Bonds and Options Valuation using a conditioning factor approach

Sankarshan Basu[1]

Abstract

This paper looks at methods to calculate prices (or approximate prices) of bonds (zero coupon as well as coupon bearing) where the interest rates follow a log-normal distribution using two different ways - the first method makes use of a conditioning variable, while the second method (only applicable in case of the zero - coupon bond case) is by making use of a a direct expansion technique.

Further, the conditioning factor method is used to value European options on assets with stochastic volatility.

1 Introduction

Bond valuation has been one of the most important aspects of finance, especially with stochastic interest rates. Another important problem that deserves particular attention is the issue of pricing of options on assets with stochastic volatility. In this paper, we look at some problems related to these issues.

The first of the two problems is that of pricing a bond (first a zero coupon bond and then two cases of the coupon bearing bond - non defaultable as well as defaultable) with non-negative interest rates. We assume a log-normal model for the interest rate, thereby ensuring non-negative interest rates. Thus, the instantaneous rate of interest r_s (interest rate process) is defined by

$$r_s = be^{X_s}, \quad \text{and} \quad X_s = \mu_s + Y_s$$

where b is a scaling constant, $\{Y_s; 0 \leq s \leq T\}$ is a Gaussian process with zero mean and μ_s is the drift of X_s. The price of a zero coupon bond is given by

$$E(e^{-b\int_0^1 e^{X_s} ds}), \tag{1}$$

where X_s is as defined earlier. The exponential nature of the model ensures that interest rates do not go negative since negative interest rates are unrealistic and could lead to undesirable consequences,

---

[1] Indian Institute of Management, Bangalore, India. Email: sankarshanb@iimb.ernet.in

as outlined by Rogers (1995). This can be put in the framework of Heath, Jarrow and Morton (1992) and is also an extension of Black and Karasinski (1991) and Black, Derman and Toy (1990).

The second problem is that of valuing European call options on assets with stochastic volatility. Thus, we have

$$dX_t = rX_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\rho dB_t^{(1)} + \sqrt{1-\rho^2} dB_t^{(2)}], \tag{2}$$

$$dV_t = \mu dt + dB_t^{(1)}, \tag{3}$$

$$\underline{\text{or}} \quad dV_t = -aV_t dt + dB_t^{(1)} \tag{4}$$

where X_t is the price process and V_t is the volatility process. r is the rate of interest and $B_t^{(1)}$ and $B_t^{(2)}$ are two independent standard Brownian motions. The volatility process V_t can follow a simple Brownian motion (equation (3)), μ being the drift of the Brownian motion or an Ornstein - Uhlenbeck process (equation (4)), a being the force of mean reversion the Ornstein - Uhlenbeck process. Further, ρ is the correlation between V_t and the logarithm of X_t. In this situation, the interest rate r is treated as a constant. Here, we are interested in the price of an European call option given by

$$X_0\{e^{-r}E(e^{Y_T} - b)^+\}, \tag{5}$$

where b is the strike price of the option is calculated, X_0 is the current price of the asset and $Y_t = ln(\frac{X_t}{X_0})$.

The common strand is that both problems essentially involve the evaluation of some functions of integrals of log-normal processes. We now look at the ways to solve these problems. The most standard way to solve these problems have been by the use of numerical solutions to the relevant set of partial differential equations. This may not only be inaccurate but also time consuming and hence if possible should be avoided. We thus look at a new way of solving these problems - we make use of a conditioning factor approach.

The paper is structured as follows: we first look at some of the work done already in this area and then briefly discuss the concept of the conditioning factor used and then go on to identify the use of the most appropriate conditioning factor. Having identified the conditioning factor, we look at pricing of the zero coupon bonds followed by pricing of coupon bearing bonds. We finally discuss the issue of pricing of the European options. The results of each of the cases are detailed in the various tables given at the end.

2 Previous Work

The last 30 years has seen a lot of work relevant to what we are discussing here. Some of the more important ones are briefly outlined here. Notable work on modelling interest rates and pricing of bonds have been carried out by Vasicek (1977), Black, Derman and Toy (1990), Black and Karasinski (1991), Hull and White (1990, 1993, Fall 1994, Winter 1994, 1996) and Heath, Jarrow and Morton (1992). All these papers mentioned above model the interest rate as either a normal distribution or a log-normal distribution. The choice of a log-normal distribution of interest rates have also been used by Goldys, Musiela and Sondermann (1994), Sandermann, Sondermann and Miltersen (1994) and Brace, Gatarek and Musiela (1997). However, the basis of research in this field has not been restricted only to the Gaussian set-up - the most significant work looking at the term structure of interest rates in a non- Gaussian framework is by Cox, Ingersoll and Ross (1985). Most of the contributions referred above deal with the *one - factor* model. However, work has been done on the *multi - factor* model as well; prominent among them are Duffie and Kan (1994, 1996) and Longstaff and Schwartz (1992a, 1992b).

In terms of research in option pricing, one of the earliest pioneering works in this field has been by Black and Scholes (1973) followed by Merton (1973), Rubenstein (1976), Hull and White (1987), Rogers and Shi (1995), Heston (1993), Jarrow and Rudd (1982), Stein and Stein (1991), Wiggins (1987), Willard (1996) and Romano and Touzi (1997). Note that while Black and Scholes, Merton and Rubenstein assumed a constant volatility of the price process, this assumption may not be the most realistic assumption - in most practical situations, the volatility of the price process is stochastic in nature (either a Brownian motion or an Ornstein - Uhlenbeck process). This general framework was introduced by Vasicek (1977). Baxter and Rennie (1996) outlines a number of modifications to the volatility process used. The work of Harrison and Kreps (1979) and Harrison and Pliska (1981) on the use of martingales and stochastic integrals in financial applications, especially in the securities market and in continuous trading is also very important.

3 Conditioning factor

We now discuss the concept of conditioning factor that we are going to use to solve the problems defined earlier in equation (1) and equation (5). The idea of a conditioning factor was first proposed by Rogers and Shi (1995). For any convex function *f*,

$$E(f(Y)) = E(E(f(Y)|Z)) \geq E(f(E(Y|Z))). \tag{6}$$

The first part of equation (6) is trivially true while the second part is Jensen's inequality. Thus, one can easily obtain the lower bound to the function - in the cases we discuss, the function, *f*, is the price of the relevant product.

This is similar to Rogers and Shi and in their case, the function *f* was *f(x) = max(x-k, 0).* The main concern here is about the choice of Z, the conditioning factor - we discuss ways to select Z later in this section. The Z used by Rogers and Shi is of the form

$$Z = \int_0^T B_s ds. \tag{7}$$

According to Rogers and Shi, they had investigated numerically several possible choices for Z, some of them bivariate. However, they found that the best choice was the one defined by equation (7).

Now, the lower bound on equation (6) is not guaranteed to be good. However, the estimate of the error can be made using the following approach. We have, for any random variable U,

$$0 \leq E(U^+) - E(U)^+ \leq \frac{1}{2}\sqrt{\mathrm{Var}(U)}.$$

Thus, using this, one can find the upper bound to the price. As a follow up to Rogers and Shi's work, Thompson (1999) has developed a method to refine the upper bound to the price of the Asian option.

Basu (1999) has provided with a mathematical justification to the choice of Z, the conditioning factor. It is indeed a fact that the form of Z defined by Rogers and Shi does work out to be the most accurate conditioning factor in terms of the error committed (the error is the least by using this form of a conditioning factor). In fact, in the general case, the conditioning factor, Z can be written as

$$Z \propto \int_0^T Y_s ds \qquad \Rightarrow \qquad Z = \frac{\int_0^T Y_s ds}{\sqrt{\mathrm{Var}(\int_0^T Y_s ds)}}, \tag{8}$$

where $\{Y_s; 0 \leq s \leq T\}$ is a Gaussian process with zero mean and variance of 1; i.e. $Z \sim N(0,1)$ distribution. We shall use this form of the conditioning factor as defined in equation (8) throughout the rest of the paper. In some cases, we might need to make some modification on the form of Z - we shall highlight that in the relevant cases. Also, for all calculations in the paper we assume, without loss of generality, $T = 1$.

4 Pricing of Bonds

4.1 Zero Coupon Bonds

We first look at pricing of zero coupon bonds - bonds that make only one payment at maturity. The primary method of pricing is based on using the conditioning factor as defined in equation (8). However, we also look at the alternative method by a direct expansion technique.

4.1.1 Pricing using conditioning factor

We adopt a log-normal model for interest rates similar to the approach of Goldys, Musiela and Sondermann (1994), Sandermann, Sondermann and Miltersen (1994) and Brace, Gatarek and Musiela (1997). The log-normal model ensures that the interest rates cannot go negative.

Let the instantaneous rate of interest r_t be given by

$$r_t = be^{\mu_t + Y_t}$$

where Y_t is a Gaussian process with zero mean and the variance - covariance given by

$$\mathrm{Cov}(Y_u, Y_v) = \sigma_{uv};$$

μ_t is the drift of Y_t and is deterministic in nature. Also, b is a scaling factor whose importance will become apparent later. This can be put in the framework of Heath, Jarrow and Morton (1992) as shown by Baxter and Rennie (1996) and is also an extension of Black and Karasinski (1991), Black, Derman and Toy (1990), Hull and White (1990) as well as a modification of Vasicek (1977).

A generalized version of the problem is the calculation of

$$E\left[f\left(\int_0^1 \{Y_s + \mu_s\}\, ds\right)\right],$$

where, f is a convex function. Thus, in particular the price of the bond (f(x) $= e^{-bx}$) is given by

$$E\left(\exp\left\{-b\int_0^1 \exp\left(Y_s + \mu_s\right)\right\} ds\right). \qquad (9)$$

We look at pricing the bond by using the conditioning factor described in equation (8) - in effect we calculate the lower and the upper bound of the price of a zero coupon bond explicitly; the true price has to be between the bounds and if the bounds merge than the common value is the true price. Now, using equation (6) and the conditioning factor Z as defined in equation (8), we can obtain the lower bound to the price of the bond. In case of the zero coupon bonds, the upper bound to the price can be easily obtained as shown later.

