Credit Risk Modeling

References:

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- **Credit Risk** by Duffie and Singleton, *New Age International Publishers*, 2005
- **Credit Risk Modeling and Valuation: An Introduction**, by Kay Giesecke, http://www.stanford.edu/dept/MSandE/people/faculty/giesecke/introduction.pdf, 2004
- Options, Futures, and Other Derivatives, Hull, *Prentice Hall India*

The Basics of Credit Risk Management

- **Loss Variable** $\tilde{L} = EAD \times SEV \times L$
- **Exposure at Default (EAD)** $= OUTST + \gamma \, COMM$

  Basel Committee on banking supervision: 75% of off-balance sheet amount. **Ex.** Committed line of one billion, current outstandings 600 million, $EAD = 600 + 75\% \times 400 = 900$.
- **Loss Given Default (LGD)** $= E[SEV]$
  - Quality of collateral
  - Seniority of claim
- $L = 1_D$, $P(D) = DP$: Probability of Default
  - Calibration from market data, Ex. KMV Corp.
  - Calibration from ratings, Ex. Moodys, S & P, Fitch, CRISIL : Statistical tools + Soft factors
  - Ratings $\rightarrow$ DP: Fit "curve" to RR *vs* average DP plot

- **Expected Loss (EL)** $E[\tilde{L}] = EAD \times LGD \times DP$
- **Unexpected Loss (UL)** $= \sqrt{V(\tilde{L})}$

$$= EAD \times \sqrt{V(SEV) \times DP^2 + LGD^2 \times DP(1 - DP)}$$

Portfolio: $\tilde{L}_{PF} = \sum_{i=1}^{m} EAD_i \times SEV_i \times L_i$

- $EL_{PF} = \sum_{i=1}^{m} EAD_i \times LGD_i \times DP_i$
- $UL_{PF}^2 = \sum_{i,j=1}^{m} EAD_i \times EAD_j \times Cov(SEV_i \times L_i, SEV_j \times L_j)$
- **Constant Severities**

$$= \sum_{i,j=1}^{m} EAD_i \times EAD_j \times LGD_i \times LGD_j \times \sqrt{DP_i(1 - DP_i)DP_j(1 - DP_j)}\, \rho_{ij}$$

- **Value at Risk (VaR):** q_α

$$q_\alpha : \inf\{q > 0 : P[\tilde{L}_{PF} \leq q] \geq \alpha\}$$

- **Economic Capital (EC_α)** $= q_\alpha - EL_{PF}$
- **Expected Shortfall:**

$$TCE_\alpha = E[\tilde{L}_{PF} \mid \tilde{L}_{PF \geq q_\alpha}]$$

- **Economic Capital based on Shortfall Risk:** $TCE_\alpha - EL_{PF}$
- **Loss Distribution**
  - Monte-Carlo Simulation
  - Analytical Approximation: Credit Risk$^+$
- **Today's Industry Models**
  - Credit Metrics and KMV-Model
  - Credit Risk$^+$
  - CreditPortfolio View
  - Dynamic Intensity Models

Credit Metrics and the KMV-Model

- **Asset Price Process:** $A_t^{(i)}$
- **Valuation Horizon:** T

$$L_i = 1_{\{A_T^{(i)} < C_i\}} \sim B(1; P(A_T^{(i)} < C_i))$$

$$r_i = \log\left(\frac{A_T^{(i)}}{A_0^{(i)}}\right) = R_i \Phi_i + \epsilon_i, \qquad i = 1, 2, \ldots, m$$

- **Firm's composite factor** Φ_i is a superposition of systematic influences (industry and country indices)
- ϵ_i : **Firm specific or idiosyncratic part**
- $R_i^2 =$ portion of the volatility in r_i explained by volatility in Φ_i

$$r_i \sim N(0,1); \qquad \Phi_i \sim N(0,1); \qquad \epsilon_i \sim N(0, 1 - R_i^2)$$

Global Correlation Model

Industry and Country Indices: $\Psi_j = \sum_{n=1}^{N} b_{j,n}\Gamma_n + \delta_n, \qquad j = 1, \ldots, J$

