SCHRÖDINGER PROPAGATOR AND THE DUNKL LAPLACIAN

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Abstract. We establish Strichartz estimates for a generalized Hermite–Schrödinger equation associated to a family of differential-difference operators involving the Dunkl Laplacian and unbounded potentials. This family includes the Hermite and Laguerre differential operators in particular. The study relies on the analysis of the so-called (k, a)-generalized semigroup studied in [1]. Moreover, we prove that homogeneous Strichartz estimates for the Schrödinger equation associated to the Dunkl Laplacian can be obtained from those for the generalized Hermite–Schrödinger equation.

1. Introduction

In this paper, we establish *Strichartz type estimates* for the Schrödinger propagator associated to a wide class of differential-difference operators on $\mathbb{R}^N$. This class includes, in particular, the Hermite operator on $\mathbb{R}^N$, the Laguerre differential operator on $\mathbb{R}^+$ etc., as observed in [2, 1].

For any self-adjoint differential operator $\mathscr{L}$ on $\mathbb{R}^N$, having the spectral decomposition $\mathscr{L} = \int_E \lambda \, dP_\lambda$, we can associate a one parameter oscillatory group $\{e^{-it\mathscr{L}} : t \in \mathbb{R}\}$ defined by

$$e^{-it\mathscr{L}} = \int_E e^{-it\lambda} dP_\lambda.$$

Here dP_λ denotes the spectral projection for $\mathscr{L}$; i.e., a projection valued 'measure' on the spectrum E of $\mathscr{L}$. The spectrum may be continuous, discrete or a combination of both, in general.

Of special importance are operators $\mathscr{L}$ with discrete spectrum $\{\lambda_k\}$ in which case the above integral reduces to

$$e^{-it\mathscr{L}} = \sum_{k=1}^{\infty} e^{-it\lambda_k} P_k. \tag{1.1}$$

Here P_k's are the projections onto the eigenspace corresponding to the eigenvalue λ_k and these projections are orthogonal since $\mathscr{L}$ is self-adjoint.

The operators with discrete spectrum includes some of the important examples like perturbation of the Euclidean Laplacian with a potential V unbounded near infinity; for instance, the Hermite operator $-\Delta_{\mathbb{R}^N} + \|x\|^2$ on $\mathbb{R}^N$, the special Hermite operator $-\Delta_{\mathbb{C}^N} + \|z\|^2 + i\sum_{j=1}^N (x_j\partial_{y_j} - y_j\partial_{x_j})$ on $\mathbb{C}^N$, the Laguerre differential operator on $\mathbb{R}^+$ (see e.g. [27] for the definition), etc.

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Oscillatory group of the form (1.1) arises as solution operator for the initial value problem for the time dependent Schrödinger equation associated to $\mathscr{L}$. More precisely, for $f \in L^2(\mathbb{R}^N)$, the function $u(x,t) := e^{-it\mathscr{L}} f(x) = \sum_{k=1}^{\infty} e^{-it\lambda_k} P_k f(x)$ is the unique solution to the following Cauchy problem

$$\begin{cases} i\partial_t u(x,t) - \mathscr{L} u(x,t) = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.2}$$

The one parameter group $\{e^{-it\mathscr{L}} : t \in \mathbb{R}\}$ which maps the initial data to the data at time t is called the Schrödinger propagator for $\mathscr{L}$.

Notice that $\mathscr{L}$ being self-adjoint, the operators $e^{-it\mathscr{L}}$ are unitary on $L^2(\mathbb{R}^N)$. Thus, $e^{-it\mathscr{L}}$ fails to map $L^2(\mathbb{R}^N)$ into the Sobolev spaces $W^s_{\mathscr{L}}(\mathbb{R}^N)$ defined by

$$W^s_{\mathscr{L}}(\mathbb{R}^N) = \left\{ f \in L^2(\mathbb{R}^N) : \mathscr{L}^s f \in L^2(\mathbb{R}^N),\ s > 0 \right\},$$

where $\mathscr{L}^s$ is defined using the spectral theory; $\mathscr{L}^s f = \sum_{k=1}^{\infty} \lambda_k^s P_k f$. Consequently, $e^{-it\mathscr{L}}$ has no regularizing effect in terms of the Sobolev space $W^s_{\mathscr{L}}(\mathbb{R}^N)$.

Quite in contrast to the above phenomenon, in 1977, R. S. Strichartz proved an interesting result for the Schrödinger propagator for the Laplacian on $\mathbb{R}^N$ (see [24]). To recall Strichartz's result, consider the Cauchy problem for the inhomogeneous Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = F(x,t), & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ u(x,0) = f(x), \end{cases}$$

with $f \in L^2(\mathbb{R}^N)$ and $F \in L^{\frac{2(N+2)}{N+4}}(\mathbb{R}^N \times \mathbb{R})$. Strichartz showed the following estimate which have come to bear his name:

$$\|u\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}^N\times\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R}^N)} + \|F\|_{L^{\frac{2(N+2)}{N+4}}(\mathbb{R}^N\times\mathbb{R})}.$$

Since then various authors have published similar estimates for solutions to Schrödinger's equation, with a wide class of bounded potentials V, for more general spaces with different exponents in space and time (see especially [29], [12], [7], [8]). See also the famous paper by M. Keel and T. Tao [13] for a far reaching generalization of Strichartz estimates.

An interesting case of an unbounded potential can be found in [15, 16, 20], where the authors consider the quadratic potential $V(x) = \|x\|^2$. The equation $i\partial_t u(x,t) + \Delta u(x,t) - \|x\|^2 u(x,t) = F(x,t)$ may be viewed as the Schrödinger equation for the Hermite operator $-\Delta + \|x\|^2$ and the proof in [15] relies on the harmonic analysis of the Hermite operator.

Our aim in this paper is to establish a local Strichartz type estimate for Schrödinger propagator for a wide class of differential-difference operators $\mathscr{L}$ having discrete spectrum, arising in the study of the Dunkl Laplacian on $\mathbb{R}^N$, generalizing [15]. The crucial tool is the integral representation of a generalized Laguerre semigroup studied in [1].

To be more specific, let G be a finite reflection group on $\mathbb{R}^N$ with root system $\mathscr{R}$. For a G-invariant real function $k = (k_\alpha)_{\alpha\in\mathscr{R}}$ *(multiplicity function)* on $\mathscr{R}$ we write Δ_k for the Dunkl Laplacian on $\mathbb{R}^N$ (see (2.3)).

For each $a > 0$ and a multiplicity function k, consider the operator

$$\Delta_{k,a} := \frac{1}{a}\Big(\|x\|^a - \|x\|^{2-a}\Delta_k\Big), \tag{1.3}$$

where $\|x\|^a$ in the right hand side of the formula stands for the multiplication operator by $\|x\|^a$. There is a nice structure theory associated to this operator, and in particular, it has discrete spectrum [1]. Note that when $a = 2$ and $k \equiv 0$, we have

$$2\Delta_{0,2} = \sum_{j=1}^{N} x_j^2 - \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2},$$

which is the classical Hermite operator on $\mathbb{R}^N$.

Consider the Cauchy problem for the inhomogeneous Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) - \Delta_{k,a} u(x,t) = F(x,t), & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u(x,0) = f(x), \end{cases} \tag{1.4}$$

with $f \in L^2_{k,a}(\mathbb{R}^N)$. Here $L^p_{k,a}(\mathbb{R}^N)$, for $1 \le p < \infty$, denotes the space of L^p-functions with respect to the weight

$$\vartheta_{k,a}(x) := \|x\|^{a-2} \prod_{\alpha \in \mathscr{R}} |\langle \alpha, x\rangle|^{k_\alpha}.$$

We may think of the Cauchy problem (1.4) as a generalized Hermite–Schrödinger equation.

The unique solution of (1.4) can be written in the form

$$u(x,t) = e^{-it\Delta_{k,a}} f(x) - i \int_0^t e^{-i(t-s)\Delta_{k,a}} F(\cdot, s)(x) ds \tag{1.5}$$

via Duhamel's principle. The first term is the solution of the homogeneous problem, that is $F = 0$ in (1.4). The second term is the solution of the inhomogeneous problem with initial data $f = 0$.

In [1] the authors established an integral formula for the so-called (k,a)-generalized Laguerre semigroup $e^{-z\Delta_{k,a}}$, for $\mathrm{Re}(z) \ge 0$, where a series expansion for the integral kernel is given. For $a = 1, 2$ the series is expressed more compactly (see (3.5–3.6)).

We introduce the mixed L^p spaces over $(-\pi/2, \pi/2) \times \mathbb{R}^N$ as the solution to the homogeneous problem is going to be periodic in t. Let $L^q L^p_{k,a} = L^q((-\pi/2, \pi/2), L^p_{k,a}(\mathbb{R}^N))$ be the space of functions h on $(-\pi/2, \pi/2) \times \mathbb{R}^N$ such that

$$\|h\|_{L^q L^p_{k,a}} := \Big\| \, \|h(\cdot, t)\|_{L^p_{k,a}(\mathbb{R}^N)} \, \Big\|_{L^q((-\pi/2, \pi/2))} < \infty.$$

A pair (p,q) is called *admissible* if $(\frac{1}{p}, \frac{1}{q})$ belongs to the trapezoid

$$\frac{1}{2}\left(\frac{2\gamma + N - 2}{2\gamma + N + a - 2}\right) < \frac{1}{p} \le \frac{1}{2} \text{ and } \frac{1}{2} \le \frac{1}{q} \le 1, \tag{1.6 a}$$

or

$$0 \le \frac{1}{q} < \frac{1}{2} \text{ and } \frac{1}{q} \ge \left(\frac{2\gamma + N + a - 2}{a}\right)\left(\frac{1}{2} - \frac{1}{p}\right), \tag{1.6 b}$$

where $\gamma = \frac{1}{2}\sum_{\alpha \in \mathscr{R}} k_\alpha$.