Now, conditionally on Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z, \tag{10}$$

$$\text{where} \quad k_u = \text{Cov}(Y_u, Z) = \frac{\int_0^1 \text{Cov}(Y_u, Y_s)ds}{\sqrt{\text{Var}(\int_0^1 Y_s ds)}} \tag{11}$$

$$\text{and} \quad \text{Cov}(Y_u, Y_v|Z) = \sigma_{uv} - k_u k_v = w_{uv} \quad \text{say.} \tag{12}$$

We are interested in calculating a lower bound (LB_1) and the corresponding upper bound (UB_1). We do that by considering the following argument. There exists some random variable ξ such that

$$E(f(X)) = E[f(E(X|Z))] + E[(X - E(X|Z))f'(E(X|Z))] + \frac{1}{2}E[(X - E(X|Z))^2 f''(\xi)],$$

$$\Rightarrow E[f(E(X|Z))] \leq E(f(X)) \leq E[f(E(X|Z))] + \frac{1}{2}E(X - E(X|Z))^2 \sup_{x \geq 0} f''(x).$$

Thus, in the case where f(x) = e^{-bx}, a lower bound is given by

$$\text{LB}_1 = E[f(E(X|Z))] \tag{13}$$

and an upper bound is given by

$$\text{UB}_1 = \text{LB}_1 + \frac{1}{2}b^2 \text{E}(\text{Var}(X|Z)), \tag{14}$$

since $\sup_{x \geq 0} f''(x) = b^2$. Also, here $X = \int_0^1 e^{Y_s + \mu_s} ds$. Thus,

$$E[\text{Var}(\int_0^1 e^{Y_s+\mu_s} ds|Z)] = \int_0^1 \int_0^1 \exp\left(\frac{1}{2}[k_u + k_v]^2 + \frac{1}{2}[w_{uu} + w_{vv}]\right)(e^{w_{uv}} - 1)\, dudv. \tag{15}$$

Let us define

$$h(z) = E(\int_0^1 e^{Y_s+\mu_s} ds|Z = z) = \int_0^1 e^{k_u z + \frac{1}{2}w_{uu}} du. \tag{16}$$

where $Z \sim N(0,1)$. The lower bound to the price of the bond is given by

$$\text{LB}_1 = \int_{-\infty}^{\infty} e^{-bh(z)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{17}$$

and the corresponding upper bound is given by

$$\text{UB}_1 = \text{LB}_1 + \frac{b^2}{2} \int_0^1 \int_0^1 \exp\left(\frac{1}{2}[k_u + k_v]^2 + \frac{1}{2}[w_{uu} + w_{vv}]\right)(e^{w_{uv}} - 1)\, dvdu. \tag{18}$$

Finally, to calculate the bounds defined by equations (17) and (18), we make use of a numerical integration procedure.

We present the exact form of k_u and w_{uv} (where k_u and w_{uv} is defined by equations (11) and (12) respectively) for three special cases; first the Geometric Brownian Motion, then an exponential function of an Ornstein - Uhlenbeck process with the initial value known and finally when the initial value of the Ornstein - Uhlenbeck process has a stationary distribution. Once these are known for each of the cases, the corresponding bounds can be easily obtained by substituting in equations (17) and (18) and finally carrying out a numerical integration.

Simple Brownian Motion case: In this case, we have,

$$r_t = be^{at+Y_t} \qquad \text{and} \qquad Y_t = \sigma B_t, \tag{19}$$

where B_t is a standard Brownian motion, $t = 1$ and $b = r_0$ is the initial value of the interest rate. The bond price is

$$E\left[\exp\left(-b\int_0^1 \exp\{\sigma B_s + as\}ds\right)\right]. \tag{20}$$

The conditioning factor is $Z = \frac{\int_0^1 B_s ds}{\sqrt{\text{Var}(\int_0^1 B_s ds)}}$, where B_s is a standard Brownian Motion. Here, $\sigma_{uv} = \sigma^2(u \wedge v)$, $\Rightarrow \sigma_{uu} = \sigma^2 u$. Also, $\mu_u = au$.

Now,

$$\text{Var}(\int_0^1 B_s ds) = \frac{1}{3} \quad \Rightarrow \quad k_u = \text{Cov}(B_u, Z) = \sqrt{3}\sigma\int_0^u (1-s)ds = \sqrt{3}\sigma(u - \frac{u^2}{2}). \tag{21}$$

Conditioning on Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = \mu u + k_u Z, \tag{22}$$

$$\text{and} \quad \text{Cov}(Y_u, Y_v|Z) = \sigma^2(u \wedge v) - k_u k_v = w_{uv}. \tag{23}$$

Ornstein - Uhlenbeck Process - Initial Value is known: Now, let us consider the case where the interest rate $\{r_s; 0 \leq s \leq 1\}$ is governed by an exponential function of the Ornstein Uhlenbeck process $\{Y_s; 0 \leq s \leq 1\}$ with the initial value Y_0 known and assumed to be 0. The interest rate model is thus defined as

$$r_t = be^{Y_t},$$

where Y_t is the solution of the stochastic differential equation

$$dY_t = -aY_t dt + \sigma dB_t \qquad \Rightarrow \qquad Y_t = \sigma\int_0^t e^{-a(s-u)}dB_u. \tag{24}$$

B_t is a standard Brownian motion and $t = 1 \Rightarrow \{r_t = be^{Y_t} = e^{lnb+Y_t}; 0 \leq t \leq 1\}$. Thus, *ln* b is the long term mean of the logarithm of the interest rate process; hence, $be^{\frac{1}{2}\frac{\sigma^2}{2a}}$ is the long term value of the interest rate. Also, $b = r_0$, the initial value of the interest rate. In this case, $\sigma_{uv} = \frac{\sigma^2}{2a}[e^{a|u-v|} - e^{-a(u+v)}]$. Further, $\mu_u = 0$ and the conditioning factor is $Z = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}$. We thus have,

$$\mathrm{Var}(\int_0^1 Y_s ds) = \frac{\sigma^2}{2a}\frac{2a + 4e^{-a} - e^{-2a} - 3}{a^2} = V, \quad \text{say} \tag{25}$$

and

$$k_u = \mathrm{Cov}(Y_u, Z) = \frac{1}{\sqrt{V}}\frac{\sigma^2}{2a}\left\{\frac{1-e^{-au}}{a} + \frac{1-e^{-a(1-u)}}{a} - \frac{e^{-au} - e^{-a(1+u)}}{a}\right\}. \tag{26}$$

So, conditional on Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z, \tag{27}$$

$$\text{and} \quad \mathrm{Cov}(Y_u, Y_v|Z) = \frac{\sigma^2}{2a}[e^{a|u-v|} - e^{-a(u+v)}] - k_u k_v = w_{uv}. \tag{28}$$

Ornstein Uhlenbeck process - Initial value has a stationary distribution: The initial value of the process has a stationary distribution, the distribution being $N(0, \frac{\sigma^2}{2a})$. Here, Y_t is the solution of the stochastic differential equation

$$dY_t = -aY_t dt + \sigma dB_t \qquad \Rightarrow \qquad Y_t = \sigma\int_{-\infty}^{t} e^{-a(s-u)} dB_u \tag{29}$$

where B_t is a standard Brownian motion and t = *1*. Now,

$$r_t = be^{Y_t} = e^{lnb + Y_t}.$$

Thus, *ln b* is the long term mean of the logarithm of the interest rate process. Hence, $be^{\frac{1}{2}\frac{\sigma^2}{2a}}$ is the long term value of the interest rate. Also, $\sigma_{uv} = \frac{\sigma^2}{2a}e^{-a|u-v|}$ and $\mu_u = 0$ and the conditioning factor is $Z = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}$. Note that in this case Y_s ranges between $(-\infty, s)$ unlike (0, s) in the earlier case of the non-stationary Ornstein Uhlenbeck process.. Thus, we have

$$\mathrm{Var}(\int_0^1 Y_s ds) = \frac{\sigma^2}{a}\frac{a + e^{-a} - 1}{a^2} = V_1 \quad \text{say}, \tag{30}$$

and

$$k_u = \mathrm{Cov}(Y_u, Z) = \frac{1}{\sqrt{V_1}}\frac{\sigma^2}{2a}[\frac{1-e^{-au}}{a} + \frac{1-e^{-a(1-u)}}{a}]. \tag{31}$$

Once again, we have that given Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z, \tag{32}$$

$$\text{and} \quad \mathrm{Cov}(Y_u, Y_v|Z) = \frac{\sigma^2}{2a}e^{-a|u-v|} - k_u k_v = w_{uv}, \quad \text{say}. \tag{33}$$

Note: In all these three cases, once we have the values for k_u, $E(Y_u|Z)$ and $\mathrm{Var}(Y_u, Y_v|Z)$ and further $\mu_t = at$ for the Brownian motion case, we can easily calculate the bounds to the price of the bond. The results are shown later.

4.1.2 Pricing via direct expansion

To compare the results that we obtain by using the conditioning factor, we calculate the bounds to the price of a zero coupon bond using a direct method for finding bounds. In this case, we use a Taylor series expansion and the fact that for $x \geq 0$, we have $e^{-x} > 1 - x$, $e^{-x} < 1 - x + \frac{x^2}{2}$, $e^{-x} > 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$, and so on. We will use the last two inequalities as the bounds suggested are very close to each other. Here, we have,

$$1 - bI_1 + \frac{1}{2}b^2 I_2 - \frac{1}{6}b^3 I_3 \leq E[e^{-b\int_0^1 e^{Y_s+\mu_s} ds}] \leq 1 - bI_1 + \frac{1}{2}b^2 I_2, \tag{34}$$

where, I_k, is a k_{th} order integral and is given by

$$I_k = E\left[\int_0^1 \text{........} \int_0^1 \exp\left(Y_s + \mu_s\right) ds\right]^k, \quad k = 0, 1, 2, \ldots.$$

Thus, the lower bound is given by

$$1 - bI_1 + \frac{1}{2}b^2 I_2 - \frac{1}{6}b^3 I_3 \tag{35}$$

and the corresponding upper bound is

$$1 - bI_1 + \frac{1}{2}b^2 I_2 \tag{36}$$

We obtain the expressions of I_1, I_2 and I_3 for the same cases used earlier and in this case denote the upper bound by UB_2 and the lower bound by LB_2. The results are shown later.

