On Merton's paradigm for assessing credit risk via option pricing theory

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Merton proposed a paradigm which gives a method to asses Value-at-Risk associated with a loan to a listed company.

Consider a company that has a fixed debt, D due at time T. If the worth of its assets (at time t) is modeled by $A(t)$ (via a stochastic process), then default occurs at time T if $A(T) < D$.

Question of interest:

$$\text{What is } P(A(T) < D) \text{ ?}$$

The process $A(t)$ is not observed as the assets include all tangible and intangible assets.

While $A(t)$ is not observed, we have indirect observations on $A(t)$ via the stock price $S(t)$ of the company. Under the efficient market hypothesis, $S(t)$ incorporates all the publicly available information about the company.

How to model the relationship between $A(t)$ and $S(t)$?

Merton proposed that we view the (consolidated) equity as a call option on the underlying assets with the debt being taken as the striking price.

The rationale: the worth of the stock of the company at time T equals

$$\text{Maximum}((A(T) - D), 0).$$

Compare this to payout from a European call option on the stock $\{S(t)\}$ with striking price K, terminal time T

$$\text{Maximum}((S(T) - K), 0).$$

Thus, the equity is like a call option on the underlying assets and thus if we have a model for the underlying assets (with some parameters) and an appropriate option pricing formula, we can equate the same to observed call option prices (in this case this means observed stock prices) to estimate parameters of the underlying asset and thereby compute default probability as well as Value-at-Risk.

Now, it is common in the literature to take geometric Brownian motion as model for the underlying asset:

$$dA(t) = \mu A(t)dt + \sigma A(t)dW(t)$$

Where μ is the expected continuously compounded return on A , σ is the volatility of firm value (on logrithm scale) and W is a standard Weiner process.

This means

$$A(t) = \exp\left\{\sigma W(t) + (\mu - \frac{1}{2}\sigma^2)t\right\}.$$

Once we have taken geometric Brownian motion as model for underlying assets, one jumps to use the Black-Scholes option pricing formula.

Recall: Price of European call option with terminal time T, Strike rice K on

$$S(t) = \exp\{\sigma W(t) + (\mu - \frac{1}{2}\sigma^2)t\}.$$

where rate of interest is r is given by

$$p = S_0\Phi(a) - K\exp^{-rT}\Phi(b)$$

where

$$a = \frac{\log(S_0) - \log(K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$b = \frac{\log(S_0) - \log(K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Equating the option price coming out of the Black-Scholes formula and the observed price, one can estimate the volatility σ of the underlying assets process, as well as current value of assets.

So the procedure:

Observe the equity E and its volatility σ_E. Let A denote the worth of the assets and σ_A denote its volatility (on log scale). Recall: there is a single debt D payable at time T.

Equations:

$$E = A\Phi(a) - D\exp^{-rT}\Phi(b) \tag{1}$$

where

$$a = \frac{\log(A) - \log(D) + (r + \frac{1}{2}\sigma_A^2)T}{\sigma_A\sqrt{T}}$$

$$b = \frac{\log(A) - \log(D) + (r - \frac{1}{2}\sigma_A^2)T}{\sigma_A\sqrt{T}}$$

It can be shown that

$$E\sigma_E = \Phi(a)A\sigma_A \tag{2}$$

Using data on E and computing σ_E, we solve the two non-linear equations to get A and σ_A.

$$E = A\Phi(a) - D\exp^{-rT}\Phi(b)$$

$$E\sigma_E = \Phi(a)A\sigma_A$$

where

$$a = \frac{\log(A) - \log(D) + (r + \frac{1}{2}{\sigma_A}^2)T}{\sigma_A\sqrt{T}}$$

$$b = \frac{\log(A) - \log(D) + (r - \frac{1}{2}{\sigma_A}^2)T}{\sigma_A\sqrt{T}}$$

Then DfD- distance to default is defined by

$$DfD = \frac{A - D}{A\sigma_A}$$

(recall: volatility of log of A is σ_A.)

The basis for all the derivation has been the assumption that A has log-normal distribution. So it should be relatively simple matter to translate DfD to a probability. Indeed, One should define

$$z = \frac{\log(A) - \log(D)}{\sigma_A}$$

and then the required probability could be read from Normal tables.

Does the model work?

There is a large literature on the subject that shows that as given above the model is inadequate and needs significant fine tuning. In fact, it is recorded that several firms with *distance-to-default* 6 or above have failed (whereas under the normality assumptions, this should be very rare).

