D.E

§2 NOETHERIAN RINGS AND MODULES

Let A be a ring.

(2.1) PROPOSITION. *For an A-module M the following three conditions are equivalent :*

(i) *Every submodule of M is finitely generated.*

(ii) *M satisfies the "ascending chain condition" for submodules, i.e. if $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ is any sequence of submodules of M then there exists a positive integer n such that $M_n = M_{n+1} = M_{n+2} = \ldots$.*

(iii) *Every non-empty family of submodules of M has a maximal element.*

PROOF. (i) $\Rightarrow$ (ii): Let $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ be a sequence of submodules of M. Let $N = \cup_{i=1}^{\infty} M_i$. Then N is a submodule of M and is therefore generated by a finite number of elements, say $x_1, \cdots, x_r$. There exists a positive integer n such that $x_1, \cdots, x_r \in M_n$. Therefore we have $M_n = N$, so that $M_n = M_{n+1} = \cdots$.

(ii) $\Rightarrow$ (iii): Let $\mathcal{F}$ be a non-empty family of submodules of M. Suppose $\mathcal{F}$ does not have a maximal element.Choose any $M_1 \in \mathcal{F}$. Suppose there exist $M_2, \cdots, M_n \in \mathcal{F}$ such that $M_1 \subset M_2 \subset \cdots \subset M_n$. Then, since M_n is not maximal, there exists $M_{n+1} \in \mathcal{F}$ such that $M_n \subset M_{n+1}$. Thus by induction on n we get an infinite sequence $M_1 \subset M_2 \subset M_3 \subset \cdots$ such that $M_n \neq M_{n+1}$ for every n. This contradicts (ii).

(iii) $\Rightarrow$ (i): Let N be a submodule of M. Let $\mathcal{F}$ be the family of all finitely generated submodules of N. Since $0 \in \mathcal{F}$, $\mathcal{F}$ is non-empty. Therefore $\mathcal{F}$ has a maximal element, say N'. If $N' \neq N$ then there exists $x \in N$, $x \notin N'$. The submodule $N' + Ax$ of N is finitely generated and contains N' properly. This is a contradiction. Therefore $N' = N$ and N is finitely generated. ■

(2.2) DEFINITION. An A-module M is said to be **noetherian** if it satisfies the equivalent conditions of the above Proposition. A ring A is a **noetherian ring** if it is noetherian as an A-module.

(2.3) EXAMPLES. Let K be a field.

(2.3.1) A vector space V over K is noetherian if and only $\mathrm{Dim}_K(V) < \infty$.

(2.3.2) Every princicipal ideal domain is noetherian. In particular, the polynomial ring $K[X]$ and the formal power series ring $K[[X]]$ are noetherian.

(2.3.3) If A is a noetherian ring and $\underline{a}$ is an ideal of A. Then $A/\underline{a}$ is a noetherian ring.

(2.4) PROPOSITION. *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of A-modules. Then M is noetherian if and only if both M' and M'' are noetherian.*

PROOF. Suppose M is noetherian. Since M' is isomorphic to a submodule of M, M' is noetherian. Let N'' be a submodule of M''. Then $g^{-1}(N'')$ is a submodule of M. Therefore there exist $x_1, \cdots, x_r \in g^{-1}(N'')$ such that $g^{-1}(N'')$ is generated by $x_1, \cdots, x_r$.

Since g is surjective, we have $N'' = g(g^{-1}(N''))$. It follows that N'' is generated by $g(x_1), \cdots, g(x_r)$. Thus M'' is noetherian.

Conversely, suppose M' and M'' are noetherian. Let N be a submodule of M. Then $g(N)$ is a submodule of M''. Therefore there exist $x_1, \cdots, x_r \in N$ such that $g(x_1), \cdots, g(x_r)$ generate $g(N)$. Next, $f^{-1}(N)$ is a submodule of M'. Therefore there exist $y_1, \cdots, y_s \in f^{-1}(N)$ such that $f^{-1}(N)$ is generated by $y_1, \cdots, y_s$. We claim that N is generated by $x_1, \cdots, x_r, f(y_1), \cdots, f(y_s)$. For, let $z \in N$. Then $g(z) = \sum_{i=1}^{r} a_i g(x_i)$ with $a_1, \cdots, a_r \in A$. Let $z' = z - \sum_{i=1}^{r} a_i x_i$. Then $z' \in N \cap \mathrm{Ker}(g) = N \cap \Im(f)$. Therefore $z' = f(x')$ with $x' \in f^{-1}(N)$. There exist $b_1, \cdots, b_s \in A$ such that $x' = \sum_{j=1}^{s} b_j y_j$. Thus $z = \sum_{i=1}^{r} a_i x_i + \sum_{j=1}^{s} b_j f(y_j)$. This proves our claim. ∎

(2.5) COROLLARY. *Let N be a submodule of an A-module M. Then M is noetherian if and only if both N and M/N are noetherian.*