Simple Brownian Motion case: Here $\sigma_{ss} = \sigma^2 s$, $\sigma_{us} = \sigma^2(u \wedge s)$ and $\mu_s = as$. Thus,

$$I_1 = \int_0^1 \exp\left(as + \frac{1}{2}\sigma^2 s\right) ds, \quad I_2 = 2\int_0^1 \int_0^u \exp\left(as + au + \frac{3}{2}\sigma^2 s + \frac{1}{2}\sigma^2 u\right) dsdu$$

$$\text{and} \quad I_3 = 6\int_0^1 \int_0^u \int_0^v \exp\left(au + av + as + \frac{1}{2}\sigma^2 u + \frac{1}{2}\sigma^2 v + \frac{1}{2}\sigma^2 s + \sigma^2 v + 2\sigma^2 s\right) dsdvdu.$$

Ornstein - Uhlenbeck Case - Initial value following a stationary distribution: In this case, $\mu_s = 0$, $\mathrm{Var}(Y_s) = \frac{\sigma^2}{2a} = \sigma_{ss}$, and $\mathrm{Cov}(Y_u, Y_v) = \frac{\sigma^2}{2a} e^{-a|u-v|} = \sigma_{uv}$. Thus

$$I_1 = e^{\frac{1}{2}\frac{\sigma^2}{2a}}, \qquad I_2 = 2e^{\frac{\sigma^2}{2a}} \int_0^1 (1 - w) e^{\frac{\sigma^2}{2a} e^{-aw}} dw,$$

$$\text{and} \quad I_3 = 6e^{\frac{3}{2}\frac{\sigma^2}{2a}} \int_0^1 (1-r) \int_0^r \exp\left(\frac{\sigma^2}{2a}\left[e^{-ar} + e^{-aw} + e^{-a(r-w)}\right]\right) dwdr.$$

Ornstein - Uhlenbeck Case - Initial value is known: This is the non-stationary Ornstein - Uhlenbeck case with $\mu_s = 0$, $\mathrm{Var}(Y_s) = \frac{\sigma^2}{2a}(1-e^{-2as}) = \sigma_{ss}$ and $\mathrm{Cov}(Y_u, Y_v) = \frac{\sigma^2}{2a}[e^{a|u-v|} - e^{-a(u+v)}] = \sigma_{uv}$. So,

$$I_1 = \int_0^1 e^{\frac{1}{2}\sigma^2\frac{1-e^{-2au}}{2a}} du, \quad I_2 = 2\int_0^1 \int_0^u e^{\frac{1}{2}\sigma^2\frac{1-e^{-2au}}{2a} + \frac{1}{2}\sigma^2\frac{1-e^{-2av}}{2a} + \sigma^2\frac{e^{a(u-v)} - e^{-a(u+v)}}{2a}} dvdu$$

$$\text{and} \quad I_3 = \int_0^1 \int_0^1 \int_0^1 e^{\frac{1}{2}(\sigma_{ss}+\sigma_{vv}+\sigma_{uu})+\sigma_{uv}+\sigma_{us}+\sigma_{vs}} dsdvdu.$$

For all these cases, once we have I_1, I_2 and I_3 we can easily calculate the bounds to the prices using equations (35) and (36).

4.1.3 Comments of Zero Coupon Bond Pricing

The lower bounds to the price of the bonds calculated by using the conditioning factor are so close to the actual price (in some cases, the simulated prices were lower than the lower bounds) that they can be regarded as a very good approximation to the true value. This is true for all situations.

An advantage of using a conditioning factor in the calculation of the bond prices is that the method works even for large values of σ. This is not the case when using the direct expansion method; here, for higher values of σ, the values start diverging quite fast, thereby causing the whole system to break down. Further, the method using conditioning factors can be easily modified to calculate the value of a contingent payment defined on the price of a bond which is not possible in the case of the direct expansion technique.

4.2 Coupon Bearing Bonds

We now look at the situation of the bond making coupon payments during the life of the bond. Note that the coupon is payable at a continuous rate. We look at two cases in particular:

- Non-defaultable Bonds: These are essentially sovereign bonds in domestic currency, which pay all coupons during the life of the bond as well as the principal on maturity - in other words the default risk is zero.
- Defaultable Bonds: These are bonds with some positive probability of default - essentially the corporate bonds in the markets.

We use a conditioning factor as defined in equation (8) to find a lower bound of the price of the bond. The interest rate is assumed to be governed by a stochastic process - here, we assume the stochastic process to be an Ornstein - Uhlenbeck process where the initial value is known. The results are very similar for the interest rate process following any other stochastic process, the methodology being exactly the same.

4.2.1 Pricing of Non-defaultable bonds

Here we want to calculate,

$$E\left[C\int_0^T e^{-\int_0^s r_u du}ds + e^{-\int_0^T r_u du}\right] = E\left[C\int_0^T e^{-\int_0^s r_u du}ds\right] + E\left[e^{-\int_0^T r_u du}\right], \qquad (37)$$

where, $E\left[C\int_0^T e^{-\int_0^s r_u du}ds\right]$ is the value of the coupon and $E\left[e^{-\int_0^T r_u du}\right]$ is the value of the principal.

As before,

$$r_t = be^{\sigma Y_t} \qquad \text{and} \qquad Y_t = \int_0^t e^{-a(t-s)}dB_s,$$

where, r_t is the instantaneous rate of interest, σ the instantaneous variance and Y_t is an Ornstein - Uhlenbeck process with the initial value known and assumed to be 0. b is a scaling constant. Also, $b = r_0$, the initial value of the interest rate and $be^{\frac{1}{2}\frac{\sigma^2}{2a}}$ is the long-term value of the interest rate. Further, C is the coupon rate and b is the discount factor.

Here, the two quantities that we want to calculate are; the value of the coupon payments and the principal. The calculation of the value of the principal is exactly the same as calculating the value of a zero coupon bond, the details are discussed in the previous section. To calculate the value of the coupon payments we again make use of a conditioning factor, slightly adjusted from the form described in equation (8) - essentially adjusting for the continuous coupon payments and is given by and with $T = 1$, we have

$$Z_1 = \frac{\int_0^1\int_0^t Y_s dsdt}{\sqrt{\text{Var}(\int_0^1\int_0^t Y_s dsdt)}}.$$

As stated earlier, $\{Y_s; 0 \le s \le 1\}$ is an Ornstein - Uhlenbeck process with the initial value $Y_0 = 0$.

Calculation of interest payments: Once we have obtained the conditioning factor as above, we can then easily calculate the value of the coupon payment. We have,

$$\text{Var}(\int_0^1\int_0^s Y_u duds) = \frac{\sigma^2}{6a^5}\left[3 - 3e^{-2a} - 12ae^{-a} - 6a^2 + 6a\right] = V_{NS} \quad say.$$

Further, Z_1 is distributed as a standard normal variable. Conditionally on Z_1, Y_u is a Gaussian process with

$$E(Y_u|Z_1) = k_u Z_1 \tag{38}$$

$$\text{where} \quad k_u = \text{Cov}(Y_u, Z_1) = \frac{1}{\sqrt{V_{NS}}} \frac{\sigma^2}{2a} \frac{e^{-a(1-u)} + 2a(1-u) - e^{-a(1+u)} - 2ae^{-au}}{a^2}. \tag{39}$$

$$\text{Also,} \qquad \text{Cov}(Y_u, Y_v|Z_1) = \text{Cov}(Y_u, Y_v) - k_u k_v = \frac{\sigma^2}{2a}\left[e^{a|u-v|} - e^{-a(u+v)}\right] - k_u k_v = w_{uv} \quad \text{say.} \tag{40}$$

Once we have these values, then we can easily calculate the the value of the coupon payments. So, conditionally on Z_1, we have the lower bound of the value of the intermediate payment given as

$$C\int_0^1 \exp\left(-b\int_0^u \exp\left[k_s Z_1 + \frac{1}{2}w_{ss}\right] ds\right) du = Ch_1(Z_1) \quad \text{say.} \tag{41}$$

Finally, the lower bound of the value of the bond is given as

$$C\int_{-\infty}^{\infty} h_1(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} dz + \text{Value of Principal} = CH_1 + H_2,$$

where C is the coupon rate, CH_1 is the value of coupon payments and H_2 is the value of the final payments.

4.2.2 Pricing of defaultable bonds

Next, we discuss the case where there is a non-zero probability of default taking place; however, as is observed in practice, the probability of default is generally quite small. Work in this area has been done by, among others, Lando (1997) and Duffie and Singleton (1995). The assumption here is that in case of a default all payments cease (including coupons) and a certain percentage of the value of the bond at maturity (known in advance) is paid out, else all coupons as well as the full value is paid on maturity (in case no default happens). The analysis here has been based on coupon paying bonds; zero coupon bonds can default only at the time of final maturity and can be treated as a special case of coupon paying bonds.

Here, we are interested in calculating

$$E\left[D\int_0^T (e^{-rs-\int_0^s \lambda_u du}\lambda_s)ds + (e^{-r}e^{-b\int_0^T \lambda_s ds}) + C\int_0^T (\int_0^s (e^{-ru}du)e^{-\int_0^s \lambda_u du}\lambda_s ds) + C\int_0^T e^{-ru}du(e^{-\int_0^T \lambda_u du})\right], \tag{42}$$

$$\text{where} \qquad \lambda_t = be^{\sigma Y_t} \qquad \text{and} \qquad Y_t = \int_0^t e^{-a(t-s)}dB_s.$$

Here, λ_t is the rate of default and Y_t is a non-stationary Ornstein - Uhlenbeck process. r is the interest rate which is assumed to be constant, σ is the instantaneous variance. Further, D is the percentage paid out in case default occurs, C is the rate of coupon payments during the life of the bond and b is a scaling factor, representing the discount rate. The terms in equation (42) represent the following:

$E\left[D\int_0^T e^{-rs-\int_0^s \lambda_u du}\lambda_s ds\right]$ = Payment at default.

$E\left[e^{-r}e^{-b\int_0^T \lambda_s ds}\right]$ = Final payment on maturity, when no default takes place.

$E\left[C\int_0^T (\int_0^s e^{-ru}du)e^{-\int_0^s \lambda_u du}\lambda_s ds\right]$ = Coupon payments in case of default.

$E\left[C\int_0^T e^{-ru}du(e^{-\int_0^T \lambda_u du})\right]$ = Coupon payments in case no default occurs.

Like earlier, taking, $T = 1$ equation (42) can be rewritten as

$$E\Bigg[(D-C)\int_0^1 e^{-rs}e^{-b\int_0^s e^{\sigma Y_u}du}be^{\sigma Y_s}ds + \frac{C}{r}\int_0^1 e^{-b\int_0^s e^{\sigma Y_u}du}be^{\sigma Y_s}ds + (1-\frac{C}{r})e^{-r}e^{-b\int_0^1 e^{\sigma Y_u}du} + \frac{C}{r}e^{-b\int_0^1 e^{\sigma Y_u}du}\Bigg]. \tag{43}$$

Now, $$\frac{C}{r}\int_0^1 e^{-b\int_0^s e^{\sigma Y_u}du}be^{\sigma Y_s}ds = \frac{C}{r}\left(1-e^{-b\int_0^1 e^{\sigma Y_u}du}\right).$$

Substituting this in equation (43), we have

$$E\left[(D-C)\int_0^1 e^{-rs}e^{-b\int_0^s e^{\sigma Y_u}du}be^{\sigma Y_s}ds + (1-\frac{C}{r})e^{-r}e^{-b\int_0^1 e^{\sigma Y_u}du} + \frac{C}{r}\right]. \tag{44}$$

What we are interested in calculating is the first term of equation (44) - the value of the payment that is made in case of default. The second term of equation (44) gives the value of the bond, assuming no default - that is calculated using the same approach as used earlier in the case of the non-defaultable bonds without any coupon payments. Now, to calculate the value of the payment if default occurs, we need to calculate the first integral of equation (44).