Independent Global Facors: Γ_n

$$\Phi_i = \sum_{j=1}^{J} w_{ij}\Psi_j$$

$$\begin{aligned}
L_i &= 1_{\{r_i<c_i\}} \sim B(1, P(r_i < c_i)) \\
r_i < c_i &\equiv \epsilon_i < c_i - R_i\Phi_i \\
p_i = P(r_i < c_i) &\Rightarrow c_i = N^{-1}(p_i) \\
p_i(\Phi_i) &= N\left[\frac{N^{-1}(p_i) - R_i\Phi_i}{\sqrt{1-R_i^2}}\right] \\
p_i(\Phi_i) &= N\left[\frac{N^{-1}(p_i) - R_i\sum_{j=1}^{J} w_{ij}\Psi_j}{\sqrt{1-R_i^2}}\right]
\end{aligned}$$

- **Simulate** a realization of Ψ_j $\rightarrow$ **Simulate** realization of L_i $\rightarrow$ **One** realization of the loss
- **Loss Distribution**

- KMV tool *GCorr* computes asset correlations
- KMV provides the weights and asset correlations to its customers
- Can use these correlations with heavy tailed copulas to obtain stronger tail dependencies:
  - F_n Univariate $t-$distribution with n d.f.
  - $F_{n,\Gamma}$ Multivariate $t-$distribution with n d.f. and correlation matrix Γ.
  - $C_{n,\Gamma}(u_1, \ldots, u_m) = F_{n,\Gamma}(F_n^{-1}(u_1), \ldots, F_n^{-1}(u_m))$
  - $\Phi_{n,\Gamma}(x_1, \ldots, x_m) = C_{n,\Gamma}(N(x_1), \ldots, N(x_m))$

Two Differences Between KMV-Model and Credit Metrics

- Credit Metrics uses equity price correlations, whereas KMV carries out the complicated translation from equity and market information to asset values
- Credit Metrics uses indices referring to a combination of some industry in some particular country, whereas KMV considers industries and countries separately

CreditPortfolio View

- **Default and rating migrations** are subject to random fluctuations that depend on the **economic cycle**
- **Unconditional migration matrix** $\bar{M} = (\bar{m}_{ij}), \qquad i, j = 1, \ldots, K$: rating categories
- $\bar{m}_{iK}$: one year historic probability of default in rating category i
- S risk segments that react differently to the economic conditions

1. Simulate a segment specific conditional default probability p_s, $s = 1, \dots, S$.
   Aggregated Second Level Scenario

2. Define the **risk index**

$$r_s = \frac{p_s}{\bar{p}_s}$$

   $\bar{p}_s$ unconditional default probability of segment s

3. **Conditional migration matrix** $M^{(s)}$:

$$m_{ij}^s = \alpha_{ij}(r_s - 1) + \bar{m}_{ij}$$

   The **shift matrix** (α_{ij}) satisfying $\sum_j \alpha_{ij} = 0$ must be calibrated by the user

$$\alpha_{ij} \geq 0, \qquad i < j, \qquad \alpha_{ij} \leq 0, \qquad i > j$$

   $M^{(s)}$ applies to all obligors in segment s. Some entries may turn out to be negative. Set equal to 0 and renormalize.

$$m_{ij}^s = \alpha_{ij}(r_s - 1) + \bar{m}_{ij}$$

- $r_s < 1:$ expanding economy, lower possibility of downgrades and higher number of upgrades
- $r_s = 1:$ average macroeconomic scenario
- $r_s > 1:$ recession, downgrades more likely

For each realization of the default probabilities, simulate the defaults and loss. Repeat simulation several times to generate the loss distribution.