(2.6) COROLLARY. *Finite direct sum of noetherian modules is noetherian.*

(2.7) COROLLARY. *Let A be a noetherian ring and let M be a finitely generated A-module. Then M is noetherian.*

PROOF. . Suppose M is generated by $x_1, \cdots, x_r$. We prove the assertion by induction on r. First suppose $r = 1$. Let $g : A \to M$ be the map defined by $g(a) = ax_1$. Then g is a surjective homomorphism and it follows from Proposition (2.4) that M is noetherian. Now, suppose $r \geq 2$. Let $M' = Ax_r$. Let $g : M \to M/M'$ be the natural surjection. Then M/M' is generated by $g(x_1), \cdots, g(x_{r-1})$. Therefore by induction both M' and M/M' are noetherian. Therefore M is noetherian by Corollary (2.5). ∎

(2.8) PROPOSITION. *Let S be a multiplicative subset of A and let M be a noetherian A-module. Then $S^{-1}M$ is a noetherian $S^{-1}A$-module.*

PROOF. Let N be an $S^{-1}A$-submodule of $S^{-1}M$. Then $N \cap M$ is an A-submodule of M and therefore generated by finitely many elements, say $x_1, \cdots, x_r$. Since $S^{-1}(N \cap M) = N$, it follows that N is generated as an $S^{-1}A$-module by $x_1/1, \cdots, x_r/1$. ∎

(2.9) COROLLARY. *Let S be a multiplicative subset of a noetherian ring A. Then $S^{-1}A$ is noetherian. In particular, the localization of a noetherian ring at a prime ideal is noetherian.*

(2.10) HILBERT'S BASIS THEOREM. *Let A be a noetherian ring. Then the polynomial ring $A[X_1, \cdots, X_n]$ in n variables over A is also noetherian.*

PROOF. . By induction on n, it is sufficient to prove the theorem for $n = 1$, i.e. that the polynomial ring $B = A[X]$ in one variable is noetherian. Let $\underline{b}$ be any ideal of B. We will show that $\underline{b}$ is finitely generated. we may assume that $\underline{b} \neq 0$. Suppose that $\underline{b}$ is not finitely generated. Then choose $f_1, f_2, f_3, \cdots$ inductively such that f_n is of smallest degree in $\underline{b} \setminus \sum_{i=1}^{n-1} Bf_i$. Let $d_n := deg(f_n)$ and $a_n :=$ leading coefficient of f. Then $d_1 \leq d_2 \leq \cdots$. Since A is noetherian , There exists a positive integer m such that $a_m \in \sum_{i=1}^{m-1} Aa_i$. Write $a_m = \sum_{i=1}^{m-1} \alpha_i a_i$ with $\alpha_i \in A$. Let $g := f_m - \sum_{i=0}^{m-1} \alpha_i X^{d_m - d_i} f_i$. Then $g \in \underline{b} \setminus \sum_{i=1}^{m-1} Bf_i$ and $deg(g) < d_m$. This contradicts the choice of f_m. Therefore $\underline{b}$ is finitely generated. ∎

(2.11) COROLLARY. *Let A be a noetherian ring and B a finitely generated A-algebra. Then B is noetherian.*

PROOF. Since every finitely generated A-algebra is a quotient of a polynomial ring $A[X_1, \cdots, X_n]$, the Corollary follows from the above theorem and example (2.3.3). ∎

(2.12) EXERCISES. Let M be an A-module.

(2.12.1) Let B be a subring of A, so that M is also a B-module. If M is noetherain as a B-module then M is noetherian as an A-module.

(2.12.2) If M is a noetherian A-module. Show that any surjective A-endomorphism of M is an isomorphism.

(2.12.3) If M is a noetherian A-module then $A/ann_A(M)$ is a noetherian ring.

(2.12.4) Let A be a non-noetherian ring and let $\mathcal{F}$ be the set of ideals in A which are not finitely generated. Show that $\mathcal{F}$ has maximal elements and that the maximal elements of $\mathcal{F}$ are prime ideals. Therefore deduce that:

(I. S. Cohen) A ring A is noetherian if and only if every prime ideal of A is finitely generated.

(2.12.5) If $A_{\underline{\mathbf{p}}}$ is noetherian for every $\underline{\mathbf{p}} \in \text{Spec}(A)$ then is A necessarily noetherian ?

(2.12.6) If B is a noetherian faithfully flat A-algebra then show that A is noetherian.

(2.12.7) Let $B = A[[X]]$ be the formal power series ring over A. Let $\underline{\mathbf{P}} \in \text{Spec}(B)$. and $\underline{\mathbf{p}} = \{f(0) \mid f \in \underline{\mathbf{P}}\}$. Show that $\underline{\mathbf{p}}$ is a prime ideal of A and if $\underline{\mathbf{p}}$ is generated by r elements then $\underline{\mathbf{P}}$ can be generated by $r+1$ elements.

Deduce that, if A is noetherian then the formal power series ring $B = A[[X_1, \cdots, X_n]]$ in n variables over A is also noetherian. (Hint: Use (2.12.4).)