We need to use a suitable conditioning factor (similar to the one defined in equation (8)) for each of the two integrals shown in equation (44). For the second integral, the conditioning factor is exactly the same as that in the zero coupon case. This is given by

$$Z^* = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}.$$

The conditioning factor for the first integral in equation (44) is given by (for details see Basu (1999))

$$Z^{**} = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}.$$

Note that Z^* and Z^{**} are exactly the same and thus the same conditioning factor (Z) can be used for both the integrals, where,

$$Z = Z^* = Z^{**}.$$

Calculations for defaultable bonds: Once we have obtained the conditioning factor as above, we can then easily calculate the value of the interim payments. The conditioning factor Z, given above, is exactly the same as the one in the zero coupon case. Now, conditionally on Z, Y_u is a Gaussian process with

$$E(Y_u|Z) = k_u Z \tag{45}$$

$$k_u = \mathrm{Cov}(Y_s, Z) = \frac{1}{\sqrt{V}}\frac{\sigma^2}{2a}\{\frac{1-e^{-au}}{a} + \frac{1-e^{-a(1-u)}}{a} - \frac{e^{-au}-e^{-a(1+u)}}{a}\}, \tag{46}$$

where V is defined as in equation (25) and

$$\mathrm{Cov}(Y_u, Y_v|Z) = \frac{\sigma^2}{2a}[e^{a|u-v|} - e^{-a(u+v)}] - k_u k_v = w_{uv}. \tag{47}$$

Once we have these values, then we can easily calculate the the value of the first integral.

So, conditionally on Z, we have

$$\int_0^1 e^{-ru}\left\{\exp\left(-b\int_0^u \exp\left[k_s Z + \frac{1}{2}w_{ss}\right]ds\right)\right\} b\left\{\exp\left(k_u Z + \frac{1}{2}w_{uu}\right)\right\} du = h_1(Z) \quad \text{say.} \tag{48}$$

Finally, using equation (44) an approximation to the price of the bond with non-zero probability of default is

$$\left\{\left(\left[D - \frac{C}{r}\right]\int_{-\infty}^{\infty} h_1(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right) + \left(1 - \frac{C}{r}\right)H_2 + \frac{C}{r}\right\} = \left\{\left(D - \frac{C}{r}\right)H_1 + \left(1 - \frac{C}{r}\right)H_2 + \frac{C}{r}\right\}$$

where H_1 is the expectation of $h_1(Z)$ with respect to Z and $H_2$is the value of the second integral of equation (44) [similar to valuing a zero coupon bond, as discussed earlier].

Note that the term $(D - \frac{C}{r})$ can become negative depending on the choices of D, C and r. That is why the price obtained using this term will not be a lower bound to the price - but just an approximation to the price. However, as is evident from the results the approximation is a very accurate one.

4.2.3 Comments on Pricing Coupon Bearing Bonds

The lower bounds to the price or the approximation to the prices calculated using the conditioning factor are so close to the actual price that they can be regarded as a very good approximation to the true value. This is true of both the situations discussed - bonds having a zero probability of default as well as bonds having a non-zero probability of default. Note that in these cases, the values could not have been calculated by a direct expansion.

5 Pricing of European Options

We now look at the problem of pricing European call options on assets with stochastic volatility. Problems of this nature were addressed by, amongst others, Hull and White (1987). They observed that using a simple log - normal model, as used by Black - Scholes (1973), frequently overprices the price of the asset. The price of an asset with stochastic volatility, according to Hull and White, under an equivalent martingale measure [see Harrison and Krepps (1979) and Harrison and Pliska (1981)] follows the following stochastic process :

$$dX_t = rX_t dt + \sigma e^{\frac{kV_t}{2}} X_t[\rho dB_t^{(1)} + \sqrt{1-\rho^2} dB_t^{(2)}], \tag{49}$$

$$dV_t = \mu dt + dB_t^{(1)}, \tag{50}$$

where X_t is the price process, σ is the instantaneous variance of the price process and r is the rate of interest, which is a constant. V_t is the volatility process and μ is the drift of the Brownian motion defining the volatility process. The volatility process could also follow an Ornstein - Uhlenbeck process (as used by Stein and Stein (1991)) and is represented as

$$dV_t = -aV_t dt + dB_t^{(1)}, \tag{51}$$

where a is the force of mean reversion of the Ornstein - Uhlenbeck process, $B_t^{(1)}$ and $B_t^{(2)}$ are two independent standard Brownian motions and ρ is the correlation between V_t and the logarithm of X_t.

We want to calculate the prices of European call options on assets with stochastic volatility. Mathematically, it is given by

$$X_0\{e^{-r}E(e^{Y_T} - b)^+\} = f(Y_T) \quad \text{say}, \tag{52}$$

where b is the strike price of the option, r is rate of interest, X_0 is the current price of the asset and $Y_t = ln(\frac{X_t}{X_0})$, where X_t is the price process described by equation (49).

To calculate the price of the call option, we use a conditioning factor approach similar to Rogers and Shi (1995) and Basu (1999). The form of the conditioning factor used is as described in equation (8). Note that in this case the function f defined by equation (52) is not convex and hence Jensen's inequality cannot be used. We however proceed with the process and what we obtain is an approximation to the price of the call option itself, rather than the lower bound to the price of the option.

We look at two cases of the volatility process - the first is when the volatility process follows a Brownian Motion and when it follows an Ornstein - Uhlenbeck process. In all cases, we take $T = 1$ and $Y_0 = 0$.

5.1 Volatility following a Brownian Motion

In this case, the stochastic volatility process and the price process is explicitly defined as in equations (49) and (50) and $\mu = 0$. We are interested in finding the value of the function defined by equation (52). Thus, carrying on from equation (49) we have

$$Y_1 = r + \sigma \int_0^1 \rho e^{\frac{kB_s^{(1)}}{2}} dB_s^{(1)} + \sigma \int_0^1 \sqrt{1-\rho^2} e^{\frac{kB_s(1)}{2}} dB_s^{(2)} - \frac{1}{2}\sigma^2 \int_0^1 e^{kB_s^{(1)}} ds. \tag{53}$$

Conditionally on the paths of $\{B_s^{(1)}, 0 \leq s \leq 1\}$, we have $\sigma \int_0^1 \sqrt{1-\rho^2} e^{\frac{kB_s^{(1)}}{2}} dB_s^{(2)}$ following a normal distribution with zero mean and variance $\left(\sigma^2(1-\rho^2) \int_0^1 e^{kB_s^{(1)}} ds\right)$ and Y_1 following a normal distribution with mean $\left(r - \frac{1}{2}\sigma^2 P + \rho\sigma Q\right)$ and variance $\left(\sigma^2(1-\rho^2)P\right)$, where

$$P = \int_0^1 e^{kB_s^{(1)}} ds \quad \text{and} \quad Q = \int_0^1 e^{\frac{kB_s^{(1)}}{2}} dB_s^{(1)}.$$

Note, Q consists of a stochastic integral and to calculate the stochastic integral we need to express it terms of time integrals. Using Itô calculus, we have (for details see Basu (1999))

$$Q = \int_0^1 \exp\left(\frac{kB_s^{(1)}}{2}\right) dB_s^{(1)} = \left\{ \frac{\exp\left(\frac{kB_1}{2}\right) - 1}{\frac{k}{2}} - \frac{1}{2}\frac{k}{2} \int_0^1 \exp\left(\frac{kB_s^{(1)}}{2}\right) ds \right\}. \tag{54}$$

The second term of equation (54) is similar to P, the only difference being that in the exponent k is replaced by $\frac{k}{2}$ and thus it can be calculated exactly the same way as P, replacing k by $\frac{k}{2}$.

We suggest an approximation approach as given by the following lemma :

Lemma: *Let P, Q and Z be random variables. Also, let σ and ρ be constants. Then, assuming*

1. *σ is small*
2. *$\Psi(\sigma^2 P, \rho\sigma Q)$ is a function such that it is at least twice differentiable and piecewise continuous*
3. *Z is used as a conditioning factor and is suitably normalised*

we have

$$E(\Psi(\sigma^2 P, \rho\sigma Q)) = E[\Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))]$$

$$+\frac{1}{2}\rho\sigma^2 E\{\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\mathrm{Var}(Q|Z)\} + O(\sigma^3). \quad (55)$$

For the proof to the lemma, see Basu (1999). Note that Ψ_{QQ} indicates the second derivative with respect to the second argument of Ψ.

In this case, let us define

$$\Psi(\sigma^2 P, \rho\sigma Q) = (e^{Y_1} - b)^+ = \max[(e^{Y_1} - b), 0],$$

where Y_1, P, Q, σ and ρ are defined earlier. Also, $\Psi(\sigma^2 P, \rho\sigma Q)$ is piecewise continuous and differentiable and hence the second derivative of $\Psi(\sigma^2 P, \rho\sigma Q)$ exists. We are interested in finding

$$E\left[\Psi(\sigma^2 P, \rho\sigma Q)\right] = E(e^{Y_1} - b)^+ = E[\max((e^{Y_1} - b), 0)] =$$

$$\exp\left(r - \frac{1}{2}\sigma^2\rho^2 P + \rho\sigma Q\right)\Phi\left(\frac{r + \frac{1}{2}\sigma^2 P(1-2\rho^2) + \rho\sigma Q - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right) - b\Phi\left(\frac{r - \frac{1}{2}\sigma^2 P + \rho\sigma Q - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right). \quad (56)$$

Equation (56) represents the first term approximation to the price of the call option. To calculate $E[\Psi(\sigma^2 P, \rho\sigma Q)]$, we make use of the *Lemma* - we first calculate $\Omega(Z)$, where

$$\Omega(Z) = \Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z)).$$

However, the first term alone does not approximate the price well enough. So, we need the second term in *Lemma* - we call that term the *Correction Factor*. This term involves the second derivative of $\Psi(\sigma^2 P, \rho\sigma Q)$ with respect to Q and is given by

$$\Psi_{QQ}(\sigma^2 P, \rho\sigma Q) = \left\{\exp\left(r + \rho\sigma Q - \frac{1}{2}\sigma^2\rho^2 P\right)\Phi\left(\frac{r + \rho\sigma Q + \frac{1}{2}\sigma^2(1-2\rho^2)P - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right)\right.$$

$$\left. + \frac{r + \rho\sigma Q - \frac{1}{2}\sigma^2\rho^2 P}{\sqrt{2\sigma^2\pi(1-\rho^2)P}}\exp\left(-\frac{(r + \rho\sigma Q + \frac{1}{2}\sigma^2(1-2\rho^2)P - lnb)^2}{2\sigma^2(1-\rho^2)P}\right)\right\}. \quad (57)$$

To obtain the correction factor we define $\Theta(Z)$ as

$$\Theta(Z) = \frac{1}{2}\rho^2\sigma^2\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\mathrm{Var}(Q|Z).$$

This is exactly the same as the second term in the *Lemma*.