CPV supports two modes of calibration:

- **CPV Macro:** default and rating migrations are explained by a macroeconomic regression model. Macroeconomic model is calibrated by means of times series of empirical data.

$$Y_{s,t} = w_{s,0} + \sum_{j=1}^{M} w_{s,j} X_{s,j,t} + \epsilon_{s,t}, \qquad \epsilon_{s,t} \sim N(0, \sigma_{s,t}^2)$$

$$X_{s,j,t} = \theta_{j,0} + \sum_{k=1}^{t_0} \theta_{j,k} X_{s,j,t-k} + \eta_{s,j,t}$$

$$p_{s,t} = \frac{1}{1 + \exp(-y_{s,t})}$$

- **CPV Direct:** p_s drawn from a gamma distribution. Need only to calibrate the two parameters of the gamma distribution for each s. p_s can turn out to be larger than 1.

Dynamic Intensity Models

- **Basic Affine or Intensity Process**

$$d\lambda(t) \;=\; \kappa(\theta - \lambda(t))\,dt + \sigma\sqrt{\lambda(t)}\,dB(t) + \Delta J(t)$$

- $J(t)$: pure jump process independent of the BM $B(t)$ with jumps arriving according to a Poisson process with rate ℓ and jump sizes $\Delta J(t) \sim \exp(\mu)$
- $\kappa =$ mean-reversion rate; $\sigma =$ diffusive volatility; $\bar{m} = \theta + \ell\mu/\kappa$ long-run mean
- **Unconditional Default Probability** $q(t) = E\left[e^{-\int_0^t \lambda(u)\,du}\right]$
- **Correlated defaults** $\lambda_i = X_c + X_i$
  X_c, X_i basic affine processes with parameters $(\kappa, \theta_c, \sigma, \mu, \ell_c)$ and $(\kappa, \theta_i, \sigma, \mu, \ell_i)$ representing the **common performance aspects** and the **idiosyncratic risk**
- λ_i : basic affine process with parameters $(\kappa, \theta_c + \theta_i, \sigma, \mu, \ell_c + \ell_i)$

$$dX_p(t) = \kappa(\theta_p - X_p(t))\,dt + \sigma\sqrt{X_p(t)}\,dB^p(t) + \Delta J^p(t), \qquad p = c, i$$

$$\begin{aligned} d(X_c + X_i)(t) &= \kappa((\theta_c + \theta_i) - (X_c + X_i)(t))\,dt \\ &\quad + \sigma(\sqrt{X_c(t)}\,dB^c(t) + \sqrt{X_i(t)}\,dB^i(t)) \; + \; \Delta(J^c + J^i)(t) \end{aligned}$$

$$dW_t = \sqrt{\frac{X_t^c}{X_t^c + X_t^i}}\,dB^c(t) + \sqrt{\frac{X_t^i}{X_t^c + X_t^i}}\,dB^i(t)$$

$$\begin{aligned} d(X_c + X_i)(t) &= \kappa((\theta_c + \theta_i) - (X_c + X_i)(t))\,dt + \sigma\sqrt{(X^c + X^i)(t)}\,dW(t) \\ &\quad + \Delta(J^c + J^i)(t) \end{aligned}$$

Conditioned on a realization of $\lambda_i(t),\ 0 \le t \le T$, the default time of obligor i is the first arrival in a non-homogenous Poisson process with rate $\lambda_i(\cdot)$

Conditional Probability of No Default $= \exp(-\int_0^T \lambda(s)ds)$

The Credit Risk^{+} Model

- Introduced in 1997 by CSFB
- **Actuarial Model**
- One of the most widely used credit portfolio models
- **Advantages:**
  - Loss Distribution can be computed analytically
  - Requires no Monte-Carlo Simulations
  - Explicit Formulas for Obligor Risk Contributions
- Numerically stable computational procedure (Giese, 2003)

The Standard CR$^+$ Model

- Choose a suitable basic unit of currency ΔL
- Adjusted exposure of obligor A, $\nu_A = \lfloor E_A/\Delta L \rfloor$
- Smaller number of **Exposure Bands**
- p_A expected default probability
- The total portfolio loss $L = \sum_A \nu_A \, N_A$.
- $N_A \in Z_+$ Default of obligor A
- PGF of the Loss Distribution $G(z) = \sum_{n=0}^{\infty} P(L = n) \, z^n$.