We now state our main results on the regularity of the solution of the homogeneous and inhomogeneous Schrödinger equation (1.4).

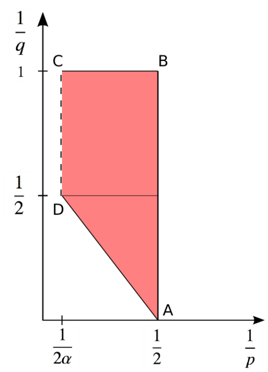

Figure 1. Admissible trapezoid

Theorem A (Homogeneous Strichartz estimate)**.** *Suppose $a = 1, 2$ and k is a non-negative multiplicity function such that*

$$a + 2\gamma + N - 2 > 0. \tag{1.7}$$

Let (p, q) be an admissible pair (see (1.6 a–b)) and $u = e^{-it\Delta_{k,a}} f$ be the solution to the homogeneous problem (1.4) *(i.e., $F \equiv 0$) with $f \in L^2_{k,a}(\mathbb{R}^N)$. Then we have the estimate*

$$\|e^{-it\Delta_{k,a}} f\|_{L^q L^p_{k,a}} \lesssim \|f\|_{L^2_{k,a}}. \tag{1.8}$$

Observe that (1.7) is automatically satisfied if $N \geq 2$.

Regarding the inhomogeneous part of the solution u (see (1.5)), we prove the following result.

Theorem B (Retarded estimate)**.** *Suppose that $a = 1, 2$ and that the non-negative multiplicity function k satisfies the inequality* (1.7)*. Let (p, q) be an admissible pair and $F \in L^{q'}((-\pi/2, \pi/2), L^{p'}_{k,a}(\mathbb{R}^N))$. Then the function,*

$$\tilde{F}(x, t) := \int_{t_0}^{t} e^{-i(t-s)\Delta_{k,a}} F(x, s) ds$$

belongs to $L^{\tilde{q}}((-\pi/2, \pi/2), L^{\tilde{p}}_{k,a}(\mathbb{R}^N))$ for every admissible pair $(\tilde{p}, \tilde{q})$. Moreover, the following estimate holds

$$\left\|\tilde{F}\right\|_{L^{\tilde{q}} L^{\tilde{p}}_{k,a}} \lesssim \|F\|_{L^{q'} L^{p'}_{k,a}}. \tag{1.9}$$

In view of Theorems A and B, we obtain the following estimate for the general solution to the inhomogeneous Schrödinger equation (1.4).

Theorem C (Inhomogeneous Strichartz estimate)**.** *Suppose $a = 1, 2$ and k is a non-negative multiplicity function satisfying the inequality* (1.7)*. Let (p, q) and $(\tilde{p}, \tilde{q})$ be two admissible pairs. The solution*

$$u(x, t) = e^{-it\Delta_{k,a}} f(x) - i \int_0^t e^{-i(t-s)\Delta_{k,a}} F(\cdot, s)(x) ds$$

to the initial value problem (1.4) *satisfies*

$$\|u(x, t)\|_{L^q L^p_{k,a}} \lesssim \|f\|_{L^2_{k,a}} + \|F\|_{L^{\tilde{q}'} L^{\tilde{p}'}_{k,a}}, \tag{1.10}$$

where $f \in L^2_{k,a}(\mathbb{R}^N)$ and $F \in L^{\tilde{q}'}((-\pi/2, \pi/2), L^{\tilde{p}'}_{k,a}(\mathbb{R}^N))$.

In the last part of the paper we turn our attention to the Schrödinger equation associated to the differential-difference part of $\Delta_{k,a}$ (see (1.3)). In other words, we consider the initial value problem

$$i\partial_t \upsilon(x,t) + \frac{1}{a}\|x\|^{2-a}\Delta_k \upsilon(x,t) = 0, \qquad \upsilon(x,0) = f(x). \tag{1.11}$$

Our goal is to prove that mixed norm estimates for the Schrödinger equation (1.11) can be obtained from those in Theorem A and vice-versa.

Theorem D. *Let* $1 \le p, q \le \infty$ *such that*

$$\left(\frac{2\gamma + N + a - 2}{a}\right)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{q} = 0. \tag{1.12}$$

For all $f \in L^2_{k,a}(\mathbb{R}^N)$ *we have*

$$\|e^{-it\Delta_{k,a}} f\|_{L^q((0,\pi/2), L^p_{k,a}(\mathbb{R}^N))} = \|e^{i\frac{t}{a}\|x\|^{2-a}\Delta_k} f\|_{L^q((0,\infty), L^p_{k,a}(\mathbb{R}^N))}. \tag{1.13}$$

We will see that the two mixed norms on the left hand sides of (1.8) and (1.13) are proportional for real functions f. By combining Theorem A with Theorem D, we deduce the homogeneous Strichartz estimate for the Schrödinger equation (1.11) when the equality in (1.12) holds. Observe that an admissible pair (p,q) satisfying (1.12) reads $(\frac{1}{p}, \frac{1}{q})$ belongs to the line AD in Figure 1.

The paper is organized as follows: In the next section we state the background material to define the Dunkl Laplacian. Section 3 is concerned with a priori estimates for the (k,a)-generalized Laguerre semigroup introduced in [1]. Using these estimates, we complete the proof of Theorem A, Theorem B and Theorem C in section 4. In section 5 we prove a relation between the operators $e^{-it\Delta_{k,a}}$ and $e^{i\frac{t}{a}\|x\|^{2-a}\Delta_k}$ which implies Theorem D.

2. Previous results on Dunkl operators

Let $\langle\cdot,\cdot\rangle$ be the standard Euclidean scalar product in $\mathbb{R}^N$. We shall use the same notation for its bilinear extension to $\mathbb{C}^N \times \mathbb{C}^N$. For $x \in \mathbb{R}^N$, denote by $\|x\| = \langle x, x\rangle^{1/2}$.

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we write r_α for the reflection with respect to the hyperplane $\langle\alpha\rangle^\perp$ orthogonal to α defined by

$$r_\alpha(x) := x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha, \qquad x \in \mathbb{R}^N.$$

We say a finite set $\mathscr{R}$ in $\mathbb{R}^N \setminus \{0\}$ is a (reduced) *root system* if:

(R1) $r_\alpha(\mathscr{R}) = \mathscr{R}$ for all $\alpha \in \mathscr{R}$,
(R2) $\mathscr{R} \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \mathscr{R}$.

In this article, we do not impose crystallographic conditions on the roots, and do not require that $\mathscr{R}$ spans $\mathbb{R}^N$. However, we shall assume $\mathscr{R}$ is reduced, namely, (R2) is satisfied.

The subgroup $G \subset O(N, \mathbb{R})$ generated by the reflections $\{r_\alpha \mid \alpha \in \mathscr{R}\}$ is called the finite Coxeter group associated with $\mathscr{R}$. The Weyl groups such as the symmetric group $\mathfrak{S}_N$ for the type A_{N-1} root system and the hyperoctahedral group for the type B_N root system are typical examples. In addition, H_3, H_4 (icosahedral groups) and $I_2(n)$ (symmetry group of the regular n-gon) are also the Coxeter groups. We refer to [11] for more details on the theory of Coxeter groups.

Definition 2.1. *A multiplicity function for G is a function $k : \mathscr{R} \to \mathbb{C}$ which is constant on G-orbits.*

Setting $k_\alpha := k(\alpha)$ for $\alpha \in \mathscr{R}$, we have $k_{g\alpha} = k_\alpha$ for all $g \in G$ from definition. We say k is non-negative if $k_\alpha \geq 0$ for all $\alpha \in \mathscr{R}$. The $\mathbb{C}$-vector space of non-negative multiplicity functions on $\mathscr{R}$ is denoted by $\mathscr{K}^+$.

For $\xi \in \mathbb{C}^N$ and $k \in \mathscr{K}^+$, Dunkl [2] introduced a family of first order differential-difference operators $T_\xi(k)$ (*Dunkl's operators*) by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \langle \alpha, \xi\rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle}, \qquad f \in C^1(\mathbb{R}^N).$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. Thanks to the G-invariance of the multiplicity function, this definition is independent of the choice of the positive subsystem $\mathscr{R}^+$.

It is shown in [3] that for any non-negative root multiplicity function k, there is a unique linear isomorphism V_k (*Dunkl's intertwining operator*) on the space $\mathcal{P}(\mathbb{R}^N)$ of polynomial functions on $\mathbb{R}^N$ such that:

(I1) $V_k(\mathcal{P}_m(\mathbb{R}^N)) = \mathcal{P}_m(\mathbb{R}^N)$ for all $m \in \mathbb{N}$,
(I2) $V_k|_{\mathcal{P}_0(\mathbb{R}^N)} = \mathrm{id}$,
(I3) $T_\xi(k)V_k = V_k\partial_\xi$ for all $\xi \in \mathbb{R}^N$.