To get the value of the option, we need to obtain the unconditional value of $\Omega(Z)$ and $\Theta(Z)$. Note that Z is the conditioning factor defined by equation (8) and has a standard normal distribution. The exact form of Z used in this case is defined as

$$Z = \frac{\int_0^1 B_s ds}{\sqrt{\mathrm{Var}(\int_0^1 B_s ds)}} \tag{58}$$

To calculate the value of the option, we need to calculate $E(P|Z)$ and $E(Q|Z)$ as well as $\mathrm{Var}(Q|Z)$ to be able to obtain $\Omega(Z)$ and $\Theta(Z)$.

Now $\mathrm{Var}(\int_0^1 B_s ds) = \frac{1}{3}$. Thus, we have $E(B_u|Z) = j_u Z$ where

$$j_u = \mathrm{Cov}(B_u, Z) = \sqrt{3}(u - \frac{u^2}{2}) \quad \text{and} \quad \mathrm{Cov}(B_u, B_v|Z) = (u \wedge v) - j_u j_v = s_{uv}. \tag{59}$$

Once we have these values, then conditionally on Z, we can then easily get the expected values of P and Q. We have

$$E(P|Z) = \int_0^1 \exp\left(k j_u Z + \frac{k^2}{2} s_{uu}\right) du \tag{60}$$

$$E(Q|Z) = \left\{ \frac{\exp\left(\frac{k}{2}\frac{\sqrt{3}}{2}Z + \frac{k^2}{4}\frac{1}{8}\right) - 1}{\frac{k}{2}} - \frac{k}{4}\int_0^1 \exp\left(\frac{k}{2} j_u Z + \frac{k^2}{8} s_{uu}\right) du \right\} \tag{61}$$

$\Rightarrow$ Conditionally on Z, $\Omega(Z) = \Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$

$$= \exp\left(r - \frac{1}{2}\sigma^2\rho^2 E(P|Z) + \rho\sigma E(Q|Z)\right) \Phi\left(\frac{r + \frac{1}{2}\sigma^2 E(P|Z)(1-2\rho^2) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right)$$

$$-b\Phi\left(\frac{r - \frac{1}{2}\sigma^2 E(P|Z) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right). \tag{62}$$

To calculate the price of the option, we also need $\Theta(Z)$ for which we need to calculate $\mathrm{Var}(Q|Z)$ and $\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$. Now, continuing from equation (57) and equation (59), we have

$\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$

$$= \left\{ \exp\left(r + \rho\sigma E(Q|Z) - \frac{1}{2}\sigma^2\rho^2 E(P|Z)\right) \Phi\left(\frac{r + \rho\sigma E(Q|Z) + \frac{1}{2}\sigma^2(1-2\rho^2)E(P|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right)\right.$$

$$\left. + \frac{r + \rho\sigma E(Q|Z) - \frac{1}{2}\sigma^2\rho^2 E(P|Z)}{\sqrt{2\sigma^2\pi(1-\rho^2)E(P|Z)}} \exp\left(-\frac{(r + \rho\sigma E(Q|Z) + \frac{1}{2}\sigma^2(1-2\rho^2)E(P|Z) - lnb)^2}{2\sigma^2(1-\rho^2)E(P|Z)}\right)\right\}.$$

Also,

$$\mathrm{Var}(Q|Z) = \left\{ \mathrm{Var}(\frac{e^{\frac{kB_1}{2}} - 1}{\frac{k}{2}}|Z) + \frac{k^2}{16}\mathrm{Var}(\int_0^1 e^{\frac{kB_s^{(1)}}{2}} ds|Z) - \frac{k}{2}\mathrm{Cov}(\frac{e^{\frac{kB_1}{2}} - 1}{\frac{k}{2}}, \int_0^1 e^{\frac{kB_s^{(1)}}{2}} ds|Z) \right\},$$

$$\text{where,}\quad \mathrm{Var}\left(\frac{e^{\frac{kB_1}{2}}-1}{\frac{k}{2}}|Z\right) = \frac{4}{k^2}\left[e^{\frac{\sqrt{3}Zk}{2}}\left(e^{\frac{k^2}{8}}-e^{\frac{k^2}{16}}\right)\right],$$

$$\mathrm{Var}\left(\int_0^1 e^{\frac{kB_s}{2}}ds|Z\right) = \int_0^1\int_0^1 \exp\left(\frac{k}{2}(j_u+j_v)Z+\frac{k^2}{8}(s_{uu}+s_{vv})\right)\left[\exp\left(\frac{k^2}{4}s_{uv}\right)-1\right]dudv$$

$$\text{and}\quad \mathrm{Cov}\left(e^{\frac{kB_1}{2}},\int_0^1 e^{\frac{kB_s^{(1)}}{2}}ds|Z\right) = \int_0^1 \exp\left(\frac{k}{2}(j_u+j_1)Z+\frac{k^2}{8}(s_{uu}+s_{11})+\frac{k^2}{4}s_{1u}\right)du$$
$$-\left[\exp\left(\frac{\sqrt{3}Zk}{2}+\frac{k^2}{32}\right)\int_0^1 \exp\left(\frac{k}{2}j_uZ+\frac{k^2}{8}s_{uu}\right)du\right].$$

Having obtained these values, we can easily find the value of the correction factor $\Theta(Z)$, conditionally on Z, given by

$$\Theta(Z) = \frac{1}{2}\rho^2\sigma^2\Psi_{QQ}(\sigma^2E(P|Z),\rho\sigma E(Q|Z))\mathrm{Var}(Q|Z).$$

Finally, to calculate the value of the option we calculate the sum of the expectations of $\Omega(Z)$ and $\Theta(Z)$ with respect to Z adjusted for the current asset price and the interest rate i.e. we calculate

$$100e^{-r}\left(\left\{\int_{-\infty}^{\infty}\Omega(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right\}+\left\{\int_{-\infty}^{\infty}\Theta(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right\}\right) = 100e^{-r}(H_1+H_2) \tag{63}$$

where H_1 is the first term approximation to the price, H_2 is the associated correction factor and X_0 is the current price of the asset (we assume $X_0 = 100$).

5.2 The Ornstein - Uhlenbeck Case

In this case, we have the volatility process following an Ornstein Uhlenbeck process and the price and the volatility processes are defined by equations (49) and (51) respectively. The rest of the parameters are similar to the case of the Brownian motion case - the only additional term being a - the mean reversion force of the Ornstein - Uhlenbeck process. As before, we are interested in finding

$$X_0\{e^{-r}E(e^{Y_1}-b)^+\},$$

where b is the strike price, X_0 is the current price of the asset and $Y_t = ln(\frac{X_t}{X_0})$, X_t being the price process. This implies

$$Y_1 = r+\sigma\int_0^1 \rho e^{\frac{kV_s}{2}}dB_s^{(1)}+\sigma\int_0^1\sqrt{1-\rho^2}e^{\frac{kV_s}{2}}dB_s^{(2)}-\frac{1}{2}\sigma^2\int_0^1 e^{kV_s}ds. \tag{64}$$

Again, conditionally on the paths of $\{B_s^{(1)}, 0 \leq s \leq 1\}$, we have $\sigma \int_0^1 \sqrt{1-\rho^2} e^{\frac{kV_s}{2}} dB_s^{(2)}$ following a normal distribution with zero mean and variance $\sigma^2(1-\rho^2)\int_0^1 e^{kV_s} ds$ and Y_1 follows a normal distribution with mean A and variance Σ^2, where

$$A = r - \frac{1}{2}\sigma^2 \int_0^1 e^{kV_t} dt + \sigma \int_0^1 \rho e^{\frac{kV_t}{2}} dB_t^{(1)} \tag{65}$$

$$\Sigma^2 = \sigma^2(1-\rho^2) \int_0^1 e^{kV_t} dt \tag{66}$$

Let us, as in the case of the Brownian motion, define

$$P = \int_0^1 e^{kV_t} dt \quad \text{and} \quad Q = \int_0^1 e^{\frac{kV_t}{2}} dB_t^{(1)}.$$

Thus, $A = r - \frac{1}{2}\sigma^2 P + \rho\sigma Q$ and $\Sigma^2 = \sigma^2(1-\rho^2)P$. We again make use of Itô calculus to express Q, a stochastic integral, in terms of time integrable terms. We thus have,

$$Q = \int_0^1 \exp\left(\frac{kV_t}{2}\right) dB_t^{(1)} = \left\{ \frac{\exp\left(\frac{kV_1}{2}\right) - 1}{\frac{k}{2}} - \frac{k}{4}\int_0^1 \exp\left(\frac{kV_t}{2}\right) dt + a \int_0^1 V_t \exp\left(\frac{kV_t}{2}\right) dt \right\}. \tag{67}$$

Also,as before, let us define

$$\Psi(\sigma^2 P, \rho\sigma Q) = (e^{Y_1} - b)^+ = \max[(e^{Y_1} - b), 0]$$

where Y_1 is given by equation (65). Again, we are interested in finding

$$E[\Psi(\sigma^2 P, \rho\sigma Q)] = E(e^{Y_1} - b)^+ = \exp\left(A + \frac{\Sigma^2}{2}\right) \Phi\left(\frac{A + \Sigma^2 - lnb}{\sqrt{\Sigma^2}}\right) - b\Phi\left(\frac{A - lnb}{\sqrt{\Sigma^2}}\right)$$

$$= \exp\left(r - \rho^2 P + Q\right)\Big) \Phi\left(\frac{r + \frac{1}{2}\sigma^2 P(1-2\rho^2) + \rho\sigma Q - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right) - b\Phi\left(\frac{r - \frac{1}{2}P + \rho\sigma Q - lnb}{\sqrt{\sigma^2(1-\rho^2)P}}\right). \tag{68}$$

Equation (68) represents the first term approximation to the price of the option. To calculate $E[\Psi(\sigma^2 P, \rho\sigma Q)]$, we make use of *Lemma.* Thus, we first calculate $\Omega(Z)$, where

$$\Omega(Z) = \Psi(\sigma^2 E(P|Z), \rho\sigma E(Q|Z)).$$

However, as stated earlier, the first term alone does not approximate the price well enough. Thus, we also need the second term of *Lemma* - in effect the *Correction Factor* $\Theta(Z)$ defined as

$$\Theta(Z) = \frac{1}{2}\rho^2\sigma^2 \Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z)) \text{Var}(Q|Z).$$

