as: Find $v_\varepsilon \in H^1(\mathcal{O}_\varepsilon)$ such that

$$\begin{aligned} &\int_{\mathcal{O}_\varepsilon}\left(\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right)\bar{A}\begin{bmatrix}\frac{\partial v_\varepsilon}{\partial r} \\ \frac{\partial v_\varepsilon}{\partial\theta}\end{bmatrix}\begin{bmatrix}\frac{\partial\psi}{\partial r} \\ \frac{\partial\psi}{\partial\theta}\end{bmatrix} + (k'(u_\varepsilon)v_\varepsilon + v_\varepsilon)\psi\right) r\,dr\,d\theta \\ &= \int_{\mathcal{O}_\varepsilon}\left(\varepsilon^2\chi_{\mathcal{I}_\varepsilon} + \chi_{\mathcal{C}_\varepsilon} + \chi_{\mathcal{O}^-}\right) B\nabla u_\varepsilon\nabla\psi\,dx, \end{aligned} \tag{24}$$

for all $\psi \in H^1(\mathcal{O}_\varepsilon)$ with $\bar{A} = [\bar{a}_{ij}]_{2\times 2} = X^t A X$ and $\bar{B} = [\bar{b}_{ij}]_{2\times 2} = X^t B X$ where

$$X = \begin{bmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{bmatrix} = \begin{bmatrix} \frac{x_1}{|x|} & -\frac{x_2}{|x|^2} \\ \frac{x_2}{|x|} & \frac{x_1}{|x|^2} \end{bmatrix} = \begin{bmatrix} \frac{x}{|x|} & \frac{\bar{x}}{|x|^2} \end{bmatrix}.$$

Now we pass to the limit as $\varepsilon \to 0$ using the polar unfolding operator and we reach the following limit problem and convergences in polar coordinates.

Adjoint limit equation polar from: Find $(v, v_1) \in V(\mathcal{O}) \times V(\mathcal{O}_U)$ such that

$$\begin{cases} \int_{\mathcal{O}^+} a_0 \frac{\partial v}{\partial r}\frac{\partial \psi}{\partial r} + |Y_{\mathcal{C}}(r)| \left(k'(u)v + v\right)\psi \\ \quad + \int_{\Omega_{\mathcal{I}}} \bar{a_{22}} \frac{\partial v_1}{\partial \tau}\frac{\partial \psi}{\partial \tau} + (k'(u+u_1)(v+v_1) + v + v_1)(\psi + \psi_1) \\ \quad + \int_{\mathcal{O}^-} A\nabla v \nabla \psi + (k'(u)v + v)\psi \\ = \int_{\mathcal{O}^+} b_0 \frac{\partial u}{\partial r}\frac{\partial \psi}{\partial r} + \int_{\mathcal{O}_{\mathcal{I}}} \bar{b}_{22} \frac{\partial u_1}{\partial \tau}\frac{\partial \psi}{\partial \tau} + \int_{\mathcal{O}^-} B\nabla u \nabla \psi, \end{cases} \tag{25}$$

for all $(\psi, \psi_1) \in V(\mathcal{O}) \times V(\mathcal{O}_U)$, where

$$a_0 = Y_{\mathcal{C}}(r)\left(\frac{\det \bar{A}}{\bar{a}_{22}}\right) \quad \text{and} \quad b_0 = Y_{\mathcal{C}}(r)\left(\bar{B}\begin{bmatrix} 1 \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}} \end{bmatrix}\begin{bmatrix} 1 \\ -\frac{\bar{a}_{21}}{\bar{a}_{22}} \end{bmatrix}\right).$$