Here, $\mathcal{P}_m(\mathbb{R}^N)$ denotes the space of homogeneous polynomials of degree m. It is known that V_k induces a homeomorphism of $C(\mathbb{R}^N)$ and also that of $C^\infty(\mathbb{R}^N)$ (cf. [27]).

For arbitrary finite reflection group G, and for any non-negative multiplicity function k, Rösler [21] proved that there exists a unique positive Radon probability-measure μ_x^k on $\mathbb{R}^N$ such that

$$V_k f(x) = \int_{\mathbb{R}^N} f(\xi) d\mu_x^k(\xi). \tag{2.1}$$

The measure μ_x^k depends on $x \in \mathbb{R}^N$ and its support is contained in the ball $B(\|x\|) := \left\{\xi \in \mathbb{R}^N \mid \|\xi\| \leq \|x\|\right\}$. In view of the Laplace type representation (2.1), Dunkl's intertwining operator V_k can be extended to a larger class of function spaces.

Aside from the development of the general theory of the Dunkl operators, we note that explicit formulas for V_k have been known for only a few cases: $G = \mathbb{Z}_2^N$, $G = S_3$, and the equal parameter case for the Weyl group of B_2 (see [4] for the recent survey by C. Dunkl).

Let $\{\xi_1, \ldots, \xi_N\}$ be an orthonormal basis of $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$. The Dunkl Laplacian operator is defined as

$$\Delta_k := \sum_{j=1}^N T_{\xi_j}(k)^2. \tag{2.2}$$

The definition of Δ_k is independent of the choice of an orthonormal basis of $\mathbb{R}^N$. In fact, it is proved in [2] that Δ_k is expressed as

$$\Delta_k f(x) = \Delta f(x) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \left\{ \frac{2\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \|\alpha\|^2 \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle^2} \right\}, \tag{2.3}$$

where ∇ denotes the usual gradient operator.

For $k \equiv 0$, the Dunkl–Laplace operator Δ_k reduces to the Euclidean Laplacian Δ.

Remark 2.2. *The Dunkl Laplacian arise as the radial part of the Laplacian on the tangent space of a Riemannian symmetric spaces. Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, and let $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of restricted roots, and m_α the multiplicity of $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. We may consider $\Sigma(\mathfrak{g}, \mathfrak{a})$ to be a subset of $\mathfrak{a}$ by means of the Killing form of $\mathfrak{g}$. The Killing form endows $\mathfrak{p}$ with a flat Riemannian symmetric space structure, and we write $\Delta_{\mathfrak{p}}$ for the (Euclidean) Laplacian on $\mathfrak{p}$. Put $\mathscr{R} := 2\Sigma(\mathfrak{g}, \mathfrak{a})$ and $k_\alpha := \frac{1}{2}\sum_{\beta\in\Sigma^+\cap\mathbb{R}\alpha} m_\beta$. We note that the root system $\mathscr{R}$ is not necessarily reduced. Then the radial part of $\Delta_{\mathfrak{p}}$, denoted by $Rad(\Delta_{\mathfrak{p}})$, (see* [9, Proposition 3.13]*) is given by*

$$Rad(\Delta_{\mathfrak{p}})f = \Delta_k f$$

for every G-invariant function $f \in C^\infty(\mathfrak{a})$.

Definition 2.3. *A k-harmonic polynomial of degree m $(m \in \mathbb{N})$ is a homogeneous polynomial p on $\mathbb{R}^N$ of degree m such that $\Delta_k p = 0$.*

Denote by $\mathcal{H}_k^m(\mathbb{R}^N)$ the space of k-harmonic polynomials of degree m. Let $d\sigma$ be the standard measure on the unit sphere $\mathbb{S}^{N-1}$, and d_k the normalizing constant defined by

$$d_k := \left(\int_{\mathbb{S}^{N-1}} \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha, \omega\rangle|^{2k_\alpha} d\sigma(\omega)\right)^{-1}. \tag{2.4}$$

For $k \equiv 0$, d_k^{-1} is the volume of the unit sphere, namely,

$$d_0 = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}. \tag{2.5}$$

Thanks to Selberg, Mehta, Macdonald [17], Heckman, Opdam [18], and others, there is a closed form of d_k in terms of Gamma functions when k is a non-negative multiplicity function (see also [5]).

For $a > 0$, let $L^2_{k,a}(\mathbb{R}^N)$ be the space of square integrable functions on $\mathbb{R}^N$ against the measure

$$\vartheta_{k,a}(x) := \|x\|^{a-2} \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha, x\rangle|^{2k_\alpha}.$$

It is G-invariant and homogeneous of degree 2γ, where

$$\gamma = \sum_{\alpha\in\mathscr{R}^+} k_\alpha. \tag{2.6}$$

For $\ell, m \in \mathbb{N}$ and $p \in \mathcal{H}_k^m(\mathbb{R}^N)$, we introduce the following functions on $\mathbb{R}^N$:

$$\Phi_\ell^{(a)}(p, x) = p(x) L_\ell^{(\lambda_{k,a,m})}\left(\frac{2}{a}\|x\|^a\right) \exp\left(-\frac{1}{a}\|x\|^a\right), \tag{2.7}$$

where $L_\ell^{(\lambda)}$ denotes the Laguerre polynomial, and $\lambda_{k,a,m} := \frac{1}{a}(2m + 2\gamma + N - 2)$. It is shown in [1] that for $k \in \mathscr{K}^+$ and $a > 0$, such that $a + 2\gamma + N - 2 > 0$, the following vector space

$$W_{k,a}(\mathbb{R}^N) := \mathbb{C}\text{-span}\left\{\Phi_\ell^{(a)}(p, \cdot) \mid \ell \in \mathbb{N}, m \in \mathbb{N}, p \in \mathcal{H}_k^m(\mathbb{R}^N)\right\}$$

is a dense subspace of $L^2_{k,a}(\mathbb{R}^N)$.

Throughout the paper we will use the notation $f \lesssim g$ to denote $f \leq Cg$ with a positive constant C varying line by line and independent of significant quantities.

3. Estimates for the generalized Laguerre semigroup

For $a > 0$, we have already introduced the differential-difference operator

$$\Delta_{k,a} = \frac{1}{a}\Big(\|x\|^a - \|x\|^{2-a}\Delta_k\Big),$$

where $\|x\|^a$ in the right hand side of the formula stands for the multiplication operator by $\|x\|^a$. It is proved in [1] that for $a > 0$ and $k \in \mathscr{K}^+$ such that $a + 2\gamma + N - 2 > 0$, $\Delta_{k,a}$ extends to a self-adjoint on $L^2_{k,a}(\mathbb{R}^N)$. Moreover, there is no continuous spectrum of $\Delta_{k,a}$, and all the discrete spectra are positive. More specifically the set of discrete spectra is given by

$$\Big\{(2a\ell + 2m + 2\gamma + N + a - 2)/a, \quad \ell, m \in \mathbb{N}\Big\}, \qquad (N \geq 2),$$
$$\Big\{(2a\ell + 2\gamma + a \pm 1)/a, \quad \ell \in \mathbb{N}\Big\}, \qquad (N = 1).$$

Note that when $a = 2$ and $k \equiv 0$, the above discussions reduces to the case of Hermite operator $\sum_{j=1}^N x_j^2 - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ on $L^2(\mathbb{R}^N)$, as mentioned before.

In [1], the authors have studied the so-called (k,a)-*generalized Laguerre* semigroup $\mathcal{I}_{k,a}(z)$ with infinitesimal generator $\Delta_{k,a}$, that is

$$\mathcal{I}_{k,a}(z) := \exp(-z\Delta_{k,a}) \tag{3.1}$$

for $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \geq 0$.

Henceforth we will denote by $\mathbb{C}^+$ the complex right-half plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$. We point that $\mathcal{I}_{0,2}(z)$ is the Hermite semigroup, and $\mathcal{I}_{0,1}(z)$ is the Laguerre semigroup (see [10, 14] respectively).

In the following theorem we gather the main properties of the (k,a)-generalized Laguerre semigroup $\mathcal{I}_{k,a}(z)$.

Theorem 3.1. *(see* [1, Theorem 3.40]*) Suppose $a > 0$ and $k \in \mathscr{K}^+$ satisfying the condition $a + 2\gamma + N - 2 > 0$. Then,*

(1) *The map*

$$\mathbb{C}^+ \times L^2_{k,a}(\mathbb{R}^N) \longrightarrow L^2_{k,a}(\mathbb{R}^N), \qquad (z, f) \longmapsto e^{-z\Delta_{k,a}} f$$

is continuous.

(2) *For any $p \in \mathcal{H}^m_k(\mathbb{R}^N)$ and $\ell \in \mathbb{N}$, $\Phi^{(a)}_\ell(p, \cdot)$ is an eigenfunction of the operator $e^{-z\Delta_{k,a}}$, that is*

$$e^{-z\Delta_{k,a}}\Phi^{(a)}_\ell(p, x) = e^{-z(\lambda_{k,a,m}+2\ell+1)}\Phi^{(a)}_\ell(p, x),$$

where $\lambda_{k,a,m} = \frac{1}{a}(2m + 2\gamma + N - 2)$ and $\Phi^{(a)}_\ell(p, x)$ is as in (2.7).