This is exactly the same as the second term in *Lemma.*

To get the value of the option, we need to obtain the unconditional value of $\Omega(Z)$ and $\Theta(Z)$. Note that Z is the conditioning factor defined by equation (8) and has a standard normal distribution. The exact form of Z used in this case is defined as

$$Z = \frac{\int_0^1 V_s ds}{\sqrt{\text{Var}(\int_0^1 V_s ds)}} \tag{69}$$

where

$$\text{Var}(\int_0^1 V_s ds) = \int_0^1 \left\{ \frac{1-e^{-a(1-s)}}{a} \right\}^2 ds = \frac{2a-(1-e^{-a})(3-e^{-a})}{2a^3}.$$

Thus, we have, $E(V_u|Z) = j_u Z$, where

$$j_u = \text{Cov}(V_u, Z) = \sqrt{\frac{2}{a}}[\frac{e^{-au}\{\cosh(au)+\sinh(au)\} - e^{-au} - e^{-a}\sinh(au)}{\sqrt{2a-(1-e^{-a})(3-e^{-a})}}], \tag{70}$$

$$\text{and} \qquad \text{Cov}(V_u, V_v|Z) = \text{Cov}(V_u, V_v) - j_u j_v = (\frac{e^{a|u-v|} - e^{-a(u+v)}}{2}) - j_u j_v = s_{uv} \tag{71}$$

Once we have these values, we can easily calculate the values of $E(P|Z)$ and $E(Q|Z)$ given by

$$E(P|Z) = \int_0^1 \exp\left(k j_u Z + \frac{k^2}{2} s_{uu} \right) du, \tag{72}$$

$$\begin{aligned} E(Q|Z) = \Bigg\{ & \frac{\exp\left(\frac{kLZ}{2} + \frac{k^2}{8}\{\frac{1-e^{-2a}}{2a} - L^2\}\right) - 1}{\frac{k}{2}} - \int_0^1 \frac{k}{4}\left[\exp\left(\frac{k}{2} j_u Z + \frac{k^2}{8} s_{uu}\right)\right] du \\ & + a\int_0^1 \left[j_u Z + \frac{1}{2}(\frac{1-e^{-2au}}{2a} - j_u^2)\right] \exp\left(\frac{j_u Z}{2} + \frac{1}{8}[\frac{1-e^{-2au}}{2a} - j_u^2]\right) du \Bigg\} \end{aligned} \tag{73}$$

where $L = \frac{(1-e^{-a})^2}{2a^2 B}$ and $B = \sqrt{\frac{2a-(1-e^{-a})(3-e^{-a})}{2a^3}}$.

Thus, conditionally on Z, we have

$$\begin{aligned} \Omega(Z) = \exp\left(r - \frac{1}{2}\sigma^2\rho^2 E(P|Z) + \rho\sigma E(Q|Z)\right) \Phi\left(\frac{r + \frac{1}{2}\sigma^2 E(P|Z)(1-2\rho^2) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right) \\ -b\Phi\left(\frac{r - \frac{1}{2}\sigma^2 E(P|Z) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right). \end{aligned} \tag{74}$$

To price the option, we also need conditionally on Z

$$\Theta(Z) = \frac{1}{2}\rho^2\sigma^2\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))\text{Var}(Q|Z).$$

For this, we need the terms $\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$ and $\text{Var}(Q|Z)$.

Now, $\Psi_{QQ}(\sigma^2 E(P|Z), \rho\sigma E(Q|Z))$

$$= \left[\exp\left(r + \rho\sigma E(Q|Z) - \frac{1}{2}\sigma^2\rho^2 E(P|Z)\right)\Phi\left(\frac{r + \frac{1}{2}\sigma^2 E(P|Z)(1-2\rho^2) + \rho\sigma E(Q|Z) - lnb}{\sqrt{\sigma^2(1-\rho^2)E(P|Z)}}\right)\right.$$

$$\left. + \frac{\exp\left(r + \rho\sigma E(Q|Z) - \frac{1}{2}\sigma^2\rho^2 E(P|Z)\right)}{\sqrt{2\sigma^2\pi(1-\rho^2)E(P|Z)}}\exp\left(-\frac{(r + \frac{1}{2}\sigma^2 E(P|Z)(1-2\rho^2) + \rho\sigma E(Q|Z) - lnb)^2}{2\sigma^2(1-\rho^2)E(P|Z)}\right)\right]$$

and

$$\mathrm{Var}(Q|Z) = I_1 + \frac{k^2}{16}I_2 + a^2 I_3 + 2aI_4 - \frac{k}{2}I_5 - \frac{ak}{2}I_6$$

where,

$I_1 = [\exp(kLz)\{\exp(\frac{k^2}{2}(\frac{1-e^{-2a}}{2a} - L^2)) - \exp(\frac{k^2}{4}[\frac{1-e^{-2a}}{2a} - L^2])\}]$,

$I_2 = \int_0^1\int_0^1 \exp(\frac{k}{2}(j_u + j_v)Z + \frac{k^2}{8}[s_{uu} + s_{vv}])\{\exp(\frac{k^2}{4}s_{uv}) - 1\}dudv$,

$I_3 = \Big[\int_0^1\int_0^1 \exp(\frac{k}{2}[j_t + j_u]Z + \frac{k^2}{8}[s_{tt} + s_{uu}])[\exp(\frac{k^2}{4}s_{tu})s_{tu} - 1]dtdu$

$\quad + \int_0^1\int_0^1 \exp(\frac{k}{2}[j_t + j_u]Z + \frac{k^2}{8}[s_{tt} + s_{uu}] + \frac{k^2}{4}s_{tu})\{j_t Z + \frac{k}{2}(s_{tt} + s_{tu})\}\{j_u Z + \frac{k}{2}(s_{uu} + s_{tu})\}dtdu\Big]$,

$I_4 = \Big[\int_0^1 \exp(\frac{k}{2}j_t Z + \frac{k^2}{8}s_{tt})\{(j_t Z + \frac{k}{2}(s_{tt} + s_{1t}))\exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11} + \frac{k^2}{4}s_{1t}) - \exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11})\}dt\Big]$,

$I_5 = \Big[\frac{1}{\frac{k^2}{2}}\int_0^1 \exp(\frac{k}{2}j_t Z + \frac{k^2}{8}s_{tt})\{\exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11} + \frac{k^2}{4}s_{1u}) - \exp(\frac{k}{2}j_1 Z + \frac{k^2}{8}s_{11})\}dt\Big]$,

$I_6 = \Big[\int_0^1\int_0^1 \exp(\frac{k}{2}[j_s + j_t]Z + \frac{k^2}{8}[s_{ss} + s_{tt}])\{[\exp(\frac{k^2}{4}s_{st})(j_t Z + \frac{k}{2}\{s_{ss} + s_{ts}\})] - [j_t Z + \frac{k}{2}s_{tt}]\}dsdt\Big]$.

Here $L = \frac{(1-e^{-a})^2}{2a^2M}$ and $M = \sqrt{\mathrm{Var}(\int_0^1 V_s ds)} = \sqrt{\frac{2a-(1-e^{-a})(3-e^{-a})}{2a^3}}$. Knowing the values of I_1, I_2, I_3, I_4, I_5 and I_6, we can easily calculate $\mathrm{Var}(Q|Z)$. Further, knowing $E(P|Z)$ (equation(72)) and $E(Q|Z)$ (equation(73)), we find the value of the correction factor $\Theta(Z)$.

Finally, to calculate the value of the option we calculate the sum of the expectations of $\Omega(Z)$ and $\Theta(Z)$ with respect to Z adjusted for the current asset price and the interest rate i.e. we calculate

$$100e^{-r}\left(\left\{\int_{-\infty}^{\infty}\Omega(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right\} + \left\{\int_{-\infty}^{\infty}\Theta(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right\}\right) = 100e^{-r}(H_1 + H_2) \tag{75}$$

where H_1 is the first term approximation to the price, H_2 is the associated correction factor and X_0 is the current price of the asset (we assume $X_0 = 100$).

To illustrate the technique described above, we have repeated the work over a number of scenarios with volatility following both a Brownian Motion as well as an Ornstein - Uhlenbeck process for a host of strike prices, as well as values of ρ. Some of them are highlighted in the tables later.

5.3 Comments on Pricing European Options on Stochastically Volatile Assets

A look at the output from this method show that in all the cases, the calculated value of the option, including the correction factor, is very close to the simulated value. The values are, in general, more accurate for the case when the volatility process follows a Brownian motion. In the case of the volatility process following an Ornstein - Uhlenbeck process, the lower the value of a the closer agreement of the calculated values with the simulated values. Also, higher the value of ρ, i.e. the closer ρ is to ± 1, the greater the contribution of the correction factor to the corrected calculated price.

The biggest advantage of this method is that one can do away with the restrictive assumption of independence of the price and the volatility processes. In fact, in practice, price and volatility are independent of each other. Also, this method is quite fast to use for different values of the strike price.

Another justification of use of the correction factor is in the approximation carried out during conditioning. In the case of the volatility processes, conditioning B_1 on $\int_0^1 B_s ds$ (or V_1 on $\int_0^1 V_s ds$) does not work so well and leads to an error. One probable reason for this is the fact that B_1 and $\int_0^1 B_s ds$ (or V_1 on $\int_0^1 V_s ds$). Thus, in both cases, the correction factor is needed to rectify that error.

6 Conclusion and Remarks

The methods described here can be used for pricing bonds (both zero coupon as well as coupon bearing) as well as European options on stochastically volatile assets. In both cases, the solution is not heavily dependant on any numerical methods - hence the level of accuracy is generally higher and so is the speed of calculation. Also, most of the calculation can be easily done on simple machine and no high-end sophisticated machines are required. Also, all the solutions, though not entirely closed form, is semi-closed form and hence more mathematically tractable in terms of error analysis and related analysis.

Finally, some of the results obtained by this method along with some comparative values (obtained through other means) are given in the tables at the end. The accuracy can be easily gauged from the closeness of the results shown in the tables.

For the bond case, the approach can be easily extended to the case of a “portfolio of bonds”. This is particularly important as the “portfolio” can then be looked upon as the set of multiple drivers to even a single bond value - something that does happen quite often in practice.

On the option framework, the work can be extended to the case where the interest rate can also be stochastic in nature and there is correlation between the interest rate process and the volatility process, the interest rate process and the price process apart from the volatility and the price process. The work can also be extended to the case when the option is of the American type - though the calculations in that case might become very complicated and more amount of numerical procedures might be needed.

7 Tables

Here we present some tables with numerical examples of the methods described above. Most of the tables are self explanatory with appropriate headings and foot notes.

First we present the results for the zero coupon bond case; table 1 presents the Brownian motion, table 2 presents results for the stationary Ornstein - Uhlenbeck process and table 3 is the case of non-stationary Ornstein - Uhlenbeck process. LB_1 and UB_1 refer to the bounds calculated using the conditioning factor while LB_2 and UB_2 are the bounds obtained through direct expansion.