Indeed, here are some pages from Moodys-KMV : a Global market leader in the business of risk assessment.

Moodys KMV computes EDF - empirical default frequency based on the data it has about defaults of companies. It presumably computes DfD for each company (using log-normal model /Black-Scholes formula / Merton Paradigm) and then computes empirical default frequency across companies having similar DfD!

If we see that the conclusions from a model do not seem to be valid, instead of only changing last step, we must discard the model and start from scratch!

Let us examine the ingredients:

- **Merton Paradigm: Equity as call option on Assets.**
- **Geometric Brownain motion model for assets.**
- **Black-Scholes option formula**

Let us examine the validity of the third ingredient.

The basis of Black-Scholes option formula is No Arbitrage principle.

Quick recap (for an artificial discrete model):

We will now consider a concrete example and use it to illustrate the ideas that play an important role in option pricing. This is an artificial example. Its role is only to explain the notions such as No Aribitrage, hedging strategy, complete markets, etc.

We avoid technicalities since we are considering a discrete model.

We will be considering an ideal situation with two simplifying assumptions : (1) there are no transaction costs (in buying or selling shares) (2) : the rate of interest on investments is same as that on loans or there is a security called bond which carries a fixed rate of return available for trade.

We will consider a company whose shares are trading at the initial time ($t = 0$) @ Rs S_0 per share. We assume that no trading is allowed in the share for a period of one year, at the end of which the price is Rs. S_1. Again, no trading is allowed for another year when the price becomes S_2. Let us assume that the bonds carry interest at 10% per year.

Let the stochastic model for (S_0, S_1, S_2) be given by

$$P(S_0 = 4000) = 1 \tag{3}$$

$$P(S_1 = 4950) = 0.5 \tag{4}$$

$$P(S_1 = 3850) = 0.5 \tag{5}$$

$$P(S_2 = 9680 \mid S_1 = 4950) = 0.1 \tag{6}$$

$$P(S_2 = 8470 \mid S_1 = 4950) = 0.4 \tag{7}$$

$$P(S_2 = 3630 \mid S_1 = 4950) = 0.5 \tag{8}$$

$$P(S_2 = 6655 \mid S_1 = 3850) = 0.5 \tag{9}$$

$$P(S_2 = 3630 \mid S_1 = 3850) = 0.5 \tag{10}$$

Suppose that also selling in the market is European call option on these shares, with terminal time $T = 2$ years, stike price $K = 6050$. At what price should this option be traded in a market in equilbrium (which means enough buyers and sellers will be there in the market at this price)?

At a first glance it would appear, at least to those familiar with probability theory, that the option price must be the discounted expected return.

In this case, if at the end of two years, S_2 is more than 6050, the gain is $(S_2 - 6050)$ (an invester who has bought the option can buy a share @ Rs. 6050 and sell it at S_2); whereas if $S_2 < 6050$, the gain is 0 (the buyer of the option need not buy the share at all). Thus the gain is

$$\max(S_2 - 6050,\ 0)$$

This gain is due at the end of 2 years. Its worth at time zero (with rate of interest 10%) is

$$\max(S_2 - 6050,\ 0)/1.21$$

Thus the expected gain is

$$g = E(\max(S_2 - 6050,\ 0)/1.21,)$$

Here, $P(S_2 = 9680) = .05$, $P(S_2 = 8470) = .2$, $P(S_2 = 6655) = .25$, $P(S_2 = 3630) = .5$.

This leads to $g = 675$.

Can the price of the option be Rs. 675?

Suppose that options are trading @ 675 so there are buyers as well as sellers at this price.

An investor A decides to buy 100 options by investing Rs. 67500.

Another investor B also decides to invest the same amount Rs. $67500 (= x_0)$ at time 0; buy $\pi_0 = 75$ shares @ Rs. 4000 by borrowing the shortfall (short selling the bond).

At the end of the year, if the price is $S_1 = 4950$, he sells 10 shares to bring down his holding to $\pi_{11} = 65$, using the proceeds to settle part of this loan. If $S_1 = 3850$, he sells 50 shares to bring his holding to $\pi_{12} = 25$, again paying off loan with the money received.

Denoting the bonds held by the investor at time 0 by ξ_0 and at time 1 by ξ_{11} if $S_1 = 4950$, ξ_{12} if $S_1 = 3850$ (negative ξ means loan), ξ's are determined by

$$\xi_0 = x_0 - \pi_0 \times 4000 \tag{11}$$

$$\xi_{11} = (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \tag{12}$$

$$\xi_{12} = (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12}. \tag{13}$$

For the invester B, $x_0 = 67500$, $\pi_0 = 75$, $\pi_{11} = 65$, $\pi_{12} = 25$ and these equations give $\xi_0 = -232500$, $\xi_{11} = -206250$, $\xi_{12} = -63250$.