(3) *The operator norm $\|e^{-z\Delta_{k,a}}\|_{op}$ is $\exp\Big(-\frac{1}{a}(2\gamma + N + a - 2)\,\mathrm{Re}(z)\Big)$.*

(4) *If $\mathrm{Re}(z) > 0$, then $e^{-z\Delta_{k,a}}$ is a Hilbert Schmidt operator.*

(5) *If $\mathrm{Re}(z) = 0$, then $e^{-z\Delta_{k,a}}$ is a unitary operator.*

Suppose $a > 0$ and k is a non-negative multiplicity function on the root system $\mathscr{R}$ satisfying

$$a + 2\gamma + N - 2 > 0. \tag{3.2}$$

Notice that (3.2) is immediately satisfied if $N \geq 2$. In [1, Section 4], an integral expression of $\mathcal{I}_{k,a}(z)$, for $\mathrm{Re}(z) \geq 0$, is proved

$$e^{-z\Delta_{k,a}} f(x) = c_{k,a} \int_{\mathbb{R}^N} \Lambda_{k,a}(x, y; z) f(y) \vartheta_{k,a}(y) dy, \tag{3.3}$$

for all $a > 0$. Here

$$c_{k,a} = a^{-\left(\frac{2\gamma+N-2}{a}\right)} \Gamma\left(\frac{2\gamma + N + a - 2}{a}\right)^{-1} d_k, \tag{3.4}$$

where d_k is defined in (2.4). Moreover, in [1, Theorem 4.20] a series expansion for the kernel $\Lambda_{k,a}$ is given. For $a = 1, 2$ the series is expressed more compactly. Henceforth we will assume that $a = 1, 2$.

To recall the expression of the kernel $\Lambda_{k,a}(x, y; z)$, for $a = 1, 2$, we introduce the following continuous functions of $\zeta \in [-1, 1]$ with parameters $r, s > 0$ and $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$:

$$\begin{aligned} h_a(r, s; z; \zeta) \quad := \quad & \frac{\exp\left(-\frac{1}{a}(r^a + s^a)\coth(z)\right)}{\sinh(z)^{\left(\frac{2\gamma+N-1}{a}\right)}} \\ & \times \begin{cases} \Gamma\left(\gamma + \dfrac{N-1}{2}\right) \widetilde{I}_{\gamma+\frac{N-3}{2}}\left(\dfrac{\sqrt{2}(rs)^{1/2}}{\sinh(z)}(1+\zeta)^{1/2}\right), & (a = 1), \\ \exp\left(\dfrac{rs\zeta}{\sinh z}\right), & (a = 2), \end{cases} \end{aligned} \tag{3.5}$$

where γ is defined in (2.6). Here $\widetilde{I}_\lambda(w) = \left(\frac{w}{2}\right)^{-\lambda} I_\lambda(w)$ is the (normalized) modified Bessel function of the first kind.

By using the polar coordinate $x = r\omega$, $y = s\eta$, we set

$$\Lambda_{k,a}(x, y; z) := V_k^\eta h_a(r, s; z; \langle \omega, \cdot \rangle)(\eta). \tag{3.6}$$

Here V_k is the Dunkl intertwining operator, and the superscript in V_k^η denotes the relevant variable. We note that $\Lambda_{k,a}(x, y; z) = h_a(r, s; z; \langle \omega, \eta \rangle)$ if $k \equiv 0$.

Remark 3.2. *From* (3.5) *it follows that*

$$\Lambda_{k,a}(x, y; i(\mu + \pi)) = e^{-i\pi\left(\frac{2\gamma+N+a-2}{a}\right)} \Lambda_{k,a}((-1)^{\frac{2}{a}} x, y; i\mu), \tag{3.7}$$

for all $\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$.

In view of the Laplace type representation (2.1) of the intertwining operator V_k, the kernel $\Lambda_{k,a}$ can be written as

$$\Lambda_{k,a}(r\omega, s\eta; z) = \int_{\mathbb{R}^N} h_a(r, s; z; \langle \omega, \xi \rangle) d\mu_\eta^k(\xi), \tag{3.8}$$

where $\mathrm{supp}(\mu_\eta^k) \subset \{\xi \in \mathbb{R}^N \mid \|\xi\| \leq 1\}$.

Using the fact that $|\widetilde{I}_\nu(w)| \leq \Gamma(\nu+1)^{-1} e^{|\operatorname{Re}(w)|}$, for $\nu \geq -1/2$ and $w \in \mathbb{C}$, it follows that

$$\left|\widetilde{I}_{\gamma+\frac{N-3}{2}}\left(\frac{\sqrt{2}(rs)^{1/2}}{\sinh(z)}(1+\langle\omega,\xi\rangle)^{1/2}\right)\right| \leq e^{2\sqrt{rs}|\operatorname{Re}(\operatorname{csch}(z))|}$$

for $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$, $\omega \in \mathbb{S}^{N-1}$ and $\xi \in \operatorname{supp}(\mu_\eta^k)$. On the other hand, we have

$$\left|\exp\left(\frac{rs\langle\omega,\xi\rangle}{\sinh z}\right)\right| \leq e^{rs|\operatorname{Re}(\operatorname{csch}(z))|}.$$

Using $r^a + s^a \geq 2(rs)^{\frac{a}{2}}$, from (3.5), it follows that

$$|h_a(r,s;z;\langle\omega,\xi\rangle)| \leq \frac{1}{|\sinh(z)|^{\left(\frac{2\gamma+N+a-2}{a}\right)}} e^{-\frac{1}{a}(r^a+s^a)\left[\operatorname{Re}(\coth(z)) - |\operatorname{Re}(\operatorname{csch}(z))|\right]},$$

for $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$. Further, since μ_η^k is a probability measure on $\mathbb{R}^N$, we deduce from (3.8) that

$$|\Lambda_{k,a}(x,y;z)| \leq \frac{1}{|\sinh(z)|^{\left(\frac{2\gamma+N+a-2}{a}\right)}} e^{-\frac{1}{a}(\|x\|^a+\|y\|^a)\left[\operatorname{Re}(\coth(z)) - |\operatorname{Re}(\operatorname{csch}(z))|\right]}. \tag{3.9}$$

If we assume that $\operatorname{Re}(z) > 0$, then $\operatorname{Re}(\coth(z)) > |\operatorname{Re}(\operatorname{csch}(z))|$. Thus the kernel $\Lambda_{k,a}$ decays exponentially with respect to x and y. Moreover, if we write $z = \epsilon + i\mu$ with $\epsilon > 0$ and $\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$, then the addition formula

$$\operatorname{csch}(\epsilon + i\mu) = \frac{\operatorname{csch}(\epsilon)\operatorname{csch}(i\mu)}{\coth(\epsilon) + \coth(i\mu)}$$

gives $|\operatorname{csch}(\epsilon + i\mu)| < |\operatorname{csch}(i\mu)|$. Thus, for $z = \epsilon + i\mu$ with $\epsilon > 0$ and $\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$, we may rewrite the estimate (3.9) as

$$|\Lambda_{k,a}(x,y;z)| \leq \frac{1}{|\sin(\mu)|^{\left(\frac{2\gamma+N+a-2}{a}\right)}}.$$

The inequality holds true for $\epsilon = 0$. Hence we have proved:

Proposition 3.3. *For $a = 1, 2$, the kernel $\Lambda_{k,a}(x,y;z)$ satisfies the following upper estimates:*

(1) *If $\operatorname{Re}(z) > 0$, there exists a constant $C > 0$ depending on z such that*

$$|\Lambda_{k,a}(x,y;z)| \leq \frac{1}{|\sinh z|^{\left(\frac{2\gamma+N+a-2}{a}\right)}} \exp(-C(\|x\|^a + \|y\|^a)). \tag{3.10}$$

(2) *For $z = \epsilon + i\mu$, such that $\epsilon \geq 0$ and $\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$, we have*

$$|\Lambda_{k,a}(x,y;z)| \leq \frac{1}{|\sin \mu|^{\left(\frac{2\gamma+N+a-2}{a}\right)}}. \tag{3.11}$$

In view of the upper estimate (3.10), there exists a positive constant C_z independent of x and y such that $c_{k,a}\int_{\mathbb{R}^N} |\Lambda_{k,a}(x,y;z)| \vartheta_{k,a}(x)dx \leq C_z$ for a.e. $y \in \mathbb{R}^N$, and $c_{k,a}\int_{\mathbb{R}^N} |\Lambda_{k,a}(x,y;z)| \vartheta_{k,a}(y)dy \leq C_z$ for a.e. $x \in \mathbb{R}^N$. For then it follows that $e^{-z\Delta_{k,a}}$ is bounded on $L^1_{k,a}(\mathbb{R}^N)$. Moreover, as $e^{-z\Delta_{k,a}}$ is already bounded on $L^2_{k,a}(\mathbb{R}^N)$ interpolation proves that $e^{-z\Delta_{k,a}}f$ is in $L^p_{k,a}(\mathbb{R}^N)$ for all $1 \leq p \leq 2$ when $f \in L^p_{k,a}(\mathbb{R}^N)$. Duality will then takes care of the range $2 \leq p \leq \infty$. Thus we have:

Theorem 3.4. *Suppose that $a = 1, 2$ and that the non-negative multiplicity function k satisfies the inequality* (3.2). *Let $f \in L^p_{k,a}(\mathbb{R}^N)$ for $1 \leq p \leq \infty$. For* $\mathrm{Re}(z) > 0$, *the* (k, a)*-generalized Laguerre semigroup $e^{-z\Delta_{k,a}}$ is in $L^p_{k,a}(\mathbb{R}^N)$, and*

$$\|e^{-z\Delta_{k,a}} f\|_{L^p_{k,a}} \leq C_z \|f\|_{L^p_{k,a}}$$

for some constant $C_z > 0$.