Convergences: Let $(u_\varepsilon, v_\varepsilon)$ and $((u+u_1), (v+v_1))$, be solutions of (22),(23) respectively. Then as $\varepsilon \to 0$ we have the following convergences weakly in $L^2(\mathcal{O}^+)$:

$$\begin{aligned} \widetilde{v_\varepsilon} &\rightharpoonup |Y(r)|v + \int_{Y_{\mathcal{I}}(r)} v_1, \\ \chi_{\mathcal{C}_\varepsilon}\left(\frac{\partial v_\varepsilon}{\partial r}\right) &\rightharpoonup |Y_{\mathcal{C}}(r)|\left(\frac{\partial v}{\partial r}\right), \\ \chi_{\mathcal{C}_\varepsilon}\left(\frac{\partial v_\varepsilon}{\partial \theta}\right) &\rightharpoonup |Y_{\mathcal{C}}(r)|\left(\gamma\left(\frac{\partial u}{\partial r}\right) - \eta\left(\frac{\partial v}{\partial r}\right)\right), \\ \chi_{\mathcal{I}_\varepsilon}\left(\frac{\partial v_\varepsilon}{\partial r}\right) &\rightharpoonup \mathbf{0}, \\ \chi_{\mathcal{I}_\varepsilon}\left(\frac{\partial v_\varepsilon}{\partial \theta}\right) &\rightharpoonup \int_{Y_{\mathcal{I}}(r)} \frac{\partial v_1}{\partial \tau}, \end{aligned}$$

where

$$\gamma = \frac{\bar{a}_{22}\bar{b}_{21} - \bar{a}_{21}\bar{b}_{22}}{\bar{a}_{22}^2} \quad \text{and} \quad \eta = \frac{\bar{a}_{21}}{\bar{a}_{22}}.$$

Also we have $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega^-)$. Now use the polar transformations

$$r\frac{\partial v}{\partial r} = x \cdot \nabla v \quad \text{and} \quad \frac{\partial v}{\partial \theta} = \bar{x} \cdot \nabla v$$

to complete the proof. □

Remark 3 Here, instead of considering the ε^2-diffusion coefficient, one can use α_ε^2-coefficient, where $\{\alpha_\varepsilon\}_\varepsilon \to 0$ as $\varepsilon \to 0$. The limiting behavior is determined by $l \doteq$

$\lim_{\varepsilon\to 0}\frac{\alpha_\varepsilon}{\varepsilon}$. Depending on whether $l=0$, $l=\infty$, or $l\in(0,\infty)$, different homogenized problems arise.

In this article, we focus on the critical case $l\in(0,\infty)$ with $l=1$. The homogenized problems for the other two cases, $l=0$ and $l=\infty$ can be derived similarly with a minor modification in the present approach. When $\alpha_\varepsilon\gg\varepsilon$, that is $l=\infty$, the problem is analogous to studying the given system with a diffusion coefficient in $\mathcal{I}_\varepsilon$ as 1. Due to this, in the homogenized problem, the terms involving u_1 will disappear. For $\alpha_\varepsilon\ll\varepsilon$, that is $l=0$, following the same approach, one can derive the limiting problem. However, in this case, in the homogenized problems, terms involving $\nabla_y u_1$ will not be present as $\alpha_\varepsilon T^\varepsilon(\nabla u_\varepsilon)\rightharpoonup 0$ weakly in $L^2(\Omega^u_{\mathcal{I}})$ as $\varepsilon\to 0$. Accordingly, solution spaces for the limit problems need to be modified.

Remark 4 While our initial objective was to apply control in any part of the interior, due to the inherent non-linearity of the problem, we are restricted to applying control only on the lower fixed part. The primary obstacle is the lack of a corrector result(Theorem 2). Specifically, when control is applied to the oscillating part, we do not achieve strong convergence of the control, which prevents the establishment of a corrector result for both the ε-level optimal state and the adjoint state. As demonstrated in our analysis, the result in Theorem 2 is essential for identifying the limit of the semi-linear term. Extending the approach to distributed control on the oscillating part remain open problem and will be considered in future work.

Author Contributions All authors have contributed equally for this research work.

Declarations

Ethics Declaration: The authors declare that they have no conflict of interest.

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