Table 1 : The interest rate follows a geometric Brownian motion.

a	σ	LB1	UB1	LB2	UB2
-0.5	0.1	94.657	94.657	94.629	94.636
	0.5	94.368	94.374	94.342	94.347
	0.75	93.965	93.979	93.943	93.951
	1	93.35	93.375	93.334	93.352
-0.2	0.1	93.86	93.86	93.839	93.843
	0.5	93.514	93.52	93.497	93.503
	0.75	93.034	93.047	93.021	93.033
	1	92.303	92.328	92.297	92.328
0	0.1	93.239	93.239	93.224	93.23
	0.5	92.849	92.855	92.838	92.847
	0.75	92.308	92.322	92.303	92.32
	1	91.49	91.514	91.491	91.538
0.2	0.1	92.534	92.534	92.526	92.534
	0.5	92.094	92.1	92.091	92.104
	0.75	91.486	91.5	91.489	91.513
	1	90.57	90.595	90.581	90.649
0.5	0.1	91.291	91.291	91.297	91.31
	0.5	90.765	90.771	90.777	90.798
	0.75	90.041	90.055	90.061	90.102
	1	88.962	88.986	88.987	89.11

Table 2 : The interest rate follows an exponential function of a stationary Ornstein - Uhlenbeck process with a = *1*.

σ	LB1	UB1	LB2	UB2
0.1	93.239	93.25	93.223	93.223
0.5	92.859	92.898	92.844	92.853
0.75	92.342	92.382	92.326	92.343
1	91.576	91.608	91.561	91.597

Table 3 : The interest rate follows an exponential function of a non-stationary Ornstein - Uhlenbeck process with a = *1*.

σ	LB1	UB1	LB2	UB2
0.1	93.245	93.246	92.227	93.233
0.5	93.029	93.031	92.939	92.948
0.75	92.736	92.749	92.557	92.575
1	92.308	92.331	92.001	92.043

Note : In some cases in tables 1,2 and 3, lower bounds calculated using one approach are slightly higher than the upper bounds calculated by the other method. This is due to small inaccuracies in the numerical integration procedures and indicates how close they are to the actual price.

Also, in our case the direct expansion works due to the fact that σ is small - it shall break down for large values of σ.

The next set of tables looks at non-defaultable as well as defaultable bonds. The coupon rate, C is taken as 5% and the payout in case of default is 50%. Also, r in case of the defaultable bonds is taken as 5%. Tables 4.1 and 4.2 depict the case of non-defaultable and defaultable bonds respectively with a short life (1 year) while tables 5.1 and 5.2 give the same results for bonds with longer lives (10 years).

Table 4.1 : Table showing the calculated values of the total payments of coupon paying bonds along with the simulated values and their standard errors where the term of the bond is 1 year and the coupon rate is 5%.

σ	a	b	Calculated	Simulated	S.E.
0.1	1	0.07	98.07985	98.05825	0.0027
0.5	1	0.07	97.68948	97.82111	0.0145
0.75	1	0.07	97.16662	97.54738	0.023

Table 4.2 : Table showing the calculated values of the payments of bonds at default along with the simulated values and their standard errors where the term of the bond is 1 year and and the amount paid out in case of default is 50%.

σ	a	b	Calculated	Simulated	S.E.
0.1	1	0.07	95.7805	95.7549	0.0015
0.5	1	0.07	95.6761	95.652	0.0078
0.75	1	0.07	95.5354	95.4768	0.01208

Table 5.1 : Table showing the calculated values of the total payments of coupon paying bonds along with the simulated values and their standard errors where the term of the bond is 10 years and the coupon rate is 5%.

σ	a	b	Calculated	Simulated	S.E.
0.1	1	0.07	53.25209	53.17027	0.0104
$\sqrt{0.1}$	1	0.07	52.58027	52.4876	0.0334
0.5	1	0.07	51.46158	51.37964	0.0537
0.75	1	0.07	49.1404	49.1597	0.081

Table 5.2 : Table showing the calculated values of the payments of bonds at default along with the simulated values and their standard errors where the term of the bond is 10 years and and the amount paid out in case of default is 50%.

σ	a	b	Calculated	Simulated	S.E.
0.1	1	0.07	74.6547	74.65141	0.0057
$\sqrt{0.1}$	1	0.07	74.3201	74.3066	0.0179
0.5	1	0.07	73.795	73.8483	0.0288
0.75	1	0.07	72.705	72.7103	0.044

Note : To calculate the prices of the long - term (10 year) bonds, we use the same formulae as in the case of 1 year bonds. However, for calculation purposes, we take the term of the bond $T = 1$ but adjust the other parameters accordingly to represent a $T = t$ year bond. Thus, for a bond with a life of t years, σ^2 changes to $\sigma^2 t$, a changes to at and b changes to bt. In our case, $t = 10$.

The next set of three tables (6.1 - 6.3) present the results for the European option price case. Table 6.1 looks at the volatility process being a pure Brownian motion while tables 6.2 and 6.3 look at the Ornstein Uhlenbeck case. In these tables, we compare the *Corrected Calculated Price* (CCP) with the simulated price.

Table 6.1: Volatility process follows a Simple Brownian Motion with $\sigma = 0.1$, r = 0.05 and k = 1.

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.95	110	2.044508	0.779841	2.824346	2.990192	0.142218
	105	3.497231	0.816586	4.313817	4.453916	0.15819
	100	5.910315	0.672918	6.583233	6.720349	0.174477
	95	9.546253	0.370436	9.916688	10.067917	0.187047
	90	14.09038	0.200528	14.290908	14.421779	0.19195
-0.95	110	1.449652	0.515492	1.965143	1.979887	0.046055
	105	3.61698	0.664221	4.281493	4.32144	0.070859
	100	6.730394	0.656562	7.386937	7.478487	0.093149
	95	10.522814	0.560182	11.082996	11.252283	0.110801
	90	14.743085	0.443986	15.187071	15.427257	0.123809
0.75	110	2.321014	0.440063	2.761077	2.79501	0.109881
	105	3.857499	0.475741	4.33324	4.370133	0.128513
	100	6.310175	0.42701	6.737185	6.803907	0.146796
	95	9.854223	0.291273	10.145325	10.274714	0.160558
	90	14.235053	0.167333	14.402386	14.549206	0.167553
-0.75	110	1.764733	0.351344	2.116077	2.080526	0.054295
	105	3.897485	0.411583	4.309067	4.276188	0.077837
	100	6.94123	0.393316	7.334547	7.277515	0.099984
	95	10.673079	0.331616	11.004534	10.929826	0.118031
	90	14.852886	0.136765	14.989651	15.024697	0.131557
0.5	110	2.51401	0.1796	2.69361	2.677158	0.099293
	105	4.162327	0.195099	4.357426	4.349432	0.118616
	100	6.703056	0.18134	6.884301	6.887848	0.137491
	95	10.214073	0.138677	10.35275	10.389251	0.151995
	90	14.457174	0.093119	14.550293	14.596035	0.160504
-0.5	110	2.106229	0.15975	2.265979	2.249182	0.063796
	105	4.159931	0.15833	4.31826	4.309344	0.086421
	100	7.104181	0.168977	7.273158	7.214149	0.107748
	95	10.766899	0.141742	10.908641	10.812996	0.125257
	90	14.91327	0.111661	15.024931	14.920499	0.137719

Table 6.1 Continued.........

Table 6.1 Continued

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.25	110	2.573018	0.042315	2.614529	2.578021	0.089701
	105	4.329345	0.046211	4.375555	4.340218	0.109824
	100	6.968613	0.043416	7.012029	6.983028	0.129222
	95	10.490538	0.037539	10.528077	10.506015	0.144528
	90	14.660471	0.016196	14.676667	14.660233	0.154393
-0.25	110	2.359796	0.040272	2.400067	2.38458	0.072381
	105	4.320381	0.044348	4.364726	4.322913	0.094257
	100	7.162415	0.041809	7.204224	7.133907	0.114915
	95	10.766043	0.034854	10.800897	10.711205	0.131551
	90	14.897422	0.027139	14.924561	14.835279	0.143077
0	110	2.517138			2.486435	0.080825
	105	4.378393			4.326996	0.101893
	100	7.118547			7.062116	0.121822
	95	10.674126			10.61299	0.137816
	90	14.811251			14.743523	0.148634

Table 6.2: The volatility process follows an Ornstein - Uhlenbeck process with a=0.1, k=1, r=0.05, $\sigma = 0.1$, $X_0 = 100$ and $V_0 = 0$.

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.95	110	2.052779	0.7262169	2.778996	3.010594	0.1194083
	105	3.516785	0.7696407	4.286425	4.521037	0.1380072
	100	5.938746	0.6435333	6.582279	6.827903	0.1563951
	95	9.570367	0.3594557	9.929823	10.20763	0.1701949
	90	14.10503	0.1884444	14.29347	14.57596	0.1754179
-0.95	110	1.465225	0.5031988	1.968424	2.009105	0.04744766
	105	3.625432	0.6366818	4.262114	4.322088	0.07209224
	100	6.730491	0.6212323	7.351724	7.436452	0.09447091
	95	10.51867	0.5240718	11.04274	11.16606	0.1127158
	90	14.73891	0.4112558	15.15017	15.28758	0.1265338
0.75	110	2.312387	0.4122456	2.724633	2.963393	0.1109139
	105	3.858761	0.4486269	4.307388	4.549214	0.1302168
	100	6.321038	0.4044438	6.728094	6.978501	0.1488933
	95	9.86696	0.2757934	10.14276	10.39694	0.1632714
	90	14.24454	0.1568302	14.40137	14.67709	0.1701071
-0.75	110	1.769973	0.3372281	2.107201	2.154169	0.05585269
	105	3.89501	0.392482	4.287492	4.369176	0.07926726
	100	6.931021	0.3723157	7.303337	7.424602	0.1009169
	95	10.65958	0.3112112	10.97079	11.11974	0.118411
	90	14.84905	0.25439755	15.08492	15.24523	0.13155
0.5	110	2.49434	0.1686056	2.662946	2.8703595	0.1017094
	105	4.149523	0.1846999	4.334223	4.549074	0.1216292
	100	6.69702	0.1717737	6.868794	7086679	0.1407383
	95	10.21131	0.1307052	10.34202	10.56835	0.1554402
	90	14.45612	0.08677706	14.5429	14.79708	0.1635396
-0.5	110	2.098748	0.1524248	2.251173	2.309457	0.06623995
	105	4.147055	0.1704263	4.317481	4.397041	0.0884599
	100	7.085885	0.1602199	7.246105	7.371216	0.1090454
	95	10.74705	0.1333718	10.88042	11.04136	0.1257021
	90	14.89608	0.1041142	15.0002	15.18395	0.1376807

Table 6.2 Continued.........