By virtue of the inequality (3.11) we can deduce the following result.

Theorem 3.5. *Retain the assumptions of Theorem 3.4. For $z = \epsilon + i\mu$ ($\epsilon \geq 0, \mu \in \mathbb{R} \setminus \pi\mathbb{Z}$), the holomorphic semigroup $e^{-z\Delta_{k,a}}$ satisfies*

$$\|e^{-z\Delta_{k,a}} f\|_{L^p_{k,a}} \leq \frac{1}{|\sin\mu|^{2\left(\frac{2\gamma+N+a-2}{a}\right)\left(\frac{1}{2}-\frac{1}{p}\right)}} \|f\|_{L^{p'}_{k,a}},$$

for $2 \leq p \leq \infty$, where $p' = p/(p-1)$.

Proof. By the uniform estimate (3.11) of the kernel $\Lambda_{k,a}(x, y; z)$ in x and y, we get the $L^1 - L^\infty$ estimate:

$$\|e^{-z\Delta_{k,a}} f\|_{L^\infty_{k,a}} \leq \frac{1}{|\sin\mu|^{\left(\frac{2\gamma+N+a-2}{a}\right)}} \|f\|_{L^1_{k,a}},$$

for $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$. Moreover, by Theorem 3.1(3), we have

$$\|e^{-z\Delta_{k,a}} f\|_{L^2_{k,a}} \leq e^{-\mathrm{Re}(z)\left(\frac{2\gamma+N+a-2}{a}\right)} \|f\|_{L^2_{k,a}}.$$

Using Riesz-Thorin's interpolation, we get that

$$\begin{aligned} \|e^{-z\Delta_{k,a}} f\|_{L^p_{k,a}} &\leq \frac{e^{-2\,\mathrm{Re}(z)\left(\frac{2\gamma+N+a-2}{a}\right)\left(1-\frac{1}{p'}\right)}}{|\sin\mu|^{\left(\frac{2\gamma+N+a-2}{a}\right)\left(\frac{2}{p'}-1\right)}} \|f\|_{L^{p'}_{k,a}} \\ &\leq \frac{1}{|\sin\mu|^{2\left(\frac{2\gamma+N+a-2}{a}\right)\left(\frac{1}{2}-\frac{1}{p}\right)}} \|f\|_{L^{p'}_{k,a}}, \end{aligned}$$

for $2 \leq p \leq \infty$, where $1/p + 1/p' = 1$. □

In the light of Remark 3.2, the $L^p_{k,a}(\mathbb{R}^N)$ norm of $e^{-it\Delta_{k,a}} f$ is π-periodic as a function of t, and thus determined by its values for $-\pi/2 \leq t \leq \pi/2$. Henceforth we will denote the space $L^q((-\pi/2, \pi/2), L^p_{k,a}(\mathbb{R}^N))$ by $L^q L^p_{k,a}$.

Next, we shall estimate the $L^2_{k,a}$-norm of the function $\int_{-\pi/2}^{\pi/2} e^{-(\epsilon+i\mu)\Delta_{k,a}} h(\cdot, \mu)(x) d\mu$ for any $h \in L^q((-\pi/2, \pi/2), L^2_{k,a} \cap L^{p'}_{k,a}(\mathbb{R}^N))$. For this, the following lemma is needed.

Lemma 3.6. *Let $h \in L^q((-\pi/2, \pi/2), L^2_{k,a}(\mathbb{R}^N))$ for $1 \leq q \leq \infty$. Then for $\epsilon \geq 0$, the function $F(\mu, \nu, x) := e^{-(\epsilon+i\mu)\Delta_{k,a}} h(x, \mu) \overline{e^{-(\epsilon+i\nu)\Delta_{k,a}} h(x, \nu)}$ is integrable on $(-\pi/2, \pi/2)^2 \times \mathbb{R}^N$ with respect to the measure $\vartheta_{k,a}(x) dx d\mu d\nu$.*

Proof. Since $h(\cdot, \mu) \in L^2_{k,a}(\mathbb{R}^N)$, it follows that $e^{-(\epsilon+i\mu)\Delta_{k,a}} h(\cdot, \mu) \in L^2_{k,a}(\mathbb{R}^N)$ for $\epsilon \geq 0$. Using Theorem 3.1 and Cauchy-Schwartz inequality we get that

$$\int_{\mathbb{R}^N} |F(\mu, \nu, x)| \vartheta_{k,a}(x) dx \leq \|h(\cdot, \mu)\|_{L^2_{k,a}} \|h(\cdot, \nu)\|_{L^2_{k,a}}.$$

Therefore

$$\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\mathbb{R}^N}|F(\mu,\nu,x)|\vartheta_{k,a}(x)dxd\sigma(\mu)d\sigma(\nu)\leq\Big(\int_{-\pi/2}^{\pi/2}\|h(\cdot,\mu)\|_{L^2_{k,a}}d\sigma(\mu)\Big)^2 \lesssim \|h\|_{L^qL^2_{k,a}}$$

for $1\leq q\leq\infty$ and $q'=q/(q-1)$, by Hölder's inequality. □

We are ready to estimate the $L^2_{k,a}$-norm of the function $\int_{-\pi/2}^{\pi/2}e^{-(\epsilon+i\mu)\Delta_{k,a}}h(\cdot,\mu)(x)d\mu$, for $h\in L^{q'}((-\pi/2,\pi/2),L^2_{k,a}\cap L^{p'}_{k,a}(\mathbb{R}^N))$. Indeed,

$$\begin{aligned}&\Big\|\int_{-\pi/2}^{\pi/2}e^{-(\epsilon+i\mu)\Delta_{k,a}}h(\cdot,\mu)d\mu\Big\|^2_{L^2_{k,a}}\\ &=\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\Big\{\int_{\mathbb{R}^N}e^{-(\epsilon+i\mu)\Delta_{k,a}}h(x,\mu)\overline{e^{-(\epsilon+i\nu)\Delta_{k,a}}h(x,\nu)}\vartheta_{k,a}(x)dx\Big\}d\mu d\nu,\end{aligned}$$

where the interchange of the order of integration is accomplished by Lemma 3.6. On the other hand, the semigroup law $e^{-z_1\Delta_{k,a}}\circ e^{-z_2\Delta_{k,a}}=e^{-(z_1+z_2)\Delta_{k,a}}$ yields, see [1]

$$c_{k,a}\int_{\mathbb{R}^N}\Lambda_{k,a}(x,y;z_1)\Lambda_{k,a}(x,y';z_2)\vartheta_{k,a}(x)dx=\Lambda_{k,a}(y,y';z_1+z_2). \tag{3.12}$$

Now using the integral representation (3.3) of $e^{-z\Delta_{k,a}}$ and the identity (3.12) we get

$$\begin{aligned}&\int_{\mathbb{R}^N}e^{-(\epsilon+i\mu)\Delta_{k,a}}h(x,\mu)\overline{e^{-(\epsilon+i\nu)\Delta_{k,a}}h(x,\nu)}\vartheta_{k,a}(x)dx\\ =\ &c_{k,a}^2\int_{\mathbb{R}^N}\Big\{\int_{\mathbb{R}^N}\Lambda(x,y,\epsilon+i\mu)h(y,\mu)\vartheta_{k,a}(y)dy\Big\}\Big\{\int_{\mathbb{R}^N}\overline{\Lambda(x,y',\epsilon+i\nu)}\ \overline{h(y',\nu)}\vartheta_{k,a}(y')dy'\Big\}\vartheta_{k,a}(x)dx\\ =\ &c_{k,a}^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}h(y,\mu)\overline{h(y',\nu)}\Big\{\int_{\mathbb{R}^N}\Lambda(x,y,\epsilon+i\mu)\overline{\Lambda(x,y',\epsilon+i\nu)}\vartheta_{k,a}(x)dx\Big\}\vartheta_{k,a}(y')\vartheta_{k,a}(y)dydy'\\ =\ &c_{k,a}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}h(y,\mu)\overline{h(y',\nu)}\Lambda(y,y',2\epsilon+i(\mu-\nu))\vartheta_{k,a}(y')\vartheta_{k,a}(y)dydy'\\ =\ &\int_{\mathbb{R}^N}\overline{h(y',\nu)}e^{-(2\epsilon+i(\mu-\nu))\Delta_{k,a}}h(y',\mu)\vartheta_{k,a}(y')dy'.\end{aligned}$$