- **Apportion Obligor Risk** among K **Sectors (Industry, Country)** by choosing numbers g_k^A such that $\sum_{k=1}^{K} g_k^A = 1$.
- **Sectoral Default Rates** represented by non-negative variables γ_k

$$E(\gamma_k) = 1, \qquad Cov(\gamma_k, \gamma_l) = \sigma_{kl} \qquad k = 1, ..., K.$$

- Standard CR$^+$ Model assumes $\sigma_{kl} = 0, \qquad k \neq l$
- Relating Obligor default rates to sectoral default rates

$$p_A(\gamma) = p_A \sum_{k=1}^{K} g_k^A \gamma_k,$$

- $p_A(\gamma)$ default rate conditional on the sector default rates $\gamma = (\gamma_1, \ldots, \gamma_K)$.
- **Specific Sector:** $\gamma_0 \equiv 1$. **Captures Idosyncratic Risk**

- Conditional on γ default variables N_A assumed to be independent Poisson
- **Main Criticism of CR$^+$ Model. Not Fair**
  - $p_A = 0.1 \quad \rightarrow \quad P(N_A = 2) = 0.0045$
  - Need not assume N_A is Poisson, but Bernoulli
- Conditional PGF

$$G_\gamma(z) = \exp(\sum_{k=1}^{K} \gamma_k \, P_k(z)),$$

$$\begin{aligned} P_k(z) &= \sum_A g_k^A p_A (z^{\nu_A} - 1) \\ &= \sum_{m=1}^{M} \left(\sum_{\{\nu_A = m\}} g_k^A p_A \right) (z^m - 1) \end{aligned}$$

- Number of defaults in any exposure band is Poisson

- **Default correlation** between obligors arise only through their dependence on the common set of sector default rates
- **Unconditional PGF of Loss Distribution**

$$G(z) = E^{\gamma}(\exp(\sum_{k=1}^{K} \gamma_k \ P_k(z))) = M_{\gamma}(T = P(z))$$

- **MGF of Univariate Gamma Distribution** with **Mean 1 and Variance** σ_{kk} is $(1 - \sigma_{kk} t_k)^{-\frac{1}{\sigma_{kk}}}$

$$G^{CR+}(z) = \exp\left(-\sum_{k=1}^{K} \frac{1}{\sigma_{kk}} \log(1 - \sigma_{kk} P_k(z))\right)$$

- **Giese(2003): Numerically Stable Fast Recursion Scheme**

The Compound Gamma CR$^+$ Model (Giese, 2003)

- Introduce sectoral correlations via common scaling factor S
- Conditional on S γ_K is Gamma distributed with shape parameter $\hat{\alpha}_k(S) = S\alpha_k, \qquad \alpha_k > 0$, and constant scale parameter β_k.
- S follows Gamma with $E[S] = 1$ and $Var(S) = \hat{\sigma}^2$.
- $1 = E\gamma_k = \alpha_k \beta_k$
- $\sigma_{kl} = \delta_{kl}\beta_k + \hat{\sigma}^2$
- **Uniform Level of Cross Covariance** $\Rightarrow$ **Distortion of Correlation Structure**.

$$M_\gamma^{CG}(T) = \exp\left\{ -\frac{1}{\hat{\sigma}^2} \log \left[1 + \hat{\sigma}^2 \sum_{k=1}^{K} \frac{1}{\beta_k} \log(1 - \beta_k t_k) \right] \right\}$$

- **Calibration Problems**

The Two Stage CR$^+$ Model (SKI, AD)

- $Y_1, \ldots, Y_N$: **Common set of Uncorrelated Risk Drivers**

$$\gamma_k = \sum_{i=1}^{N} a_{ki}\, Y_i$$

- $Y_i \sim$ Gamma with mean 1 and variance V_{ii}
- **Principle Component Analysis of Macroeconomic Variables**
- **Factor Analysis**