Before proceeding further, let us note that a trading strategy is determined by $x_0, \pi_0, \pi_{11}, \pi_{12}$ which in tern determine $\xi_0, \xi_{11}, \xi_{12}$.

The table given below shows the net worth of the holdings of A, B in each of the five possible outcomes of (S_1, S_2): (A's assets are 100 options and no liabilities; B's assets are $\pi_{11}(\pi_{12})$ shares and a deposit of $\xi_{11}(\xi_{12})$ made at time 1 if $S_1 = 4950(S_1 = 3850)$).

Table 1 : Net worth of the holdings.

Outcome: $(S_1, S_2) =$	A	B
(4950, 9680)	3,63,000	4,02,325
(4950, 8470)	2,42,000	3,23,675
(4950, 3630)	0	9,075
(3850, 6655)	60,500	96,800
(3850, 3630)	0	21,175

Note that while both A, B made the same initial investment, namely 67500, but B has done better than A in each possible outcome of the stock prices. So whatever A was buying is ourpriced. Thus option price must be less than 675!

To see this more clearly, assume that options are priced at Rs. 675 and there are enough buyers and sellers at this price.

An investor C devises a strategy as follows: Sell 100 options @ 675 per option to collect Rs. 67500 and then follow the strategy of B : $x_0 = 67500$, $\pi_0 = 75$, $\pi_{11} = 65$, $\pi_{12} = 25$.

Then the net worth of C's holding is given by the difference of the 3rd column and 2nd column in Table 1: Thus Net Worth of C's holdings is given by

Outcome (S_1, S_2)	C
(4950, 9680)	39,325
(4950, 8470)	81,675
(4950, 3630)	9,075
(3850, 6655)	26,300
(3850, 3630)	21,175

Thus, C would make a profit in each of the five outcomes without making any initial investment. Clearly, every investor would like to follow this strategy and make money without taking any risk. This in turns would disturb the equilibrium as soon there would be no buyers for the option. Thus the equilibrium price has to be less than Rs 675.

The strategy of C is an example of an Arbitrage opportunity.

Arbitrage opportunity is a strategy that involves no initial investment and for which the net worth of holdings (at some time in future) is non-negative for each possible outcome and strictly positive for at least one possible outcome.

As explained above, if an Arbitrage opportunity exists, it would disturb the equlibrium as all investors would like to replicate the same. Thus, in a market in equilibrium, Arbitrage opportunities do not exist. This is known as the principle of No Arbitrage abbriviated as NA.

In this example we can conclude

$$NA \Rightarrow p < 675$$

(where p is the price of the option).

Now, let us consider another invester D's strategy : $x_0 = 30000$, $\pi_0 = 50$, $\pi_{11} = 45$, $\pi_{12} = 15$. The equations

$$\begin{aligned} \xi_0 &= x_0 - \pi_0 \times 4000 \\ \xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\ \xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12}. \end{aligned}$$

yield : $\xi_0 = -1,70,000\, \xi_{11} = -162,250, \xi_{12} = -52,250.$ In each of the outcomes, the net worth of D's holding is given by

Table 2 : Worth of D's holding

Outcome $(S_1, S_0) =$	Worth of D's holding
(4950, 9680)	$9680\ \pi_{11} + 1.1\xi_{11} = 2,57,125$
(4950, 8470)	$8470\ \pi_{11} + 1.1\xi_{11} = 2,02,675$
(4950, 3630)	$3630\pi_{11} + 1.1\xi_{11} = -15,125$
(3850, 6655)	$6655\ \pi_{12} + 1.1\xi_{12} = 42,350$
(3850, 3630)	$3630\ \pi_{12} + 1.1\xi_{12} = -3,025$

Table 3 : Worth of 100 options

Outcome $(S_1, S_0) =$	Worth of 100 options
(4950, 9680)	(9680 -6050)*100 = 3,63,000
(4950, 8470)	(8470 -6050)*100 = 2,42,000
(4950, 3630)	0
(3850, 6655)	(6655 -6050)*100 = 60,500
(3850, 3630)	0

Note that for each outcome, D's holdings are worth less than the worth of100 options. As a consequence, the price of 100 options is more than the investment at time zero that is needed for D's strategy, namely 30000. Thus, one option is worth more than 300.