Therefore

$$\begin{aligned}\Big\|\int_{-\pi/2}^{\pi/2}e^{-(\epsilon+i\mu)\Delta_{k,a}}h(\cdot,\mu)d\mu\Big\|^2_{L^2_{k,a}} &= \int_{-\pi/2}^{\pi/2}\int_{\mathbb{R}^N}\overline{h(y',\nu)}\Big(\int_{-\pi/2}^{\pi/2}e^{-(2\epsilon+i(\mu-\nu))\Delta_{k,a}}h(y',\mu)d\mu\Big)\vartheta_{k,a}(y')dy'd\nu\\ &\lesssim \Big\|\int_{-\pi/2}^{\pi/2}e^{-(2\epsilon+i(\mu-\nu))\Delta_{k,a}}h(y',\mu)d\mu\Big\|_{L^qL^p_{k,a}}\|h\|_{L^{q'}L^{p'}_{k,a}},\end{aligned}$$

for all $p,q\geq 1$, by Holder's inequality. Here p' and q' are the conjugate exponents of p and q respectively. Moreover, we claim that

$$\Big\|\int_{-\pi/2}^{\pi/2}e^{-(2\epsilon+i(\mu-\nu))\Delta_{k,a}}h(y',\mu)d\mu\Big\|_{L^qL^p_{k,a}}\lesssim\|h\|_{L^{q'}L^{p'}_{k,a}}. \tag{3.13}$$

Indeed, by Minkowski's inequality [6], and Theorem 3.5, we have

$$\begin{aligned} \Big\| \int_{-\pi/2}^{\pi/2} e^{-(2\epsilon+i(\mu-\nu))\Delta_{k,a}} h(y',\mu)d\mu \Big\|_{L^p_{k,a}} &\leq \int_{-\pi/2}^{\pi/2} \Big\| e^{-(2\epsilon+i(\mu-\nu))\Delta_{k,a}} h(\cdot,\mu) \Big\|_{L^p_{k,a}} d\mu \\ &\leq \int_{-\pi/2}^{\pi/2} \frac{\|h(\cdot,\mu)\|_{L^{p'}_{k,a}}}{|\sin(\mu-\nu)|^{2\left(\frac{2\gamma+N+a-2}{a}\right)\left(\frac{1}{2}-\frac{1}{p}\right)}} d\mu, \end{aligned}$$

for $2 \leq p \leq \infty$. Let us set $\kappa(\mu) := |\sin(\mu)|^{-2\left(\frac{2\gamma+N+a-2}{a}\right)\left(\frac{1}{2}-\frac{1}{p}\right)}$. Thus we may rewrite the above second integral as a convolution of $\|h(\cdot,\cdot)\|_{L^{p'}_{k,a}}$ with the kernel κ. Moreover, the function κ belongs to the weak $L^r_W((-\pi/2,\pi/2))$ space for $r > 1$ such that $2r\left(\frac{2\gamma+N+a-2}{a}\right)\left(\frac{1}{2}-\frac{1}{p}\right) \leq 1$. That is $1 < r \leq \frac{ap}{(p-2)(2\gamma+N+a-2)}$. Hence we can apply [19, Lemma 2] to deduce that

$$\Big\| \int_{-\pi/2}^{\pi/2} e^{-(2\epsilon+i(\mu-\nu))\Delta_{k,a}} h(y',\mu)d\mu \Big\|_{L^q L^p_{k,a}} \lesssim \|h\|_{L^{q'} L^{p'}_{k,a}}$$

for $q = 2r$, with r as above, and for $1 \leq q \leq 2$. This finishes the proof of the claim (3.13). Hence, we established the following theorem.

Theorem 3.7. *Let (p,q) be an admissible pair (see (1.6 a-b) for the definition). Then, for all $h \in L^{q'}((-\pi/2,\pi/2), L^{p'}_{k,a} \cap L^2_{k,a}(\mathbb{R}^N))$, we have*

$$\Big\| \int_{-\pi/2}^{\pi/2} e^{-(\epsilon+i\mu)\Delta_{k,a}} h(\cdot,\mu)d\mu \Big\|_{L^2_{k,a}} \lesssim \|h\|_{L^{q'} L^{p'}_{k,a}}, \tag{3.14}$$

for $\epsilon \geq 0$. Here p' and q' denote the conjugate of p and q respectively.

4. Proofs of Theorems A, B and C

Theorem A is a special case of the following theorem, which concerns the Strichartz type estimate for the (k,a)-generalised Laguerre semigroup $\mathcal{I}_{k,a}(z)$ (see (3.1)).

Theorem 4.1. *Suppose $a = 1,2$ and k is a non-negative multiplicity function such that $a + 2\gamma + N - 2 > 0$. Let (p,q) be an admissible pair. Then for every $\epsilon \geq 0$, we have*

$$\|e^{-(\epsilon+i\mu)\Delta_{k,a}} f\|_{L^q L^p_{k,a}} \lesssim \|f\|_{L^2_{k,a}}. \tag{4.1}$$

Proof. If $\|f\|_{L^2_{k,a}} = \infty$ the estimate is trivial. If $\|f\|_{L^2_{k,a}} < \infty$ choose $h \in L^{q'}((-\pi/2,\pi/2), L^{p'}_{k,a} \cap L^2_{k,a}(\mathbb{R}^N))$. By the estimate (3.14), we have

$$\begin{aligned} \left| \int_{-\pi/2}^{\pi/2} \langle e^{-(\epsilon+i\mu)\Delta_{k,a}} f, h(\cdot,\mu)\rangle_{L^2_{k,a}} d\mu \right| &= \left| \int_{-\pi/2}^{\pi/2} \langle f, e^{-(\epsilon-i\mu)\Delta_{k,a}} h(\cdot,\mu)\rangle_{L^2_{k,a}} d\mu \right| \\ &\leq \|f\|_{L^2_{k,a}} \Big\| \int_{-\pi/2}^{\pi/2} e^{-(\epsilon-i\mu)\Delta_{k,a}} h(x,\mu)d\mu \Big\|_{L^2_{k,a}} \\ &\lesssim \|f\|_{L^2_{k,a}} \|h\|_{L^{q'} L^{p'}_{k,a}}, \end{aligned}$$

for an admissible pair (p,q). Above we have used the fact that $(e^{-z\Delta_{k,a}})^* = e^{-\bar{z}\Delta_{k,a}}$ (see [1, p. 33]). Now the theorem follows from the density of $L^{q'}((-\pi/2,\pi/2),L^{p'}_{k,a}\cap L^2_{k,a}(\mathbb{R}^N))$ in $L^{q'}((-\pi/2,\pi/2),L^{p'}_{k,a}(\mathbb{R}^N))$. □

Proof of Theorem B. Let (p,q) be an admissible pair and $F\in L^{q'}((-\pi/2,\pi/2),L^{p'}_{k,a}(\mathbb{R}^N))$. Let G belongs to a dense space in $L^{\tilde{q}'}((-\pi/2,\pi/2),L^{\tilde{p}'}_{k,a}(\mathbb{R}^N))$, for some admissible pair $(\tilde{p},\tilde{q})$. We consider the bilinear form

$$T(F,G)=\int_{-\pi/2}^{\pi/2}\int_0^t \langle e^{is\Delta_{k,a}}F(\cdot,s),e^{it\Delta_{k,a}}G(\cdot,t)\rangle_{L^2_{k,a}}\,dsdt.$$

By Hölder's inequality, we get

$$|T(F,G)|\le \sup_{t\in(-\pi/2,\pi/2)}\Big\|\int_0^t e^{is\Delta_{k,a}}F(x,s)ds\Big\|_{L^2_{k,a}}\|e^{it\Delta_{k,a}}G\|_{L^1L^2_{k,a}}. \tag{4.2}$$

Using Theorem 3.7 we deduce that

$$\Big\|\int_0^t e^{is\Delta_{k,a}}F(x,s)ds\Big\|_{L^2_{k,a}}\lesssim \|F\|_{L^{q'}L^{p'}_{k,a}}.$$

On the other hand, arguing as in the proof of Theorem 3.7, for the last term in (4.2), we show that

$$\|e^{it\Delta_{k,a}}G\|_{L^1L^2_{k,a}}\lesssim \|G\|_{L^{\tilde{q}'}L^{\tilde{p}'}_{k,a}}$$

holds for any admissible pair $(\tilde{p},\tilde{q})$. Thus inequality (4.2) becomes

$$|T(F,G)|\lesssim \|F\|_{L^{q'}L^{p'}_{k,a}}\|G\|_{L^{\tilde{q}'}L^{\tilde{p}'}_{k,a}}.$$

This shows that if $F\in L^{q'}((-\pi/2,\pi/2),L^{p'}_{k,a}(\mathbb{R}^N))$ for some admissible pair (p,q), then the function $\tilde{F}(t,x)=\int_0^t e^{i(s-t)\Delta_{k,a}}F(s,x)ds$ belongs to $L^{\tilde{q}}((-\pi/2,\pi/2),L^{\tilde{p}}_{k,a}(\mathbb{R}^N))$ for any admissible pair $(\tilde{p},\tilde{q})$, proving the theorem. □

Proof of Theorem C. Recall that

$$u(x,t)=e^{-it\Delta_{k,a}}f(x)-i\int_0^t e^{-i(t-s)\Delta_{k,a}}F(x,s)ds$$

is the unique solution to the inhomogeneous Cauchy problem (1.4). By Theorem A, we have

$$\|u(x,t)\|_{L^qL^p_{k,a}}\lesssim \|f\|_{L^2_{k,a}}+\Big\|\int_0^t e^{-i(t-s)\Delta_{k,a}}F(\cdot,s)(x)ds\Big\|_{L^qL^p_{k,a}},$$

for any admissible pair (p,q). If in addition $F\in L^{\tilde{q}'}((-\pi/2,\pi/2),L^{\tilde{p}'}_{k,a}(\mathbb{R}^N))$ for some admissible pair $(\tilde{p},\tilde{q})$, then by Theorem B we also have the following estimate

$$\Big\|\int_0^t e^{-i(t-s)\Delta_{k,a}}F(\cdot,s)(x)ds\Big\|_{L^qL^p_{k,a}}\lesssim \|F\|_{L^{\tilde{q}'}L^{\tilde{p}'}_{k,a}},$$

and hence the theorem. □

5. A link between solutions of the generalized Hermite–Schrödinger and Schrödinger equations

Assume that $a>0$ and that k is a non-negative multiplicity function such that $a+2\gamma+N-2>0$. For $x,y\in\mathbb{R}^N$ let

$$B_{k,a}(x,y) := e^{i\pi\left(\frac{2\gamma+N+a-2}{2a}\right)}\Lambda_{k,a}\left(x,y;i\frac{\pi}{2}\right),$$

where $\Lambda_{k,a}(x,y;z)$ is defined in (3.6). The kernel $B_{k,a}(\cdot,\cdot)$ is defined in [1] as the integral kernel of a (k,a)-generalized Fourier transform, which reduces to the Dunkl transform if $a=2$, and gives rise to a Dunkl analogue of the Hankel transform if $a=1$. We refer to [1] for more details and properties on $B_{k,a}(\cdot,\cdot)$.