$$
\begin{aligned}
G(z) &= E^{\gamma}(exp(\sum_{k=1}^{K} \gamma_k \; P_k(z))) = E^{Y}(exp(\sum_{k=1}^{K}(\sum_{i=1}^{N} a_{ki} \; Y_i) \; P_k(z))) \\
&= E^{Y}(exp(\sum_{i=1}^{N}(\sum_{k=1}^{K} a_{ki} \; P_k(z)) \; Y_i)) \\
&= E^{Y}(exp(\sum_{i=1}^{N} Y_i \; Q_i(z))) = M_Y(T = Q(z))
\end{aligned}
$$

$$
Q_i(z) = \sum_{k=1}^{K} a_{ki} \; P_k(z)
$$

$$
G(z) = \exp\left\{ -\sum_{i=1}^{N} \frac{1}{\sigma_{ii}} \log(1 - \sigma_{ii} Q_i(z)) \right\}
$$

Model Comparison

- Giese (2003) had pointed out deficiencies in the earlier attempt to incorporate correlations due to Burgisser *et al*
- We compare the compound gamma and the two stage gamma models
- Test portfolio made up of $K = 12$ sectors, each containing 3,000 obligors
- Obligors in sectors 1 to 10 belong in equal parts to one of three classes with adjusted exposures $E_1 = 1$, $E_2 = 2.5$, and $E_3 = 5$ monetary units and respective default probabilities $p_1 = 5.5\%$, $p_2 = .8\%$, $p_3 = .2\%$.
- For the three obligor classes in sectors 11 and 12, we assume the same default rates but twice as large exposures ($E_1 = 2$, $E_2 = 5$, $E_3 = 10$)
- $\sigma_{kk} = 0.04, k = 1, \ldots, 10 \qquad \sigma_{11,11} = \sigma_{12,12} = 0.49$
- Correlation between sectors 11 and 12 is 0.5 whereas correlations between all the other sectors are set equal to 0
- $\gamma_i = Y_i$, $i = 1, \ldots, 11$, $\gamma_{12} = 0.5(Y_{11} + Y_{12})$, with $Var(Y_{11}) = 0.49$ $Var(Y_{12}) = 1.47$, and Var(Y_i) = 0.04 for $i = 1, \ldots, 10$

	Standard CR+	Compound Gamma Model	Two-Stage Model
Expected Loss	1%	1%	1%
Std Deviation	0.15%	0.17%	0.17%
99% Quantile	1.42%	1.48%	1.53%
99.5% Quantile	1.49%	1.55%	1.62%
99.9% Quantile	1.64%	1.71%	1.84%

Table 1: Comparison of the loss distributions from the standard CR^+, compound gamma and two stage models for the test portfolio in example 1. All loss statistics are quoted as percentage of the total adjusted exposure.

- $\hat{\sigma}^2 = 0.013$. This translates to a much lower correlation of 0.0265 (instead of 0.5) between sectors 11 and 12

Risk Contributions

- **Value at Risk** *VAR* ℓ_q
- **Economic Capital** $\ell_q - E[L]$
- **Expected Shortfall** $E[L|L \geq \ell_q]$
- **Quantile Contribution** QC_A

$$QC_A = \nu_A E(N_A|L = \ell_q) = p_A \nu_A \frac{\sum_{k=1}^{K} g_A^k D^{(\ell_q - \nu_A)} G_k(z)}{D^{(\ell_q)} G(z)}$$

$$G_k(z) = \frac{\partial}{\partial t_k} M_Y(T = Q(z))$$

$$\begin{aligned} G_k(z) &= G(z) \sum_{i=1}^{N} \frac{a_{ki}}{1 - \sigma_{ii} Q_i(z)} \\ &= G(z) \, (\sum_{i=1}^{N} a_{k,i} exp(-log(1 - \sigma_{ii} Q_i(z)))). \end{aligned}$$

Sector	CR^+	Compound Gamma Model	Two-Stage Model
$1, 2$	24.25%	21.71%	27.42%
$3, \ldots, 10$	0.37%	1.64 %	0.2 %
$11, 12$	24.25 %	21.71%	21.59 %

Table 2: Aggregated risk contributions (in percent). Contributions to the loss variance for the risk-adjusted breakdown of VaR (on a 99.9% confidence level).

- Compound gamma model can't pick up differing correlations among sectors that are otherwise similar.