It indeed option price is 300 or less, then the strategy consisting of buying 100 options and $\pi_0 = -50$, $\pi_{11} = -45$, $\pi_{12} = -15$ would be an arbitrage opportunity (note that the π's are (-1) times the corresponding π in D's strategy. Thus

$$NA \Rightarrow p > 300.$$

Can we narrow the interval $(300, 675)$ for the option price any further?

We need to list all possible trading strategies an investtor might follow.

It can be seen that a trading strategy consists of $(x_0, \pi_0, \pi_{11}, \pi_{12})$. Then $\xi_0, \xi_{11}, \xi_{12}$ are determined by

$$\begin{aligned} \xi_0 &= x_0 - \pi_0 \times 4000 \\ \xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\ \xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12}. \end{aligned}$$

It is clear from the preceeding discussion that if for a trading strategy $(x_0, \pi_0, \pi_{11}, \pi_{12})$ (with $\xi_0, \xi_{11}, \xi_{12}$ determined by equations on preceeding page)

$$9680\pi_{11} + 1.1\xi_{11} \geq 3,63,000 \quad (14)$$

$$8470\pi_{11} + 1.1\xi_{11} \geq 2,42,000 \quad (15)$$

$$3630\pi_{11} + 1.1\xi_{11} \geq 0 \quad (16)$$

$$6655\pi_{12} + 1.1\xi_{12} \geq 60,500 \quad (17)$$

$$3630\pi_{12} + 1.1\xi_{12} \geq 0. \quad (18)$$

then $p \leq \frac{x_0}{100}$. Further, If any of the inequality $(12) - (16)$ is a strict inequality, then $p < \frac{x_0}{100}$.

Likewise if $(x_0, \pi_0, \pi_{11}, \pi_{12})$, (with $\xi_0, \xi_{11}, \xi_{12}$) determined as before) satisfy

$$9680\pi_{11} + 1.1\xi_{11} \leq 3,63,000 \tag{19}$$

$$8470\pi_{11} + 1.1\xi_{11} \leq 2,42,000 \tag{20}$$

$$3630\pi_{11} + 1.1\xi_{11} \leq 0 \tag{21}$$

$$6655\pi_{12} + 1.1\xi_{12} \leq 60,500 \tag{22}$$

$$3630\pi_{12} + 1.1\xi_{12} \leq 0. \tag{23}$$

then $p \geq \frac{x_0}{100}$ and if any of (17)-(21) is a strict inequality, $p > \frac{x_0}{100}$.

Thus, the optimum value x^+ for the linear programming problem (I)

$$\text{minimize } \ x_0$$

$$\text{subject to } (9) - (16)$$

is an upper bound for 100p; and if for the optimum solution, even one of (12)-(16) is strict inequality then $100p < x^+$.

(Here the variables $x_0, \pi_0, \pi_{11}, \pi_{12}, \xi_0, \xi_{11}, \xi_{12}$ are unrestrained.)

The optimum value x^- for the linear programming problem (II)

$$\textbf{maximize } x_0$$

$$\textbf{subject to } (9) - (11) \textbf{ and } (17) - (21)$$

is a lower bound for $100p$; and if for the optimum solution, even one of (17)-(21) is strict inequality then $x^- < 100p$.

The optimum solution to the problem (I) is $x_0 = 50,000,$ with (13) being a strict inequality. Thus $100p < 50000$.

The optimum solution to the problem (II) is $x_0 = 42,500$ with (17) being a strict inequality. Thus $100p > 42500$ we thus conclude that

$$425 < p < 500.$$

Let us explore an alternate scenerio. Suppose that instead of $P(S_2 = 9680 \mid S_1 = 4950) = 0.1$ and $P(S_2 = 8470 \mid S_1 = 4950) = 0.4$ one has

$$P(S_2 = 9680 \mid S_1 = 4950) = 0.5$$

In this case, the upper bound x^+ is solution to the problem III:

$$\text{minimize } x_0$$

subject to (9), (10), (11), (12), (14), (15), (16). (One constrain is removed from problem I).

problem III:

$$\text{minimize} \quad x_0$$

Subject to

$$\begin{aligned}
\xi_0 &= x_0 - \pi_0 \times 4000 \\
\xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\
\xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12} \\
9680\pi_{11} + 1.1\xi_{11} &\geq 3,63,000 \\
3630\pi_{11} + 1.1\xi_{11} &\geq 0 \\
6655\pi_{12} + 1.1\xi_{12} &\geq 60,500 \\
3630\pi_{12} + 1.1\xi_{12} &\geq 0.
\end{aligned}$$

In this case the optimum is attained by the same strategy that optimised problem I, $x_0 = 50,000$, $\pi_0 = 80$, $\pi_{11} = 60$, $\pi_{12} = 20$, $\xi_0 = -270000$, $\xi_{11} = -198000$ and $\xi_{12} = -66000$.