In [1, Theorem 4.20], the authors obtained the following expansion formula for $B_{k,a}(\cdot,\cdot)$:

$$B_{k,a}(x,y) = e^{i\pi\left(\frac{2\gamma+N+a-2}{2a}\right)}a^{\left(\frac{2\gamma+N-2}{2a}\right)}\Gamma\left(\frac{2\gamma+N+a-2}{a}\right)\sum_{m=0}^{\infty}\sum_{j=1}^{d(m,N)}\Lambda_{k,a}^{(m)}(\|x\|,\|y\|)Y_{m,j}(\omega)Y_{m,j}(\omega'),$$

where

$$\Lambda_{k,a}^{(m)}(\|x\|,\|y\|) = e^{-i\pi\left(\frac{2m+2\gamma+N+a-2}{2a}\right)}(\|x\|\|y\|)^{-\gamma-\frac{N}{2}+1}J_{\lambda_m}\left(\frac{2}{a}\|x\|^{\frac{a}{2}}\|y\|^{\frac{a}{2}}\right).$$

In the above equation $\lambda_m=\frac{2m+2\gamma+N-2}{a}$, $\gamma=\sum_{\alpha\in\mathscr{R}^+}k_\alpha$, J_δ is the Bessel function of type δ and $Y_{m,j}$ are the k-spherical harmonics on $\mathbb{S}^{N-1}$ which form an orthonormal basis for the Hilbert space $L^2(\mathbb{S}^{N-1},d_k\vartheta_k(\omega)d\sigma(\omega))$ where d_k is defined in (2.4).

For $t>0$ let us define

$$\Gamma_{k,a}(x,y;t) = c_{k,a}^2\int_{R^N}e^{-t\frac{\|\xi\|^a}{a}}B_{k,a}(x,\xi)B_{k,a}((-1)^{\frac{2}{a}}y,\xi)\vartheta_{k,a}(\xi)d\xi, \tag{5.1}$$

where the constant $c_{k,a}$ is defined by (3.4). Then $\Gamma_{k,a}$ is the heat kernel so that the function

$$u(x,t) := \int_{\mathbb{R}^N}f(y)\Gamma_{k,a}(x,y;t)\vartheta_{k,a}(y)dy$$

solves the heat equation

$$\partial_t u(x,t) = \frac{1}{a}\|x\|^{2-a}\Delta_k u(x,t), \qquad u(x,0)=f(x). \tag{5.2}$$

This is due to the fact that

$$\|x\|^{2-a}\Delta_k^x B_{k,a}(x,\xi) = -\|\xi\|^a B_{k,a}(x,\xi), \tag{5.3}$$

where the superscript in Δ_k^x indicates the relevant variable (see [1, Theorem 5.7]).

In order to get a good formula for the heat kernel we proceed as follows. First we need to calculate the integral

$$\int_{\mathbb{S}^{N-1}}B_{k,a}(x,\xi)B_{k,a}((-1)^{\frac{2}{a}}y,\xi)\prod_{\alpha\in\mathscr{R}^+}|\langle\alpha,\eta\rangle|^{2k_\alpha}d\sigma(\eta)$$

appearing in the definition of the heat kernel. (In the above and in what follows we have written $\xi = \|\xi\|\eta$.) In view of the orthogonality of the k-spherical harmonics $Y_{m,j}$, we obtain:

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}} B_{k,a}(x,\xi)B_{k,a}((-1)^{\frac{2}{a}}y,\xi)\prod_{\alpha\in\mathscr{R}^+}|\langle\alpha,\eta\rangle|^{2k_\alpha}d\sigma(\eta)\\
&= d_k^{-1}e^{i\pi\left(\frac{2\gamma+N+a-2}{a}\right)}a^{2\left(\frac{2\gamma+N-2}{a}\right)}\Gamma\left(\frac{2\gamma+N+a-2}{a}\right)^2\\
&\quad\sum_{m=0}^{\infty}\sum_{j=1}^{d(m,N)}(-1)^{\frac{2m}{a}}\Lambda_{k,a}^{(m)}(\|x\|,\|\xi\|)\Lambda_{k,a}^{(m)}(\|y\|,\|\xi\|)Y_{m,j}(\omega)Y_{m,j}(\omega')\\
&= d_k^{-1}a^{2\left(\frac{2\gamma+N-2}{a}\right)}\Gamma\left(\frac{2\gamma+N+a-2}{a}\right)^2(\|x\|\,\|y\|)^{-\gamma-N/2+1}\|\xi\|^{-2\gamma-N+2}\\
&\quad\sum_{m=0}^{\infty}\sum_{j=1}^{d(m,N)}Y_{m,j}(\omega)Y_{m,j}(\omega')J_{\lambda_m}\left(\frac{2}{a}\|x\|^{\frac{a}{2}}\|\xi\|^{\frac{a}{2}}\right)J_{\lambda_m}\left(\frac{2}{a}\|y\|^{\frac{a}{2}}\|\xi\|^{\frac{a}{2}}\right)\\
&= d_k c_{k,a}^{-2}(\|x\|\,\|y\|)^{-\gamma-N/2+1}\|\xi\|^{-2\gamma-N+2}\\
&\quad\sum_{m=0}^{\infty}\sum_{j=1}^{d(m,N)}Y_{m,j}(\omega)Y_{m,j}(\omega')J_{\lambda_m}\left(\frac{2}{a}\|x\|^{\frac{a}{2}}\|\xi\|^{\frac{a}{2}}\right)J_{\lambda_m}\left(\frac{2}{a}\|y\|^{\frac{a}{2}}\|\xi\|^{\frac{a}{2}}\right).
\end{aligned}$$

Using the above we can rewrite the heat kernel $\Gamma_{k,a}(x,y;t)$ as

$$\Gamma_{k,a}(x,y;t) = d_k(\|x\|\,\|y\|)^{-\gamma-N/2+1}\sum_{m=0}^{\infty}\sum_{j=1}^{d(m,N)}Y_{m,j}(\omega)Y_{m,j}(\omega')I_m(t,\|x\|,\|y\|),$$

where

$$\begin{aligned}
I_m(t,\|x\|,\|y\|) &= \int_0^\infty e^{-\frac{t}{a}u^a}J_{\lambda_m}\left(\frac{2}{a}\|x\|^{\frac{a}{2}}u^{\frac{a}{2}}\right)J_{\lambda_m}\left(\frac{2}{a}\|y\|^{\frac{a}{2}}u^{\frac{a}{2}}\right)u^{a-1}du\\
&= \frac{2}{a}\int_0^\infty e^{-\frac{t}{a}v^2}J_{\lambda_m}\left(\frac{2}{a}\|x\|^{\frac{a}{2}}v\right)J_{\lambda_m}\left(\frac{2}{a}\|y\|^{\frac{a}{2}}v\right)vdv\\
&= \frac{1}{t}\exp\left(-\frac{\|x\|^a+\|y\|^a}{ta}\right)I_{\lambda_m}\left(2\frac{\|x\|^{\frac{a}{2}}\|y\|^{\frac{a}{2}}}{ta}\right).
\end{aligned}$$

The above third equality is due to the following formula

$$\int_0^\infty e^{-p^2t^2}J_\lambda(at)J_\lambda(bt)tdt = \frac{1}{2p^2}\exp\left(-\frac{a^2+b^2}{4p^2}\right)I_\lambda\left(\frac{ab}{2p^2}\right),$$

(see [28, Exercise 68]). Therefore, using [1, formulas (4.40–4.41)], we obtain

$$\begin{aligned}\Gamma_{k,a}(x,y;t) &= \frac{d_k}{a^{\left(\frac{2\gamma+N-2}{a}\right)}}\frac{1}{t^{\left(\frac{2\gamma+N-2}{a}+1\right)}}\exp\left(-\frac{\|x\|^a+\|y\|^a}{ta}\right)\\ &\quad \sum_{m=0}^{\infty}\sum_{j=1}^{d(m,N)}\left(\frac{\|x\|^{\frac{a}{2}}\|y\|^{\frac{a}{2}}}{ta}\right)^{\frac{2m}{a}}\widetilde{I}_{\lambda_m}\left(2\frac{\|x\|^{\frac{a}{2}}\|y\|^{\frac{a}{2}}}{ta}\right)Y_{m,j}(\omega)Y_{m,j}(\omega')\\ &= \frac{c_{k,a}}{t^{\left(\frac{2\gamma+N-2}{a}+1\right)}}\exp\left(-\frac{\|x\|^a+\|y\|^a}{ta}\right)\\ &\quad\times\begin{cases}\Gamma\left(\gamma+\frac{N-1}{2}\right)\widetilde{V}_k\left(\widetilde{I}_{\gamma+\frac{N-3}{2}}\left(\frac{\sqrt{2\|x\|\|y\|(1+\langle\omega,\cdot\rangle)}}{t}\right)\right)(\omega'), & (a=1),\\ \widetilde{V}_k\left(\exp\left(\frac{\|x\|\|y\|}{t}\langle\omega,\cdot\rangle\right)\right)(\omega'), & (a=2).\end{cases}\end{aligned}$$

Let us record the above calculations in the following theorem.