Table 6.2 Continued

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.25	110	2.548681	0.04003625	2.588718	2.755464	0.09290375
	105	4.309619	0.04386708	4.353486	4.534631	0.1133543
	100	6.952337	0.04116863	6.993506	7.176826	0.1328749
	95	10.47678	0.03287068	10.50966	10.71232	0.1479818
	90	14.65015	0.02367886	14.67383	14.90507	0.1571716
-0.25	110	2.342994	0.03831577	2.381316	2.472027	0.07555303
	105	4.301222	0.04222176	4.343443	4.436763	0.09709313
	100	7.140569	0.0396711	7.18024	7.314232	0.1170629
	95	10.74409	0.03283458	10.77692	10.95539	0.13306
	90	14.8789	0.02532481	14.90423	15.1089	0.1440347
0	100	2.494862			2.447907	0.08430024
	105	4.356859			4.48986	0.1052663
	100	7.097289			7.257128	0.1248408
	95	10.65413			10.83947	0.1405096
	90	14.79503			15.01577	0.1505679

Table 6.3: The volatility process follows an Ornstein - Uhlenbeck process with a = 10, k = 1, r = 0.05, $V_0 = 0$, $X_0 = 100$ and $\sigma = 0.1$.

ρ	b	Calculated	C.F.	CCP	Simulated	S.E.
0.95	110	2.145946	0.085411	2.231357	2.308187	0.073264
	105	3.928759	0.098936	4.027696	4.066879	0.095204
	100	6.610466	0.093486	6.703952	6.723736	0.115949
	95	10.192842	0.069991	10.262832	10.29781	0.131655
	90	14.454283	0.04308	14.497363	14.543108	0.140696
-0.95	110	1.991431	0.090238	2.081669	2.078354	0.060331
	105	3.922141	0.098474	4.020615	3.999647	0.083325
	100	6.751405	0.087625	6.83903	6.853382	0.104518
	95	10.386169	0.065568	10.473794	10.507584	0.121012
	90	14.596767	0.044268	14.641035	14.695058	0.132083
0.75	110	2.180094	0.053217	2.23332	2.299371	0.072509
	105	3.983075	0.061118	4.044764	4.094231	0.094412
	100	6.678632	0.057421	6.736053	6.773408	0.115251
	95	10.256393	0.04322	10.299613	10.334223	0.131357
	90	14.499057	0.027137	14.526194	14.5503	0.140927
-0.75	110	2.058914	0.055518	2.114432	2.099698	0.061669
	105	3.977592	0.060891	4.038484	4.022176	0.084489
	100	6.787376	0.054626	6.842002	6.826005	0.106001
	95	10.406935	0.041081	10.448016	10.438221	0.12277
	90	14.611664	0.027673	14.639337	14.625447	0.133612
0.5	110	2.208241	0.023676	2.231917	2.279903	0.07128
	105	4.035839	0.026927	4.062766	4.123707	0.093146
	100	6.74748	0.025112	6.772597	6.803292	0.1202188
	95	10.322142	0.018987	10.341129	10.364329	0.130605
	90	14.544878	0.012187	14.557065	14.555246	0.140691
-0.5	110	2.128028	0.024329	2.152356	2.152001	0.064217
	105	4.031225	0.026861	4.058086	4.051619	0.086781
	100	6.818265	0.024318	6.842583	6.809635	0.108321
	95	10.421298	0.018368	10.439667	10.399439	0.125018
	90	14.620672	0.012319	14.632991	14.596948	0.13546

Table 6.3 Continued ....

Table 6.3 Continued

ρ	b	Calculated	C.F.	CCP	Simulated	S.E
0.25	110	2.218443	0.005933	2.224377	2.2814	0.069658
	105	4.068162	0.006689	4.074851	4.128462	0.091961
	100	6.796636	0.006197	6.802833	6.807212	0.113399
	95	10.371454	0.0047	10.376154	10.373525	0.12969
	90	14.579721	0.00307	14.582791	14.56239	0.139915
-0.25	110	2.178563	0.006012	2.184576	2.209668	0.664102
	105	4.06566	0.006681	4.072341	4.075665	0.088959
	100	6.831474	0.0061	6.837573	6.810703	0.110319
	95	10.420737	0.004623	10.42536	10.37738	0.12704
	90	14.617989	0.003084	14.621073	1.4585143	0.137122
0	110	2.208792			2.255537	0.068118
	105	4.078353			4.107981	0.090602
	100	6.824676			6.811075	0.112042
	95	10.404252			10.372034	0.128603
	90	14.604139			14.571978	0.138699

8 References

- Basu, S. (1999); "Approximating functions of integrals of Log- Gaussian processes: Applications in Finance" Ph.D. Thesis, *London School of Economics, University of London.*

- Baxter, M. and Rennie, A. (1996); "*Financial Calculus*", Cambridge University Press.

- Black, F. and Scholes, M. (1973); "The pricing of options and corporate liabilities"; *Journal of Political Economy*, **81**, 637 - 659.

- Black, F., Derman, E. and Toy, W. (1990); "A one factor model of interest rates and its applications to treasury bond options"; *Financial Analysts Journal*, **January / February**, 33 - 39.

- Black, F. and Karasinski, P. (1991); "Bond and option pricing when short rates are lognormal"; *Financial Analysts Journal*, **July / August**, 52 - 59.

- Brace, A., Gatarek, D. and Musiela, M. (1997); "The market model of interest rate dynamics"; *Mathematical Finance*, **7(1)**, 127 - 154.

- Cox, J. C., Ingersoll (Jr.), J. E. and Ross, S. A. (1985); "A theory of term structure of interest rates"; *Econometrica*, **53(2)**, 385 - 407.

- Duffie, D. and Kan, R. (1994); "Multi-factor interest rate models"; *Philosophical Transactions of the Royal Society, London, Series A*, **347**, 577 - 586.

- Duffie, D. and Kan, R. (1996); "A yield-factor model of interest rates"; *Mathematical Finance*, **6**, 379 - 406.

- Duffie, D and Singleton, K. (1995); "Modelling term structures of deaultable bonds"; Working Paper, Stanford University.

- Duffie, D., Schroder, M. and Skiadas, C. (1996); " Recursive valuation of defaultable securities and the timing of resolution of uncertainty"; *The Annals of Applied Probability*, **6(4)**, 1075 - 1090.

- Goldys, B., Musiela, M. and Sondermann, D. (1994); "Lognormality of rates and term structure models"; Working Paper, University of New South Wales, Australia.

- Harrison, J.M. and Kreps, D. (1979); “Martingales and arbitrage in multiperiod securities market”; *Journal of Economic Theory*, **20**, 381 - 408.
- Harrison, J.M. and Pliska, S. (1981); “Martingales and stochastic integrals in the theory of continuous trading”; *Stochastic Processes and Their Applications*, **11**, 215 - 260.
- Heath, D., Jarrow, R. and Morton, A. (1992); “Bond pricing and term structure of interest rates : A new methodology for contingent claim valuation ”; *Econometrica*, **60(1)**, 77 - 105.
- Heston, S. L. (1993); “A closed-form solution for options with stochastic volatility with applications to bond and currency options”; *The Review of Financial Studies*, **6(2)**, 327 - 343.
- Hull, J. and White, A. (1987); “ The pricing of options on assets with stochastic volatilities”; *The Journal of Finance*, **XLII (2)**, 281 - 300.
- Hull, J. and White, A. (1990); “Pricing interest rate derivative securities ”; *The Review of Financial Studies*, **3(4)**, 573 - 592.
- Hull, J. and White, A. (1993); “One factor interest rate models and the valuation of interest rate derivative securities”; *Journal of Financial and Quantitative Analysis*, **28 (2)**, 235 - 255.
- Hull, J. and White, A. (Fall, 1994); “Numerical procedures for implementing term structure models - I : Single factor models”; *Journal of Derivatives*, **2(1)**, 7 - 16.
- Hull, J. and White, A. (Winter, 1994); “Numerical procedures for implementing term structure models - II : Two factor models”; *Journal of Derivatives*, **2(2)**, 37 - 48.
- Hull, J. and White, A. (Spring 1996); “Using Hull - White interest rate trees”; *Journal of Derivatives*, 26 - 36.
- Jarrow, R. and Rudd, A. (1982); “ Approximate option valuation for arbitrary stochastic processes”; *Journal of Financial Economics*, **10**, 347 - 369.
- Lando, D (1997); “Modelling bonds and derivatives with default risk”; *Mathematica of Derivative Securities*; ed. Dempster and Pliska, 369 - 393, Cambridge University Press.
- Longstaff, F.A. and Schwartz, E.S. (1992a); “Interest rate volatility and the term structure : a two-factor general equilibrium model”; *Journal of Finance*, **47**, 1259 - 1282.

- Longstaff, F.A. and Schwartz, E.S. (1992b); “A two-factor interest rate model and contingent claims valuation”; *Journal of Fixed Income*, **3**, 16 - 23.
- Merton, Robert C. (1973); “Theory of rational option pricing”; *Bell Journal of Economics and Management Science*, **4**, 141 - 183.
- Rogers, L. C. G. and Shi, Z. (1995); “The value of an Asian Option”; *Journal of Applied Probability*, **32(4)**, 1077 - 1088.
- Rogers, L. C. G. (1995); “ Which model for term structure should one use”; *Mathematical Finance*, ed. Davis, Duffie, Fleming, Shreve, Springer - Verlag, 93 - 115.
- Romano, Marc and Touzi, Nizar (1997); “Contingent claims and market completeness in a stochastic volatility model”; *Mathematical Finance*, **7(4)**, 399 - 412.
- Rubinstein, Mark (1976); “The valuation of uncertain income streams and the pricing of options”; *Bell Journal of Economics*, **7**, 407 - 425.
- Sandermann, K., Sondermann, D. and Miltersen, K. (1994); “Closed form term structure derivatives in a *Heath Jarrow Morton* model with log-normal annual compounded interest rates”; *Proceedings of the Seventh Annual Futures Research Symposium, Bonn*, 145 - 165.
- Stein, E. M. and Stein, J. (1991); “Stock price distributions with stochastic volatility : An analytic approach”; *The Review of Financial Studies*, **4(4)**, 727 - 752.
- Thompson, G. W. P. (1999); “Topics in Mathematical Finance”; Ph.D. thesis, University of Cambridge, Cambridge, U.K.
- Vasicek, O. (1977); “ An equilibrium characterisation of the term structure”; *Journal of Financial Economics*, **5**, 177 - 188.
- Wiggins, J. B. (1987); “Option values under stochastic volatility : Theory and empirical estimates”; *Journal of Financial Economics*, **19**, 351 - 372.
- Willard, G. A. (1996); “Calculating prices and sensitivities for path-independent derivative securities in multifactor models”; Ph.D. thesis, John M. Olin School of Business, Washington University in Saint Louis, Missouri, USA.