In this case, it can be seen that all the inequalities are equalities.

The lower bound x^- is the solution to problem IV:

$$\text{maximize } \; x_0$$

subject to (9), (10), (11), (17), (19), (20), (21). (same constraines as problem III with $\geq$ replaced by $\leq$). Thus the optimum solution is the same as the one for problem III with $x_0 = 50000$.

Thus $x^- = 50000, x^+ = 50000$. It follows that the option price must be 500.

In both problems III and IV, all the constrains are equalities for the optimum solution. Thus, with intial investment $x_0 = 50000$, there exists a strategy : $\pi_0 = 80, \pi_{11} = 60$ and $\pi_{12} = 20$ for which the net worth of the holdings at the end of 2 years in exactly the same as the worth of 100 options for all possible outcomes of the share prices. Such a strategy is called a hedging strategy for the options.

Likewise, if instead of $P(S_2 = 9680 \mid S_1 = 4950) = 0.1$ and
$P(S_2 = 8470 \mid S_1 = 4950) = 0.4$ one has

$$P(S_2 = 8470 \mid S_1 = 4950) = 0.5$$

then again, the upper and lower bounds agree and a hedging strategy exists : $x_0 = 42,500$, $\pi_0 = 65$, $\pi_{11} = 50$, $\pi_{12} = -66,000$. Thus in this case, the option price is $p = 425$.

Note that in the fomulation of the Linear Programming problems, the probabilites of the outcomes did not appear at all.

For the model

$$\begin{aligned}
P(S_0 = 4000) &= 1 && (24)\\
P(S_1 = 4950) &= p_1 && (25)\\
P(S_1 = 3850) &= (1 - p_1) && (26)\\
P(S_2 = 9680 \mid S_1 = 4950) &= p_{11} && (27)\\
P(S_2 = 3630 \mid S_1 = 4950) &= (1 - p_{11}) && (28)\\
P(S_2 = 6655 \mid S_1 = 3850) &= p_{12} && (29)\\
P(S_2 = 3630 \mid S_1 = 3850) &= (1 - p_{12}) && (30)
\end{aligned}$$

then the option price is 500 whatever be p_1, p_{11}, p_{12} as long as $p_1, p_{11}, p_{12} \in (0, 1)$.

For the model

$$
\begin{aligned}
P(S_0 = 4000) &= 1 && (31)\\
P(S_1 = 4950) &= p_1 && (32)\\
P(S_1 = 3850) &= (1 - p_1) && (33)\\
P(S_2 = 8470 \mid S_1 = 4950) &= p_{11} && (34)\\
P(S_2 = 3630 \mid S_1 = 4950) &= (1 - p_{11}) && (35)\\
P(S_2 = 6655 \mid S_1 = 3850) &= p_{12} && (36)\\
P(S_2 = 3630 \mid S_1 = 3850) &= (1 - p_{12}) && (37)
\end{aligned}
$$

then the option price is 425 whatever be p_1, p_{11}, p_{12} as long as $p_1, p_{11}, p_{12} \in (0, 1)$.

We have seen that the option price did not depend upon the probabilities of the various outcomes, but it depended upon the set of possible outcomes. This is so because we are matching the returns for each outcome and so it doesn't matter as to with what probability an outcome occurs. Thus expected value of the (discounted) gain can be more or less than the option price.

Same phenomenon is seen for the Geometric Brownian motion model: the Option price given by Black Scholes formula does not depend on the drift parameter but depends only on the volatility.

So that went wrong with the reasoning price = expected gain. The reason is that along with the option, another comodity, namely the shares of the same company, one also available in the market and of course, the shares are correlated with the option - and thus we need to valuate the option in terms of a basket consisting of money and shares. If the shares of the company were not being traded but only the options were being sold, then perhaps the expected (discounted) gain can be taken as the price (if the utility is taken as linear).

Coming back to Merton's paradigm and Credit risk, since the underlying assets are not being traded, one should not use Black-Scholes formula which is based on No Arbitrage argument.

We should choose a utility function and then the expected (discounted) gain can be taken as the price of the option.