Theorem 5.1. *For $x,y\in\mathbb{R}^N$ and $t>0$, the heat kernel $\Gamma_{k,a}$ is given by*

$$\begin{aligned}\Gamma_{k,a}(x,y;t) &= \frac{c_{k,a}}{t^{\left(\frac{2\gamma+N-2}{a}+1\right)}}\exp\left(-\frac{\|x\|^a+\|y\|^a}{ta}\right)\\ &\quad\times\begin{cases}\Gamma\left(\gamma+\frac{N-1}{2}\right)\widetilde{V}_k\left(\widetilde{I}_{\gamma+\frac{N-3}{2}}\left(\frac{\sqrt{2\|x\|\,\|y\|(1+\langle\omega,\cdot\rangle)}}{t}\right)\right)(\omega'), & (a=1),\\ \widetilde{V}_k\left(\exp\left(\frac{\|x\|\,\|y\|}{t}\langle\omega,\cdot\rangle\right)\right)(\omega'), & (a=2).\end{cases}\end{aligned}\tag{5.4}$$

Remark 5.2. *For $a=2$, the expression of $\Gamma_{k,2}(x,y;t)$ was previously proved in* [22].

It is obvious from the above theorem that $\Gamma_{k,a}$ extends naturally to complex time arguments, where $z\in\mathbb{C}\setminus\{z\in\mathbb{R}\mid z\le 0\}$ for $a=2$, and $z\in\mathbb{C}^*$ for $a=1$.

Now, we consider the initial value problem for the Schrödinger equation

$$\begin{cases}i\partial_t\upsilon(x,t)+\frac{1}{a}\|x\|^{2-a}\Delta_k\upsilon(x,t)=0, & (x,t)\in\mathbb{R}^N\times(0,\infty),\\ \upsilon(x,0)=f(x),\end{cases}\tag{5.5}$$

where Δ_k denotes the Dunkl Laplacian (see (2.3)).

Formally, the Schrödinger equation (5.5) is obtained by the transformation $t\mapsto it$ of the heat equation (5.2) to 'imaginary time'. In other words, for all $f\in L^2_{k,a}(\mathbb{R}^N)$, the function

$$e^{i\frac{t}{a}\|x\|^{2-a}\Delta_k}f(x):=\int_{\mathbb{R}^N}f(y)\Gamma_{k,a}(x,y;it)\vartheta_{k,a}(y)dy$$

solves (5.5). In the remaining part of this section we will compare the operators $e^{-it\Delta_{k,a}}$ and $e^{i\frac{t}{a}\|x\|^{2-a}\Delta_k}$ by searching for a link between their kernels $\Lambda_{k,a}(x,y;it)$ and $\Gamma_{k,a}(x,y;it)$ respectively.

We recall from (3.5) that

$$h_a(r,s;it;\zeta) \;:=\; \frac{\exp\Big(\frac{i}{a}(r^a+s^a)\cot(t)\Big)}{i^{\left(\frac{2\gamma+N+a-2}{a}\right)}\sin(t)^{\left(\frac{2\gamma+N+a-1}{a}\right)}} \times \begin{cases} \Gamma\Big(\gamma+\dfrac{N-1}{2}\Big)\widetilde{I}_{\gamma+\frac{N-3}{2}}\left(\dfrac{\sqrt{2}(rs)^{1/2}}{i\sin(t)}(1+\zeta)^{1/2}\right), & (a=1),\\ \exp\left(\dfrac{rs\zeta}{i\sin(t)}\right), & (a=2). \end{cases} \tag{5.6}$$

That is the kernel $\Lambda_{k,a}(x,y;it)$ of $e^{-it\Delta_{k,a}}$ satisfies

$$\Lambda_{k,a}(x,y;-it) = \overline{\Lambda_{k,a}(x,y;it)}.$$

Furthermore, by Remark 3.2 we have

$$\Lambda_{k,a}(x,y;i(t+\pi)) = e^{-i\pi\left(\frac{2\gamma+N+a-2}{a}\right)}\Lambda_{k,a}((-1)^{\frac{2}{a}}x,y;it).$$

For real functions f, it follows that the $L^p_{k,a}(\mathbb{R}^N)$ norm of $e^{-it\Delta_{k,a}}f$ is even and π-periodic as a function of t, and thus determined by its values for $0<t\le\frac{\pi}{2}$. Hence the two mixed norms $\|e^{-it\Delta_{k,a}}f\|_{L^q((-\pi/2,\pi/2),L^p_{k,a}(\mathbb{R}^N))}$ and $\|e^{-it\Delta_{k,a}}f\|_{L^q((0,\pi/2),L^p_{k,a}(\mathbb{R}^N))}$ are proportional for real functions f.

Using the change of variable $s=\tan(t)$ with $t\in(0,\pi/2)$, we get

$$\Lambda_{k,a}(x,y;i\arctan s) = c_{k,a}^{-1}(1+s^2)^{\left(\frac{2\gamma+N+a-2}{2a}\right)}\exp\left(-is\frac{\|x\|^a}{a}\right)\Gamma_{k,a}\Big((1+s^2)^{\frac{1}{a}}x,y;is\Big).$$

The following is then immediate.

Theorem 5.3. *For all $f\in L^2_{k,a}(\mathbb{R}^N)$ and all $s>0$, we have*

$$e^{-i\arctan(s)\Delta_{k,a}}f(x) = (1+s^2)^{\left(\frac{2\gamma+N+a-2}{2a}\right)}\exp\left(-is\frac{\|x\|^a}{a}\right)e^{i\frac{s}{a}\|x\|^{2-a}\Delta_k}f((1+s^2)^{\frac{1}{a}}x).$$

Now we are ready to prove Theorem D.

Proof of Theorem D. Assuming $p,q<\infty$, we obtain

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}\Big\{\int_{\mathbb{R}^N}\Big|e^{-it\Delta_{k,a}}f(x)\Big|^p\vartheta_{k,a}(x)dx\Big\}^{q/p}dt\\
&=\int_0^{\infty}\Big\{\int_{\mathbb{R}^N}\Big|e^{-i\arctan(s)\Delta_{k,a}}f(x)\Big|^p\vartheta_{k,a}(x)dx\Big\}^{q/p}\frac{ds}{1+s^2}\\
&=\int_0^{\infty}\Big\{\int_{\mathbb{R}^N}\Big|e^{i\frac{s}{a}\|x\|^{2-a}\Delta_k}f((1+s^2)^{\frac{1}{a}}x)\Big|^p\vartheta_{k,a}(x)dx\Big\}^{q/p}(1+s^2)^{q\left(\frac{2\gamma+N+a-2}{2a}\right)-1}ds\\
&=\int_0^{\infty}\Big\{\int_{\mathbb{R}^N}\Big|e^{i\frac{s}{a}\|x\|^{2-a}\Delta_k}f(x)\Big|^p\vartheta_{k,a}(x)dx\Big\}^{q/p}(1+s^2)^{q\left\{\left(\frac{2\gamma+N+a-2}{2a}\right)-\frac{1}{q}-\frac{N}{ap}-\frac{1}{ap}(2\gamma+a-2)\right\}}ds.
\end{aligned}$$

If in addition the pair (p,q) satisfies

$$\left(\frac{2\gamma+N+a-2}{a}\right)\frac{1}{p}+\frac{1}{q}=\left(\frac{2\gamma+N+a-2}{2a}\right), \tag{5.7}$$

then

$$\int_0^{\frac{\pi}{2}} \Big\{ \int_{\mathbb{R}^N} \Big| e^{-it\Delta_{k,a}} f(x) \Big|^p \vartheta_{k,a}(x) dx \Big\}^{q/p} dt = \int_0^{\infty} \Big\{ \int_{\mathbb{R}^N} \Big| e^{i\frac{s}{a}\|x\|^{2-a}\Delta_k} f(x) \Big|^p \vartheta_{k,a}(x) dx \Big\}^{q/p} ds.$$

The cases when p or q is infinite are similar. □

As an immediate consequence of Theorem A and Theorem D we get a homogeneous Strichartz inequality for the Schrödinger equation (5.5) when the equality in (5.7) holds. We point out that an admissible pair (p, q) satisfying (5.7) reads $(\frac{1}{p}, \frac{1}{q})$ belongs to the line AD in Figure 1.

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